

Chapter 2

Asymptotic Estimates for Integrals

Mechanical problems can be described by differential equations, the solutions of which often cannot be expressed by elementary functions, but have an integral representation. In this chapter, we discuss asymptotic estimates of integrals of the form

$$F(\mu) = \int_{a(\mu)}^{b(\mu)} f(z, \mu) dz, \quad (2.1)$$

where μ is a small parameter and $z \in \mathbb{C}$. Sometimes, it is more convenient to use a large parameter λ and consider the integral

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} f(z, \lambda) dz. \quad (2.2)$$

With obvious adjustments, all statements which are valid for integrals hold also for (2.2). In the general case, asymptotic estimates are given as $\mu \rightarrow 0$ and $\lambda \rightarrow \infty$.

2.1 Series Expansion of Integrands

Suppose the integrand in (2.1) can be expanded in an asymptotic series of the form

$$f(z, \mu) = \sum_{n=0}^N a_n(z) \mu^{p_n} + O(\mu^{p_{N+1}}), \quad p_0 < p_1 < \cdots < p_N < \cdots < +\infty, \quad (2.1.1)$$

uniformly in z over the interval $[a(\mu), b(\mu)]$. Then this series can be integrated term-by-term and

$$F(\mu) = \sum_{n=0}^N b_n(\mu) \mu^{p_n} + O([b(\mu) - a(\mu)] \mu^{p_{N+1}}), \quad b_n(\mu) = \int_{a(\mu)}^{b(\mu)} a_n(z) dz. \quad (2.1.2)$$

The form of expansion (2.1.2) is simplest if the integrand can be expanded in Taylor series.

Example 1

Find the asymptotic expansion of the function

$$F(\mu) = \int_0^1 \frac{\sin \mu x}{x} dx, \quad x \in \mathbb{R}.$$

The function $\sin \mu x$ has the asymptotic expansion

$$\sin \mu x = \sum_{n=0}^N \frac{(-1)^n x^{2n+1}}{(2n+1)!} \mu^{2n+1} + O(\mu^{2N+3}),$$

which is uniform over the interval of integration $[0, 1]$. Therefore,

$$\begin{aligned} F(\mu) &= \sum_{n=0}^N \int_0^1 \left[\frac{(-1)^n x^{2n}}{(2n+1)!} \mu^{2n+1} + O(\mu^{2N+3}) \right] dx \\ &= \sum_{n=0}^N \frac{(-1)^n}{(2n+1)!(2n+1)} \mu^{2n+1} + O(\mu^{2N+3}). \end{aligned}$$

When the limits of the interval of integration depend on a small parameter, the change of variable

$$t = \frac{z - a(\mu)}{b(\mu) - a(\mu)} \quad (2.1.3)$$

transforms integral (2.1) to an integral of the form

$$F(\mu) = \int_0^1 f_1(t, \mu) dt.$$

A similar problem is solved for integrals with large parameters.

Example 2

Find an asymptotic expansion for the function

$$F(\mu) = \int_0^\mu x^{-3/4} \exp(-x) dx.$$

Substitution (2.1.3) transforms the given integral into

$$F(\mu) = \mu^{1/4} \int_0^1 t^{-3/4} \exp(-\mu t) dt.$$

The function $\exp(-\mu t)$ has the asymptotic representation

$$\exp(-\mu t) = \sum_{n=0}^N \frac{(-1)^n}{n!} \mu^n t^n + O(\mu^{N+1}),$$

that is uniform for all t in the interval of integration $[0, 1]$. Therefore,

$$\begin{aligned} F(\mu) &= \mu^{1/4} \sum_{n=0}^N \int_0^1 \left[\frac{(-1)^n}{n!} t^{n-3/4} \mu^n + O(\mu^{N+1}) \right] dt \\ &= \sum_{n=0}^N \frac{(-1)^n}{n!(n+1/4)} \mu^{n+1/4} + O(\mu^{N+5/4}) \\ &= 4\mu^{1/4} - \frac{4}{5}\mu^{5/4} + \frac{2}{9}\mu^{9/4} + O(\mu^{13/4}). \end{aligned}$$

In some cases, the integrand does not have a uniform asymptotic representation on the entire interval of integration. In this case one should try to split the initial interval into parts such that, in some of them, the function has a uniform asymptotic expansion and in others the integral could be calculated by means of quadratures or estimated.

Example 3

Find an asymptotic expansion for the function

$$F(\mu) = \int_{\mu}^{\infty} \exp(-x^2) dx.$$

The function $\exp(-x^2)$ has the uniform asymptotic expansion

$$\exp(-x^2) = \sum_{n=0}^N (-1)^n \frac{x^{2n}}{n!} + O(x^{2N+2})$$

in any segment $[0, x_0]$. Therefore, we write the given integral as

$$\int_{\mu}^{\infty} \exp(-x^2) dx = \int_0^{\infty} \exp(-x^2) dx - \int_0^{\mu} \exp(-x^2) dx.$$

The first integral in the right side can be easily calculated after a change of Cartesian coordinates to polar coordinates [49] and it is equal to $\sqrt{\pi}/2$. For the second integral, the expansion of the integrand is uniform in the interval of integration $[0, 1]$. Therefore,

$$\begin{aligned} F(\mu) &= \sqrt{\pi}/2 - \sum_{n=0}^N \mu \int_0^1 \left[\frac{(-1)^n t^{2n}}{n!} \mu^{2n} + O(\mu^{2N+2}) \right] dt \\ &= \sqrt{\pi}/2 - \sum_{n=0}^N \frac{(-1)^n \mu^{2n+1}}{n!(2n+1)} + O(\mu^{2N+2}). \end{aligned}$$

2.1.1 Exercises

2.1.1. Find an asymptotic expansion of the function

$$F(\mu) = \int_0^{\pi/2} \sqrt{1 - \mu \sin^2 x} \, dx.$$

2.1.2. The function F is defined by the integral

$$F(\mu) = \int_0^\mu \frac{dx}{\sqrt{1 - m \sin^2 x}}, \quad m \in \mathbb{N}.$$

The Jacobi elliptic functions are defined as

$$\begin{aligned} \operatorname{sn}(x) &= \sin(F^{-1}(x)), \\ \operatorname{cn}(x) &= \cos(F^{-1}(x)), \\ \operatorname{am}(x) &= F^{-1}(x). \end{aligned}$$

Find the first two terms of the asymptotic expansions for the Jacobian functions.

2.1.3. Find an asymptotic expansion for the function

$$F(\mu) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \mu \sin^2 x}}.$$

2.1.4. Find the first two terms of the asymptotic expansions of the function

$$F(\lambda) = \int_0^\lambda \sqrt{1 + x^2} \, dx$$

and the order of the correction,

- (a) by calculating the given integral by means of quadratures,
- (b) by expanding the integrand in a series.

2.2 Integration by Parts

Integration by parts is the next method to find asymptotic estimates for integrals. Let the integrand in (2.1) be a product of two functions:

$$f(z, \mu) = g(z, \mu)h(z, \mu) \quad (2.2.1)$$

under the assumption that $h(z, \mu)$ is analytic in z in some domain containing the path of integration and $g(z, \mu)$ is integrable in z .

Denote

$$\begin{aligned} h_0(z, \mu) &\equiv h(z, \mu), & h_n(z, \mu) &\equiv \frac{\partial^n}{\partial z^n} h(z, \mu), \\ g_0(z, \mu) &\equiv g(z, \mu), & g_{-n+1}(z, \mu) &\equiv \frac{\partial}{\partial z} g_{-n}(z, \mu). \end{aligned} \quad (2.2.2)$$

Integrating (2.1) n times by parts with respect to z over $[a, b]$, we get the following expression

$$\begin{aligned} F(\mu) = \sum_{k=0}^{n-1} (-1)^k [g_{-k-1}(b, \mu)h_k(b, \mu) - g_{-k-1}(a, \mu)h_k(a, \mu)] \\ + (-1)^n \int_{a(\mu)}^{b(\mu)} g_{-n}(z, \mu)h_n(z, \mu) dz \end{aligned} \quad (2.2.3)$$

If

- (1) relation (2.2.3) holds for all $n \in \mathbb{N}$,
- (2) the terms under the summation sign in (2.2.3) form an asymptotic sequence,
- (3) for all $n \in \mathbb{N}$, the remainder of the series satisfies the condition in the definition of asymptotic expansion,

then (2.2.3) is the required asymptotic expansion.

The main problem in the application of the method is a successful choice of the functions g and h .

Example 1

Find an asymptotic expansion of the function

$$F(\lambda) = \int_1^\lambda \frac{e^x}{x} dx.$$

We write the integrand as $f(x) = g(x)h(x)$, where $g(x) = e^x$ and $h(x) = 1/x$. Then, by formula (2.2.3)

$$F(\lambda) = \sum_{k=1}^n \left(e^\lambda \frac{k!}{\lambda^k} - e k! \right) + n! \int_1^\lambda \frac{e^x}{x^{n+1}} dx.$$

The second term under the summation sign has order $O(1)$ and for the integral the following estimates are valid:

$$\begin{aligned} n! \int_1^\lambda \frac{e^x}{x^{n+1}} dx &= n! \int_1^{\lambda/2} \frac{e^x}{x^{n+1}} dx + n! \int_{\lambda/2}^\lambda \frac{e^x}{x^{n+1}} dx \\ &< n! \int_1^{\lambda/2} e^x dx + n! \int_{\lambda/2}^\lambda e^x \left(\frac{2}{\lambda} \right)^{n+1} dx = O\left(\frac{e^\lambda}{\lambda^{n+1}} \right). \end{aligned}$$

Hence,

$$F(\lambda) = \sum_{k=1}^n e^\lambda \frac{k!}{\lambda^k} + O\left(\frac{e^\lambda}{\lambda^{n+1}} \right).$$

The numerical (solid line) and the asymptotic (dashed line) values of the integral versus μ are plotted in Fig. 2.1 for $n = 8$.

