

# Preface after Fifty years

This book launched my career as a theoretical physicist fifty years ago. I am most fortunate to have this opportunity for reflecting on its influence and status today. Let me begin with the title *Space-Time Algebra*. John Wheeler's first comment on the manuscript about to go to press was "Why don't you call it *Spacetime Algebra*?" I have followed his advice, and *Spacetime Algebra* (STA) is now the standard term for the mathematical system that the book introduces.

I am pleased to report that STA is as relevant today as it was when first published. I regard nothing in the book as out of date or in need of revision. Indeed, it may still be the best quick introduction to the subject. It retains that first blush of compact explanation from someone who does not know too much. From many years of teaching I know it is accessible to graduate physics students, but, because it challenges standard mathematical formalisms, it can present difficulties even to experienced physicists.

One lesson I learned in my career is to be bold and explicit in making claims for innovations in science or mathematics. Otherwise, they will be too easily overlooked. Modestly presenting evidence and arguing a case is seldom sufficient. Accordingly, with confidence that comes from decades of hindsight, I make the following **Claims** for STA as formulated in this book:

- (1) STA enables a unified, **coordinate-free** formulation for all of relativistic physics, including the Dirac equation, Maxwell's equation and General Relativity.
- (2) Pauli and Dirac matrices are represented in STA as **basis vectors** in space and spacetime respectively, with no necessary connection to spin.

- (3) STA reveals that the **unit imaginary** in quantum mechanics has its origin in spacetime geometry.
- (4) STA reduces the mathematical divide between classical, quantum and relativistic physics, especially in the use of **rotors** for rotational dynamics and gauge transformations.

Comments on these claims and their implications necessarily refer to contents of the book and ensuing publications, so the reader may wish to reserve them for later reference when questions arise.

Claim (2) expresses the crucial secret to the power of STA. It implies that the physical significance of Dirac and Pauli algebras stems entirely from the fact that they provide algebraic representation of geometric structure. Their representations as matrices are irrelevant to physics—perhaps inimical, because they introduce spurious complex numbers without physical significance. The fact that they are spurious is established by claim (3).

The crucial geometric relation between Dirac and Pauli algebras is specified by equations (7.9) to (7.11) in the text. It is so important that I dubbed it **space-time split** in later publications. Note that the symbol  $i$  is used to denote the *pseudoscalar* in (6.3) and (7.9). That symbol is appropriate because  $i^2 = -1$ , but it should not to be confused with the imaginary unit in the matrix algebra.

Readers familiar with Dirac matrices will note that if the gammas on the right side of (7.9) are interpreted as Dirac's  $\gamma$ -matrices, then the objects on the left side must be Dirac's  $\alpha$ -matrices, which, as is seldom recognized, are  $4 \times 4$  matrix representations of the  $2 \times 2$  Pauli matrices. That is a distinction without physical or geometric significance, which only causes unnecessary complications and obscurity in quantum mechanics. STA eliminates it completely.

It may be helpful to refer to STA as the “Real Dirac Algebra”, because it is isomorphic to the algebra generated by Dirac  $\gamma$ -matrices over the field of real numbers instead of the complex numbers in standard Dirac theory. Claim (3) declares that only the real numbers are needed, so standard Dirac theory has a superfluous degree of freedom. That claim is backed up in Section 13 of the text where Dirac's equation is given two different but equivalent formulations within STA. In equation (13.1) the role of unit imaginary is played by the pseudoscalar  $i$ , while, in equation (13.13) it is played by a spacelike bivector.

Thus, the mere reformulation of the Dirac equation in terms of STA automatically assigns a geometric meaning to the unit imaginary in quantum mechanics! I was stunned by this revelation, and I set out immediately to ascertain what its physical implications might be. Before the ink was dry as the STA book went to press, I had established in [1] that (13.13) was the most significant form for the Dirac equation, with the bivector unit imaginary related to spin in an intriguing way. This insight has been a guiding light for my research into geometric foundations for quantum mechanics ever since. The current state of this so-called “Real Dirac Theory” is reviewed in [2, 3].

Concerning Claim (4): Rotors are mathematically defined by (16.7) and (16.8), but I failed to mention there what has since become a standard name for this important concept. Rotors are used for an efficient coordinate-free treatment of Lorentz transformations in Sections 16, 17 and 18. That provides the foundation for the *Principle of Local Relativity* formulated in Section 23. It is an essential gauge principle for incorporating Dirac spinors into General Relativity. That fact is demonstrated in a more general treatment of gauge transformations in Section 24.

