

Chapter 2

The Higgs Boson in the Standard Model of Particle Physics

2.1 The Principle of Gauge Symmetries

The principle of gauge symmetries can be motivated by the Lagrangian density of the free *Dirac* field, which is covariant under global $U(1)$ gauge transformations of the complex phase of the spinor fields, ψ :

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = e^{i\vartheta} \psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{-i\vartheta} \\ \mathcal{L}'(x) &= \bar{\psi}'(i\gamma^\mu \partial_\mu - m) \psi'(x) = \bar{\psi} e^{-i\vartheta} (i\gamma^\mu \partial_\mu - m) e^{i\vartheta} \psi(x) \quad (2.1) \\ &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi(x) = \mathcal{L}(x)\end{aligned}$$

The $U(1)$ transformation, which only acts on the components of the spinor and not on its arguments is usually called an internal symmetry of the field. The global character of this transformation is imposed by the fact that the phase ϑ does not depend on $\mathbf{x}$ or t . The free choice of ϑ , which is inherent to the equations of motion of the *Dirac* field still requires the phase to be the same at any point in space-time. Extending this *global* symmetry to a *local* symmetry, where ϑ is allowed to be different in any coordinate in space-time ($\vartheta \rightarrow \vartheta(x)$) appears natural, but breaks the covariance of the equation, due to the derivative that appears in the *Dirac* equation:

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = e^{i\vartheta(x)} \psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) e^{-i\vartheta(x)} \\ \mathcal{L}'(x) &= \bar{\psi}'(i\gamma^\mu \partial_\mu - m) \psi'(x) = \bar{\psi} e^{-i\vartheta} (i\gamma^\mu \partial_\mu - m) e^{i\vartheta} \psi(x) \quad (2.2) \\ &= \bar{\psi} (i\gamma^\mu (\partial_\mu + i\partial_\mu \vartheta) - m) \psi(x) \neq \mathcal{L}(x)\end{aligned}$$

The covariance can however be restored and in fact enforced, by replacing the normal partial derivative by the *covariant derivative* $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$,¹ where an additional degree of freedom is introduced by the gauge field A_μ , into which the covariance breaking term, $i\partial_\mu\vartheta$, can be absorbed. From the imposed covariance requirement on the Lagrangian density

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = e^{i\vartheta(x)}\psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\vartheta(x)} \\ D_\mu\psi &\rightarrow (D_\mu\psi)'(x) = e^{i\vartheta(x)}(D_\mu\psi)(x)\end{aligned}\tag{2.3}$$

the transformation behavior of both the covariant derivative, D_μ , and the gauge field, A_μ , can be derived:

$$\begin{aligned}(D_\mu\psi)'(x) &= (\partial_\mu + ieA'_\mu)e^{i\vartheta(x)}\psi = e^{i\vartheta(x)}(\partial_\mu + i\partial_\mu\vartheta(x) + ieA'_\mu)\psi \\ &\equiv e^{i\vartheta(x)}(D_\mu\psi)(x) = e^{i\vartheta(x)}(\partial_\mu + ieA_\mu)\psi\end{aligned}\tag{2.4}$$

$$\begin{aligned}A_\mu &\rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\vartheta(x) \\ D_\mu &\rightarrow D'_\mu = D_\mu - i\partial_\mu\vartheta(x)\end{aligned}$$

The transformation behavior of the gauge field, A_μ , is known from electrodynamics. In the physics interpretation A_μ can be identified with a mediating particle, that introduces an interaction between fermions with a coupling constant e . It mediates the information of change in phase of the *Dirac* spinors between two different points x_μ and x'_μ in the four dimensional space-time, and thus obtains a geometrical interpretation. With the picture of electrodynamics in mind, A_μ can be identified with the photon field.

To obtain a dynamic field the Lagrangian density needs to be completed by a term that describes the dynamic behavior of the gauge field. This term should be gauge and *Lorentz* covariant, which is true for the square of the field strength tensor, defined as $\mathcal{L}_{\text{kin}}$ in:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{i}{e}[D_\mu, D_\nu]\tag{2.5}$$

¹Note that in classical formulations D_μ is sometimes introduced as $D_\mu = \partial_\mu - ieA_\mu$. Here it will be introduced with a “+” sign to keep consistency with the canonical formulation of the SM later on.

The *Lorentz* invariance of $\mathcal{L}_{\text{kin}}$ is obvious from the contraction of the *Lorentz* indices. The gauge invariance is demonstrated below:

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \partial_\mu A_\nu - \frac{1}{e} \partial_\mu \partial_\nu \vartheta - \partial_\nu A_\mu + \frac{1}{e} \partial_\nu \partial_\mu \vartheta = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

It should be pointed out that the field strength tensor can be obtained from the commutator, $[\cdot, \cdot]$, of the covariant derivative, as shown in Eq. (2.5). It should also be pointed out that A_μ only appears up to second order in $\mathcal{L}_{\text{kin}}$. This has the important consequence that for the electric field, that will be derived from A_μ in the *Dirac* equation, the fundamental principle of linear superposition is obeyed. The full Lagrangian density of a fermion field with interaction reads as:

$$\begin{aligned} \mathcal{L}(x) &= \bar{\psi} (i\gamma^\mu D_\mu - m) \psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= \bar{\psi} (i\gamma^\mu (\partial_\mu + ieA_\mu) - m) \psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= \underbrace{\bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi(x)}_{\text{free fermion field}} - \underbrace{e\bar{\psi}\gamma^\mu \psi A_\mu}_{\text{IA}} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{gauge}} \end{aligned} \quad (2.6)$$

where, after some re-ordering of terms, the first part of the Lagrangian density corresponds to the propagation of the free fermion field, the second term (labeled as “IA”) to the interaction with the photon field and the third term (labeled as “gauge”) to the propagation of the free photon field. The canonical variation just of the kinetic term, $\mathcal{L}_{\text{kin}}$, of the gauge field returns the *Lorentz* covariant formulation of the *Maxwell* equations in the absence of matter

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

varied by the field A_μ (2.7)

$$\partial_\mu \left(\frac{\delta \mathcal{L}_{\text{kin}}}{\delta \partial_\mu A_\nu} \right) - \frac{\delta \mathcal{L}_{\text{kin}}}{\delta A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = 0$$

which in the *Lorentz* gauge ($\partial_\mu A^\mu = 0$) further on leads to the *Klein-Gordon* equation for the propagation of a free and massless boson field:

$$\begin{aligned} \partial_\nu \partial^\nu A^\mu - \underbrace{\partial^\nu \partial_\mu A^\mu}_{=0} &= 0; \quad (\partial_\nu \partial^\nu - 0) A^\mu = 0 \\ \partial_\mu A^\mu &= 0 \end{aligned} \quad (2.8)$$

From the transformation behavior $A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \vartheta(x)$ it is evident that a mass term for the gauge boson, A_μ , of type

$$\frac{m}{2} A'_\mu A'^\mu \equiv \frac{m}{2} A_\mu A^\mu - \frac{m}{e} A^\mu \partial_\mu \vartheta(x) + \frac{m}{2e^2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) \neq \frac{m}{2} A_\mu A^\mu \quad (2.9)$$

in the Lagrangian density breaks local gauge invariance. This is a manifestation of the deeper truth that in naive gauge theories the gauge fields *a priori* have to be massless. Note that the mass term for the fermion field, ψ , of type $\bar{\psi} m \psi$ in Eq. (2.6) does not break gauge invariance.

2.1.1 Extension to Non-Abelian Gauge Symmetries

The above explanations refer to the special case of the $U(1)$ symmetry, corresponding to an *Abelian* symmetry group, for which the ordering of the operators is irrelevant. Any extension of this symmetry group, e.g. to higher dimensions, leads to *non-Abelian* symmetry transformations, for which this is not the case any more. The $SU(2)$ symmetry (which is isomorphic to the well known $\mathcal{O}(3)$ symmetry group of three dimensional rotations) or the $SU(3)$ symmetry are typical examples in particle physics: the $SU(3)_C$ color symmetry leads to the formulation of *Quantum Chromodynamics (QCD)*; the $SU(2)_L$ flavor symmetry of the electroweak isospin will be discussed in the following sections. A general representation of higher dimensional isospin vectors is

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \quad n \geq N$$

which in principle can be of any dimension $n \geq N$ (where N is the dimension of the symmetry group). In the examples given above, the isospin vectors would be $\psi_q = (q_r \ q_g \ q_b)^\top$ for a quark triplet in the case of the strong color isospin, where the components correspond to the three degrees of freedom representing the color charge of the quark (red, green or blue). It would be $\psi_\ell = (\nu_\ell \ \ell)^\top$ for a lepton doublet, in the case of the weak isospin, where ν_ℓ corresponds to a neutrino and ℓ to the corresponding charged lepton field.

To investigate the extension of local gauge symmetries to *non-Abelian* symmetry groups the further discussion will be concentrated on the $SU(2)_L$, with a spinor, ψ , with the components ψ_α and a transformation matrix $G \in SU(2)_L$. The matrix G has an adjoint matrix $G^\dagger : GG^\dagger = 1_2$, which corresponds to the transposed and complex conjugate. As G is an element of a *Lie* group, it can be expressed by its tangential space, spanned by the $(N^2 - 1) = 3$ generators $t_a : G = e^{i\vartheta_a t_a}$, where

ϑ_a are three continuous parameters.² A typical irreducible representation of these generators in the minimal dimension, two, are the *Pauli* matrices:

$$t_1 = \frac{\sigma_1}{2} = \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad t_2 = \frac{\sigma_2}{2} = \frac{1}{2} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad t_3 = \frac{\sigma_3}{2} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the *non-Abelian* character of the transformation group is resembled by the fact, that the generator matrices t_a are non-commuting. For the $SU(2)$ the commutation relations take the general form:

$$[t_a, t_b] = t_a t_b - t_b t_a = i \epsilon_{abc} t_c \neq 0 \quad (2.10)$$

where the totally anti-symmetric *Levi-Civita* tensor ϵ_{abc} corresponds to the structure constants of the $SU(2)$. For different (e.g. higher dimensional) representations also different matrices could be defined, with the only requirement that these new representations should be irreducible and fulfill the commutation relations of Eq. (2.10). For infinitesimal transformations $\vartheta \equiv \vartheta_a t_a$, the matrices G and $G^\dagger$ can be expressed by

$$\begin{aligned} G &= 1_2 + i \vartheta_a t_a = 1_2 + i \vartheta \\ G^\dagger &= 1_2 - i \vartheta_a t_a = 1_2 - i \vartheta \end{aligned}$$

which corresponds to the lowest non-trivial *Taylor* expansion terms of the exponential function. It should be noted that for *non-Abelian* formulations from this point on ϑ corresponds to a matrix composed as a linear combination of the generators t_a with continuous parameters ϑ_a , while for the *Abelian* case ϑ was just a single continuous parameter. The proper choice has to be derived from the context. In the *non-Abelian* case the gauge transformations of Eq. (2.1) take the form:

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = G(x) \psi(x) = e^{i \vartheta_a(x) t_a} \psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) G^\dagger(x) = \bar{\psi}(x) e^{-i \vartheta_a(x) t_a} \\ \mathcal{L}'(x) &= \bar{\psi}' (i \gamma^\mu \partial_\mu - m) \psi'(x) = \bar{\psi} e^{-i \vartheta_a(x) t_a} (i \gamma^\mu \partial_\mu - m) e^{i \vartheta_a(x) t_a} \psi(x) \quad (2.11) \\ &\equiv \bar{\psi} (i \gamma^\mu D_\mu - m) \psi(x) = \mathcal{L}(x) \end{aligned}$$

which now contains the transformation matrix G , which itself can depend on the spacial coordinates x via the continuous parameters $\vartheta_a(x)$. Note that the spinor, ψ , now has components α (taking e.g. the values 0 and 1) and that trivial terms like $i \gamma^\mu$ or m implicitly have to be extended by a unit matrix 1_2 . The covariant derivative takes the form $\partial_\mu \rightarrow D_\mu = \partial_\mu + i g t_a A_\mu^a = \partial_\mu + i g A_\mu$, where $A_\mu \equiv t_a A_\mu^a$ again

²Note that the book follows the convention to sum over re-appearing identical indices as in the case of the *Einstein* sum convention.

corresponds to a 2×2 -matrix, built from a linear combination of the *Pauli* matrices. From the covariance requirement on the Lagrangian density