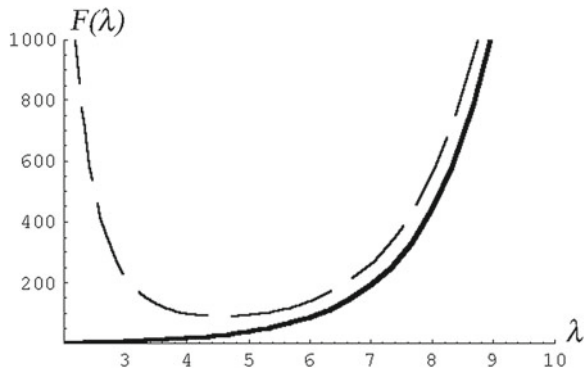
Example 2

Find an asymptotic expansion of the function

$$F(\lambda) = \int_\lambda^\infty \frac{e^{-x}}{x^2} dx.$$

We assume that $f(x) = g(x)h(x)$, where $g(x) = e^{-x}$ and $h(x) = 1/x^2$. Then an integration by parts gives

Fig. 2.1 Numerical values (solid line) and asymptotic values (dashed line) for Example 1



$$F(\lambda) = -e^{-x} \frac{1}{x^2} \Big|_{\lambda}^{\infty} - 2 \int_{\lambda}^{\infty} \frac{e^{-x}}{x^3} dx.$$

Integrating by parts again we obtain

$$F(\lambda) = \sum_{k=2}^n (-1)^k e^{-\lambda} \frac{(k-1)!}{\lambda^k} + n!(-1)^{n-1} \int_{\lambda}^{\infty} \frac{e^{-x}}{x^{n+1}} dx.$$

For $\lambda \leq x < \infty$,

$$x^{n+1} \geq \lambda^{n+1} \quad \text{and} \quad \frac{1}{x^{n+1}} \leq \frac{1}{\lambda^{n+1}},$$

and, therefore,

$$\int_{\lambda}^{\infty} \frac{e^{-x}}{x^{n+1}} dx \leq \frac{1}{\lambda^{n+1}} \int_{\lambda}^{\infty} e^{-x} dx = \frac{e^{-\lambda}}{\lambda^{n+1}}.$$

Hence, we obtain the asymptotic expansion

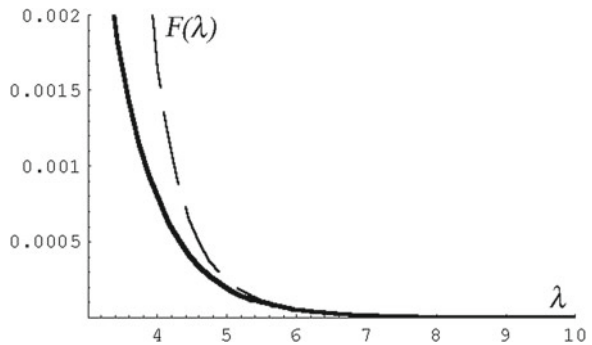
$$F(\lambda) = e^{-\lambda} \sum_{k=2}^n (-1)^k \frac{(k-1)!}{\lambda^k} + O\left(\frac{e^{-\lambda}}{\lambda^{n+1}}\right).$$

This series diverges since the ratio of the m th and the $(m-1)$ th terms goes to $-\infty$ as $m \rightarrow \infty$:

$$\lim_{m \rightarrow \infty} \frac{(-1)^{m-1} m! \lambda^m}{\lambda^{m+1} (-1)^{m-2} (m-1)!} = - \lim_{m \rightarrow \infty} \frac{m}{\lambda} = -\infty.$$

The numerical (solid line) and asymptotic (dashed line) values of the integral versus μ are plotted in Fig. 2.2 for $n = 8$.

Fig. 2.2 Numerical values (solid line) and asymptotic values (dashed line) for Example 1



2.2.1 Exercises

Find the asymptotic expansions of the following functions as $\lambda \rightarrow \infty$.

$$2.2.1. F(\lambda) = \int_0^\infty \frac{e^{-x}}{\lambda + x} dx.$$

$$2.2.2. F(\lambda) = \int_\lambda^\infty \frac{\cos(x - \lambda)x}{x} dx.$$

$$2.2.3. F(\lambda) = \int_\lambda^\infty \frac{e^{-x}x}{x} dx.$$

2.3 Laplace Method

Consider the integral

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} f(x, \lambda) dx, \quad x \in \mathbb{R}.$$

If the function $f(x, \lambda)$ has a sharp maximum at the point $x_0 \in I$, $I = [a, b]$, which is sharper when λ is larger, then it is good to calculate the integral by means of Laplace method. This method, which is most conveniently applied to integrals of the form

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} \varphi(x) e^{\lambda S(x)} dx, \quad x \in \mathbb{R}, \quad (2.3.1)$$

is called a *Laplace integral* [12].

We look for the asymptotic expansion of $F(\lambda)$ as $\lambda \rightarrow \infty$. A vast literature is devoted to this question, for example the books [14, 20, 22, 23, 66] and handbooks [1, 35]. We mention, in particular, the three volumes of the monograph by Riekstynsh [54], in which integrals of type (1.3.1) and integrals of more general types are studied.

We start with several particular cases to illustrate the application of the method.

Example 1

Let $S(x) = -x$ in (2.3.1). In a neighborhood of the point $x = 0$, we expand the function $\varphi(x)$ in a Taylor series

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} x^n,$$

and consider the integral

$$F(\lambda) = \int_0^\infty \varphi(x) e^{-\lambda x} dx. \quad (2.3.2)$$

Integrating (2.3.2) n times by parts, one gets the asymptotic expansion

$$F(\lambda) \simeq \sum_{n=0}^N \frac{\varphi^{(n)}(0)}{\lambda^{n+1}} + O\left(\frac{1}{\lambda^{N+2}}\right), \quad \text{as } \lambda \rightarrow \infty.$$

Example 2

Let $S(x) = -x$ in (2.3.1). If in a neighborhood of the point $x = 0$, the function $\varphi(x)$ admits the convergent asymptotic expansion

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \quad \alpha > -1,$$

then the integral

$$F(\lambda) = \int_0^b \varphi(x) e^{-\lambda x} dx, \quad b > 0, \quad (2.3.3)$$

has the asymptotic expansion

$$F(\lambda) \simeq \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha + 1)}{\lambda^{n+\alpha+1}} a_n, \quad \text{as } \lambda \rightarrow \infty, \quad (2.3.4)$$

where $\Gamma(z)$ is the Gamma-function.

Indeed, an integration by parts in (2.3.3) gives

$$F(\lambda) = \sum_{n=0}^{\infty} a_n F_n(\lambda), \quad F_n(\lambda) = \int_0^b x^{n+\alpha} e^{-\lambda x} dx.$$

Integrating by parts, one can show that

$$\int_b^{\infty} x^{n+\alpha} e^{-\lambda x} dx = o\left(\lambda^{-N}\right),$$

where N is any natural number. Therefore,

$$\begin{aligned} F_n(\lambda) &= \int_0^{\infty} x^{n+\alpha} e^{-\lambda x} dx - \int_b^{\infty} x^{n+\alpha} e^{-\lambda x} dx \\ &= \int_0^{\infty} x^{n+\alpha} e^{-\lambda x} dx + o\left(\lambda^{-N}\right). \end{aligned}$$

Taking

$$\int_0^\infty x^{n+\alpha} e^{-\lambda x} dx = \frac{1}{\lambda^{n+\alpha+1}} \int_0^\infty z^{n+\alpha} e^{-z} dz = \frac{\Gamma(n+\alpha+1)}{\lambda^{n+\alpha+1}},$$

into account, we obtain formula (2.3.4).

Note, that depending on a_n , the series (2.3.4) may diverge or converge. However, in the last case, it may not converge to the function $F(\lambda)$, but gives only its asymptotic expansion. For example, applying formula (2.3.4) with $\alpha = 0$ to the integral

$$F(\lambda) = \int_0^{\pi/2} e^{-\lambda x} \sin x dx,$$

we get

$$F(\lambda) \simeq \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^{2k+2}} = \frac{1}{1+\lambda^2}. \quad (2.3.5)$$

The exact solution,

$$F(\lambda) = \frac{1}{1+\lambda^2} \left(1 - \lambda e^{-\pi\lambda/2}\right),$$

differs from the function to which series (2.3.5) converges. This is, as in similar further cases, because the series (2.3.4), or (2.3.5), contributes to the integral only in a neighborhood of the point of maximum of the function $e^{\lambda h(x)}$ while exponentially small terms are neglected.

Example 3

Let

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n, \quad S(x) = -x^2, \quad a < 0, \quad b > 0. \quad (2.3.6)$$

Then integral (2.3.1) has the expansion

$$\int_a^b \varphi(x) e^{-\lambda x^2} dx \simeq \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\lambda^{n+1/2}} a_{2n}, \quad \text{as } \lambda \rightarrow \infty. \quad (2.3.7)$$

Series (2.3.7) is a contribution to integral of the neighborhood of the point $x = 0$, where the function $e^{-\lambda x^2}$ attains its maximum.

Now we go back to the general integral (2.3.1). For that integral one cannot assume $u = \varphi(x)$ and $dv = e^{\lambda S(x)} dx$ to integrate by parts since the last expression cannot be integrated in the general case. We suppose that $S'(x) \neq 0$ in the interval (a, b) , and let

$$u = \frac{\varphi(x)}{S'(x)}, \quad dv = e^{\lambda S(x)} S'(x) dx.$$

Then

$$F(\lambda) = \frac{e^{\lambda S(x)} \varphi(x)}{\lambda S'(x)} \Big|_a^b - \frac{1}{\lambda} \int_a^b e^{\lambda S(x)} \left[\frac{\varphi(x)}{S'(x)} \right]' dx$$

or, as $\lambda \rightarrow \infty$,

$$F(\lambda) \simeq \frac{e^{\lambda S(b)} \varphi(b)}{\lambda S'(b)} - \frac{e^{\lambda S(a)} \varphi(a)}{\lambda S'(a)} + O\left(\frac{1}{\lambda^2}\right).$$

If $S(a) \neq S(b)$, then the asymptotic behavior is determined by the end point where $S(x)$ is larger. Suppose, for definiteness, that $S(a) > S(b)$. Continuing the process one gets

$$F(\lambda) = - \sum_{k=0}^N \frac{M^k \left(P(a) e^{\lambda S(a)} \right)}{\lambda^{k+1}} + O\left(\frac{e^{\lambda S(a)}}{\lambda^{N+2}} \right), \quad (2.3.8)$$

where

$$M^k = \left(\frac{-1}{S'(x)} \frac{d}{dx} \right)^k, \quad P(x) = \frac{\varphi(x)}{S'(x)}.$$

Thus far, everything that was said allows us to conclude that the point $x = c \in (a, b)$ corresponding to the largest value of $S(x)$ makes the main contribution in $F(\lambda)$, whether it is an end point or not.

Assume that $c = a$ and in the neighborhood on this point

$$\varphi(x) \simeq \varphi_0 (x - a)^\alpha, \quad S'(a) = 0, \quad S''(a) < 0,$$

i.e.

$$S(x) \simeq S(a) + \frac{S''(a)(x - a)^2}{2!}.$$

For the integral to exist, it is necessary that $\alpha > -1$. Then

$$F(\lambda) \simeq \varphi_0 e^{\lambda S(a)} \int_a^{a+\delta} (x - a)^\alpha e^{\lambda S''(a)(x-a)^2/2!} dx.$$

Since $S''(a) < 0$, the error is exponentially small if $a + \delta$ is replaced by ∞ . After the change of variables

$$\tau = - \frac{\lambda S''(a)(x - a)^2}{2!},$$

one gets $\tau = 0$ for $x = a$ and

$$F(\lambda) \simeq \frac{\varphi_0}{2} \left[\frac{2}{-\lambda S''(a)} \right]^{(\alpha+1)/2} e^{\lambda S(a)} \int_0^\infty \tau^{(\alpha-1)/2} e^{-\tau} d\tau,$$

or

$$F(\lambda) \simeq \frac{\varphi_0}{2} \left[\frac{2}{-\lambda S''(a)} \right]^{(\alpha+1)/2} e^{\lambda S(a)} \Gamma\left(\frac{\alpha+1}{2}\right).$$

Let now $a < c < b$ and $S''(c) < 0$. As above, one obtains

$$F(\lambda) \simeq \varphi_0 \left[\frac{2}{-\lambda S''(c)} \right]^{(\alpha+1)/2} e^{\lambda S(c)} \Gamma\left(\frac{\alpha+1}{2}\right).$$

In a similar manner we consider cases where a larger number of derivatives are equal to zero at $x = c$ (or $x = a$). Here we consider the case when $\alpha = 0$.