The most general gauge invariant derivative for a spinor field in STA is given by equations (24.6) and (24.12). The “C connection” is the coupling to the gravitational field, while “D connection” in (24.16) was tentatively identified with strong interactions. I was very suspicious of that tentative identification at the time. Later, when the electroweak gauge group became well established, I reinterpreted the D connection as electroweak [4]. That has the great virtue of grounding electroweak interactions in the spacetime geometry of STA. The issue is most thoroughly addressed in [5]. However, a definitive argument or experimental test linking electroweak interactions to geometry in this way remains to be found.

Finally, let me return to Claim (1) touting STA as a unified mathematical language for all spacetime physics. The whole book makes the case for that Claim. However, though STA provides coordinate-free formulations for the most fundamental equations of physics, solving those equations with standard coordinate-based methods required taking them apart and thereby losing the advantage of invariant formulation. To address that problem, soon after this book was published,

I set forth on the huge task of reformulating standard mathematical methods. That was a purely mathematical enterprise (but with one eye on physics). I generalized STA to create a coordinate-free *Geometric Algebra* (GA) and *Geometric Calculus* (GC) for all dimensions and signatures. There were already plenty of clues in STA on how to do it. In particular, Section 22 showed how to define a vector derivatives and integrals that generalize the concept of differential form. Another basic task was to reformulate linear algebra to enable coordinate-free calculations with GA. The outcome of this initiative, after two decades, is the book [6]. Also during this period I demonstrated the efficiency of GA in introductory physics and Classical Mechanics, as presented in my book [7].

By fulfilling the four Claims just discussed, STA initiated developments of GA into a comprehensive mathematical system with a vast range of applications that is still expanding today. These developments fall quite neatly into three Phases. Phase I covers the first two decades, when I worked alone with assistance of my students, principally Garret Sobczyk, Richard Gurtler and Robert Hecht-Nielsen. This work attracted little notice in the literature, with the exception of mathematician Roget Boudet, who promoted it enthusiastically in France.

Phase I was capped by my two talks [8, 9] at a NATO conference on Clifford Algebras organized by Roy Chisholm. That conference also initiated Phase II and a steady stream of similar conferences still flowing today. Let me take this opportunity to applaud the contribution of Chisholm and the other conference organizers who selflessly devote time and energy to promoting the flow of scientific ideas. This essential social service to science gets too little recognition.

The high point of Phase II was the publication of *Gauge Theory Gravity* by Anthony Lasenby, Chris Doran and Steve Gull [10]. I see this as a fundamental advance in spacetime physics and a capstone of STA [11]. Phase II was capped with the comprehensive treatise [12] by Doran and Lasenby.

Phase III was launched by my presentation of *Conformal Geometric Algebra* (CGA) in July 1999 at a conference in Ixtapa, Mexico [13]. After stimulating discussions with Hongbo Li, Alan Rockwood and Leo Dorst, diverse applications of GA published during Phase II

had congealed quite quickly in my mind into a sharp formulation of CGA. Response to my presentation was immediate and strong, initiating a steady stream of CGA applications to computer science [14] and engineering (especially robotics). In physics, CGA greatly simplifies the treatment of the crystallographic space groups [15], and applications are facilitated by the powerful Space Group Visualizer created by Echard Hitzer and Christian Perwass [16].

I count myself as much mathematician as physicist, and I see development of GA from STA to GC and CGA as reinvigorating the mathematics of Hermann Grassmann and William Kingdon Clifford with an infusion of twentieth century physics. The history of this development and the present status of GA has been reviewed in [17]. I am sorry to say that few mathematicians are prepared to recognize the central role of GA in their discipline, because it has become increasingly insular and divorced from physics during the last century.

Personally, I dedicate my work with GA to the memory of my mathematician father, Magnus Rudolph Hestenes, who was always generous with his love but careful with his praise [9].

## References

- [1] D. Hestenes, Real Spinor Fields, *J. Math. Phys.* **8**, 798–808 (1967).
- [2] D. Hestenes, Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics. *Am. J. Phys.* **71**, 104–121 (2003).
- [3] D. Hestenes, Spacetime Physics with Geometric Algebra, *Am. J. Phys.* **71**, 691–714 (2003).
- [4] D. Hestenes, Space-Time Structure of Weak and Electromagnetic Interactions, *Found. Phys.* **12**, 153–168 (1982).
- [5] D. Hestenes, Gauge Gravity and Electroweak Theory. In: H. Klei-  
nert, R.T. Jantzen and R. Ruffini (Eds.), *Proceedings of the Ele-  
venth Marcel Grossmann Meeting on General Relativity* (World  
Scientific: Singapore, 2008), pp. 629–647.
- [6] D. Hestenes and G. Sobczyk, *CLIFFORD ALGEBRA to GEO-  
METRIC CALCULUS, A Unified Language for Mathematics and  
Physics* (Kluwer: Dordrecht/Boston, 1984).