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = G(x)\psi(x) = e^{i\vartheta_a(x)t_a}\psi(x) \\ \bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x)G^\dagger(x) = \bar{\psi}(x)e^{-i\vartheta_a(x)t_a} \\ D_\mu\psi &\rightarrow (D_\mu\psi)'(x) = G(x)(D_\mu\psi)(x) = e^{i\vartheta_a(x)t_a}(D_\mu\psi)(x)\end{aligned}\tag{2.12}$$

again the modification of the transformation behavior of the covariant derivative, D_μ , and of the gauge field, A_μ , can be obtained:

$$\begin{aligned}(D_\mu\psi)'(x) &= (\partial_\mu + igA'_\mu)\psi'(x) = (\partial_\mu + igA'_\mu)G\psi(x) \\ &= (\partial_\mu G + G\partial_\mu + igA'_\mu G)\psi(x) \\ &\equiv G(D_\mu\psi)(x) = G(\partial_\mu + igA_\mu)\psi(x) = (G\partial_\mu + igGA_\mu)\end{aligned}$$

Note that on the left-hand side of the “ $\equiv$ ” sign in the above equation the non-commuting operator G is multiplied from the right, while on the right-hand side of the equation, it is multiplied from the left. The transformation behavior of A_μ turns out to be:

$$\begin{aligned}A_\mu &\rightarrow A'_\mu = GA_\mu G^\dagger + \frac{i}{g}(\partial_\mu G)G^\dagger \\ A_\mu &\rightarrow A'_\mu = A_\mu + i[\vartheta, A_\mu] - \frac{1}{g}\partial_\mu\vartheta\end{aligned}\tag{2.13}$$

where the second formulation of Eq.(2.13) corresponds to infinitesimal transformations ϑ and $[\cdot, \cdot]$ again to the commutator. Note that the first term $GA_\mu G^\dagger$ acts like a coordinate transformation in the $SU(2)$ isospace that mixes the components of A_μ . This transformation behavior is called *adjoint representation*. The transformation behavior for the covariant derivative D_μ also follows the *adjoint representation*

$$\begin{aligned}D_\mu &\rightarrow D'_\mu = GD_\mu G^\dagger \\ D_\mu &\rightarrow D'_\mu = D_\mu + i[\vartheta, D_\mu]\end{aligned}\tag{2.14}$$

where again the second formulation corresponds to infinitesimal transformations. The easiest way to understand this is via the relation $GG^\dagger = 1$ as outlined below:

$$\begin{aligned}(D_\mu\psi)' &= GD_\mu\psi = GD_\mu \underbrace{G^\dagger G}_1 \psi = \underbrace{GD_\mu G^\dagger}_{D'_\mu} \underbrace{G\psi}_{\psi'} = D'_\mu\psi' \\ &\equiv 1_2 \quad D'_\mu \quad \psi'\end{aligned}$$

The last missing piece is to determine how the description of the dynamic part of the gauge field changes. For this purpose the field strength tensor as introduced via

the commutator in Eq. (2.5) (right) is used:

$$\begin{aligned} F_{\mu\nu} &= -\frac{i}{g}[D_\mu, D_\nu] = -\frac{i}{g}[(\partial_\mu + igA_\mu), (\partial_\nu + igA_\nu)] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \end{aligned} \quad (2.15)$$

Compared to the *Abelian* case, the field strength tensor gets an additional term from the commutator of A_μ . The easiest way to derive its transformation behavior in the *non-Abelian* case again is via the commutator relation:

$$\begin{aligned} F_{\mu\nu} \rightarrow F'_{\mu\nu} &= -\frac{i}{g}[D'_\mu, D'_\nu] = -\frac{i}{g}(GD_\mu G^\dagger GD_\nu G^\dagger - GD_\nu G^\dagger GD_\mu G^\dagger) \\ &= G\left(-\frac{i}{g}[D_\mu, D_\nu]\right)G^\dagger = GF_{\mu\nu}G^\dagger \end{aligned}$$

The kinetic term in the Lagrangian density is required to be an $SU(2)$ singlet and a *Lorentz* scalar. These requirements can be matched by the *ansatz*:

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \\ &= \text{Tr}\left(t_a F_{\mu\nu}^a \cdot t_b F^{\mu\nu b}\right) = F_{\mu\nu}^a F^{\mu\nu b} \text{Tr}(t_a \cdot t_b) = F_{\mu\nu}^a F^{\mu\nu b} \frac{1}{2} \delta_{ab} = \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} \end{aligned}$$

where Tr is the trace in the $SU(2)$ isospace and the *Lorentz* covariance is explicit from the contraction of the *Lorentz* indices. A test of the transformation behavior in the isospace demonstrates that this *ansatz* is justified:

$$\begin{aligned} \mathcal{L}'_{\text{kin}} &= \text{Tr}(F'_{\mu\nu}F'^{\mu\nu}) = \text{Tr}(GF_{\mu\nu}G^\dagger GF^{\mu\nu}G^\dagger) = \text{Tr}(GF_{\mu\nu}F^{\mu\nu}G^\dagger) \\ &= \text{Tr}(F_{\mu\nu}F^{\mu\nu}) = \mathcal{L}_{\text{kin}} \end{aligned}$$

At the end of this section, the equations of motion for the *non-Abelian* $SU(2)$ shall be given. The Lagrangian density reads as:

$$\begin{aligned} \mathcal{L}(x) &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) \\ &= \bar{\psi}(i\gamma^\mu(\partial_\mu + igA_\mu) - m)\psi(x) - \frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) \\ &= \bar{\psi}_\alpha\left(i\gamma^\mu(\partial_\mu + igt_a A_\mu^a) - m\right)\psi_\alpha(x) - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} \end{aligned} \quad (2.16)$$

where in the last term the individual components have been spelled out explicitly. Like in the *Abelian* case the gauge fields are massless. Mass terms of type $\frac{m^2}{2}A_\mu A^\mu$ would break the gauge invariance for the same arguments as given for Eq. (2.9). The variation of $\mathcal{L}(x)$ by ψ (resp. $\bar{\psi}$) reveals the equations of motion for fermions:

$$(i\gamma^\mu \partial_\mu - m)\psi_\alpha = g\gamma^\mu t_a A_\mu^a \psi_\alpha$$

Note that this is a system of two correlated equations for $\alpha = 0, 1$, the spinors ψ_α are objects with the minimal dimension 4 and corresponding behavior under *Lorentz* transformations and the t_a are 2×2 matrices in the corresponding $SU(2)$ isospace, which turns these equations of motion into a rather complex system of equations. The variation by A_μ^a reveals the equations of motion for the gauge bosons, which are again given for the gauge fields in absence of fermion fields to allow easy comparison with the *Abelian* example given in Eq. (2.7):

$$\partial_\nu F_a^{\mu\nu} = -g\epsilon_{abc}A_b^\mu t_c$$

Here the term, $-g\epsilon_{abc}A_b^\mu t_c$ on the right-hand side of the equation introduces the self-coupling of the gauge bosons with the same coupling strength, g , as to the fermions. The fact that g is the same as for the coupling to fermions requires that the coupling to the fermions is universal, which is often referred to as lepton universality. In the $SU(2)$ there are three such equations corresponding to the three generators t_a , ($a = 1, 2, 3$) and the associated gauge bosons. They are coupled via the structure constants of the $SU(2)$. Due to the self-coupling the gauge bosons never are freely propagating fields in contrast to the *Abelian* case.

Table 2.1 Comparison of the most important characteristics of a (left) *Abelian* and (right) *non-Abelian* gauge theory: (first line) the transformation behavior of the (spinor) field, (second line) the covariant derivative, the transformation behavior of the (third line) covariant derivative and (fourth line) gauge field, (fifth line) the transformation behavior of the field strength tensor and (last line) the Lagrangian density

Main Characteristics of Gauge Field Theories	
Abelian	Non-abelian
$\psi(x) \rightarrow \psi'(x) = e^{i\vartheta(x)}\psi(x)$	$\psi(x) \rightarrow \psi'(x) = e^{i\vartheta_a(x)t_a}\psi(x)$ $= G(x)\psi(x)$
$\partial_\mu \rightarrow D_\mu = (\partial_\mu + ieA_\mu)$	$\partial_\mu \rightarrow D_\mu = \left(\partial_\mu + ig t_a A_\mu^a\right)$ $= (\partial_\mu + ig A_\mu)$
$D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu\vartheta(x)$	$D_\mu \rightarrow D'_\mu = D_\mu + i[\vartheta, D_\mu]$ $= G D_\mu G^\dagger$
$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\vartheta$	$A_\mu \rightarrow A'_\mu = A_\mu + i[\vartheta, A_\mu] - \frac{1}{g}\partial_\mu\vartheta$ $= G A_\mu G^\dagger + \frac{i}{g}(\partial_\mu G)G^\dagger$
$F_{\mu\nu} \equiv -\frac{i}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$ $F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu}$	$F_{\mu\nu} \equiv -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ $F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu} + i[\vartheta, F_{\mu\nu}]$ $= G F_{\mu\nu} G^\dagger$
$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$	$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}$ $= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})$

Note that the *Abelian* transformation behavior can be obtained from the transformation behavior of the *non-Abelian* case, by requiring that the commutators for the gauge fields is 0. This can be trivially seen for A_μ and $F_{\mu\nu}$. The only non-obvious case of the covariant derivative, D_μ , is briefly explained in the text

A summary of the most important characteristics of the *Abelian* and *non-Abelian* gauge theories, side by side, is given in Table 2.1. Note that the transformation behavior of the *Abelian* case can be obtained from the *non-Abelian* transformations by the requirement that the commutators for the gauge fields are 0, which can be trivially seen for A_μ and $F_{\mu\nu}$. The only non-obvious check of consistency is the transformation behavior of the covariant derivative, D_μ for which care has to be taken, when applying the commutator as an operator. The calculation is demonstrated below:

$$\begin{aligned}
 D'_\mu &= D_\mu + i (\vartheta D_\mu - D_\mu \vartheta) \\
 &= D_\mu + i (\vartheta (\partial_\mu + i g A_\mu) - (\partial_\mu + i g A_\mu) \vartheta) \\
 &= D_\mu + i \underbrace{(\vartheta (\partial_\mu + i g A_\mu) - \vartheta (\partial_\mu + i g A_\mu))}_{\equiv 0} - i \partial_\mu \vartheta \\
 &= D_\mu - i \partial_\mu \vartheta
 \end{aligned}$$

2.2 The Electroweak Gauge Theory

In 1914, *James Chadwick* established that the energy spectrum of the radioactive β decay is not discrete as in the case of α radiation but continuous [1]. It took another 16 years for *Wolfgang Pauli* to postulate the existence of the neutrino, ν , as another product of the β decay besides to the lepton, which could explain this continuous energy spectrum [2]. The first theory of the weak interactions was formulated by *Enrico Fermi* in 1933 [3]. In this theory the interaction was described by a four fermion coupling of type

$$\mathcal{H}_{\text{IA}} = G \int d^3x (\bar{p}(x) \gamma^\mu n(x)) (\bar{e}(x) \gamma^\mu \nu(x)) + h.c.$$

where $\mathcal{H}_{\text{IA}}$ corresponds to the *Hamiltonian* function, $\bar{p}(x)$ to the proton, $n(x)$ to the neutron, $\bar{e}(x)$ to the electron and $\nu(x)$ to the neutrino spinor. It followed suit the structure of *quantum electrodynamics*, with a key modification after the discovery that the weak interaction violates parity. Until today the weak interaction is the only interaction that we know of with this peculiar behavior. In 1958 *Richard Feynman* and *Murray Gell-Mann* introduced a model of the weak interaction, in which parity was maximally violated:

$$\mathcal{H}_{\text{IA}} = \int d^3x \frac{G}{\sqrt{2}} (\bar{p} \gamma^\mu (1 - \gamma^5) n) (\bar{e} \gamma^\mu (1 - \gamma^5) \nu) \quad (2.17)$$

It takes into account that on the elementary particle level only the left-handed part of particles and the right-handed part of anti-particles take part in the (flavor

changing) charged current weak interaction. The special matrix operator γ^5 is defined as $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and has the following characteristics:

$$\{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1_4 \quad (\gamma^5)^\dagger = \gamma^5$$

where $\{\cdot, \cdot\}$ is the anti-commutator. The second term $(\gamma^5)^2 = 1_4$ becomes obvious when written out in matrix notation (in the *Dirac* representation):

$$\gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$

where the elements in this notation correspond to blocks of 2×2 sub-matrices. Applied to a spinor with four components γ^5 swaps the first two elements with the last two elements. When applied twice the spinor will retain its original form. The terms $\frac{1}{2}(1 \pm \gamma^5)$ are projection operators which project general states on to their right-handed (+) and left-handed (−) components. They have the properties

$$\begin{aligned} \left(\frac{1}{2}(1 \pm \gamma^5)\right)^2 &= \frac{1}{2}(1 \pm \gamma^5) \\ \frac{1}{2}(1 + \gamma^5) \cdot \frac{1}{2}(1 - \gamma^5) &= 0 \end{aligned} \quad (2.18)$$

where the first term in Eq. (2.18) is the defining characteristic for a projection operator and the second term indicates, that the two operators are orthogonal to each other. The fact that a single factor $\frac{1}{2}(1 - \gamma^5)$ in the Lagrangian density is sufficient to project out the left-handed state for both the electron and of the neutrino field is demonstrated below:

$$\begin{aligned} \bar{e}\gamma^\mu \left(\frac{1 - \gamma^5}{2}\right) \nu &= \bar{e}\gamma^\mu \left(\frac{1 - \gamma^5}{2}\right)^2 \nu = \bar{e} \left(\frac{1 + \gamma^5}{2}\right) \gamma^\mu \left(\frac{1 - \gamma^5}{2}\right) \nu \\ &= \overline{\left(\left(\frac{1 - \gamma^5}{2}\right) e\right)} \gamma^\mu \left(\frac{1 - \gamma^5}{2}\right) \nu = \bar{e}_L \gamma^\mu \nu_L \end{aligned}$$