Let

- (1) $\varphi(x), S(x) \in C([a, b])$,
- (2) $\max_{x \in I} S(x)$ is attained only at the point x_0 ,
- (3) $\varphi(x), S(x) \in C^\infty$ for x close to x_0 ,

then the following statement is valid [54]:

(A) If the maximum is attained for $x \in (a, b)$ and

$$S^{(j)}(x_0) = 0, \quad 1 \leq j \leq 2m-1, \quad S^{(2m)}(x_0) \neq 0, \quad m \geq 1,$$

then the main term of the asymptotic expansion has the form

$$F(\lambda) = \frac{\Gamma(1/2m) e^{\lambda S(x_0)}}{m \lambda^{1/2m}} \left[-\frac{(2m)!}{S^{(2m)}(x_0)} \right]^{1/2m} \left[\varphi(x_0) + O\left(\frac{1}{\lambda^{1/2m}}\right) \right]. \quad (2.3.9)$$

(B) If the maximum is attained at one end of the interval ($x_0 = a$) and

$$S^{(j)}(x_0) = 0, \quad 1 \leq j \leq m-1, \quad S^{(m)}(x_0) \neq 0, \quad m \geq 2,$$

then the main term of the asymptotic expansion has the form

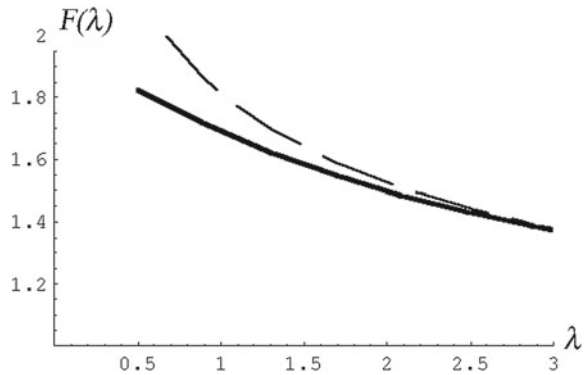
$$F(\lambda) = \frac{\Gamma(1/m) e^{\lambda S(x_0)}}{m \lambda^{1/m}} \left[-\frac{(m)!}{S^{(m)}(x_0)} \right]^{1/m} \left[\varphi(x_0) + O\left(\frac{1}{\lambda^{1/m}}\right) \right]. \quad (2.3.10)$$

Example 4

Find an asymptotic expansion of the function

$$F(\lambda) = \int_a^b e^{-\lambda x^4} dx, \quad a \leq 0, \quad b > 0.$$

Fig. 2.3 Exact values (*solid line*) and asymptotic values (*dashed line*) of the integral $F(\lambda)$ of Example 4



For $a < 0$ and $b > 0$ we should use the asymptotic formula (2.3.9), which defines the main term of the asymptotic expansion

$$\frac{1}{2} \Gamma\left(\frac{1}{4}\right) \lambda^{-1/4}.$$

For $a = -\infty$ and $b = \infty$, the main term of the integral coincides with the exact value of the integral. In Fig. 2.3, we compare the exact (solid line) and asymptotic (dashed line) values of the integral for $a = -1$, $b = 1$.

For $a = 0$, $b > 0$ one should use formula (2.3.10), which defines the main term of the asymptotic expansion

$$\frac{1}{4} \Gamma\left(\frac{1}{4}\right) \lambda^{-1/4}.$$

As expected, the value of the integral is twice smaller than in the first case.

2.3.1 Exercises

2.3.1. Find the first two terms of the asymptotic expansion of the function

$$F(\lambda) = \int_0^{\pi^2/4} \exp(\lambda \cos \sqrt{x}) dx.$$

Find the first term of the asymptotic expansion of the following functions.

2.3.2. $F(\lambda) = \int_0^1 \exp(-1/x - \lambda x) dx.$

2.3.3. $F(\lambda) = \int_0^\infty \exp(-\alpha x^{-\alpha} - \lambda x) dx, \quad \alpha > 0.$

2.3.4. $F(\lambda) = \int_{-1}^1 (1 - x^2)^\lambda x^2 dx.$

$$2.3.5. F(\lambda) = \int_{-1}^1 (1 - x^2)^\lambda dx.$$

$$2.3.6. F(\lambda) = \int_0^1 (1 - x^2)^\lambda x dx.$$

$$2.3.7. F(\lambda) = \int_0^\pi x^\lambda \sin x dx, \quad \lambda \in \mathbb{N}.$$

$$2.3.8. F_\nu(\lambda) = \int_0^\infty \exp(-\lambda \cosh x) \cosh(\nu x) dx.$$

2.4 Stationary Phase Method

Consider integrals of the form

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} \varphi(x) \exp(i\lambda S(x)) dx, \quad x \in \mathbb{R}, \quad (2.4.1)$$

where $\varphi(x)$ is a complex valued function and $S(x)$ a real valued function called *phase function*, and, as before, $\lambda > 0$ is a large parameters.

2.4.1 Integrals Without Stationary Points

Consider a finite interval $I = [a, b]$, in which

$$\varphi(x) \in C^{N+1}(I), \quad S(x) \in C^{N+2}(I), \quad S'(x) \neq 0, \quad (2.4.2)$$

i.e. $\varphi(x)$ and $S(x)$ have $N + 1$ and $N + 2$ continuous derivatives. Then

$$F(\lambda) = \sum_{k=0}^N \frac{1}{(i\lambda)^{k+1}} \left[M^k(P(x)) \exp(i\lambda S(x)) \right] \Big|_a^b + O\left(\frac{1}{\lambda^{N+2}}\right), \quad (2.4.3)$$

where

$$M = \frac{-1}{S'(x)} \frac{d}{dx}, \quad P(x) = \frac{\varphi(x)}{S'(x)}.$$

In this case, the main term of the asymptotic expansion is

$$F(\lambda) = \frac{1}{i\lambda} \left[\frac{\varphi(x)}{S'(x)} \exp(i\lambda S(x)) \right] \Big|_a^b + O\left(\frac{1}{\lambda^2}\right).$$

Formula (2.4.3) can be proved by integration by parts and the Riemann–Lebesgue lemma [14, 53]

$$\int_a^b \varphi(x) e^{i\lambda x} dx = o(1), \quad \text{as } \lambda \rightarrow \infty$$

for a piecewise continuous function $\varphi(x)$.

Example 1

Find the first term of the asymptotic expansion and an estimate of the error for the function

$$F(\lambda) = \int_0^{\pi/4} \sin(\lambda \sin x) x dx.$$

Write the given integral in the form

$$F(\lambda) = \Im \int_0^{\pi/4} \exp(i\lambda \sin x) x dx.$$

Applying formula (2.4.3) we get

$$\begin{aligned} F(\lambda) &= \Im \left(\frac{\pi}{4i\lambda} \frac{2}{\sqrt{2}} \exp\left(i\lambda \frac{\sqrt{2}}{2}\right) + O\left(\frac{1}{\lambda^2}\right) \right) \\ &= -\frac{\pi}{2\sqrt{2}\lambda} \cos\left(\lambda \frac{\sqrt{2}}{2}\right) + O\left(\frac{1}{\lambda^2}\right). \end{aligned}$$

In Fig. 2.4, the numerical (solid line) and asymptotic (dashed line) values for the integral are compared.

From relation (2.4.3) and under assumption (2.4.2), it follows that the asymptotic expansion $F(\lambda)$ depends only on the values of the functions $\varphi(x)$ and $S(x)$ and their derivatives at the end of the interval of integration. In neighborhoods of other points, oscillation interference occurs. In Fig. 2.5, the graph of the function $\Re(\exp(S(x)))$ is plotted where $S(x) = i(x - 6)^2$ and $S'(x) \neq 0$ in the interval $[-3, 5]$.

If $\varphi(x)$ and $S(x)$ or their derivatives have a finite number of points of discontinuity, the interval of integration $[a, b]$ may be split in parts in such a way that the points of

Fig. 2.4 Numerical values (solid line) and asymptotic values (dashed line) of the function $F(\lambda)$ of Example 1

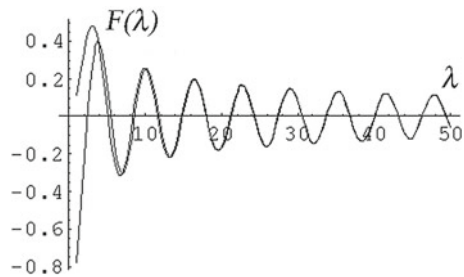
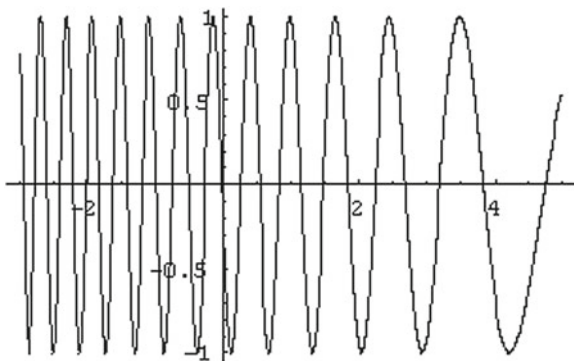


Fig. 2.5 Graph of the function $\Re(\exp(S(x)))$ of Example 1



discontinuity coincide with the ends of these parts and then apply formula (2.4.3) to each of the parts.

If $b = \infty$, conditions (2.4.2) are still satisfied and

$$M^k(P(x)) = o(1) \quad \text{as } x \rightarrow \infty, \quad \frac{d}{dx} M^k(P(x)) \in L_1[0, \infty],$$

then

$$F(\lambda) = \sum_{k=0}^N \frac{1}{(i\lambda)^{k+1}} \left[M^k(P(x)) \exp(i\lambda S(x)) \right] \Big|_a + O\left(\frac{1}{\lambda^{N+2}}\right). \quad (2.4.4)$$

Example 2

Find the first two terms of the expansion and the estimate of the error for the integral

$$F(\lambda) = \int_0^\infty \frac{e^{i\lambda x} dx}{(1+x)^\alpha}, \quad \alpha > 0.$$

In this case, $S(x) = x$, $S'(x) = 1$, $M^k(x) = (-1)^k \frac{d^k}{dx^k}$,

$$P(x) = (1+x)^{-\alpha}, \quad M^0(P(x)) = (1+x)^{-\alpha},$$

$$M^k(P(x)) = (\alpha + k - 1) M^{k-1}(P(x)) (1+x)^{-1}.$$

Therefore formula (2.4.4) holds and

$$F(\lambda) = -\left[\frac{1}{i\lambda} + \alpha \frac{1}{(i\lambda)^2} \right] + O\left(\frac{1}{\lambda^3}\right).$$

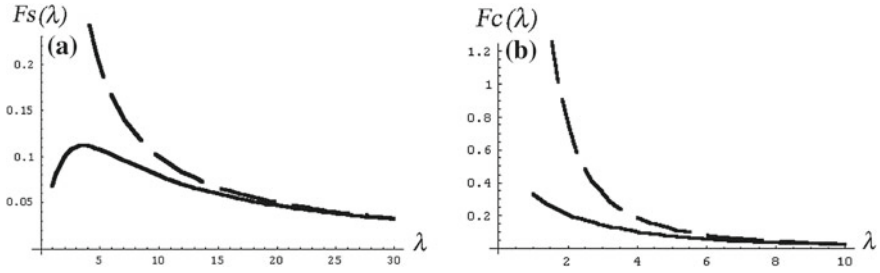


Fig. 2.6 Numerical values (solid line) and asymptotic values (dashed line) of $F_s(\lambda)$ and $F_c(\lambda)$ of Example 2

Separating the real and imaginary parts, we obtain

$$F_s(\lambda) = \Im(F(\lambda)) = \int_0^\infty \frac{\sin \lambda x \, dx}{(1+x)^\alpha} = \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right),$$

$$F_c(\lambda) = \Re(F(\lambda)) = \int_0^\infty \frac{\cos \lambda x \, dx}{(1+x)^\alpha} = \frac{\alpha}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right).$$

In Fig. 2.6, the numerical (solid line) and asymptotic (dashed line) values of (a) the function $F_c(\lambda) = \Re(F(\lambda))$ and (b) $F_s(\lambda) = \Im(F(\lambda))$ are compared for $\alpha = 3$.