- [7] D. Hestenes, *New Foundations for Classical Mechanics* (Kluwer: Dordrecht/Boston, 1986).
- [8] D. Hestenes, A Unified Language for Mathematics and Physics. In: J.S.R. Chisholm and A.K. Common (Eds.), *Clifford Algebras and their Applications in Mathematical Physics* (Reidel: Dordrecht/Boston, 1986), pp. 1–23.
- [9] D. Hestenes, Clifford Algebra and the Interpretation of Quantum Mechanics. In: J.S.R. Chisholm and A.K. Common (Eds.), *Clifford Algebras and their Applications in Mathematical Physics* (Reidel: Dordrecht/Boston, 1986), pp. 321–346.
- [10] A. Lasenby, C. Doran and S. Gull, Gravity, gauge theories and geometric algebra, *Phil. Trans. R. Lond. A* **356**, 487–582 (1998).
- [11] D. Hestenes, Gauge Theory Gravity with Geometric Calculus, *Foundations of Physics* **36**, 903–970 (2005).
- [12] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge: Cambridge University Press, 2003).
- [13] D. Hestenes, Old Wine in New Bottles: A new algebraic framework for computational geometry. In: E. Bayro-Corrochano and G. Sobczyk (Eds), *Advances in Geometric Algebra with Applications in Science and Engineering* (Birkhäuser: Boston, 2001), pp. 1–14.
- [14] L. Dorst, D. Fontijne and S. Mann, *Geometric Algebra for Computer Science* (Morgan Kaufmann, San Francisco, 2007).
- [15] D. Hestenes and J. Holt, The Crystallographic Space Groups in Geometric Algebra, *Journal of Mathematical Physics* **48**, 023514 (2007).
- [16] E. Hitzer and C. Perwas: <http://www.spacegroup.info>
- [17] D. Hestenes, Grassmann’s Legacy. In: H.-J. Petsche, A. Lewis, J. Liesen and S. Russ (Eds.), *From Past to Future: Grassmann’s Work in Context* (Birkhäuser: Berlin, 2011).

# Preface

This book is an attempt to simplify and clarify the language we use to express ideas about space and time. It was motivated by the belief that such is an essential step in the endeavor to simplify and unify our theories of physical phenomena. The object has been to produce a “space-time calculus” which is ready for physicists to use. Particular attention has been given to the development of notation and theorems which enhance the geometric meaning and algebraic efficiency of the calculus; conventions have been chosen to differ as little as possible from those in formalisms with which physicists are already familiar.

Physical concepts and equations which have an intimate connection with our notions of space-time are formulated and discussed. The reader will find that the “space-time algebra” introduces novelty of expression and interpretation into every topic. This naturally suggests certain modifications of current physical theory; some are pointed out in the text, but they are not pursued. The principle objective here has been to formulate important physical ideas, not to modify or apply them.

The mathematics in this book is relatively simple. Anyone who knows what a vector space is should be able to understand the algebra and geometry presented in chapter I. On the other hand, appreciation of the physics involved in the remainder of the book will depend a great deal on the reader’s prior acquaintance with the topics discussed.

This work was completed in 1964 while I was at the University of California at Los Angeles. It was supported by a grant from the National Science Foundation—for which I am grateful.

I wish to make special mention of my debt to Marcel Riesz, whose trenchant discussion of Clifford numbers and spinors (8, 9) provided an initial stimulus and some of the key ideas for this work. I would

also like to express thanks to Professors V. Bargmann, J. A. Wheeler and A. S. Wightman for suggesting some improvements in the final manuscript and to Richard Shore and James Sock for correcting some embarrassing errors.

DAVID HESTENES  
Palmer Laboratory  
Princeton University





<http://www.springer.com/978-3-319-18412-8>

Space-Time Algebra

Hestenes, D.

2015, XXIV, 102 p., Hardcover

ISBN: 978-3-319-18412-8

A product of Birkhäuser Basel