Chapter II

Electrodynamics

8 Maxwell's Equation

The electromagnetic field can be written as a single bivector field F in $\mathscr{D}$. Representing the charged current density by a vector J in $\mathscr{D}$, Maxwell's equation can be written[1]

$$\Box F = J. \tag{8.1}$$

Using (7.15), we can separate (8.1) into vector and pseudovector parts:

$$\Box \cdot F = J, \tag{8.2a}$$

$$\Box \wedge F = 0. \tag{8.2b}$$

If these equations are expressed in terms of a tensor basis, it is found that the coefficients show Maxwell's equation in familiar tensor form.

Alternatively, we can re-express Maxwell's equation as a set of four equations in the Pauli algebra, by singling out a particular time-like direction γ_0. Using (7.12) we can write

$$F = \mathbf{E} + i\mathbf{B} \tag{8.3a}$$

where

$$\mathbf{E} = \tfrac{1}{2}(F - F^*), \tag{8.3b}$$

$$i\mathbf{B} = \tfrac{1}{2}(F + F^*), \tag{8.3c}$$

[1] This form of Maxwell's equation is due to Marcel Riesz [9].

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and we can write J in the form

$$J = (J\gamma_0)\gamma_0 = (J \cdot \gamma_0 + J \wedge \gamma_0)\gamma_0 \equiv (\rho + \mathbf{J})\gamma_0. \tag{8.4}$$

Now, multiplying (8.1) by γ_0 and recalling (7.19), we have[2]

$$(\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + i\mathbf{B}) = \rho - \mathbf{J} \tag{8.5}$$

or

$$\partial_0\mathbf{E} + \boldsymbol{\nabla}\mathbf{E} + i(\partial_0\mathbf{B} + \boldsymbol{\nabla}\mathbf{B}) = \rho - \mathbf{J}.$$

Equating separately the scalar, vector, bivector and pseudoscalar parts in $\mathscr{P}$, we get the four equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \tag{8.6a}$$
$$\partial_0\mathbf{E} + i\boldsymbol{\nabla} \wedge \mathbf{B} = -\mathbf{J}, \tag{8.6b}$$
$$i\partial_0\mathbf{B} + \boldsymbol{\nabla} \wedge \mathbf{E} = 0, \tag{8.6c}$$
$$i\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{8.6d}$$

Using the definition (6.13) of cross product we easily convert (8.6) into a form of Maxwell's equation which is too familiar to require comment.

Equation (8.1) implies that J is a conserved current, for

$$\Box^2 F = \Box J = \Box \cdot J + \Box \wedge J. \tag{8.7}$$

Since $\Box^2$ is a scalar operator, the left side of (8.7) is a bivector, so

$$\Box^2 F = \Box \wedge J \tag{8.8}$$

and

$$\Box \cdot J = 0 = \partial_0\rho + \boldsymbol{\nabla} \cdot \mathbf{J}. \tag{8.9}$$

We can, if we wish, express F as the gradient of a vector potential,

$$F = \Box A = \Box \cdot A + \Box \wedge A. \tag{8.10a}$$

But, since F is a bivector, (8.10a) says that

$$F = \Box \wedge A \tag{8.10b}$$

and

$$\Box \cdot A = 0. \tag{8.10c}$$

[2]Equation (8.5) is equivalent to the old quaternion form for Maxwell's equation. I do not know who first invented it. A convenient early reference is [12].

Substituting (8.10a) into (8.1), we arrive at the wave equation for A:

$$\Box^2 A = J. \tag{8.11}$$

The vector potential is not unique, for we can add to it the gradient of any scalar χ for which $\Box^2\chi = 0$. Thus, if

$$A' = A + \Box\chi, \tag{8.12}$$

then

$$\Box A' = \Box A + \Box^2\chi = \Box A.$$

If we wish, we can solve for $\mathbf{E}$ and $\mathbf{B}$ in terms of potentials. Let

$$A\gamma_0 = A_0 + \mathbf{A}. \tag{8.13}$$

Then

$$\begin{aligned} F = \mathbf{E} + i\mathbf{B} = \Box A &= (\Box\gamma_0)(\gamma_0 A) \\ &= (\partial_0 - \boldsymbol{\nabla})(A_0 - \mathbf{A}) \\ &= -(\partial_0\mathbf{A} + \boldsymbol{\nabla} A_0) + \boldsymbol{\nabla}\wedge\mathbf{A}. \end{aligned}$$