2.4.2 Erdélyi's Lemma [20]

Consider the integral

$$F(\lambda) = \int_0^a x^{\beta-1} \varphi(x) \exp(i\lambda x^\alpha) \, dx. \quad (2.4.5)$$

If

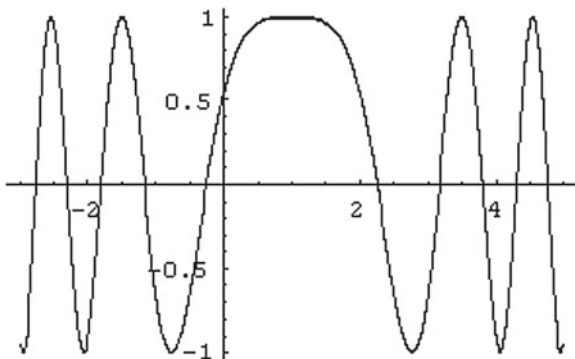
$$\varphi(x) \in C^\infty((0, a]), \quad \alpha \geq 1, \quad \beta > 0, \quad \forall k \quad \varphi^{(k)}(a) = 0,$$

then the following expansion holds

$$F(\lambda) = \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{\lambda^{(k+\beta)/\alpha}} \Gamma\left(\frac{k+\beta}{\alpha}\right) \frac{\varphi^{(k)}(0)}{k!} \exp\left(\frac{i\pi(k+\beta)}{2\alpha}\right). \quad (2.4.6)$$

Erdélyi's lemma defines the contribution of the point $x_0 = 0$ in the asymptotic expansion of the integral.

Fig. 2.7 Graph of $\Re(\exp(S(x)))$ with stationary point $x = 1$



2.4.3 Integrals with Stationary Points

The points $x = c$, where $S'(x) = 0$, are called *stationary points*. With stationary points, expansion (2.4.3) does not hold. In Fig. 2.7 we plot the graph of the function $\Re(\exp(S(x)))$, where $S(x) = i(x-1)^2$, which has the stationary point $x = 1$ in the interval $[-3, 5]$.

To illustrate the input of the stationary point in the value of the integral we start with an example with only one stationary point, $x = 0$, where $S''(0) \neq 0$,

$$F(\lambda) = \int_{-1}^3 \exp(i\lambda x^2) dx.$$

The function $S(x) = x^2$ attains its minimum at the point $x_0 = 0$ and $x_0 \in I$. Represent the initial integral in the form

$$\int_{-1}^3 \exp(i\lambda x^2) dx = \int_{-\infty}^{+\infty} \exp(i\lambda x^2) dx - J_1 - J_2,$$

where

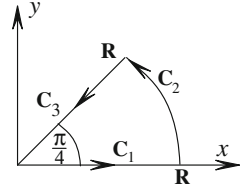
$$J_1 = \int_{-\infty}^{-1} \exp(i\lambda x^2) dx, \quad J_2 = \int_3^{+\infty} \exp(i\lambda x^2) dx.$$

It may be shown by integration by parts that the integrals J_1 and J_2 have order $O(\lambda^{-1})$, i.e.

$$F(\lambda) = \int_{-\infty}^{+\infty} \exp(i\lambda x^2) dx + O(\lambda^{-1}) = 2 \int_0^{+\infty} \exp(i\lambda x^2) dx + O(\lambda^{-1}).$$

We evaluate this integral by Cauchy's theorem, which says that the circulation integral of an analytic function of a complex variable, in a domain limited by a closed contour, is equal to zero. The main idea for calculating the integral $F(\lambda)$ is in selecting the

Fig. 2.8 Integral along the closed path $C_1 C_2 C_3$



contour in such a way as to transform the initial Fourier integral into a Laplace integral. The integral along the closed path $C_1 C_2 C_3$ (see Fig. 2.8) is equal to 0.

Besides that it can be shown that for $R \rightarrow \infty$ the integral over C_2 converges to 0. Therefore, assuming that $x^2 = z^2 \exp(i\pi/2)$ and using Laplace's method we get

$$\int_0^{+\infty} \exp(i\lambda x^2) dx = \exp(i\pi/4) \int_0^{+\infty} \exp(-\lambda z^2) dz = \frac{\exp(i\pi/4)}{2} \sqrt{\frac{\pi}{\lambda}}$$

or

$$F(\lambda) = \exp(i\pi/4) \sqrt{\frac{\pi}{\lambda}} + O(\lambda^{-1}).$$

In Fig. 2.9a, b, the numerical (solid line) and asymptotic (dashed line) values of functions $F_c(\lambda) = \Re(F(\lambda))$ and $F_s(\lambda) = \Im(F(\lambda))$ are compared.

If the function $S(x)$ has a finite number of isolated stationary points x_i in the interval $[a, b]$, then the asymptotic expansion of integral (2.4.1) consists in a sum of contributions of the stationary points of $F(\lambda, x_i)$.

As a rule, using the stationary phase method, one limits oneself to the calculation of the first term of an asymptotic expansion.

We define the contribution of a stationary point.

Let $I = [a, b]$ be a finite segment where $S(x) \in C(I)$. Assume that $S(x)$ attains its maximum, $\max_{x \in I} S(x)$, at only the point x_0 and $\varphi(x) \approx C(x - x_0)^\alpha$, $\alpha > -1$, in a neighborhood of the point x_0 . Then

- (A) If $a < x_0 < b$ and $S^{(j)}(x_0) = 0$, $1 \leq j \leq 2m - 1$, $S^{(2m)}(x_0) \neq 0$, $m \geq 1$, then the main term of the asymptotic expansion has the form

$$\begin{aligned} F(\lambda, x_0) = & m^{-1} \Gamma\left(\frac{1+\alpha}{2m}\right) \left[\frac{(2m)!}{|S^{2m}(x_0)|} \right]^{(1+\alpha)/2m} \lambda^{-(1+\alpha)/2m} \\ & \times \exp\left(i\lambda S(x_0) + \frac{i\pi(1+\alpha)}{4m} \operatorname{sgn}(S^{2m}(x_0))\right) \left[\varphi(x_0) + O(\lambda^{-1/2m}) \right]. \end{aligned} \quad (2.4.7)$$

- (B) If $x_0 = a$ and $S^{(j)}(x_0) = 0$, $1 \leq j \leq m - 1$, $S^{(m)}(x_0) \neq 0$, $m \geq 2$, then the main term of the asymptotic expansion has the form

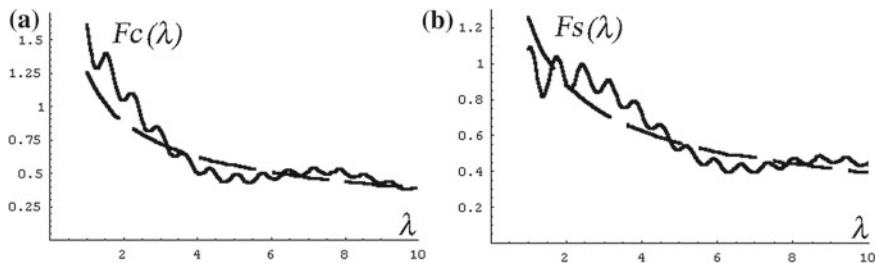


Fig. 2.9 Numerical values (*solid line*) and asymptotic values (*dashed line*) of functions F_c and F_s

$$F(\lambda, x_0) = m^{-1} \Gamma\left(\frac{1+\alpha}{m}\right) \left[\frac{(m)!}{|S^m(x_0)|} \right]^{(1+\alpha)/m} \lambda^{-(1+\alpha)/m} \\ \times \exp\left[i\lambda S(x_0) + \frac{i\pi(1+\alpha)}{2m} \operatorname{sgn}(S^m(x_0)) \right] \left[\varphi(x_0) + O(\lambda^{-1/m}) \right]. \quad (2.4.8)$$

Example 3

Find the first term of the asymptotic expansion of the integral

$$F(\lambda) = \int_0^1 \frac{\exp(i\lambda x^3)}{\sqrt{x}} dx$$

The integrand has only the critical point $x_0 = 0$, the contribution of which is evaluated by Erdélyi's lemma. For $\alpha = 3$, $\beta = 1/2$, $\varphi(x) \equiv 1$, one gets

$$F(\lambda) = \frac{1}{3} \lambda^{-1/6} \Gamma\left(\frac{1}{6}\right) \exp(i\pi/12).$$

In Fig. 2.10 the numerical (solid line) and asymptotic (dashed line) values of the functions (a) $F_c(\lambda) = \Re(F(\lambda))$ and (b) $F_s(\lambda) = \Im(F(\lambda))$ are compared.

Example 4

Find the first term of the asymptotic expansion and estimate the error of Bessel's function of positive integer index n , defined by the integral

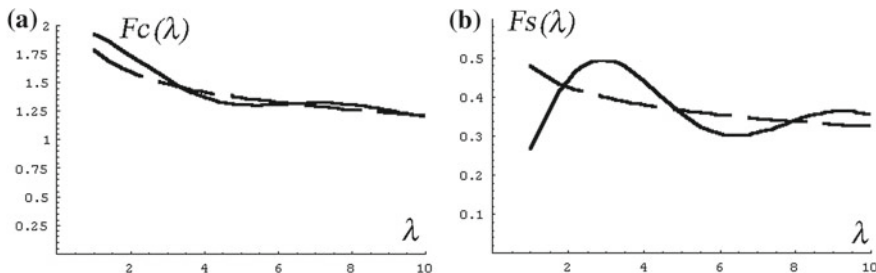
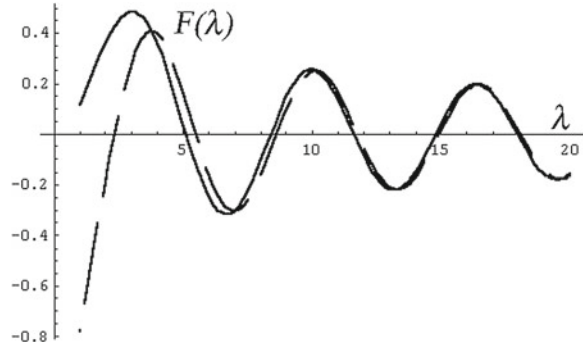


Fig. 2.10 Numerical values (*solid line*) and asymptotic values (*dashed line*) of functions F_c and F_s

Fig. 2.11 Numerical values (solid line) and asymptotic values (dashed line) of the integral $J_n(\lambda)$ of Example 4



$$J_n(\lambda) = \pi^{-1} \int_0^\pi \cos(\lambda \sin x - nx) dx.$$

Represent the given integral as

$$J_n(\lambda) = \pi^{-1} \Re \int_0^\pi \exp(i\lambda \sin x) \exp(-inx) dx,$$

For this we have $S(x) = \sin x$, $x_0 = \pi/2$, $S(x_0) = 1$, $S''(x_0) = -1$, $m = 1$, $\varphi(x_0) = \exp(-in\pi/2)$. As a result, we obtain

$$\begin{aligned} J_n(\lambda) &= \Re \left[\pi^{-1} \Gamma\left(\frac{1}{2}\right) \left[\frac{2!}{1!}\right]^{1/2} \lambda^{-1/2} \exp\left(i\lambda - \frac{i\pi}{4}\right) \left[\exp(-in\pi/2) + O(\lambda^{-1})\right] \right] \\ &= \frac{1}{\sqrt{\lambda}} \sqrt{\frac{2}{\pi}} \cos\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(\lambda^{-1}). \quad (2.4.9) \end{aligned}$$

In Fig. 2.11 the numerical values (solid line) and asymptotic values (dashed line) of the integral are compared.