The transformation

$$\chi: \quad \psi \rightarrow \gamma^5 \psi \quad ; \quad \bar{\psi} \rightarrow (\gamma^5 \psi)^\dagger \gamma^0 = \psi^\dagger \gamma^5 \gamma^0 = -\bar{\psi} \gamma^5 \quad (2.19)$$

is called *chiral* transformation. The projections on to the left- and right-handed states are *eigenstates* of the chiral transformation with the *eigenvalues* ∓ 1 :

$$\begin{aligned} \psi_L &\rightarrow \gamma^5 \psi_L = -\psi_L \\ \psi_R &\rightarrow \gamma^5 \psi_R = +\psi_R \end{aligned} \quad (2.20)$$

Note that the terms of $\mathcal{H}_{IA}$ in Eq. (2.17) and terms of type $\bar{\psi}\gamma^\mu\partial_\mu\psi$ are invariant under chiral transformations, while terms of type $\bar{\psi}m\psi$ are not, since they lead to a sign flip. In this picture the introduction of mass terms for fermions would break the chiral symmetry of $\mathcal{H}$.

2.2.1 Extension to a Theory of Electroweak Interactions

In 1961 *Sheldon Lee Glashow* managed to develop a gauge field theory that was capable of describing weak and electromagnetic interactions in a unified approach [4]. In this section, this will be explained in detail only for the first generation of leptonic interactions with an electron and a neutrino field, for simplicity reasons. Also for simplicity reasons, this introduction is given in the unitary gauge, in which unphysical degrees of freedom in the theory do not explicitly appear any more and all remaining fields can be identified with physical degrees of freedom. Since the theory does not depend on the choice of the gauge, this choice can be made without restriction. There will be one paragraph in the following sections where the choice of the gauge will play an important role in the argumentation and two examples that will sketch the basic idea of special gauge choices. These appearances will be stated explicitly.

To construct a gauge field theory additional global symmetries have to be introduced into the Lagrangian density, for which then local gauge invariance will be enforced. This is achieved by extending the internal number of degrees of freedom in the Lagrangian density to some higher dimensional space, which, in this case, will be of dimension two. This space is usually called the space of *weak isospin*. Since only left-handed leptons take part in the weak interaction, all fermion fields will be decomposed into their left- and right-handed components. Only the left-handed part of the fields will take part in the (flavor changing) charged current weak interaction. To achieve this only the left-handed leptons will be combined into a doublet in the space of weak isospin.

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \psi_R = e_R \quad (2.21)$$

This doublet acts like a spin $1/2$ object in this hyperspace. The only difference between ν_L and e_L in the sense of the interaction will be the third component of the weak isospin doublet I_3 . The right-handed component of the electron is defined as an isospin singlet $\psi_R = e_R$, with trivial transformation behavior under $SU(2)$ transformations. Since non-trivial $SU(2)$ transformations only act upon the left-handed components of the particles, the symmetry will obtain an index L . The neutrino is still assumed to be massless and to consist only of a left-handed component for simplicity reasons. The actual electron and neutrino fields are linear combinations of the left- and right-handed components:

$$\begin{aligned} \nu &= \nu_L \\ e &= e_L + e_R \end{aligned}$$

The decomposition of a simple Lagrangian density without interaction terms and without mass terms into a left- and right-handed component takes the form

$$\begin{aligned}
 \mathcal{L}_0 &= i\bar{e}\gamma^\mu\partial_\mu e + i\bar{\nu}\gamma^\mu\partial_\mu \nu = \\
 &= i\bar{e}_L\gamma^\mu\partial_\mu e_L + \underbrace{i\bar{e}_L\gamma^\mu\partial_\mu e_R}_{\equiv 0} + \underbrace{i\bar{e}_R\gamma^\mu\partial_\mu e_L}_{\equiv 0} + i\bar{e}_R\gamma^\mu\partial_\mu e_R + i\bar{\nu}_L\gamma^\mu\partial_\mu \nu_L = \\
 &= i\bar{e}_L\gamma^\mu\partial_\mu e_L + i\bar{e}_R\gamma^\mu\partial_\mu e_R + i\bar{\nu}\gamma^\mu\partial_\mu \nu
 \end{aligned}$$

where the mixed left- and right-handed terms are 0, since these components are orthogonal to each other. This is explicitly demonstrated in the following equation:

$$i\bar{e}_L\gamma^\mu\partial_\mu e_R = i\bar{e}\left(\frac{1+\gamma^5}{2}\right)\gamma^\mu\partial_\mu\left(\frac{1+\gamma^5}{2}\right)e = i\bar{e}\left(\frac{1+\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right)\gamma^\mu\partial_\mu e = 0$$

Thus the Lagrangian density can be written in a more compact form as

$$\mathcal{L}_0 = \underbrace{\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L}_{\text{doublet}} + \underbrace{\bar{e}_R\gamma^\mu\partial_\mu e_R}_{\text{singlet}}$$

where the first term corresponds to the isospin doublet term, as defined in Eq. (2.21) and the second term corresponds to the isospin singlet term. From the isospin doublet term the flavor changing interaction will be derived. The isospin singlet term for the right-handed part of the electron will not take part in this interaction. $\mathcal{L}_0$ is invariant under global $SU(2)_L$ gauge transformations of the type:

$$\begin{aligned}
 \psi_L &\rightarrow \psi'_L = G\psi_L & G &\in SU(2)_L \\
 \psi_R &\rightarrow \psi'_R = \psi_R
 \end{aligned}$$

which have been discussed in Sect. 2.1. These transformations correspond to rotations in the weak isospace, with $(n^2 - 1) = 3$ generators. Under these transformations ψ_L transforms like a vector while ψ_R transforms like a scalar. Following the rules as outlined in Sect. 2.1 leads to the introduction of the following covariant derivative and gauge fields:

$$\begin{aligned}
 \partial_\mu &\rightarrow D_\mu = \partial_\mu + ig t_a W_\mu^a = \partial_\mu + ig W_\mu & a &= 1, 2, 3 \\
 W_\mu &\rightarrow W'_\mu = W_\mu + i[\vartheta, W_\mu] - \frac{1}{g}\partial_\mu\vartheta & & \\
 &= G W_\mu G^\dagger + \frac{i}{g}(\partial_\mu G) G^\dagger & &
 \end{aligned} \tag{2.22}$$

where the second line indicates the transformation behavior of the gauge fields. Note that the gauge field W_μ is a linear combination of the three components W_μ^a , which

correspond to the generators $t^a = \frac{\sigma^a}{2}$. The field strength tensor is also introduced in analogy to Sect. 2.1:

$$\begin{aligned} W_{\mu\nu} &= \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu] \\ W_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g\epsilon^{abc}W_\mu^b W_\nu^c \end{aligned} \quad (2.23)$$

where the second expression corresponds to the component-wise formulation, for which Eq. (2.10) has been used. This leads to a canonical definition of the Lagrangian density as

$$\begin{aligned} \mathcal{L}^{SU(2)_L} &= i\bar{\psi}_L \gamma^\mu D_\mu \psi_L + i\bar{e}_R \gamma^\mu \partial_\mu e - \frac{1}{2} \text{Tr} (W_{\mu\nu} W^{\mu\nu}) \\ &= i(\bar{\nu}_L \ \bar{e}_L) \gamma^\mu \left(\partial_\mu + ig t^a W_\mu^a \right) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{e}_R \gamma^\mu \partial_\mu e_R \\ &\quad - \frac{1}{4} \left[\left(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g\epsilon^{abc}W_\mu^b W_\nu^c \right) \right. \\ &\quad \left. \cdot \left(\partial^\mu W^{\nu a} - \partial^\nu W^{\mu a} - g\epsilon^{ade}W^{\mu d} W^{\nu e} \right) \right] \end{aligned} \quad (2.24)$$

where the last part of Eq. (2.24) corresponds to the component-wise notation of the compact form, given in the first part of the equation. Since the operators

$$\begin{aligned} t_+ &= \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \rightarrow \quad t_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ t_- &= \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \rightarrow \quad t_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (2.25)$$

act like the well known ascending and descending operators from quantum mechanics the three gauge fields W_μ^a can be rewritten as

$$\begin{aligned} W_\mu^+ &= \frac{1}{\sqrt{2}} (W_\mu^1 - iW_\mu^2) \\ W_\mu^- &= \frac{1}{\sqrt{2}} (W_\mu^1 + iW_\mu^2) \\ W_\mu^a t^a &= \frac{1}{\sqrt{2}} (W_\mu^+ t^+ + W_\mu^- t^-) + W_\mu^3 t^3 \end{aligned}$$

which leads to the following interacting term of the $SU(2)_L$ Lagrangian density for left-handed leptons:

$$\begin{aligned} \mathcal{L}_{\text{IA}}^{SU(2)_L} &= i\bar{\psi}_L \gamma^\mu D_\mu \psi_L = i\bar{\psi}_L \gamma^\mu \left(\partial_\mu + ig t^a W_\mu^a \right) \psi_L \\ &= i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L - g\bar{\psi}_L \gamma^\mu \left[\frac{1}{\sqrt{2}} (t^+ W_\mu^+ + t^- W_\mu^-) + t^3 W_\mu^3 \right] \psi_L \end{aligned}$$

$$= \underbrace{i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L}_{\text{KIN}} - \frac{g}{2} \left[\underbrace{\sqrt{2}\bar{\nu} \left(W_\mu^+ \gamma^\mu \right) e_L}_{e \rightarrow \nu} + \underbrace{\sqrt{2}\bar{e}_L \left(W_\mu^- \gamma^\mu \right) \nu}_{\nu \rightarrow e} + \underbrace{W_\mu^3 \left(\bar{\nu} \gamma^\mu \nu - \bar{e}_L \gamma^\mu e_L \right)}_{\text{NC}^*} \right]$$

where for the last term the mixing of components due to the *Pauli* matrices has been carried out explicitly. The first term in the equation (labeled by “KIN”) corresponds to the kinematic term of the freely propagating leptons (in compact notation), the second term (“ $e \rightarrow \nu$ ”) to an interaction vertex that leads to the destruction of an electron and the creation of a neutrino, the third term (“ $\nu \rightarrow e$ ”) to an interaction vertex that leads to the destruction of a neutrino and the creation of an electron. These terms can now be identified with the observed *charged current* reactions. On the other hand the last term (“NC*”) does not yet correspond to the observed electroweak *neutral current* reaction, as it only couples to the left-handed part of the electron. This means that the electromagnetic interaction is not properly included in this $SU(2)_L$ Lagrangian density. To achieve this the global $U(1)$ symmetry, which is also inherent to the Lagrangian density, is exploited and the principle of local gauge invariance is extended to a $SU(2)_L \times U(1)_Y$ ³ symmetry. This implies that in addition to the local $SU(2)_L$ symmetry, the Lagrangian density should also be invariant under local $U(1)_Y$ phase transformations. In contrast to the $SU(2)_L$ transformations, the $U(1)_Y$ transformations act on both the left-handed component, ψ_L , and the right-handed component, ψ_R , of the fields. For the left-handed component, it acts on the doublet as a whole. This additional symmetry requirement leads to one more generator for the $U(1)_Y$ symmetry group that, in turn, will lead to the introduction of another gauge field

B_μ	gauge field
$D'_\mu \equiv \partial_\mu + i g' \frac{Y}{2} B_\mu$	covariant derivative
$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$	field strength tensor

Since the $U(1)_Y$ symmetry is imposed on the isospin doublet as a whole, it is not independent from the $SU(2)_L$ symmetry and the gauge fields W_μ and B_μ will be entangled. This will become more obvious by an explicit entanglement of the coupling constant, g , of the $SU(2)_L$ with g' later. The constant Y is the hypercharge of the $SU(2)_L$ singlet or the $SU(2)_L$ doublet as a whole, which can be different for each object in the isospace, expressing the additional freedom in the *Abelian* over the *non-Abelian* gauge theory, where g is fixed to be the same for all objects in the isospace by the gauge boson self-couplings. In the further discussion the hypercharge is defined such that the electric charge q of each corresponding component of the