So

$$\mathbf{E} = -\partial_0\mathbf{A} - \boldsymbol{\nabla} A_0, \;\; \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{8.14}$$

9 Stress-Energy Vectors

From the source-free Maxwell equation we can find four conserved vector currents. From

$$\Box F = 0, \qquad \widetilde{F}\widetilde{\Box} = 0 \tag{9.1}$$

we get[3]

$$\widetilde{F}\Box F + \widetilde{F}\widetilde{\Box} F = \partial_\mu \widetilde{F}\gamma^\mu F = 0. \tag{9.2}$$

We have used the symbol $\widetilde{\Box}$ to mean $\Box$ differentiating to the left instead of to the right. Define S^μ by

$$S^\mu = \tfrac{1}{2}\widetilde{F}\gamma^\mu F. \tag{9.3}$$

[3]In (9.2) we have assumed inertial coordinates. Generalization to arbitrary coordinates is a simple matter by the techniques of chapter V.

Since $S^\mu = \widetilde{S}^\mu = -\overline{S}^\mu$, the S^μ are vectors. Let us call them the *stress-energy vectors* of the electromagnetic field. S^μ gives the flux density of electromagnetic stress-energy through the hypersurface orthogonal to γ^μ. The components of the S^μ are

$$S^{\mu\nu} = S^\mu \cdot \gamma^\nu = (S^\mu \gamma^\nu)_S = \tfrac{1}{2}(\widetilde{F}\gamma^\mu F \gamma^\nu)_S. \tag{9.4}$$

$S^{\mu\nu}$ is the so-called *energy-momentum* tensor.[4] By writing F in terms of a bivector basis, the righthand side of (9.4) can be reduced to familiar tensor form. But (9.4) is already more felicitous than the tensor form. The following properties of $S^{\mu\nu}$ are easily verified directly from (9.4):

$$S^{\mu\nu} = S^{\nu\mu}, \quad S^\mu_\mu = 0, \quad S^{00} \geqq 0. \tag{9.5}$$

We will find it more convenient to deal with the vectors S^μ rather than the scalars $S^{\mu\nu}$. To increase our familiarity with the S^μ, let us re-express S^0 by writing F in the Pauli algebra of γ_0,

$$S^0 = S_0 = \tfrac{1}{2}\widetilde{F}\gamma_0 F = -\tfrac{1}{2}F\gamma_0 F = -\tfrac{1}{2}FF^*\gamma_0, \tag{9.6}$$

$$FF^* = (\mathbf{E} + i\mathbf{B})(-\mathbf{E} + i\mathbf{B}) = -(\mathbf{E}^2 + \mathbf{B}^2) + i\mathbf{E} \wedge \mathbf{B}. \tag{9.7}$$

Hence

$$S_0 = [\tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E} \times \mathbf{B}]\gamma_0. \tag{9.8}$$

This shows that S_0 is the space-time generalization of the *Poynting vector*.

We can write (9.2) in the form

$$\partial_\mu S^\mu = 0. \tag{9.9}$$

Since the S^μ are vectors, (9.9) gives four conserved quantities. The reader may show that (9.9) can also be written in the form

$$\Box \cdot S^\mu = 0. \tag{9.10}$$

It is easy to find the modification of (9.9) produced by sources. Since

$$\tfrac{1}{2}(\widetilde{F}J + JF) = -(FJ - JF) = -F \cdot J, \tag{9.11}$$

[4] The form $(\widetilde{F}\gamma^\mu F \gamma^\nu)_S$ for the energy-momentum tensor of the electromagnetic field was first given by Marcel Riesz [8].