2.4.4 Complete Asymptotic Expansions

By van der Korput's lemma [20], the interval of integration may be split into parts to provide only one point in the interval to contribute to the value of the integral. To calculate the contribution of a point, the following complete asymptotic expansions are used:

For case (A), i.e. if the stationary point x_0 is within the interval of integration and the first $2m - 1$ derivatives of the function $S(x)$ at x_0 are equal to zero and $S^{(2m)}(x_0) \neq 0$, then

$$F(\lambda, x_0) = \frac{1}{\lambda^{(1+\alpha)/2m}} \exp\left(i\lambda S(x_0) + \frac{i\pi(1+\alpha)}{4m} \operatorname{sgn}(S^{2m}(x_0))\right) \sum_{k=0}^{\infty} a_k \frac{1}{\lambda^{k/m}}.$$

where

$$a_k = \frac{2^{2k+1/2}}{(2k)!} \Gamma\left(\frac{2k+1+\alpha}{2m}\right) \exp\left[\frac{i\pi(2k+1+\alpha)}{4m} \operatorname{sgn}(S^m(x_0))\right] \times \left[\left(h^{-1}(x, x_0) \frac{d}{dx}\right)^{2k} (\varphi(x)h(x, x_0)) \right] \Big|_{x=x_0},$$

and

$$h(x, x_0) = [2 \operatorname{sgn}(S^{2m}(x_0)) [S(x) - S(x_0)]]^{1-1/2m} / S'(x).$$

For case (B), if a stationary point x_0 coincides with an end of the interval, the first $m-1$ derivatives of the function $S(x)$ at the stationary point are equal to zero and $S^{(m)}(x_0) \neq 0$, then

$$F(\lambda, x_0) = \frac{1}{\lambda^{(1+\alpha)/m}} \exp(i\lambda S(x_0)) \sum_{k=0}^{\infty} a_k \lambda^{-k/m},$$

where

$$a_k = \frac{1}{k!m} \Gamma\left(\frac{k+1+\alpha}{m}\right) \exp\left(\frac{i\pi(1+k+\alpha)}{2m} \operatorname{sgn}(S^m(x_0))\right) \left[\left(\frac{d}{dx}\right)^k [\varphi(x)h(x, x_0)] \right] \Big|_{x=x_0},$$

and

$$h(x, x_0) = [-\operatorname{sgn}(S^m(x_0)) (S(x) - S(x_0))]^{-(k+1)/m} (x - x_0)^{k+1}.$$

So, contributions to integral (2.4.1) come only from neighborhoods of

- (i) the ends of the integration interval,
- (ii) points of discontinuity for the functions $\varphi(x)$ and $S(x)$ or their derivatives,
- (iii) the stationary points.

For example, the integral

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{1+x^2} dx = \pi e^{-\lambda}$$

admits the asymptotic expansion (2.4.3) with zero terms, since it has no points of discontinuity nor stationary points and the contributions of the ends are equal to zero since $\varphi^{(n)} \rightarrow 0$ as $x \rightarrow \pm\infty$.

2.4.5 Exercises

2.4.1. Find the first term of the asymptotic expansion and estimate the error for Bessel's function of the large real index ν for a fixed argument $t > 0$,

$$J_\nu(\nu t) = \pi^{-1} \int_0^\pi \cos(\nu(x - t \sin x)) dx \\ - \pi^{-1} \sin \nu\pi \int_0^\infty \exp(-\nu(x + t \sinh x)) dx$$

2.4.2. Find the first term of the asymptotic expansion and estimate the error of the integral

$$F(\lambda) = \int_0^1 \exp(i\lambda x^3) dx.$$

2.4.3. Find the first term of the asymptotic expansion and estimate the error of the integral

$$F_n(\lambda) = \int_0^1 \exp(i\lambda x^n) dx.$$

2.5 Saddle Point Method

2.5.1 Description of the Method

We seek an asymptotic expansion of the integral

$$F(\lambda) = \int_\gamma \varphi(z) e^{\lambda h(z)} dz, \quad \text{as } \lambda \rightarrow \infty, \quad (2.5.1)$$

where the curve γ lies in the complex plane and the functions $\varphi(z)$ and $h(z)$ are analytic in a domain S which contains γ .

The saddle point method includes two stages [20, 24, 53]:

- (1) deforming the contour γ into the contour γ_0 is most convenient to find the asymptotic estimates.
- (2) calculating asymptotics for the integral over the contour γ_0 .

As $\lambda \rightarrow \infty$, the absolute value of the integrand attains its maximum at points z where the function $\Re(h(z))$ is maximal. Assume that, among the contours with the same ends as γ , there exists a contour γ_0 where

$$\min_{\gamma} \max_{z \in \gamma} \Re(h(z)) = \max_{z \in \gamma_0} \Re(h(z)).$$

We also assume that one can deform the contour γ to contour γ_0 inside the domain S . Then, according to Cauchy's theorem,

$$F(\lambda) = \int_{\gamma_0} \varphi(z) e^{\lambda h(z)} dz. \quad (2.5.2)$$

For simplicity, let the maximum,

$$\max_{z \in \gamma_0} \Re(h(z))$$

be attained only at the point $z = c$. If $z = c$ is an inner point of γ_0 , then, from the minimax property of the contour γ_0 , it follows that c is a stationary point of the function $f(x, y) = \Re(h(z))$, where $z = x + iy$. At c , $f_x = f_y = 0$, and from the Cauchy–Riemann conditions it follows that $h'(c) = 0$.

The point $z = c$ is called a *saddle point*, and the value of $\Re(h(c))$ is the height of the saddle point. A saddle point is called *simple* if $h''(c) \neq 0$. In this case, two straight lines, $\Im(h(z) - h(c)) = 0$, intersect at right angle at point c . On these lines, the function $f(x, y)$ changes in the fastest way. One of the lines, on which $f(x, y)$ decreases away from point $z = c$, is called the *line of fastest descent*.

The contour γ_0 may be deformed such that in the neighborhood of the point $z = c$ it coincides with the line of fastest descent. Then, in this neighborhood, $\Im(h(z)) = \text{const}$ and to estimate integral (2.5.2) it is convenient to use the Laplace method.

If the contour γ_0 lies entirely in the domain of regularity of the functions $\varphi(z)$ and $h(z)$ and the maximum value, $\Re(h(z))$, on γ_0 is attained at the saddle points (or at the ends of the interval) where the contour passes through, then the asymptotic behavior of integral (2.5.1) as $\lambda \rightarrow \infty$ is given as the sum of the contributions of the saddle points (or the ends of the contour).

There exists no general algorithm to construct the contour γ_0 . For the specific integral (2.5.1) we firstly should find the saddle points and draw the lines of fastest descent through them. After that, one should try to deform the contour γ in such a way that it consists of the parts of the lines of fastest descent and maybe some other curves over which the integrals may be neglected for being asymptotically small.

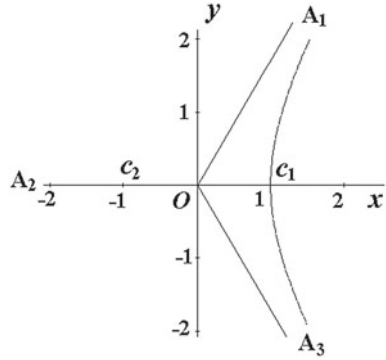
2.5.2 Asymptotics of Airy's Functions

Find the first terms of the expansions for Airy's functions. Consider the integrals [17]

$$w_k(\eta) = C_k \int_{\gamma_k} e^{z^3/3 - \eta z} dz, \quad (2.5.3)$$

where γ_k are the contours with ends going to infinity over rays OA_n where $\arg z = \frac{\pi}{3}(-1 + 2n)$, $n = 1, 2, 3$ (Fig. 2.12). Suppose the integrand in (2.5.3) converges to zero as $|z| \rightarrow \infty$ along these rays.

Fig. 2.12 Rays OA_n ,
 $n = 1, 2, 3$



The standard Airy's functions $\text{Ai}(\eta)$ and $\text{Bi}(\eta)$ are real for real η and are defined by the formulas [1, 53]:

$$\begin{aligned}\text{Ai}(\eta) &= w_1(\eta), \quad C_1 = i/2\pi, \quad \gamma_1 = A_1 O A_3, \\ \text{Bi}(\eta) &= w_2(\eta), \quad C_2 = 1/2\pi, \quad \gamma_2 = A_2 O A_1 + A_2 O A_3.\end{aligned}\quad (2.5.4)$$

Airy's functions find applications in the asymptotic integration of differential equations with parameter which contain turning points (see Chap. 5).

Consider the case $\eta > 0$. After the change of variables $z = \sqrt{\eta} z_1$, formula (2.5.3) becomes

$$w_k(\eta) = C_k \sqrt{\eta} \int_{\gamma_k} e^{\lambda h(z_1)} dz_1, \quad h(z) = \frac{z^3}{3} - z, \quad \lambda = \eta^{3/2}. \quad (2.5.5)$$

Evaluate the integral in (2.5.5) by the saddle point method. The roots of the equation $h'(c) = 0$ are the saddle points $c_{1,2} = \pm 1$. Since $h''(c_k) = 2c_k \neq 0$, the saddle points are the saddle foci for the function $f(x, y) = \Re(h(z))$, where $z = x + iy$. In Fig. 2.13a we plot the graph of the function $\Re f(x, y)$ near the point $c = 0$.

The lines of fastest descent passing through the saddle points c_k , are defined by the equation

$$\Im h(z) = \Im h(c_k). \quad (2.5.6)$$

In the case under consideration, $\Im h(c_k) = 0$ for $k = 1, 2$. Equation (2.5.6) are equivalent to the equations $y = 0$ and $x^2 - y^2/3 = 1$. The lines of fastest descent through c_1 and c_2 are shown in Fig. 2.13b with thick lines and the contour lines for function $\Re(h(z))$ with thin lines.

The paths of integration γ_1 and γ_2 are deformed into the paths $\gamma_{10} = A_1 c_1 A_3$ and $\gamma_{20} = A_2 c_2 c_1 A_1 + A_2 c_2 c_1 A_3$, consisting of the parts of the lines of fastest descent.

The paths γ_{k0} have the rays OA_n as their asymptotes on which the integrand in (2.5.5) goes to zero as $z \rightarrow \infty$. Therefore the asymptotic expansions of integrals (2.5.5) consist only of the contributions of the neighborhoods of the saddle points.