³The additional index Y is tribute to the hypercharge Y in the covariant derivative.

isospin singlet or doublet is related to the hypercharge by the *Gell-Mann-Nishijima* relation:

$$\begin{array}{ll} g' & \text{coupling constant} \\ Y & \text{hypercharge} \\ q = I_3 + \frac{Y}{2} & \text{electric charge} \end{array}$$

Note the clear distinction between the coupling constant and the charge of an object in this case, which is more obvious here as in *quantum electrodynamics*. The proper choice of $Y_L = -1$ (left-handed) and $Y_R = -2$ (right-handed) for the leptons, leads to the electric charges as observed experimentally. An overview of the values for the hypercharge and third component of the weak isospin, I_3 , for the complete first flavor generation of fermions is given in Table 2.2. This peculiar choice of hypercharges is related to the unitary gauge and has been made such that the quantum mechanical charge operator reveals the charges of the elementary particles, like the electron or the proton, as observed by experiment. It is not in contradiction to the principle of gauge invariance that such a choice has to be made. Within the theory, it is only important that a gauge can be found, in which such a choice is possible. In this formulation, the $SU(2)_L \times U(1)_Y$ Lagrangian density thus takes the form

$$\begin{aligned} \mathcal{L}^{SU(2)_L \times U(1)_Y} = & i\bar{\psi}_L \gamma^\mu \left(\partial_\mu + i g' \frac{Y_L}{2} B_\mu + i g t^a W_\mu^a \right) \psi_L + i\bar{\psi}_R \gamma^\mu \left(\partial_\mu + i g' \frac{Y_R}{2} B_\mu \right) \psi_R \\ & - \frac{1}{2} \text{Tr} \left(W_{\mu\nu}^a W^{a\mu\nu} \right) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \end{aligned} \quad (2.26)$$

Table 2.2 Summary of the hypercharge ($Y_{L/R}$), third isospin component (I_3) and electric charge (q) of the particles in the first fermion generation of weak isospin, as obtained from the *Gell-Mann-Nishiyama* relation $q = I_3 + \frac{1}{2} Y$

Particle	$SU(2)_L \times U(1)_Y$ Hypercharges			
	Left-handed		Right-handed	
	Y_L	I_3	Y_R	q
ν	-1	+1/2	—	0
e		-1/2	-2	-1
u	-1/3	+1/2	-2/3	+1/3
d		-1/2	-4/3	-2/3

with the neutral current component

$$\begin{aligned}
 \mathcal{L}^{NC} &= - \underbrace{\frac{g}{2} W_\mu^3 (\bar{\nu} \gamma_\mu \nu - \bar{e}_L \gamma_\mu e_L)}_{\text{weak IA}} - \underbrace{\frac{g'}{2} B_\mu [Y_L (\bar{\nu} \gamma_\mu \nu - \bar{e}_L \gamma_\mu e_L) + Y_R \bar{e}_R \gamma_\mu e_R]}_{\text{em IA}} \\
 &= \underbrace{\left(-\frac{g}{2} W_\mu^3 + \frac{g'}{2} B_\mu \right)}_{\propto Z_\mu} (\bar{\nu} \gamma_\mu \nu) + \underbrace{\left(\frac{g}{2} W_\mu^3 + \frac{g'}{2} B_\mu \right)}_{\propto Z_\mu} (\bar{e}_L \gamma_\mu e_L) + g' B_\mu (\bar{e}_R \gamma_\mu e_R)
 \end{aligned}$$

As the first term in the last part of this equation has to be proportional to the Z_μ field, it follows that the gauge fields W_μ^3 and B_μ do not correspond to the physical Z boson and photon fields. But it can be achieved to construct the physical fields from the original gauge fields from a rotation by the angle θ_W , which is referred to as the weak mixing angle:

$$\begin{aligned}
 \sin \theta_W &= \frac{g'}{\sqrt{g^2 + g'^2}} & \cos \theta_W &= \frac{g}{\sqrt{g^2 + g'^2}} & \tan \theta_W &= \frac{g'}{g} \\
 Z_\mu &= \frac{1}{\sqrt{g^2 + g'^2}} (g W_\mu^3 - g' B_\mu) = \cos \theta_W W_\mu^3 - \sin \theta_W B_\mu \\
 A_\mu &= \frac{1}{\sqrt{g^2 + g'^2}} (g W_\mu^3 + g' B_\mu) = \sin \theta_W W_\mu^3 + \cos \theta_W B_\mu
 \end{aligned} \tag{2.27}$$

After some arithmetic, this term leads to the *neutral current* part of the Lagrangian density in its final form

$$\begin{aligned}
 \mathcal{L}^{NC} &= - \frac{\sqrt{g^2 + g'^2}}{2} Z_\mu (\bar{\nu} \gamma_\mu \nu) \\
 &\quad + \frac{\sqrt{g^2 + g'^2}}{2} \left[(\cos^2 \theta_W - \sin^2 \theta_W) Z_\mu + 2 \sin \theta_W \cos \theta_W A_\mu \right] (\bar{e}_L \gamma_\mu e_L) \\
 &\quad + \frac{\sqrt{g^2 + g'^2}}{2} \left[-2 \sin^2 \theta_W Z_\mu + 2 \sin \theta_W \cos \theta_W A_\mu \right] (\bar{e}_R \gamma_\mu e_R)
 \end{aligned} \tag{2.28}$$

As can be seen from Eq. (2.28), only the Z_μ couples to the neutrino. Furthermore the photon has the same coupling to the left- and right-handed part of the electron, which resembles the fact that the photon does not distinguish between left- and right-handed states. This is not the case for the Z_μ boson. The factor

$$q = \sqrt{g^2 + g'^2} \sin \theta_W \cos \theta_W = \frac{g \cdot g'}{\sqrt{g^2 + g'^2}}$$

can be identified by the electric charge q .⁴ Expressed by q and θ_W the Lagrangian density takes the form:

$$\begin{aligned}\mathcal{L}^{SU(2)_L \times U(1)_Y} &= \mathcal{L}^{\text{kin}} + \mathcal{L}^{CC} + \mathcal{L}^{NC} + \mathcal{L}^{\text{gauge}} \\ \mathcal{L}^{\text{kin}} &= i\bar{e}\gamma^\mu\partial_\mu e + i\bar{\nu}\gamma^\mu\partial_\mu \nu \\ \mathcal{L}^{CC} &= -\frac{q}{\sqrt{2}\sin\theta_W} \left[W_\mu^+ \bar{\nu}\gamma_\mu e_L + W_\mu^- \bar{e}_L \gamma_\mu \nu \right] \\ \mathcal{L}^{NC} &= -\frac{q}{2\sin\theta_W \cos\theta_W} Z_\mu \left[(\bar{\nu}\gamma_\mu \nu) + (\bar{e}_L \gamma_\mu e_L) \right] - q \left[A_\mu + \tan\theta_W Z_\mu \right] (\bar{e}\gamma_\mu e) \\ &\quad (2.29)\end{aligned}$$

$$\mathcal{L}^{\text{gauge}} = -\frac{1}{2}\text{Tr} \left(W_{\mu\nu}^a W^{a\mu\nu} \right) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \left| \begin{array}{l} B_\mu \rightarrow A_\mu \\ W_\mu^3 \rightarrow Z_\mu \end{array} \right.$$

This Lagrangian density describes the full structure of the electroweak interaction: the kinematic terms for the leptons ($\mathcal{L}^{\text{kin}}$), the charged current ($\mathcal{L}^{CC}$), the weak and electromagnetic components of the neutral current ($\mathcal{L}^{NC}$) and the kinematic terms of the gauge fields W_μ^a and A_μ ($\mathcal{L}^{\text{gauge}}$). The fact the $SU(2)_L$ is a *non-Abelian* gauge field theory leads to a characteristic self-coupling of the weak gauge bosons. The rotation by the electroweak mixing angle implies that all types of self couplings will at the same time apply for Z bosons and photons.

The theory thus makes a prediction for the structure of the electroweak interaction, which by construction, is maximally parity violating in the leptonic interaction vertex. The obvious weakness of this theory is that mass terms of the form

$$\bar{\psi}m\psi$$

for the gauge bosons, but also for the weakly interacting leptons are explicitly not gauge invariant. For gauge bosons this has been shown in Eq.(2.9). For the lepton fields this becomes clear from the following calculation

$$\begin{aligned}\bar{e}m_e e &= \overline{(e_R + e_L)} m_e (e_R + e_L) = m_e (\bar{e}_R e_R + \bar{e}_R e_L + \bar{e}_L e_R + \bar{e}_L e_L) \\ &= m_e (\bar{e}_R e_L + \bar{e}_L e_R)\end{aligned}\quad (2.30)$$

⁴The general usage of the variable name “ e ” for the elementary charge of the electron has been replaced by “ q ” in this and the following sections to prevent misunderstandings in cases, where “ e ” is used for other objects, e.g. like the electron spinor.

where $\bar{e}_R e_R$ and $\bar{e}_L e_L$ are zero due to the orthogonality property of the projectors. Since e_R is a $SU(2)_L$ singlet while e_L is just a component of the $SU(2)_L$ doublet ψ_L , the remaining terms as such do have a non-trivial behavior under $SU(2)_L$ transformations and are not gauge invariant. This additional complication in the SM only occurs due to the splitting of the fermion fields into a left- and right-handed component. It was therefore obvious since its introduction in 1961 that the $SU(2)_L \times U(1)_Y$ gauge theory is incomplete and needs to be extended by another theoretical concept, which was suggested to be the concept of *spontaneous symmetry breaking* that will be discussed in detail in the following section.

2.3 Electroweak Symmetry Breaking and the Higgs Boson

The expression *spontaneous symmetry breaking* refers to the situation where a system, described by the Lagrangian density $\mathcal{L}$, is invariant under the transformation of a given symmetry group G , while this symmetry is broken by the energy ground state of the system. The simple example of a needle standing upright on its tip has been given in Sect. 1.2: the Lagrangian density of this system is invariant under rotations, φ , around the axis of the needle. But the system is metastable and the needle will fall into an arbitrary direction in φ to end up in the energy ground state. This situation is illustrated in Fig. 1.3 (left). The direction in which the needle will end up lying can not be predicted. All possible angles are degenerate and ignoring frictional energy losses the needle could move around the original axis without further energy costs. This is a general characteristic of the phenomenon. In the context of quantum field theories it is formalized in the *Goldstone* theorem [5], which states that in a relativistic, covariant quantum field theory, in which symmetries are spontaneously broken, particles with mass zero are created. These particles are called *Goldstone* bosons. They correspond to the degeneracy of the energy ground state in the simplistic example given in Sect. 1.2.

Goldstone bosons can be elementary particles, which are already part of the Lagrangian density, bound states, which are created within the theory (like the hydrogen atom or *Cooper* pairs) or they can be identified by unphysical excitations or artificial degrees of freedom within the gauge theory, which are usually removed by the choice of a proper gauge.