we get

$$\partial_\mu S^\mu = J \cdot F. \tag{9.12}$$

$K = F \cdot J$ is the space-time generalization of the *Lorentz force*. This is easily seen by writing it as two equations in the Pauli algebra of γ_0,

$$\begin{aligned} K\gamma_0 =& K_0 + \mathbf{K}, && (9.13)\\ F \cdot J\gamma_0 &= \tfrac{1}{2}(FJ - JF)\gamma_0 \\ &= \tfrac{1}{2}(FJ\gamma_0 - J\gamma_0 F^*) \\ &= \tfrac{1}{2}[(\mathbf{E} + i\mathbf{B})(\rho + \mathbf{J}) - (\rho + \mathbf{J})(-\mathbf{E} + i\mathbf{B})] \\ &= \rho\mathbf{E} + \mathbf{J} \cdot \mathbf{E} + \mathbf{J} \times \mathbf{B}. && (9.14) \end{aligned}$$

By separately equating the scalar and vector parts of (9.13) and (9.14), we arrive at

$$K_0 = \mathbf{J} \cdot \mathbf{E}, \tag{9.15}$$

$$\mathbf{K} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{9.16}$$

10 Invariants

The invariants of the electromagnetic field are given by F^2 The square of a bivector has only scalar and pseudoscalar parts,

$$F^2 = F \cdot F + F \wedge F. \tag{10.1}$$

To see the invariants in tensor form we must express F in terms of a bivector basis: $F = \frac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}$. Using (3.15), we find for the scalar part

$$F \cdot F = -\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu}. \tag{10.2}$$

Using (A.30), we find for the pseudoscalar part

$$F \wedge F = -F^{\alpha\beta}F^{\mu\nu}\varepsilon_{\alpha\beta\mu\nu}i. \tag{10.3}$$

Alternatively, we may express the invariants in terms of $\mathbf{E}$ and $\mathbf{B}$,

$$F^2 = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E} \cdot \mathbf{B}. \tag{10.4}$$

The decomposition $F = \mathbf{E} + i\mathbf{B}$ depends on a particular timelike vector. It would seem better to decompose F using invariants of the field. We accomplish this here for the case $F^2 \neq 0$. First we note that F can be written

$$F = fe^{i\phi} \tag{10.5}$$

where ϕ is a scalar and

$$f = \mathbf{e} + i\mathbf{b} \tag{10.6a}$$

with

$$\mathbf{e} \cdot \mathbf{b} = 0. \tag{10.6b}$$

Now we observe that when $F^2 \neq 0$ both ϕ and f^2 are determined by F^2, for

$$F^2 = f^2 e^{2i\phi} = (\mathbf{e}^2 - \mathbf{b}^2)e^{2i\phi}. \tag{10.7}$$

Here F^2 is represented as a scalar f^2 which is taken through an angle 2ϕ by a duality rotation to give the scalar and pseudoscalar parts of F^2.

Let us express the perhaps more significant invariants ϕ and f^2 in terms of $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B}$. The invariant of $F^* = -\mathbf{E} + i\mathbf{B}$ is

$$(F^*)^2 = (F^2)^* = f^2 e^{-2i\phi} = \mathbf{E}^2 - \mathbf{B}^2 - 2i\mathbf{E} \cdot \mathbf{B}. \tag{10.8}$$

Hence

$$(F^*)^2 F^2 = f^4 = (\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2. \tag{10.9}$$

So

$$f^2 = \pm[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2]^{\frac{1}{2}}. \tag{10.10}$$

We can find another interesting expression for f^2 by noting that

$$F^* F = -(\mathbf{E}^2 + \mathbf{B}^2) + 2\mathbf{E} \times \mathbf{B}. \tag{10.11}$$

Since $\mathbf{E} \times \mathbf{B}$ anticommutes with both $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{aligned} f^4 &= F^* F^* F F = (F^* F)(F^* F)^* \\ &= (\mathbf{E}^2 + \mathbf{B}^2) - 4(\mathbf{E} \times \mathbf{B})^2. \end{aligned}$$

So

$$f^2 = \pm[(\mathbf{E}^2 + \mathbf{B}^2)^2 - 4(\mathbf{E} \times \mathbf{B})^2]^{\frac{1}{2}}. \tag{10.12}$$

The expression for ϕ is found to be

$$\tan 2\phi = \frac{2\mathbf{E} \cdot \mathbf{B}}{\mathbf{E}^2 - \mathbf{B}^2}. \tag{10.13}$$