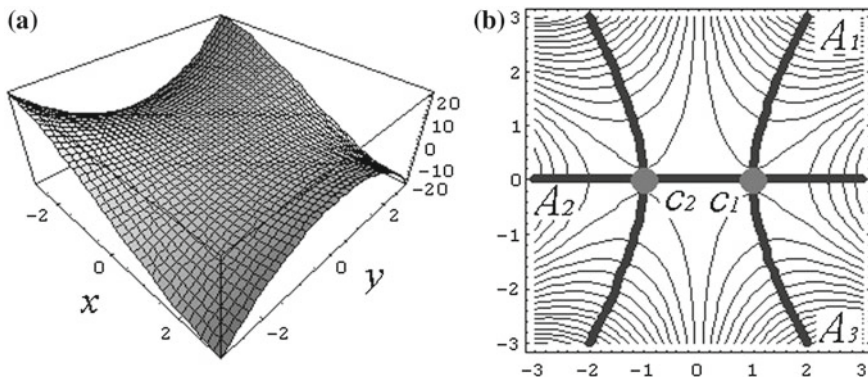


Fig. 2.13 **a** Graph of the function $\Re f(x, y)$. **b** Thick lines of fastest descent and thin contour lines

For $\text{Ai}(\eta)$, we get

$$\text{Ai}(\eta) = w_1(\eta) = C_1 \sqrt{\eta} \int_{\gamma_{10}} e^{\lambda h(z)} dz \simeq C_1 \sqrt{\eta} J_1,$$

where

$$J_1 = \int_{\gamma_{10}^*} e^{\lambda h(z)} dz$$

is the contribution of the saddle point c_1 and γ_{10}^* is a small arc of the contour γ_{10} , containing the saddle point c_1 . On the arc γ_{10}^* we have

$$\tau^2 = h(c_1) - h(z) \simeq -h''(c_1)(z - c_1)^2/2. \quad (2.5.7)$$

From (2.5.7) it follows that on γ_{10}^* the approximate equality $dz \simeq i d\tau$ holds. Hence,

$$J_1 \simeq e^{\lambda h(c_1)} \int_{\delta}^{-\delta} e^{-\lambda \tau^2} i d\tau = -i e^{\lambda h(c_1)} \int_{-\infty}^{\infty} e^{-\lambda \tau^2} d\tau = -i \sqrt{\pi/\lambda} e^{-2\lambda/3},$$

$$\text{Ai}(\eta) \simeq \frac{i\eta^{1/2}}{2\pi} J_1 = \frac{1}{2\sqrt{\pi}} \eta^{1/4} e^{-\zeta}, \quad \zeta = \frac{2}{3} \eta^{3/2}.$$

In a similar manner, the main term of the asymptotic expansion can be found for the function $\text{Bi}(\eta)$ as $\eta \rightarrow \infty$,

$$\text{Bi}(\eta) \simeq \frac{1}{\sqrt{\pi}} \eta^{1/4} e^{\zeta}.$$

A calculation shows that the contribution of the point c_2 is doubled since the path γ_{20} goes through this point twice and the contribution of the point c_1 may be neglected since the height of the saddle point c_1 is smaller than the height of the point c_2 .

To find asymptotics for Airy's function as $\eta \rightarrow -\infty$ we change the variable $z = \sqrt{-\eta}z_1$ in (2.5.3). Then, formula (2.5.3) becomes

$$w_k(-\eta) = C_k \sqrt{\eta} \int_{\gamma_k} e^{\lambda h(z_1)} dz_1, \quad h(z_1) = \frac{z_1^3}{3} + z_1, \quad \lambda = \eta^{3/2}. \quad (2.5.8)$$

The approximate expressions for integrals (2.5.8) may be obtained by means of the saddle point method (see the solution for Exercise 2.5.1).

2.5.3 Exercises

2.5.1. Find the first terms of the asymptotic expansions for Airy's functions as $\eta \rightarrow -\infty$.

Find the first term of the asymptotic expansions of the following integrals as $\lambda \rightarrow \infty$.

2.5.2. $F(\lambda) = \int_0^\infty \exp(\lambda(x + ix - x^3)) dx.$

2.5.3. $F(\lambda) = \int_{-\infty}^\infty \exp(i\lambda x)(1 + x^2)^{-\lambda} dx.$

2.5.4. $F(\lambda) = \int_{-\infty}^{+\infty} \exp(i\lambda(3x - x^3)) dx.$

2.6 Answers and Solutions

2.1.1. $F(\mu) = \frac{1}{2} \pi \left[1 - \frac{1}{4} \mu - \frac{3}{64} \mu^2 + O(\mu^3) \right].$

2.1.2. After a change of variables, the given function is represented in the form

$$F(\mu) = \mu \int_0^1 \frac{dt}{\sqrt{1 - m \sin^2 \mu t}}.$$

The integrand is expanded into the truncated series

$$f(t) = 1 + \frac{m}{2} \sin^2 \mu t + \frac{3m^2}{8} \sin^4 \mu t + O(\mu^6).$$

In its turn

$$\begin{aligned}\sin \mu t &= \mu t - \frac{(\mu t)^3}{6} + O(\mu^5), \quad \sin^2 \mu t = (\mu t)^2 - \frac{(\mu t)^4}{3} + O(\mu^6), \\ \sin^4 \mu t &= (\mu t)^4 + O(\mu^6).\end{aligned}$$

Hence,

$$f(t) = 1 + \frac{m}{2} (\mu t)^2 + \left(-\frac{m}{6} + \frac{3m^2}{8}\right) (\mu t)^4 + O(\mu^6)$$

and after an integration by parts we obtain

$$F(\mu) = \mu + \frac{m}{6} \mu^3 + \left(-\frac{m}{30} + \frac{3m^2}{40}\right) \mu^5 + O(\mu^7).$$

To find the inverse function F^{-1} we use the method of undetermined coefficients and represent μ as a truncated series,

$$\mu = F + aF^3 + bF^5 + O(F^7), \quad \text{as } F \rightarrow 0.$$

Equating the coefficients of F of equal powers, we get

$$a = -\frac{m}{6}, \quad b = \frac{m}{30} - \frac{m}{6} 3a - \frac{3m^2}{40}.$$

Whence $b = \frac{m^2 + 4}{120}$. Therefore

$$\text{am}(\mu) = \mu - \frac{m}{6} \mu^3 + \frac{m^2 + 4}{120} \mu^5 + O(\mu^7),$$

$$\text{sn}(\mu) = \sin(F^{-1}(\mu)) = \mu - \frac{m+1}{6} \mu^3 + \frac{m^2 + 14m + 1}{120} \mu^5 + O(\mu^7),$$

$$\text{cn}(\mu) = \cos(F^{-1}(\mu)) = 1 - \frac{1}{2} \mu^2 + \frac{4m+1}{24} \mu^4 + O(\mu^6).$$

2.1.3. $F(\mu) = \frac{\pi}{2} \left[1 + \frac{1}{4} \mu + \frac{9}{64} \mu^2 \right] + O(\mu^3).$

2.1.4.

(a) Using the substitution $t = \sinh x$ one gets

$$\begin{aligned}\int \sqrt{1+x^2} dx &= \int \sinh^2 t dt = \frac{t}{2} + \frac{\sinh 2t}{4} \\ &= \frac{1}{2} \ln(x + \sqrt{x^2+1}) + \frac{x\sqrt{x^2+1}}{2}.\end{aligned}$$

Then

$$F(\lambda) = \frac{1}{2} \ln(\lambda + \sqrt{\lambda^2+1}) + \frac{\lambda\sqrt{\lambda^2+1}}{2} = \frac{1}{2} \ln(2\lambda) + \frac{\lambda^2}{2} + \frac{1}{4} + O(\lambda^{-2}).$$

(b) The integrand can be expanded in a series for $x > 0$

$$\sqrt{1+x^2} = x \left[1 + \frac{1}{2x^2} + O(x^{-2}) \right] = x + \frac{1}{2x} + O(x^{-1}).$$

Write the given integral in the form

$$\int_0^\lambda \sqrt{1+x^2} dx = \int_0^1 \sqrt{1+x^2} dx + \int_1^\lambda \sqrt{1+x^2} dx.$$

For the first integral in the right side the estimate $O(1)$ holds. Substitute in the second integral the expansion for the integrand and integrate term by term,

$$F(\lambda) = \frac{\lambda^2}{2} + \frac{1}{2} \ln(\lambda) + O(1).$$

$$\mathbf{2.2.1.} \quad F(\lambda) = \sum_{n=1}^N \frac{(-1)^{n-1}(n-1)!}{\lambda^n} + O\left(\frac{1}{\lambda^{N+1}}\right).$$

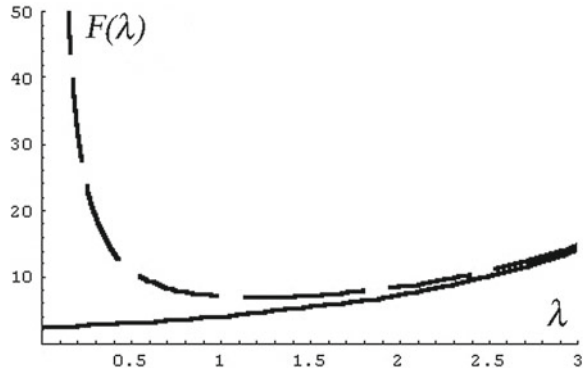
$$\mathbf{2.2.2.} \quad F(\lambda) = \sum_{n=1}^N \frac{(-1)^{n-1}(2n-1)!}{\lambda^{2n}} + O\left(\frac{1}{\lambda^{2N+2}}\right).$$

$$\mathbf{2.2.3.} \quad F(\lambda) = e^{-\lambda} \left[\sum_{n=1}^N \frac{(-1)^{n-1}(n-1)!}{\lambda^n} + O\left(\frac{N!}{\lambda^{N+1}}\right) \right].$$

2.3.1. In this case $S(x) = \cos(\sqrt{x})$, $S'(x) = -\frac{\sin(\sqrt{x})}{2\sqrt{x}}$, $S'(x) \neq 0$ for $x \in (0, \frac{\pi^2}{4}]$, $S'(+0) = -\frac{1}{2}$. Therefore we may use formula (2.3.6). Calculate

$$\frac{d}{dx} \left(\frac{1}{S'(x)} \right) = \frac{\frac{\sin \sqrt{x}}{\sqrt{x}} - \cos \sqrt{x}}{\sin^2 \sqrt{x}}.$$

Fig. 2.14 Numerical solution (solid line) and one-term asymptotic approximation (dashed line)



Hence,

$$F(\lambda) = -\frac{1}{\lambda S'(0)} e^{\lambda S(0)} + \frac{1}{\lambda^2 S'(0)} \frac{d}{dx} \left(\frac{1}{S'(x)} \right) \Big|_{x=0} e^{\lambda S(0)} + O\left(\frac{1}{\lambda^3}\right).$$

Calculating $\frac{d}{dx} \left(\frac{1}{S'(x)} \right)_{x=0}$ and substituting $\sin \sqrt{x}$ with $\sqrt{x} - (\sqrt{x})^3/6$ and $\cos \sqrt{x}$ with $1 - x/2$, we obtain

$$F(\lambda) = e^{\lambda} \left[\frac{2}{\lambda} + \frac{2}{3\lambda^2} + O\left(\frac{1}{\lambda^3}\right) \right].$$

In Fig. 2.14 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation.

2.3.2. The substitution $x = z\lambda^{-1/2}$ transforms the integral into

$$F(\lambda) = \frac{1}{\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \exp\left(-\sqrt{\lambda} \left(z + \frac{1}{z}\right)\right) dz.$$

So,

$$F(\lambda) = \lambda^{-3/4} \exp(-2\sqrt{\lambda}) \left[\sqrt{\pi} + O(\lambda^{-1}) \right].$$

In Fig. 2.15 the solid line corresponds to numerical solution and the dashed line to the one-term asymptotic approximation.