2.3.1 The Goldstone Model

The *Goldstone* model can be introduced by a field ϕ with a potential $V(\phi)$ and a Lagrangian density $\mathcal{L}$ given by

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$$

$$\begin{aligned}
V(\phi) &= -\mu^2 |\phi|^2 + \lambda |\phi|^4 \\
\mathcal{L}(\phi) &= \partial_\mu \phi \partial^\mu \phi^* - V(\phi)
\end{aligned}$$

$V(\phi)$ will be referred to as the *Goldstone* potential. For this example an illustration is given in Fig. 1.3 (right). The Lagrangian density $\mathcal{L}$ is invariant under $U(1)$ transformations $\phi \rightarrow \phi' = e^{i\theta} \phi$. The energy ground state is where the Hamiltonian

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi^*)} \partial_\mu \phi^* - \mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* + V(\phi)$$

is minimal, which is the case for $|\phi| = \sqrt{\frac{\mu^2}{2\lambda}}$. This minimum will later on be identified with the non-zero vacuum expectation value

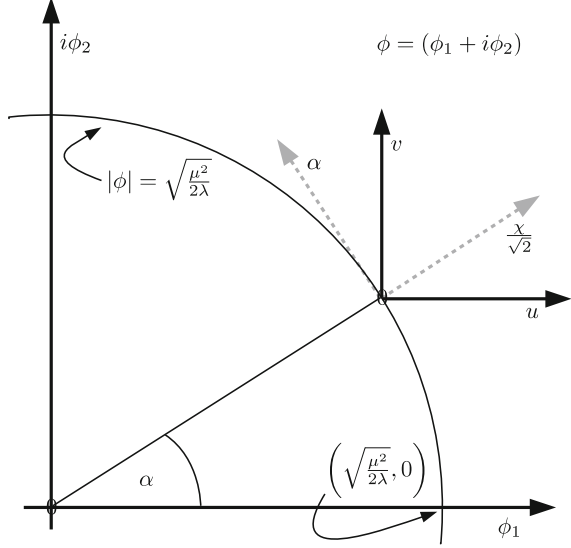
$$v \equiv \sqrt{\frac{\mu^2}{2\lambda}}. \quad (2.31)$$

The ground state is illustrated in Fig. 2.1. An expansion around the ground state in the point $\left(\sqrt{\frac{\mu^2}{2\lambda}}, 0\right)$ in *Cartesian* coordinates leads to

$$\begin{aligned}
\phi(u, v) &= \sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}} (u + i v) \\
\mathcal{L} &= \left[\partial_\mu \phi \partial^\mu \phi^* - V(\phi) \right]_{\phi=\phi(u, v)} = \frac{1}{2} \partial_\mu u \partial^\mu u + \frac{1}{2} \partial_\mu v \partial^\mu v - V'(u, v) \\
V'(u, v) &= \left[-\mu^2 |\phi|^2 + \lambda |\phi|^4 \right]_{\phi=\phi(u, v)} = -\frac{\mu^4}{4\lambda} + \mu^2 u^2 + \mu \sqrt{\lambda} u (u^2 + v^2) \\
&\quad + \frac{\lambda}{4} (u^2 + v^2)^2
\end{aligned}$$

Note that the φ symmetry around the origin of the potential is not visible any more in this expansion. At the same time a complex structure of terms containing v and u has emerged. The term $\mu^2 u^2$ formally corresponds to a mass term for the field u , which can be identified with an excitation of the field in the confining direction that leads horizontally out of the minimum, in which $V'(u, v)$ has been developed. The field v , which does not lead out of the minimum of $V'(u, v)$, does not acquire a mass term. This field corresponds to the *Goldstone* boson. Other terms lead to tri-linear and quartic self-couplings of the fields u and v . There are no terms which are linear in u and v . This is obvious from the fact that the field ϕ has been developed in the minimum of the Hamiltonian function, for which the *Taylor* expansion starts with a first non-trivial term in second order. This argument holds for any potential.

Fig. 2.1 Up-view onto the *Goldstone* potential $V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4$ as discussed in Sect. 2.3 and illustrated in Fig. 1.3. Both a *Cartesian* and the cylindrical coordinate system as discussed in the text are indicated in the figure. For the examples given in the text the potential is developed in the minimum at $(\sqrt{\frac{\mu^2}{2\lambda}}, 0)$



The symmetry of the system is better represented by cylindrical coordinates, in which the Lagrangian density $\mathcal{L}$ takes the form

$$\phi(\chi, \alpha) = e^{i\alpha} \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{\chi}{\sqrt{2}} \right)$$

$$\mathcal{L} = \left[\partial_\mu \phi \partial^\mu \phi^* - V(\phi) \right]_{\phi=\phi(\chi, \alpha)} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{\chi}{\sqrt{2}} \right)^2 \partial_\mu \alpha \partial^\mu \alpha - V'(\chi)$$

$$V'(\chi) = \left[-\mu^2|\phi|^2 + \lambda|\phi|^4 \right]_{\phi=\phi(\chi)} = -\frac{\mu^4}{4\lambda} + \mu^2\chi^2 + \mu\sqrt{\lambda}\chi^3 + \frac{\lambda}{4}\chi^4$$

the expressions for the self-coupling terms are simpler, the mass term is created for the real field χ . Even though α does not appear any more in the potential, it corresponds to the *Goldstone* boson. This is an example, where the *Goldstone* boson corresponds to a gauge degree of freedom, which has been removed by the choice of a proper gauge that inherently respects the symmetry of the problem.

2.3.2 Extension to a Gauge Theory

The extension of the *Goldstone* model to a gauge theory starts from the introduction of the covariant derivative as described in Sect. 2.1. For simplicity reasons this is

shown for the simple *Goldstone* model in cylindrical coordinates and an *Abelian* gauge symmetry:

$$\begin{aligned}
 \mathcal{L} &= [(\partial_\mu + igA_\mu)\phi][(\partial^\mu + igA^\mu)\phi]^* - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}|_{\phi(\chi,\alpha)} \\
 &= \left| \frac{1}{\sqrt{2}}\partial_\mu\chi e^{i\alpha} + ie^{i\alpha} \left(\sqrt{\frac{\mu^2}{\lambda}} + \frac{\chi}{\sqrt{2}} \right) (gA_\mu + \partial_\mu\alpha) \right|^2 - V'(\chi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
 &= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \left(\left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{\chi}{\sqrt{2}} \right) (gA_\mu + \partial_\mu\alpha) \right)^2 - V'(\chi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
 \end{aligned}$$

This leads to a quadratic mass term in the gauge field A_μ , the term $\partial_\mu\alpha$ can be absorbed into A_μ in the gauge $A_\mu + \partial_\mu\theta$ with $\theta = -\frac{1}{g}\alpha$. Via $V'(\chi)$ the field χ obtains a mass, too, as described above and appears as a physical field. In this example χ plays the role of the physical Higgs boson field in the SM, that will be introduced later. In addition the model obtains characteristic coupling terms of type $\sim \chi^2 A_\mu A^\mu$ and $\sim \chi A_\mu A^\mu$ of the Higgs boson field, χ , to the gauge field and characteristic self-coupling terms, which originate from the specific form of the *Goldstone* potential. The introduction of the *Goldstone* potential and the expansion of the field $\phi \rightarrow \phi(\chi, \alpha)$ in the energy ground state, that is shifted to a non-zero value, has dynamically generated a mass term $\frac{g^2\mu^2}{2\lambda} A_\mu A^\mu$ for the gauge field A_μ from the coupling $g^2|\phi|^2 A_\mu A^\mu$ between A_μ and ϕ . This mass term emerges from the coupling of the gauge boson, A_μ , to the vacuum expectation value $v = \sqrt{\frac{\mu^2}{\lambda}}$. As discussed before such a mass term alone would break the gauge symmetry. But the additional presence of the new Higgs boson field, χ , and of tri-linear and quartic couplings of A_μ to χ restore and protect the invariance. This leads to firm predictions of the structure of these couplings. This coupling structure is the characteristic of a Higgs boson which is different from a gauge boson or any other particle in this sense.

The field ϕ was originally complex with two degrees of freedom (ϕ_1 and ϕ_2). In the final form the field χ is real with only one degree of freedom, while the field of the *Goldstone* boson α has been completely removed from the Lagrangian density by the choice of a proper gauge. It seems as if the model had lost one degree of freedom. In fact this is not the case. It reappears as an additional degree of freedom of the gauge field A_μ : as a massless particle A_μ has only two degrees of freedom, which are usually chosen as transverse polarizations. As a massive particle it gains one more degree of freedom of longitudinal polarization. One says that the gauge field has eaten up the additional degree of freedom of the *Goldstone* boson α and has acquired mass on it. This shift of degrees of freedom from the *Goldstone* boson(s) to the gauge field(s) is referred to as the *equivalence theorem* [6]. It is a main ingredient of the Higgs mechanism.

There are a few concluding remarks on the special choice of the *Goldstone* potential, which might have appeared arbitrary on first sight:

- The *Goldstone* potential as chosen above leads to a symmetry breaking vacuum expectation value in the theory, which is a prerequisite of the model. It only depends on $|\phi|$ and does not distinguish any direction in space. Furthermore it does not lead to negative infinite energies, which is another prerequisite for the theory to be stable.
- In the example the potential has been cut at order $|\phi|^4$. This can be motivated by a dimensional analysis: in natural units the action $S = \int L d^4x \sim \hbar$ is dimensionless, x has the dimension $[x] = -1$ and the partial derivative has the dimension $[\partial_\mu] = +1$. This gives the field ϕ the dimension $[\phi] = +1$. For these reasons, the coupling constants in the potential obtain the dimension $[\mu] = 1$ and $[\lambda] = 0$. Any coupling of negative dimension would turn the theory non-renormalizable. For this reason, it makes sense to stop the power series of the *Goldstone* potential at the lowest needed dimension.

The incorporation of spontaneous symmetry breaking into a gauge theory was the last missing piece of the *Weinberg-Salam* model of electroweak interactions, which led to its completion to the SM. The electroweak sector of the SM will be summarized in the following section.

2.4 The Electroweak Sector of the Standard Model of Particle Physics

The $SU(2)_L \times U(1)_Y$ gauge model as discussed in Sect. 2.2 provides the accurate description of the weak and electromagnetic interactions. Its biggest shortcoming is that the gauge symmetry strictly requires the gauge bosons to be massless. The fact that the W and the Z boson do have a non-vanishing mass implies that the symmetry is not manifest in nature. The solution to this puzzle is to have an energy ground state of the quantum vacuum, which does not obey the symmetry and thus prevents its direct manifestation. The fact that the Lagrangian density as discussed so far does not lead to such a symmetry breaking energy ground state pointed to the existence of a hidden sector in the theory with a new particle that had not yet been observed by the time of its postulation and whose presence implements the spontaneous breaking of the $SU(2)_L$ symmetry in the quantum vacuum in one or the other way. This part of the theory is called the Higgs sector. Both the $SU(2)_L \times U(1)_Y$ gauge symmetry and the Higgs model have been introduced in the preceding sections of this chapter. In the following they will be fit together like a zip lock to result in the complete

theory of the SM as formulated since the late 1960s. This construction starts from Eq. (2.29), which is extended by an additional Lagrangian density term

$$\begin{aligned}\mathcal{L}^{\text{Higgs}} &= \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi) \\ V(\phi) &= -\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi \right)^2\end{aligned}\quad (2.32)$$

with a new field

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \quad \phi^\dagger = (\phi_+^* \ \phi_0^*) \equiv (\phi_- \ \phi_0^*) \quad (2.33)$$

which should transform like an $SU(2)_L$ isospin doublet with the coupling constant g and the hypercharge $Y_\phi = 1$ under $U(1)_Y$ transformations. For the individual components of the doublet this leads to the electric charges of $q(\phi_0) = 0$ and $q(\phi_+) = +1$ according to the *Gell-Mann-Nishijima* relation as discussed earlier in this chapter. As an $SU(2)_L$ doublet ϕ transforms like

$$\begin{aligned}\phi &\rightarrow \phi' = e^{i\vartheta'} G \phi \\ \phi^\dagger &\rightarrow \phi'^\dagger = \phi^\dagger G^\dagger e^{-i\vartheta'} \quad G = e^{i\vartheta^a t^a} \in SU(2) \quad \vartheta^a, \vartheta' \in \mathbb{R}\end{aligned}$$

(as discussed in Sect. 2.1), where ϑ' and ϑ^a are continuous parameters. Enforcing local gauge invariance of $\mathcal{L}^{\text{Higgs}}$ under $SU(2)_L \times U(1)_Y$ transformations again implies the introduction of a covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + i g' \frac{Y_\phi}{2} B_\mu + i g t^a W_\mu^a$$

as for Eq. (2.26). In the next step, ϕ will be expanded in the vicinity of its energy ground state, in the minimum $v = \sqrt{\frac{\mu^2}{2\lambda}}$ of the Higgs potential

$$\phi = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \end{pmatrix}$$

where the new field H has been introduced. Note that the choice to do the expansion around the non-vanishing vacuum expectation value in the lower component of ϕ and the assignment of Y_ϕ is a consequence of the choice of the unitary gauge that has been made at the beginning of this chapter. It does not contradict the fact that according to gauge invariance the expansion could in principle be done in any other point in the minimum of the vacuum. The important feature of the theory is that such a choice, as for the unitary gauge, can be found. Since the SM is a gauge invariant theory any other gauge would lead to the same observable quantities, but the correspondence

with the quantum mechanical operators would be non-trivial. Setting the expanded version of ϕ in the kinetic term of $\mathcal{L}^{\text{Higgs}}$ leads to

$$D_\mu \phi^\dagger D^\mu \phi = \left| \left[\frac{1}{\sqrt{2}} \partial_\mu H + \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right) \left(i g t^a W_\mu^a + i g' \frac{Y_\phi}{2} B_\mu \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2$$

In the last step of this calculation the product of the unit vector of the isospin with the t^a , ($a = +, -, 3$) matrices is evaluated in each component:

$$\begin{aligned} D_\mu \phi^\dagger D^\mu \phi = & \left| \left[\frac{1}{\sqrt{2}} \partial_\mu H - \frac{i}{2} (g W_\mu^3 - g' B_\mu) \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 \\ & + \left| \left(i \frac{g}{2} W_\mu^+ \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \end{aligned} \quad (2.34)$$

