Preface

Weak convergence of probability measures or, what is the same, convergence in distribution of random variables is arguably one of the most important basic concepts of asymptotic probability theory and mathematical statistics. The classical central limit theorem for sums of independent real random variables, a cornerstone of these fields, cannot possibly be thought of properly without the notion of weak convergence/convergence in distribution. Interestingly, this limit theorem as well as many others which are usually stated in terms of convergence in distribution remain true under unchanged assumptions for a stronger type of convergence. This type of convergence, called *stable convergence* with *mixing convergence* as a special case, originates from the work of Alfred Rényi more than 50 years ago and has been used by researchers in asymptotic probability theory and mathematical statistics ever since (and should not be mistaken for weak convergence to a stable limit distribution). What seems to be missing from the literature is a single comprehensive account of the theory and its consequences in applications, illustrated by a number of typical examples and applied to a variety of limit theorems. The goal of this book is to present such an account of stable convergence which can serve as an introduction to the area but does not compromise on mathematical depth and rigour.

In Chap. 1 we will give a detailed motivation for the study of stable convergence of real random variables and disclose some of its main features. With the exception of one crucial example this introductory chapter contains no proofs, but references to later chapters in which proofs can be found. It will be seen that stable convergence is best thought of as a notion of convergence for conditional distributions of random variables given sub-σ-fields of the σ-field of the underlying probability space on which the random variables are defined. Now conditional distributions are Markov kernels so that the theory of weak convergence of Markov kernels is the proper framework for stable convergence. Since we want to include limit theorems for (continuous-time) stochastic processes later on, it is reasonable to consider from the very start random variables with values in separable metrizable spaces. Therefore, we have to deal with the setting of Markov kernels from sample spaces of arbitrary probability spaces to separable metrizable spaces (which quite often are assumed to be polish). The required facts from the theory of weak convergence of such Markov kernels will be presented in Chap. 2.

In Chap. 3 the material from Chap. 2 is used to describe two approaches to stable convergence of random variables in separable metrizable spaces. In the first approach the limits of stably convergent sequences are always Markov kernels. In the second (essentially equivalent) approach the limit kernels are represented as conditional distributions of random variables. This approach allows for what might sometimes be considered as a somewhat more intuitive description of stable convergence results.

In Chap. 4 we demonstrate the usefulness of stable convergence in different areas. Our focus is on limit points of stably convergent sequences with an application to occupation times of Brownian motion and random index limit theorems as well as the empirical measure theorem and the δ-method.

Chapters 5–10 constitute in some sense the second part of the book in which it is shown that in a variety of known distributional limit theorems the convergence is actually stable or even mixing.

In Chap. 5 we discuss general conditions under which limit theorems in distribution are mixing. In particular, it turns out that the classical distributional limit theorems for centered and normalized partial sums and sample maxima of independent and identically distributed real random variables are automatically mixing.

Chapter 6 is devoted to martingale central limit theorems. Here, stable and mixing convergence is strongly dependent on the filtrations involved and the normalization used. Full stable convergence follows from a nesting condition of the filtrations. Illustrations concern martingales with stationary increments, exchangeable sequences, the Pólya urn and adaptive Monte Carlo estimators.

In Chap. 7 it is shown that the natural extension of Donsker's functional central limit theorem for partial sum processes of independent real random variables to martingale difference sequences holds with stable convergence in the metric space of all continuous real valued functions defined on the nonnegative real axis.

Chapter 8 contains a stable limit theorem for "explosive" processes with exponential rate. Since the increments of these processes are not asymptotically negligible, conditions of Lindeberg-type are not satisfied. Nevertheless, the limits can be normal, but quite different limits are also possible. This result is crucial for deriving stable limit theorems for some estimators in autoregressive processes of order one in Chap. 9 and in Galton-Watson branching processes in Chap. 10. From our point of view, these applications in two classical models of probability theory and mathematical statistics provide once more convincing illustrations of the importance of the concept of stable convergence.

Exercises appear throughout the book. We have supplied solutions of the exercises in Appendix B while Appendix A contains some basic facts about weak convergence of probability distributions, conditional distributions and martingales.

As is apparent from the brief description of its content this book is by no means meant as an encyclopedic account of all major stable limit theorems which have been established in the last 50 years or so. We tried to be reasonably complete in the basic Chap. 3 and in some sense also in Chaps. 4 and 6, but the selection of the material presented in other chapters is quite subjective. As far as our sources are concerned, we tried to give credit where credit is due, but we did not spend much time obtaining definite historical evidence in all cases. In addition to the published sources listed in the References, the first author benefitted considerably from a series of lectures on stable convergence given by David Scott at the University of Munich in the fall semester 1978/79. It is a pleasure to thank Holger Rootzén who made valuable comments on an earlier version of the manuscript. Our thanks also go to a referee for careful reading of the manuscript and for useful suggestions.

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