2.3.3. Substituting $x = z(\alpha/\lambda)^{1/(1+\alpha)}$ we get the integral

$$F(\lambda) = \left(\frac{\alpha}{\lambda}\right)^{1/(1+\alpha)} \int_0^{\infty} \exp\left(-\lambda^{\alpha/(\alpha+1)} \alpha^{1/(\alpha+1)} (z + z^{-\alpha})\right) dz.$$

Fig. 2.15 Numerical solution (*solid line*), one-term asymptotic approximation (*dashed line*)

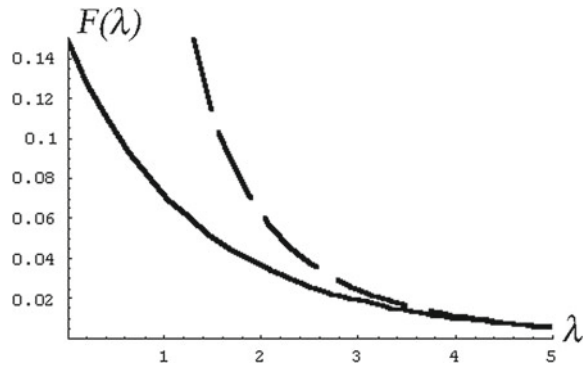
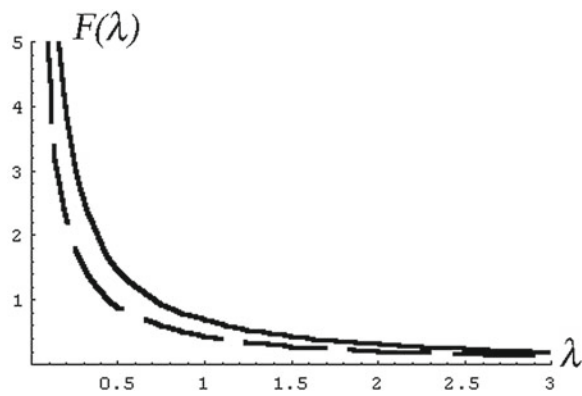


Fig. 2.16 Numerical solution (*solid line*), one-term asymptotic approximation (*dashed line*)



Therefore

$$F(\lambda) \simeq \lambda^{-\frac{\alpha+2}{2(1+\alpha)}} \alpha^{\frac{1}{1+\alpha}} \exp\left(-(\alpha+1)\alpha^{\frac{1-\alpha}{1+\alpha}} \lambda^{\frac{\alpha}{1+\alpha}}\right) \left(\sqrt{\frac{2\pi}{1+\alpha}}\right).$$

In Fig. 2.16 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation for $\alpha = 0.3$.

2.3.4. We write the given integral as $F(\lambda) = \int_{-1}^1 \exp(\lambda S(x)) x^2 dx$. In this case, $S(x) = \ln(1-x^2)$, $x_0 = 0$, $S(x_0) = 0$, $S''(x_0) = -2$, $m = 1$, $\varphi(x) = x^2$, i.e. $\alpha = 2$. Hence $F(\lambda) = \lambda^{-3/2}(\sqrt{\pi}/2 + O(\lambda^{-3/2}))$. In Fig. 2.17 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation.

2.3.5. We write the initial integral in the form

$$F(\lambda) = \int_{-1}^1 \exp(\lambda S(x)) dx.$$

Fig. 2.17 Numerical solution (*solid line*) and one-term asymptotic approximation (*dashed line*)

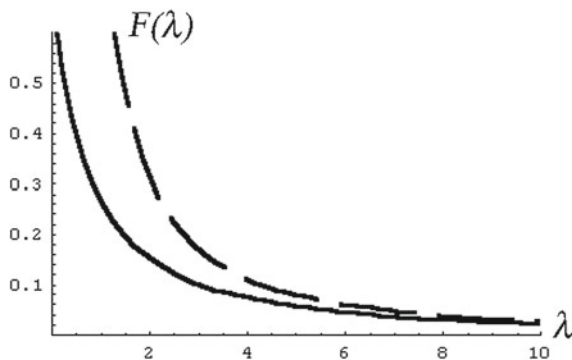
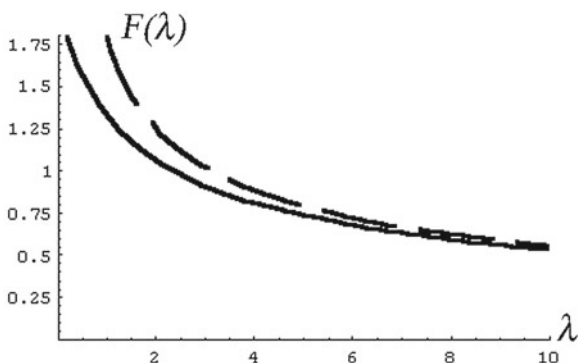


Fig. 2.18 Numerical solution (*solid line*) and one-term asymptotic approximation (*dashed line*)



In this case, $S(x) = \ln(1 - x^2)$, $x_0 = 0$, $S(x_0) = 0$, $S''(x_0) = -2$, $m = 1$, $\varphi(x) = 1$, i.e. $\alpha = 0$. Thus, $F(\lambda) = \lambda^{-1/2}(\sqrt{\pi} + O(\lambda^{-1/2}))$. In Fig. 2.18 the solid line corresponds to numerical solution and the dashed line to one-term asymptotic approximation.

2.3.6. Write the given integral as $F(\lambda) = \int_0^1 \exp(\lambda S(x)) x dx$. In this case, $S(x) = \ln(1 - x^2)$, $x_0 = 0$, $S(x_0) = 0$, $S''(x_0) = -2$, $m = 1$, $\varphi(x) = x$, i.e. $\alpha = 1$. Note that now the point of maximum lies on the boundary (this point coincides with the interval end point). So, we have $F(\lambda) = \lambda^{-1}(1/2 + O(\lambda^{-1}))$. In Fig. 2.19 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation.

2.3.7. We represent the initial integral in the form

$$F(\lambda) = \int_0^\pi \exp(nS(x)) \sin x dx.$$

In this case, $S(x) = \ln x$, $x_0 = b = \pi$ (note that the function $S(x)$ attains its maximum at the upper limit), $S(x_0) = \ln \pi$, $S'(x_0) = 1/\pi$, $m = 1$, $h(x, x_0) = x$. Hence, $\varphi(x_0) = 0$, i.e. the asymptotic expansion does not include the term for $k = 0$.

Fig. 2.19 Numerical solution (*solid line*) and one-term asymptotic approximation (*dashed line*)

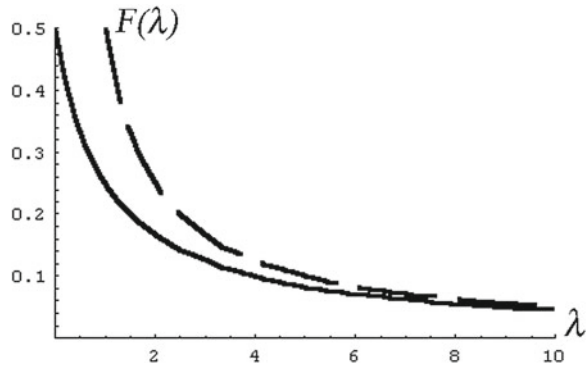
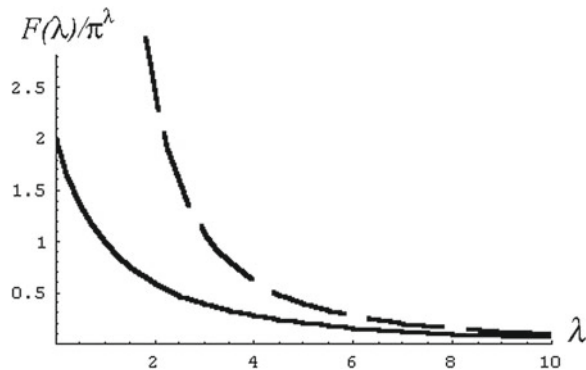


Fig. 2.20 Numerical solution (*solid line*), one-term asymptotic approximation (*dashed line*)



Calculate $x \frac{d}{dx} (x \sin x)|^{x=\pi} = -\pi^2$. Thus,

$$F(\lambda) = \lambda^{-2} \exp(\lambda \ln \pi) \left[\pi^2 + O(\lambda^{-1}) \right] = \pi^{\lambda+2} \lambda^{-2} \left[1 + O(\lambda^{-1}) \right].$$

In Fig. 2.20 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation.

2.3.8. The function $S(x) = \cosh(x)$ has a critical point at $x_0 = 0$. $S(x_0) = 1$, $S''(x_0) \neq 0$.

$$F_\nu(\lambda) = \sqrt{\frac{\pi}{2}} \frac{\exp(-\lambda)}{\sqrt{\lambda}} \left[1 + O(\lambda^{-1/2}) \right].$$

In Fig. 2.21 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation.

2.4.1. $J_\nu(\nu t) = \sqrt{\frac{2}{\pi \nu \sqrt{t^2 - 1}}} \cos \left(\nu \arccos \frac{1}{t} - \nu \sqrt{t^2 - 1} + \frac{\pi}{4} \right) + O(\nu^{-1})$. In

Fig. 2.22 the solid line corresponds to the numerical solution and the dashed line to the one-term asymptotic approximation for $t = 1.3$.

Fig. 2.21 Numerical solution (*solid line*), one-term asymptotic approximation (*dashed line*)

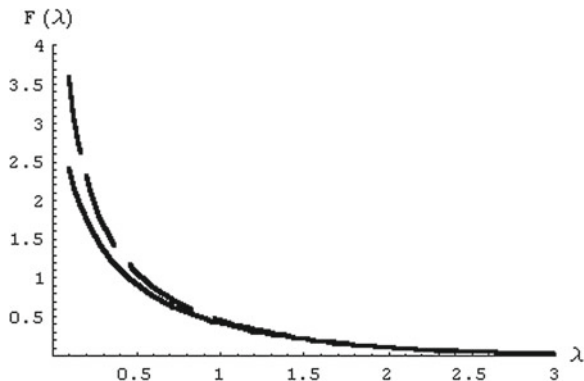
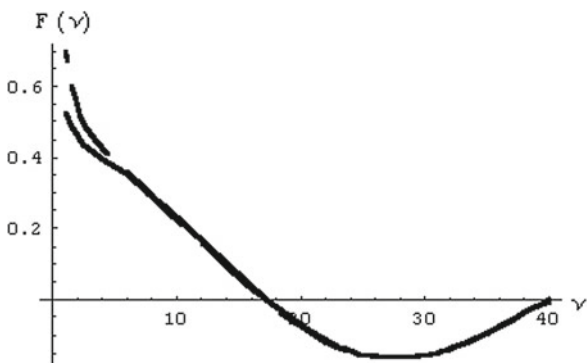


Fig. 2.22 Numerical solution (*solid line*), one-term asymptotic approximation (*dashed line*)



2.4.2. $S(x) = x^3$, $x_0 = 0$, $S''(x_0) = 0$, $S'''(x_0) = 6$, $S(x_0) = 0$. Therefore

$$F(\lambda) = \frac{1}{3} \Gamma\left(\frac{1}{3}\right) \lambda^{-1/3} \exp\left(\frac{i\pi}{6}\right) \left[1 + O\left(\lambda^{-1}\right)\right].$$

The numerical (solid line) and asymptotic (dashed line) values of the functions $F_c(\lambda) = \Re(F(\lambda))$ and $F_s(\lambda) = \Im(F(\lambda))$ are compared in Fig. 2.23.

2.4.3. $S(x) = x^n$, $x_0 = 0$, $S^{(n-1)}(x_0) = 0$, $S^{(n)}(x_0) = n!$, $S(x_0) = 0$. Therefore,

$$F(\lambda) = \frac{1}{n} \Gamma\left(\frac{1}{n}\right) \lambda^{-1/n} \exp\left(\frac{i\pi}{2n}\right) \left[1 + O\left(\lambda^{-1/n}\right)\right].$$

2.5.1. Let $h(z) = z + z^3/3$. The saddle points $c_{1,2} = \pm i$ for integrals (2.5.8) are obtained from the condition $h'(c) = 0$. The lines of fastest descent passing through the saddle points are found from Eq. (2.5.6), which, for this case, are $x^2y - y^3 + 3y = \pm 2$, where $x = \Re(z)$ and $y = \Im(z)$. In Fig. 2.24a we plot the function $\Re(h(z))$ in a neighborhood of $z = 0$, and in Fig. 2.24b the lines of fastest descent passing through the saddle points c_1 and c_2 and the contour lines of the function $\Re(h(z))$.