The following remarks should help to understand Eq. (2.34): (i) the ascending operator t^+ , which belongs to the field W_μ^+ has shifted the unit vector of the down component up; (ii) the descending operator t^- , which belongs to the field W_μ^- evaluated to the unit vector of the down component is zero; (iii) the operator t^3 evaluated on the unit vector of the down component has flipped the sign of the term associated with W_μ^3 , according to the structure of the *Pauli* matrix σ_3 . Also note that

$$(g W_\mu^3 - g' B_\mu) \equiv \sqrt{g^2 + g'^2} Z_\mu \quad (2.35)$$

according to the definitions in Eq. (2.27). The evaluation of the absolute value squared finally results in

$$\begin{aligned} D_\mu \phi^\dagger D^\mu \phi = & \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{g^2 + g'^2}{4} \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right)^2 Z_\mu Z^\mu + \\ & \frac{g^2}{4} \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right)^2 W_\mu^+ W^{\mu-} \end{aligned} \quad (2.36)$$

thus generating mass terms for the gauge fields Z_μ , W_μ^+ and W_μ^- from the coupling to the quantum vacuum, v . As in the simplified example of Sect. 2.3, a new physical field, H , emerges as a radial excitation of ϕ in the quantum vacuum. This Higgs field acquires a mass on its own given by the potential $V(\phi)$. As discussed before this is independent from the specific form of the *Goldstone* potential. The couplings of the

gauge bosons Z_μ , W_μ^+ and W_μ^- to H protect the gauge invariance in the theory. The masses of the gauge bosons can be read off from the equations to be

$$\begin{aligned} \left(\frac{g}{2}\right)^2 v^2 W_\mu^+ W^{\mu-} &\equiv m_W^2 W_\mu^+ W^{\mu-} \\ \left(\frac{\sqrt{g^2 + g'^2}}{2}\right)^2 v^2 Z_\mu Z^\mu &\equiv m_Z^2 Z_\mu Z^\mu \end{aligned} \quad (2.37)$$

which appears like a quartic coupling of the gauge bosons to the non-vanishing quantum vacuum, equivalent to the quartic Higgs coupling. Equation (2.37) illustrates how from an underlying theory, like the SM, effective phenomenological parameters like m_W and m_Z , can be further resolved to give deeper insights into the dynamic processes of nature. Equation (2.37) furthermore leads to the relation

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cdot \cos^2 \theta_W} = 1 \quad (2.38)$$

From this relation, an accurate constraint on $\cos \theta_W$ and a firm prediction of $m_Z > m_W$ arise. Note for the evaluation of the absolute value of Eq. (2.34) that the second term in the absolute value, which is proportional to W_μ^+ , is also proportional to the upper unit vector, while the first term is proportional to the lower unit vector. This is why there are no mixed terms between these two parts in the absolute value. The same is true for the kinetic term in the first part of the absolute value, which is purely real, and the second term, which is proportional to Z_μ and purely imaginary, which again implies that there are no mixed terms.

It has been discussed in the previous section how in the Higgs mechanism the *Goldstone* bosons are eaten up and commit their degrees of freedom to the gauge bosons, which in turn become massive. Though this has not been shown explicitly this is also the case here: the general complex $SU(2)_L$ doublet field ϕ has four (scalar) degrees of freedom, of which three get eaten up by the W_μ^+ , W_μ^- and Z_μ boson. One degree of freedom remains and turns into a real field: the Higgs boson field, H .

The vacuum expectation value v can be obtained from Eq. (2.37) and the relation between the mass of the W boson, m_W , and the *Fermi* constant, G_F , which has been very accurately determined from muon lifetime measurements [7]:

$$\frac{1}{2}gv = m_W = \sqrt{\frac{\sqrt{2}g^2}{8G_F}} \quad ; \quad v = \frac{1}{\sqrt{\sqrt{2}G_F}} = 246.22 \text{ GeV}$$

The value of $v = 246.22 \text{ GeV}$ [8] sets the scale of electroweak symmetry breaking. This knowledge allows the replacement of the self-coupling λ in the *Goldstone* potential, leaving only the mass of the Higgs boson m_H undetermined.

2.4.1 Custodial Symmetry

The Lagrangian density of the Higgs boson sector

$$\begin{aligned}\mathcal{L}^{\text{Higgs}} &= (D_\mu \phi)^\dagger D^\mu \phi - V(\phi) \\ V(\phi) &= -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \\ D_\mu \phi &= \left(\partial_\mu + i g' \frac{Y_\phi}{2} B_\mu + i g t^a W_\mu^a \right) \phi\end{aligned}\tag{2.39}$$

does not only have a local $SU(2)_L \times U(1)_Y$ symmetry, which has been introduced by construction, but also an approximate larger global symmetry, which happens to be present by accident. This can be seen if the Higgs doublet field, ϕ , and its charge conjugate

$$\phi_c = 2i t_2 \phi^* = \begin{pmatrix} \phi_0^* \\ -\phi_- \end{pmatrix}\tag{2.40}$$

are combined into a *bi-doublet* matrix

$$\Phi = \frac{1}{\sqrt{2}} (\phi_c \ \phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_- & \phi_0 \end{pmatrix} \quad \Phi^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_0 & -\phi_+ \\ \phi_- & \phi_0^* \end{pmatrix}\tag{2.41}$$

The definition of ϕ_c is in analogy to the charge conjugation of spinors. It obtains the hypercharge $Y_{\phi_c} = -Y_\phi = -1$. It is a feasible exercise to show, that ϕ_c has an $SU(2)_L \times U(1)_Y$ transformation behavior, which is equivalent to ϕ . In the matrix representation ϕ_c , corresponds to the first column and ϕ to the second column of the matrix Φ . Note that there is no such correspondence any more for $\Phi^\dagger$. In this notation Eq. (2.39) can be obtained from

$$\begin{aligned}\mathcal{L}^{\text{Higgs}} &= \text{Tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) - V(\Phi) \\ V(\Phi) &= -\mu^2 \text{Tr} \left(\Phi^\dagger \Phi \right) + \lambda \left(\text{Tr} \left(\Phi^\dagger \Phi \right) \right)^2 \\ D_\mu \Phi &= \left(\partial_\mu \Phi - \frac{i g'}{2} B_\mu \Phi \sigma_3 + \frac{i g}{2} W_\mu^a \sigma_a \Phi \right) \\ (D_\mu \Phi)^\dagger &= \left(\partial_\mu \Phi^\dagger + \frac{i g'}{2} B_\mu \sigma_3 \Phi^\dagger - \frac{i g}{2} W_\mu^a \Phi^\dagger \sigma_a \right)\end{aligned}\tag{2.42}$$

which can be verified when evaluating the term

$$\begin{aligned}\text{Tr} \left(\Phi^\dagger \Phi \right) &= \text{Tr} \left(\frac{1}{2} \begin{pmatrix} \phi_0^* \phi_0 + \phi_+ \phi_- & 0 \\ 0 & \phi_+ \phi_- + \phi_0^* \phi_0 \end{pmatrix} \right) \\ &= \phi_+ \phi_- + \phi_0^* \phi_0 \equiv \phi^\dagger \phi\end{aligned}\tag{2.43}$$

There is one subtlety in this notation to be noted: the fact, that $Y_{\phi_c} = -Y_\phi$ has lead to the introduction of a minus sign and the multiplication with σ_3 from the right in $D_\mu \Phi$ for the term in the covariant derivative, which is affected by the hypercharge. In this case it is important that σ_3 is multiplied from the right. In this notation the transformation behavior of $SU(2)_L \times U(1)_Y$ of Φ is given by

$$\begin{aligned} SU(2)_L : \quad \Phi &\rightarrow L \Phi \\ U(1)_Y : \quad \Phi &\rightarrow \Phi e^{-\frac{i}{2} \sigma_3 \vartheta} \end{aligned}$$

where L is equivalent to the transformation matrix G as introduced in Sect. 2.1.1 (the change from G to L in the notation will become clear in the next paragraph) and the *Pauli* matrix σ_3 has again been introduced due to the opposite hypercharges of ϕ and ϕ_c . The global $SU(2)_L$ symmetry of $\mathcal{L}^{\text{Higgs}}$ in this notation can be trivially verified from

$$\begin{aligned} \text{Tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) &\rightarrow \text{Tr} \left((D_\mu \Phi)^\dagger \underbrace{L^\dagger L}_{\equiv 1} D^\mu \Phi \right) = \text{Tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) \\ &\equiv 1 \\ V(\phi) &\rightarrow V(L\phi) = -\mu^2 \text{Tr} \left(\Phi^\dagger \underbrace{L^\dagger L}_{\equiv 1} \Phi \right) + \lambda \left(\text{Tr} \left(\Phi^\dagger \underbrace{L^\dagger L}_{\equiv 1} \Phi \right) \right)^2 = V(\phi) \\ &\equiv 1 \qquad \qquad \qquad \equiv 1 \end{aligned}$$

which is not much of a surprise, since a global $SU(2)_L$ is part of the construction of the local $SU(2)_L$ symmetry. The non trivial additional symmetry enters via the transformation

$$SU(2)_R : \quad \Phi \rightarrow \Phi R$$

where the $SU(2)$ transformation matrix is multiplied from the right, corresponding to rotations in a right-handed coordinate system. It is indeed non-trivial to show that $\mathcal{L}^{\text{Higgs}}$ is also invariant under such global transformations. It has to be done by explicitly checking the relations

$$\begin{aligned} \text{Tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) &\rightarrow \text{Tr} \left(R^\dagger (D_\mu \Phi)^\dagger D^\mu \Phi R \right) \approx \text{Tr} \left((D_\mu \Phi)^\dagger D^\mu \Phi \right) \\ V(\phi) &\rightarrow V(\phi R) = -\mu^2 \text{Tr} \left(R^\dagger \Phi^\dagger \Phi R \right) + \lambda \left(\text{Tr} \left(R^\dagger \Phi^\dagger \Phi R \right) \right)^2 = V(\phi) \end{aligned} \tag{2.44}$$

which will be done in the following paragraphs. It should first be noted that the matrix $\Phi^\dagger \Phi$ will play a special role, which is why it has been written out explicitly in Eq. (2.43). In a first step, the invariance of the term $\text{Tr} \left(\sigma_a^\dagger \Phi^\dagger \Phi \sigma_a \right)$ will be discussed, which appears several times in Eq. (2.44). Expressed in the basis of the $SU(2)_R$ generators the rotations R are linear combinations of the form $R^{(\dagger)} = 1_2 \pm \sum \vartheta_a \sigma_a$, $a = 1, 2, 3$. When multiplied from the left or right the transformation will lead to terms of type $A : 1_2 \Phi^\dagger \Phi 1_2$, $B : 1_2 \Phi^\dagger \Phi \sigma_a$, $C : \sigma_a \Phi^\dagger \Phi 1_2$,

$D : \sigma_a \Phi^\dagger \Phi \sigma_b$. Multiplication with σ_1 from the left (right) swaps the rows (columns) of $\Phi^\dagger \Phi$. The same is true for the multiplication with σ_2 apart from additional factors of i and $-i$ that will appear in the elements of the matrix. The multiplication with σ_3 from the left (right) adds a minus sign to all elements in the lower row (last column) of $\Phi^\dagger \Phi$. This completes the ingredients needed to check the invariance of $V(\Phi)$ in Eq. (2.44): terms of type A are trivially invariant; terms of type B and C will lead to a swap of rows or columns, which will shift the off-diagonal zero elements in $\Phi^\dagger \Phi$ on the diagonal, these terms have thus no effect on the trace; for terms of type D , three cases have to be distinguished: (i) if σ_3 is multiplied from left and right the minus sign applied once to the lower row and once to the last column leaves the lower right element and thus the trace unchanged; (ii) if σ_3 and $\sigma_{1,2}$ appear in the product, this will result in an effective swap of a row or a column and thus again have no effect on the trace; (iii) if combinations of $\sigma_{1,2}$ appear in the product the swap of rows and columns will swap the upper left with the lower right diagonal element and thus again leave the trace invariant. This is also true for the additional factors of i and $-i$ that might appear in the diagonal elements depending on the occurrences of σ_2 . To translate these findings into the test of $\text{Tr} \left(R^\dagger (D_\mu \Phi)^\dagger D^\mu \Phi R \right)$ the product will be written out explicitly

$$\text{Tr} \left(R^\dagger \left(\partial_\mu \Phi^\dagger + \frac{ig'}{2} B_\mu \sigma_3 \Phi^\dagger - \frac{ig}{2} \Phi^\dagger \sigma_a W_\mu^a \right) \left(\partial_\mu \Phi - \frac{ig'}{2} B_\mu \Phi \sigma_3 + \frac{ig}{2} \sigma_a W_\mu^a \Phi \right) R \right) \quad (2.45)$$

Not the whole calculation will be done here. Instead, it will be shown in a first step that the derivative part of Eq. (2.44) is exactly true for $g' = 0$. In this case Eq. (2.45) leads to three different types of traces, $A' : \text{Tr} (R^\dagger \Phi^\dagger \Phi R)$, $B' : \text{Tr} (R^\dagger \Phi^\dagger \sigma_a \Phi R)$ and $C' : \text{Tr} (R^\dagger \Phi^\dagger \sigma_a \sigma_{a'} \Phi R)$. The invariance of type A' has been demonstrated above, type C' can be mapped into A' , due to the orthogonality of the *Pauli* matrices, $\sigma_a \sigma_{a'} = \delta_{aa'}$. The only non trivial case to check are the traces of type B' . This can be done explicitly for each σ_a , $a = 1, 2, 3$ and leads to the result of $\text{Tr} (\Phi^\dagger \sigma_a \Phi) = 0$, $\forall a$. Since the rotation by R will be without influence on the trace, as demonstrated above, these terms will not contribute to the overall trace.