Evidently ϕ is determined by $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E} \cdot \mathbf{B}$ only when $F^2 \neq 0$ and then only to an additive multiple of π. It is natural to interpret ϕ as the physical *phase* of the electromagnetic field. In the next section we will do this for a free field. Other authors [13] have called the ϕ the complexion of the Maxwell field. It seems that no one has pinned down its physical manifestations. We note here that it does not contribute to the stress-energy vectors, for, since i anticommutes with γ^μ,

$$S^\mu = \tfrac{1}{2}\widetilde{F}\gamma^\mu F = \tfrac{1}{2}\widetilde{f}\gamma^\mu f. \tag{10.14}$$

11 Free Fields

The reader has probably noticed that the source free Maxwell Equation has the same form as the Dirac equation for a free neutrino field,

$$\Box F = 0. \tag{11.1}$$

It is therefore not surprising that we can describe the polarization of a photon in the same way that we describe the polarization of a neutrino. Just the same, it is worthwhile to briefly discuss photon polarization, in order to see what special insights are provided by the Dirac algebra. Consider the plane wave solutions of (11.1):

$$F(x) = fe^{ik\cdot x}. \tag{11.2}$$

Here $x = x^\mu\gamma_\mu$ is the position vector in Minkowski space-time, $k = k^\mu\gamma_\mu$ is a constant vector and f is a constant bivector. Multiply (11.1) by γ_0 to obtain an equation involving only p-numbers:

$$(\partial_0 + \boldsymbol{\nabla})F = 0. \tag{11.3}$$

Let $k\gamma_0 = k_0 + \mathbf{k}$, and substitute (11.2) into (11.3) to get the equation

$$\mathbf{k}f = k_0 f. \tag{11.4}$$

We can multiply (11.4) by $k_0 + \mathbf{k}$ to find $k_0 = \pm|\mathbf{k}|$ (i.e. $k^2 = 0$). These two solutions correspond to photons which are right or left circularly polarized, or, in different words, photons with positive or negative helicity. We can interpret the solution with $k_0 = |\mathbf{k}|$ as a particle, and the solution with $k_0 = -|\mathbf{k}|$ as an antiparticle. This allows us to identify the operation of space conjugation (defined in section 6) with antiparticle conjugation, for, applying (7.12) to (11.4), we get

$$-\mathbf{k}f^* = k_0 f^*. \tag{11.5}$$

In section 13 we will see that the same interpretation holds for the Dirac equation with mass.

When the neutrino is described by the Dirac equation a side condition is necessary to limit the number of solutions. The corresponding condition on the electromagnetic field is

$$F = \overline{F}. \tag{11.6}$$

This just follows from the fact that F is a bivector. The condition that F be a bivector can be written

$$F = \overline{F} = -\widetilde{F}. \tag{11.7}$$

We can use the bivector property of f to get more detailed information from (11.4). As in (8.3), we write

$$f = \mathbf{e} + i\mathbf{b}. \tag{11.8}$$

Equation (11.4) can be split into the two equations

$$k_0\,\mathbf{e} = i\mathbf{k}\mathbf{b}, \tag{11.9a}$$

$$k_0\,\mathbf{b} = -i\mathbf{k}\mathbf{e}. \tag{11.9b}$$

It follows that

$$\mathbf{k} = k_0\,\hat{\mathbf{e}} \times \hat{\mathbf{b}}, \tag{11.10a}$$

$$\mathbf{k}\cdot\mathbf{e} = \mathbf{k}\cdot\mathbf{b} = 0, \tag{11.10b}$$

$$\mathbf{e}\cdot\mathbf{b} = 0, \tag{11.10c}$$

$$\mathbf{e}^2 = \mathbf{b}^2. \tag{11.10d}$$

The last two conditions can be written $f^2 = 0$. This implies

$$F^2 = 0, \tag{11.11}$$

an invariant condition that F represents a field of circularly polarized radiation.

Let us close this section with a geometrical interpretation of the circularly polarized plane wave solutions (11.2). Operating on f, $e^{ik\cdot x}$ is a duality rotation which rotates the vector $\mathbf{e}$ into a bivector and rotates the bivector $i\mathbf{b}$ into a vector. Thus we get the picture of the electric and magnetic vectors spinning about the momentum vector $\mathbf{k}$ as the plane wave progresses, but this "spinning" is due to a duality transformation rather than the usual kind of spatial rotation.

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