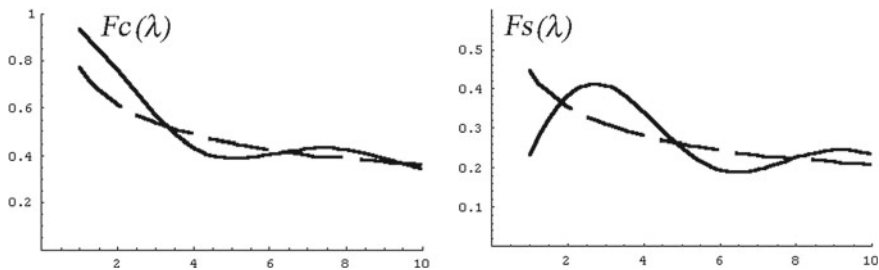


Fig. 2.23 Numerical (solid line) and asymptotic (dashed line) values of F_c and F_s

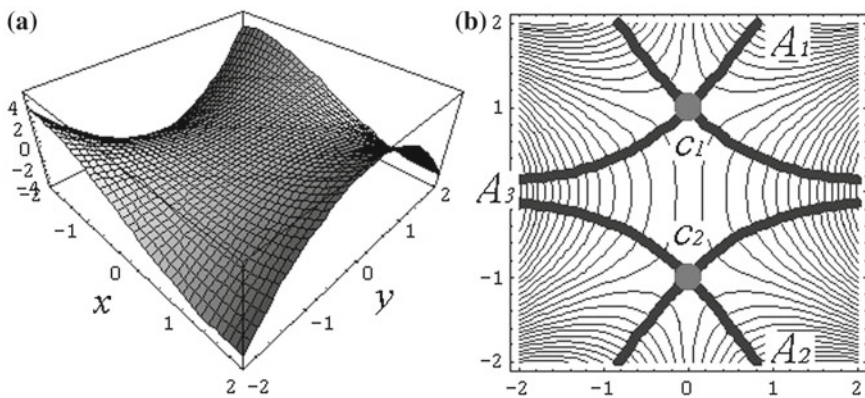


Fig. 2.24 For $\Re(h(z))$, **a** plot in a neighborhood of $z = 0$, **b** lines of fastest descent through c_1 and c_2 and contour lines

The integration paths γ_1 and γ_2 are deformed into the paths $\gamma_{10} = A_1 c_1 A_2 + A_2 c_2 A_3$ (consisting of two lines of fastest descent with the end points going to infinity) and $\gamma_{20} = A_2 c_1 A_1 + A_2 c_2 A_3$, respectively.

Let $J_k = \int_{\gamma_{k0}^*} e^{\lambda h(z)} dz$, $k = 1, 2$, be the contributions of the saddle points c_1 and c_2 , corresponding to integrations from A_2 to A_1 and from A_2 to A_3 , respectively. Taking into account that for the inverse direction the sign of the contribution J_k is reversed, one gets

$$\text{Ai}(-\eta) \simeq C_1 \sqrt{\eta} (J_2 - J_1), \quad \text{Bi}(-\eta) \simeq C_2 \sqrt{\eta} (J_2 + J_1), \quad \text{as } \eta \rightarrow \infty.$$

To calculate the contribution J_1 of the integral over the arc γ_{10}^* we change the variable (2.5.7). With $dz \simeq \exp(i\pi/4)d\tau$ one finds

$$J_1 \simeq e^{\lambda h(c_1) + i\pi/4} \int_{-\infty}^{\infty} e^{-\lambda \tau^2} d\tau = \sqrt{\pi/\lambda} e^{i(2\lambda/3 + \pi/4)}.$$

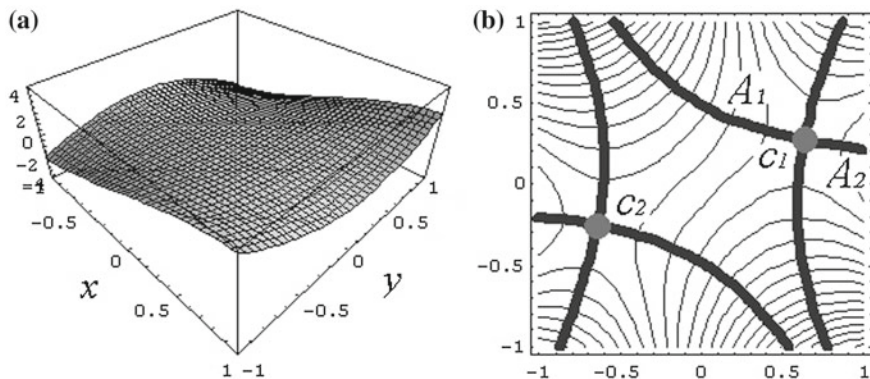
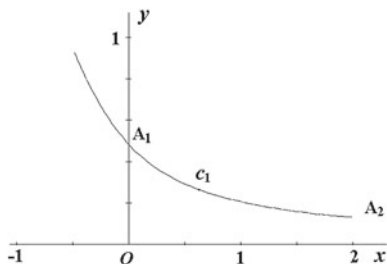


Fig. 2.25 For $\Re(h(z))$, **a** plot in a neighborhood of $z = 0$, and **b** lines of fastest descent through c_1 and c_2 and contour lines

Fig. 2.26 The path γ_0



Similarly, we obtain

$$J_2 \simeq \sqrt{\pi/\lambda} e^{-i(2\lambda/3+\pi/4)}.$$

Hence,

$$\text{Ai}(-\eta) \simeq \frac{1}{\sqrt{\pi}} \eta^{-1/4} \sin\left(\zeta + \frac{\pi}{4}\right), \quad \text{Bi}(-\eta) \simeq \frac{1}{\sqrt{\pi}} \eta^{-1/4} \cos\left(\zeta + \frac{\pi}{4}\right),$$

where $\zeta = \frac{2}{3}(-\eta)^{3/2}$ and $\lambda = (-\eta)^{3/2}$.

2.5.2. Let $h(z) = z + iz - z^3$. The equation $h'(z) = 0$ has two roots: $c_{1,2} = \pm 2^{1/4} 3^{-1/2} e^{i\pi/8}$. In Fig. 2.25a we plot the graph of the function $\Re(h(z))$ in a neighborhood of $z = 0$, and in Fig. 2.25b the lines of fastest descent through the saddle points c_1 and c_2 and the contour lines of the function $\Re(h(z))$. The equation for the line of fastest descent through c_1 has the form $y^3 - 3x^2y + x + y = 2^{7/4} 3^{-3/2} \sin(3\pi/8)$.

For the path γ_0 we chose the contour consisting of the segment OA_1 and a part of the line of fastest descent $A_1c_1A_2$ (see Fig. 2.26).

The main term of the asymptotics of $F(\lambda)$ as $\lambda \rightarrow \infty$ is equal to the contribution J_1 of the saddle point c_1 , since, compared to it, the contribution of the segment OA_1 is exponentially small.

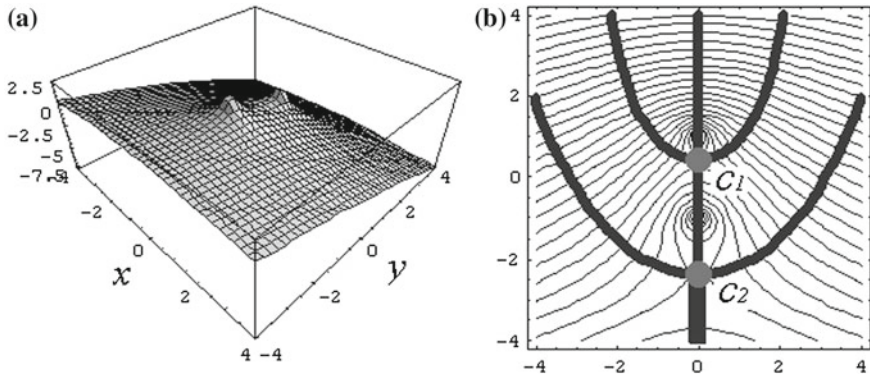


Fig. 2.27 For $\Re(h(z))$, **a** plot in a neighborhood of $z = 0$, **b** lines of fastest descent through c_1 and c_2 and contour lines

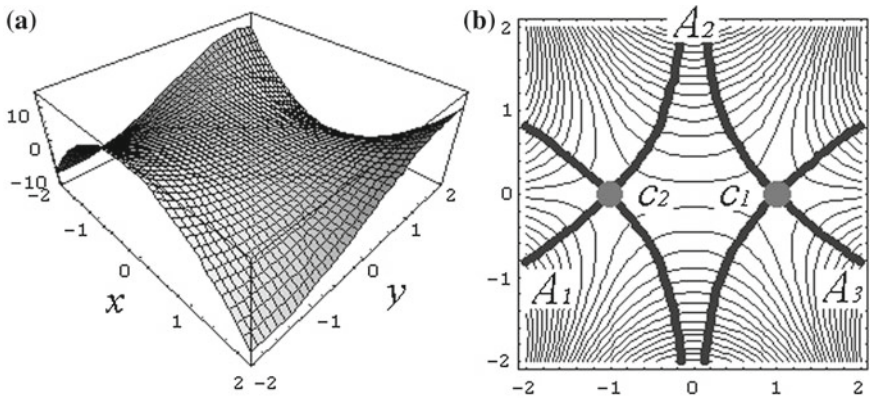


Fig. 2.28 For $\Re(h(z))$, **a** plot in a neighborhood of $z = 0$, **b** lines of fastest descent through c_1 and c_2 and contour lines

After the change of variable (2.5.7), in view of $dz \simeq 2^{-1/8} 3^{-1/4} e^{i\pi/16} d\tau$, one gets

$$F(\lambda) \simeq J_1 = \sqrt{\pi/\lambda} 2^{-1/8} 3^{-1/4} \exp \left(2^{7/4} 3^{-3/2} e^{3\pi i/8} \lambda - i\pi/16 \right).$$

2.5.3. Let $h(z) = iz - \ln(1 + z^2)$. Then $F(\lambda)$ is of the form (2.5.1) and $\varphi(z) \equiv 1$. In this case, there are two simple saddle points: $c_1 = i(\sqrt{2} - 1)$ and $c_2 = -i(\sqrt{2} + 1)$. In Fig. 2.27a we have the graph of the function $\Re(h(z))$ in a neighborhood of $z = 0$, and in Fig. 2.27b we have the lines of fastest descent passing through the saddle points c_1 and c_2 and the contour lines of the function $\Re(h(z))$. The equation for the line of fastest descent through c_1 is $\sin x(1 + x^2 - y^2) = 2xy \cos x$.

The main term of the asymptotic expansion of $F(\lambda)$ is equal to the contribution J_1 of the saddle point c_1 . Changing the variable (2.5.7) we get

$$F(\lambda) \simeq J_1 = \lambda^{-1/2} \exp(-\lambda c) (2c)^{-\lambda} [\pi(1-c)]^{1/2},$$

where $c = \sqrt{2} - 1$.

2.5.4. The saddle points for the function $h(z) = i(3z - z^3)$ are obtained from the equation $h'(z) = 0$ with roots $c_{1,2} = \pm 1$.

In Fig. 2.28a, we plot the graph of the function $\Re(h(z))$ in a neighborhood of $z = 0$, and in Fig. 2.28b the lines of fastest descent through the saddle points c_1 and c_2 and the contour lines of the function $\Re(h(z))$.

The main term of the expansion of $F(\lambda)$ is equal to the sum of the contributions of the saddle points c_1 and c_2 ,

$$F(\lambda) \simeq J_1 + J_2 = 2\pi^{1/2}(3\lambda)^{-1/2} \cos(2\lambda - \pi/4).$$

Asymptotic methods in mechanics of solids

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