In a second step, an example is given to demonstrate that this invariance is not exact in the case of $g' \neq 0$. In this case, the product of the middle terms in Eq. (2.45) will lead to a trace of type $\text{Tr} (R^\dagger \sigma_3 \Phi^\dagger \Phi \sigma_3 R)$. To give an example where the invariance is broken from the expansions of $R^{(\dagger)} = 1_2 \pm \sum \vartheta_a \sigma_a$ the term is chosen, where σ_2 is multiplied from left and right. This term leads to

$$\text{Tr} (\sigma_2 \sigma_3 \Phi^\dagger \Phi \sigma_3 \sigma_2) \propto \text{Tr} (\sigma_1 \Phi^\dagger \Phi (-1) \sigma_1) = -\text{Tr} (\Phi^\dagger \Phi)$$

The residual minus sign demonstrates that those terms containing σ_3 in general violate the exact symmetry. Since g' is small the symmetry is still approximately valid. The fact that Eq. (2.44) holds, implies that $\mathcal{L}^{\text{Higgs}}$ has an additional approximate global symmetry of type

$$SU(2)_L \times SU(2)_R : \Phi \rightarrow L \Phi R$$

When the Higgs field ϕ acquires the non-vanishing vacuum expectation value in matrix notation this is expressed by the matrix

$$\langle \Phi \rangle = \frac{1}{2} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

which breaks both global symmetries $SU(2)_L$ and $SU(2)_R$ since

$$L \langle \Phi \rangle \neq \langle \Phi \rangle \quad \langle \Phi \rangle R \neq \langle \Phi \rangle$$

But it leaves a sub-group $SU(2)_{L+R}$ unbroken in which $R = L^\dagger$ since

$$L \langle \Phi \rangle L^\dagger = \langle \Phi \rangle$$

Either $SU(2)_R$ or $SU(2)_{L+R}$ is referred to as *custodial symmetry* [9, 10]. It has been shown that this symmetry is non-trivial. It is only approximate and appears due to the special structure of Φ , which allows Eq.(2.44) to be true. Since $SU(2)$ is a three dimensional group, the number of broken generators from $SU(2)_L \times SU(2)_R$ to $SU(2)_{L+R}$ is $3 + 3 - 3 = 3$. These give rise to three *Goldstone* bosons, which in turn give mass to the W^+ , W^- and Z boson, as demonstrated with the calculations leading to Eq.(2.37). It can be shown that in the limit $g' \rightarrow 0$ the three heavy gauge bosons transform like a triplet in the three dimensional adjoint space of $SU(2)_{L+R}$, which explains why m_Z and m_W are so close to each other. In the limit of $g' \rightarrow 0$ they would even be the same and the difference only occurs due to the small violation of the exact symmetry by g' as can be seen from Eq.(2.37). The custodial symmetry also protects the relation of Eq.(2.38) from large higher-order corrections, which would move ρ away from being $\mathcal{O}(1)$. This is where the exceptional name of this symmetry originates from. A similar custodial symmetry can be found in the sector of Higgs quark *Yukawa* couplings, under the assumption that the quark masses are the same, as will be briefly discussed at the end of the following section. Indeed, the violations of these custodial symmetries by g' and by the difference between the masses of the *b*- and the *top* quark were exploited in the global parameter fits that had been used to estimate m_t and m_H from their loop contributions to the electroweak precision data taken at *LEP* as will be discussed in Sect. 3.2.

2.4.2 Giving Masses to Fermions

As has been discussed before, also naive mass terms of fermions are violating the local $SU(2)_L \times U(1)_Y$ gauge symmetry, due to their unequal splitting in left- and right-handed parts. Giving a mass to the leptons without breaking the gauge symmetry can also be achieved dynamically by adding a coupling of the lepton doublet to the

Higgs boson field, ϕ . This is demonstrated for the electron-neutrino doublet in the following paragraphs. The corresponding term in the Lagrangian density is

$$\mathcal{L}_e^{\text{Yukawa}} = -y_e \left(\bar{e}_R \phi^\dagger \psi_L \right) + y_e^* \left(\bar{\psi}_L \phi e_R \right) \quad \psi_L = \begin{pmatrix} \nu \\ e_L \end{pmatrix}$$

which corresponds to a common *Yukawa* coupling with the coupling constant y_e . The Lagrangian density $\mathcal{L}^{\text{Yukawa}}$ transforms like a $SU(2)_L \times U(1)_Y$ singlet as will be discussed with the following arguments: (i) both ϕ and ψ_L are $SU(2)_L$ doublets, but their product is a $SU(2)_L$ singlet as well as e_R . Therefore, the product of the three elements also transforms like a $SU(2)_L$ singlet; (ii) the $U(1)_Y$ transformation behavior is described by the product

$$e^{\pm i \frac{q'}{2} (Y_R + Y_\phi - Y_L) \theta'}$$

where the minus sign in front of Y_L comes from the fact that ϕ and ψ_L are always adjoint to each other, and which in the configuration $Y_L = -Y_\phi = -1$ and $Y_R = -2$, as given in Table 2.2, always equals to 1. Correspondingly, the product of the three elements will also transform like a $U(1)_Y$ singlet. The coupling constant y_e can be chosen to be real. Any complex phase could be absorbed into a phase of e_R , which is also true for the quarks that will be discussed later. Again expanding the field ϕ in its energy ground state

$$\mathcal{L}_e^{\text{Yukawa}} = -y_e \left(\sqrt{\frac{\mu^2}{2\lambda}} + \frac{H}{\sqrt{2}} \right) (\bar{e}_R e_L + \bar{e}_L e_R) = -m_e \left(1 + \frac{1}{v} \frac{H}{\sqrt{2}} \right) \bar{e} e$$

leads to a mass term for the coupling lepton. The *Yukawa* coupling is determined by $y_e = m_e/v$ and thus proportional to the mass of the lepton and the inverse of the vacuum expectation value, v . Mass terms for down-type quarks can be introduced into the theory in complete analogy, as will be demonstrated for the first flavor generation quark doublet: in this case the *Yukawa* coupling takes the form

$$\mathcal{L}_d^{\text{Yukawa}} = -y_d \left(\bar{d}_R \phi^\dagger \psi_L \right) + y_d \left(\bar{\psi}_L \phi d_R \right) \quad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad (2.46)$$

There is one more subtlety when introducing mass terms for up-type fermions in general: since ϕ is developed in its lower component, in the unitary gauge, it can only serve to give mass terms for the fields in the lower component of the doublet. Since neutrinos have been assumed to be massless throughout these considerations, this did not become apparent for the lepton doublet. For the quark doublet it can not be ignored any more. The way how to obtain mass terms for the upper components of the doublets is again not a priori given. It would be possible to introduce a second Higgs doublet field, $\tilde{\phi}$, to serve this purpose. In the minimal SM it is achieved via

the charge conjugate of ϕ , as defined in Eq.(2.40). In this formalism, the *Yukawa* coupling to the up-type quark can be introduced as

$$\mathcal{L}_u^{\text{Yukawa}} = -y_u \left(\bar{u}_R \phi_c^\dagger \psi_L \right) + y_u \left(\bar{\psi}_L \phi_c u_R \right) \quad \psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad (2.47)$$

All further considerations, including the check for the $SU(2)_L \times U(1)_Y$ transformation behavior are in analogy to the case of the lepton doublet. The hypercharges of the corresponding quark singlets and doublets are given in Table 2.2.

As discussed for the pure Higgs sector of the Lagrangian density, $\mathcal{L}^{\text{Higgs}}$, before, in Sect. 2.4.1, there is also a custodial symmetry in the Higgs *Yukawa* sector for quarks, under the assumption that the quarks in the doublet have equal mass (i.e. $y_u = y_d = y$). This can be seen if the right-handed quarks are grouped into a global $SU(2)_R$ doublet $\psi_R = (u_R \ d_R)$. In this notation, Eqs. (2.46) and (2.47) can be rewritten in a compact form as

$$\mathcal{L}_q^{\text{Yukawa}} = -\sqrt{2}y \left(\bar{\psi}_R \Phi^\dagger \psi_L \right)$$

using the matrix notation for Φ as introduced in Sect. 2.4.1. From the transformation behavior of $\psi_{L,R}$ under the global $SU(2)_{L,R}$ transformations the invariance of $\mathcal{L}_q^{\text{Yukawa}}$ under global $SU(2)_L \times SU(2)_R$ transformations can easily be seen from

$$\begin{aligned} SU(2)_L : \quad & \psi_L \rightarrow L \psi_L \quad \psi_R \rightarrow \psi_R \\ SU(2)_R : \quad & \psi_R \rightarrow R^\dagger \psi_R \quad \psi_L \rightarrow \psi_L \\ SU(2)_L \times SU(2)_R : \quad & \bar{\psi}_R \Phi^\dagger \psi_L \rightarrow \overline{(R^\dagger \psi_R)} (L \Phi R)^\dagger L \psi_L = \bar{\psi}_R \underbrace{R R^\dagger}_{\equiv 1} \Phi^\dagger \underbrace{L^\dagger L}_{\equiv 1} \psi_L = \bar{\psi}_R \Phi^\dagger \psi_L \end{aligned}$$

This completes the discussion on the first generation of weak isospin doublets including quarks. The last peculiarity of the electroweak interaction that will be discussed in this context is not crucial for the discussion of electroweak symmetry breaking and mass generation, while it adds to the distinctiveness of the electroweak interaction in general. From the observation of decays like

$$n \rightarrow p \ e^- \ \bar{\nu}_e \quad \text{and} \quad \Lambda^0 \rightarrow p \ e^- \ \bar{\nu}_e$$

it is obvious that the weak interaction allows transitions between the upper and lower components of an isospin doublet not only within the same, but also across different isospin doublets. In this case, the decay $n \rightarrow p \ e^- \ \bar{\nu}_e$, is an example for a normal transition from a d -quark to a u -quark within the same doublet. The decay $\Lambda^0 \rightarrow p \ e^- \ \bar{\nu}_e$ is an example for a transition from an s -quark to a u -quark, which is a transition across two distinct doublets. Another important experimental observation is that these transitions seem to be only allowed via the coupling to the W and between up- and down-type elements of the doublets. So called flavor

changing neutral currents (FCNC), e.g. from an s -quark to a d -quark, seem to be highly suppressed.

In the SM, this behavior can be understood if the mass *eigenstates* of the quarks are not the same as the $SU(2)_L$ flavor *eigenstates*. When only considering the two weak isospin doublets for u -, d -, s - and c -quarks, this can be achieved by a unitary rotation of the *eigenstates* against each other

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos \varphi_{12} & \sin \varphi_{12} \\ -\sin \varphi_{12} & \cos \varphi_{12} \end{pmatrix} \cdot \begin{pmatrix} d \\ s \end{pmatrix}$$

where $\varphi_{12} = 13.04(5)^\circ$ [8] corresponds to the *Cabbibo* angle, which indicates the amount by which the $SU(2)_L$ flavor basis is rotated against the basis of mass *eigenstates*. It is thereby convention to express this rotation in the down-type components of the doublets, while it would be equivalent and lead to the same predictions to express the rotation in the up-type components. This rotation also protects the theory from FCNC (at tree level) via the *Glashow-Iliopoulos-Maiani* or GIM mechanism [11], which implies that such processes can only appear at higher orders in the SM.

The extension of the fermion fields to three generations of quark doublets leads to the introduction of the *Cabbibo-Kobayashi-Maskawa* or *CKM* matrix as an extension of the simple two dimensional rotation via the *Cabbibo* angle [12]. In the standard parametrization, this rotation matrix of the complex spinor fields reads as

$$\begin{aligned} M_{\text{CKM}} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & c_{23} & -s_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta_{13}} & 0 & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \\ c_{ij} &= \cos \varphi_{ij} ; s_{ij} = \sin \varphi_{ij} \quad (ij = 12, 13, 23) \end{aligned}$$

where the three real angles φ_{ij} correspond to the *Euler* angles in three dimensions (corresponding to the three generations of weak isospin doublets) and δ_{13} corresponds to a complex phase, which remains also in the unitary gauge. The complex phase, δ_{13} , is the parameter that determines direct CP violation in the SM. If it were equal to zero, direct CP violation would not be allowed. But since δ_{13} itself is only a parameter which has to be determined by experiment it carries no further predictive

power of the SM beyond incorporating the possibility of direct CP violation in the theory. It is a peculiarity of the electroweak interaction that, in spite of the origin of the quark masses being related to the electroweak sector of the SM, the *eigenstates* of the electroweak interaction of the quarks are not the same as their mass *eigenstates*.

2.4.3 Summary and Conclusions

The complete Lagrangian density of the electroweak sector of the SM, which for reasons of simplicity is only given for the first generation of leptons, reads as:

$$\begin{aligned}
 \mathcal{L}^{\text{SM}} &= \mathcal{L}_{\text{kin}}^{\text{Lepton}} + \mathcal{L}_{\text{IA}}^{\text{CC}} + \mathcal{L}_{\text{IA}}^{\text{NC}} + \mathcal{L}_{\text{kin}}^{\text{Gauge}} + \mathcal{L}_{\text{kin}}^{\text{Higgs}} + \mathcal{L}_{V(\phi)}^{\text{Higgs}} + \mathcal{L}_e^{\text{Yukawa}} \\
 \mathcal{L}_{\text{kin}}^{\text{Lepton}} &= i\bar{e}\gamma^\mu\partial_\mu e + i\bar{\nu}\gamma^\mu\partial_\mu \nu \\
 \mathcal{L}_{\text{IA}}^{\text{CC}} &= -\frac{q}{\sqrt{2}\sin\theta_W} \left[W_\mu^+ \bar{\nu}\gamma^\mu e_L + W_\mu^- \bar{e}_L\gamma^\mu \nu \right] \\
 \mathcal{L}_{\text{IA}}^{\text{NC}} &= -\frac{q}{2\sin\theta_W \cos\theta_W} Z_\mu \left[(\bar{\nu}\gamma^\mu \nu) + (\bar{e}_L\gamma^\mu e_L) \right] - q \left[A_\mu + \tan\theta_W Z_\mu \right] (\bar{e}\gamma^\mu e) \\
 \mathcal{L}_{\text{kin}}^{\text{Gauge}} &= -\frac{1}{2}\text{Tr} \left(W_{\mu\nu}^a W^{a\mu\nu} \right) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \Bigg|_{\substack{B_\mu \rightarrow A_\mu \\ W_\mu^3 \rightarrow Z_\mu}} \\
 \mathcal{L}_{\text{kin}}^{\text{Higgs}} &= \frac{1}{2}\partial_\mu H \partial^\mu H + \left(1 + \frac{1}{v} \frac{H}{\sqrt{2}} \right)^2 m_W^2 W_\mu^+ W^{\mu-} + \left(1 + \frac{1}{v} \frac{H}{\sqrt{2}} \right)^2 m_Z^2 Z_\mu Z^\mu \\
 &\quad (2.48) \\
 \mathcal{L}_{V(\phi)}^{\text{Higgs}} &= -\frac{\mu^2 v^2}{2} + \mu^2 \left(\frac{H}{\sqrt{2}} \right)^2 + 2\frac{\mu^2}{v} \left(\frac{H}{\sqrt{2}} \right)^3 + \frac{\mu^2}{2v^2} \left(\frac{H}{\sqrt{2}} \right)^4 \\
 \mathcal{L}_e^{\text{Yukawa}} &= -\left(1 + \frac{1}{v} \frac{H}{\sqrt{2}} \right) m_e \bar{e}e
 \end{aligned}$$

The parts, which only contain the Higgs boson field, H , in $\mathcal{L}_{V(\phi)}^{\text{Higgs}} + \mathcal{L}_{\text{kin}}^{\text{Higgs}}$ correspond to the *Klein-Gordon* equation for a scalar boson with a tri-linear and a quartic self-coupling given by the explicit choice of the *Goldstone* potential. Due to the self-coupling, there is no free Higgs boson field. A few more consequences of the explicit coupling structure will be further discussed in Sect. 3.1, together with the

triviality constraint within the SM. In $\mathcal{L}_{V(\phi)}^{\text{Higgs}}$ the mass of the Higgs boson field, H , can be read off as

$$m_H^2 \equiv 2\mu^2 \quad (2.49)$$

The coupling of the Higgs boson to fermions, heavy gauge bosons and to itself, expressed by m_H and v , can be read off from the Lagrangian density to be

$$\begin{aligned} f_{H \rightarrow ff} &= i \frac{m_f}{v} & (\text{Fermions}) \\ f_{H \rightarrow VV} &= i \frac{2m_V^2}{v} & (\text{Heavy Bosons trilinear}) \\ f_{HH \rightarrow VV} &= i \frac{2m_V^2}{v^2} & (\text{Heavy Bosons quartic}) \\ f_{H \rightarrow HH} &= i \frac{3m_H^2}{v} & (H \text{ Boson trilinear}) \\ f_{HH \rightarrow HH} &= i \frac{3m_H^2}{v^2} & (H \text{ Boson quartic}) \end{aligned} \quad (2.50)$$

Note that since H is an indistinguishable particle, each appearance, n , in the scattering vertex needs to be taken into account by a combinatorial factor $1/n!$, in the description of the elementary scattering process, according to the *Feynman* rules [13]. This leads to an additional factor of $2!$ for $f_{HH \rightarrow VV}$, of $3!$ for $f_{H \rightarrow HH}$ and of $4!$ for $f_{HH \rightarrow HH}$. Both in the case of self-couplings and in the case of the coupling to gauge bosons the tri-linear and quartic couplings have the same strength, with the only difference that the quartic couplings are suppressed by one additional factor of $1/v$. Further on, in contrast to the fermionic couplings, which depend linearly on the mass of the fermions, m_f , the bosonic couplings are proportional to the masses of the bosons squared. As has been discussed before this coupling structure constitutes a characteristic property of a Higgs boson. While with more stringency for the coupling to gauge bosons than for the coupling to fermions, it constitutes a unique coupling behavior, which is non-universal among fermion generations.

The individual non-trivial steps towards the full electroweak theory are summarized below:

- In the first step, the Lagrangian density has been extended into the two-dimensional $SU(2)_L$ weak isospace, which is comprehensible only for the left-handed components of matter. The neutrino and the left-handed component of the electron have been combined into an isospin doublet $\psi_L = (\nu \ e_L)^T$, for which local $SU(2)_L$ gauge invariance has been enforced. This has led to a description of the weak theory. The right-handed part of the electron, e_R , which does not take part in the weak interaction behaves like a singlet under $SU(2)_L$ transformations.
- To also obtain a description for the electromagnetic force in addition local invariance has been required for the global $U(1)_Y$ symmetry of the Lagrangian density, which acts on the left-handed $SU(2)_L$ doublet, ψ_L , as a whole, and on the right-handed $SU(2)_L$ singlet, e_R , with the different hypercharges Y_L and Y_R . These

two local gauge invariance requirements have led to the structure of electroweak interactions and to the four gauge fields B_μ and W_μ^a , $a = 1, 2, 3$.

- To achieve that the gauge coupling to the ν is governed by only a single physical field, the Z boson field, the fields B_μ and W_μ^3 have been transformed into the fields A_μ and Z_μ . This was possible by a trivial rotation in the plane of the neutral gauge fields by the weak mixing angle θ_W .
- Up to this point the main issue of the model was, that local gauge symmetries require all gauge boson fields to be massless, while the W^+ , W^- and Z bosons of the weak interaction have been measured to be massive. This finding implies that the $SU(2)_L$ symmetry can not be manifest in the Lagrangian density. The symmetry can still be immanent, but hidden, if it is broken in the energy ground state of the system, which corresponds to the quantum vacuum.
- Since all fields and interactions in the Lagrangian density up to this point obey the $SU(2)_L$ symmetry the incorporation of an energy ground state, which breaks the symmetry implied the postulation of a new weak isospin doublet field, ϕ , with a self-coupling and a potential imparting this property to the quantum vacuum. The simplest potential, with these properties, which is bound from below, does not distinguish any direction in weak isospace and still leads to a renormalizable theory is the *Goldstone* potential defined in Eq. (2.32).
- The requirement of local $SU(2)_L$ gauge invariance to this field leads to a mass term for the three $SU(2)_L$ gauge bosons, in the energy ground state of the quantum vacuum. The mass terms appear as a coupling of the gauge bosons to the non-zero expectation value of the quantum vacuum, v . Three out of four degrees of freedom of the complex $SU(2)_L$ doublet field, ϕ , are eaten up by the gauge fields, which in turn obtain an additional degree of freedom of longitudinal polarization, each, due to the gained mass. One degree of freedom remains in the model, which in unitary gauge can be expressed by the single real field H .
- The field H is called the Higgs boson field. As a single scalar field it obeys the *Klein Gordon* equation and thus is a boson. It obtains a mass due to its self-coupling with the potential $V(\phi) \rightarrow V(H)$. This mass term, $\propto H^2$ appears naturally and irrespective of the exact form of the potential, from the first non-trivial term of the *Taylor* expansion of the Lagrangian density, when developed in the minimum of the energy ground state. The physical Higgs boson field, H , can be viewed as the radial excitation of the Higgs doublet field, ϕ , in the minimum of the *Goldstone* potential.
- The gauge invariance violating transformation behavior of the mass terms of the heavy gauge bosons is compensated by additional coupling terms of the bosons to the new physical Higgs boson field, H . This coupling is $\propto m_V^2$.
- For fermions the problem of masses is different and it only occurs due to the chiral nature of the weak interaction coupling to fermions, which requires the splitting of fermions into left-handed $SU(2)_L$ doublets and right-handed $SU(2)_L$ singlets. It is this distinction and the different behavior under $SU(2)_L$ transformations, of left- and right-handed particle components which leads to the breaking of the local $SU(2)_L$ gauge symmetry for fermion mass terms. Despite of its different nature also this problem can be solved by the coupling of the involved fermion fields to

the new Higgs boson field, ϕ , e.g. via a *Yukawa* coupling, $\propto \bar{e}_R \phi \psi_L$. As in the case of the massive gauge bosons the mass terms appear from the coupling to the non-zero vacuum expectation value, v . Also here the gauge symmetry breaking behavior of the mass terms is compensated by additional coupling terms of the fermions to the new physical Higgs boson field, H . This coupling is $\propto m_f$.

- The vacuum expectation value, v , is developed in the lower component of ϕ . This only allows to give mass terms to the lower components of the fermion doublets. In the minimal SM, mass terms for the upper components of the fermion doublets can be obtained from the charge conjugate of ϕ , ϕ_c . This is not possible in *supersymmetric* extensions of the SM, were ϕ has to fit into the structure of a larger multiplet.

These non-trivial points extend the *Weinberg-Salam* model of electroweak interactions to the electroweak sector of the SM as a complete theory. It contains the physical 21 fields for the neutrinos, the left- and right-handed components of the leptons and quarks, as constituents of matter, the gauge fields Z_μ , W_μ^+ , W_μ^- and A_μ as mediators of the electromagnetic and weak interaction, and the Higgs field H .

In this collection of fields the Higgs boson plays a special role in the theory: it is the only particle with spin 0. In the theory this is expressed by the fact that it is a scalar field with a single external degree of freedom, in contrast to a spinor- or vector field. The Higgs boson is neither a constituent of matter, like the fermion fields, nor is it a force mediating gauge field. It is omnipresent as an excitation of the non-zero expectation value in the quantum vacuum. In this sense it is similar to the omnipresent *aether* in the closing of the 19th century. This omnipresence is manifest in the non-zero masses of the particles, which couple to the non-zero vacuum expectation value, v , and to the excitations of the quantum vacuum, H , in the same way. A brief summary of all involved fields in the electroweak sector of the SM is given in Table 2.3.

Table 2.3 Summary of all fields that appear in the electroweak sector of the SM

Spin 0	Higgs Field					
	H					
Spin 1/2	Leptons [†]			Quarks [†]		
	ν_e	ν_μ	ν_τ	u	c	t
	e	μ	τ	d	s	b
Spin 1	Gauge Fields					
	$W^{+/-}$	Z	γ			

All gauge fields have spin 1, the fields corresponding to the elementary constituents of matter, the leptons and quarks have spin 1/2. [†] Note that the lepton and quark fields describing elementary particles with a finite mass have a left-handed and a right-handed component, of which only the left-handed component is taking part in the weak charged current interaction. The Higgs boson is the only field in the SM that has spin 0

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