

Chapter 2

Global Isometric Embedding of Surfaces in $\mathbb{R}^3$

Qing Han

Abstract In this note, we give a short survey on the global isometric embedding of surfaces (2-dimensional Riemannian manifolds) in $\mathbb{R}^3$. We will present associated partial differential equations for the isometric embedding and discuss their solvability. We will illustrate the important role of Gauss curvature in solving these equations.

2.1 Introduction

Isometric embedding is a classical problem in differential geometry. In this note, we present a short survey on the global isometric embedding of Riemannian manifolds in Euclidean spaces. We begin with the following question.

Question 2.1.1 Given a smooth n -dimensional Riemannian manifold (M^n, g) , does it admit a smooth isometric embedding in Euclidean space $\mathbb{R}^N$ of some dimension N ?

This is a long standing problem in differential geometry. When an isometric embedding in $\mathbb{R}^N$ is possible for sufficiently large N , there arises a further question. What is the smallest possible value for N ? Those questions have more classical local versions in which solutions are sought only in a sufficiently small neighborhood of some specific point on the manifold. Analytically it involves finding a smooth embedding $\mathbf{r} : M^n \rightarrow \mathbb{R}^N$ such that $d\mathbf{r} \cdot d\mathbf{r} = g$, or in local coordinates

$$\partial_i \mathbf{r} \cdot \partial_j \mathbf{r} = g_{ij}, \quad i, j = 1, \dots, n. \quad (2.1.1)$$

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Q. Han (✉)

Department of Mathematics, University of Notre Dame, Notre Dame 46556, USA
e-mail: qhan@nd.edu

Q. Han

Beijing International Center for Mathematical Research, Peking University,
Beijing 100871, China
e-mail: qhan@math.pku.edu.cn

This is a differential system of $n(n+1)/2$ equations for N unknowns. In general, a necessary condition for this equation to be solvable is $N \geq s_n \equiv n(n+1)/2$.

For the global isometric embedding, we have the following result.

Theorem 2.1.2 *Any smooth closed Riemannian manifold (M^n, g) admits a smooth isometric embedding in $\mathbb{R}^N$ for some $N = N(n)$.*

Theorem 2.1.2 was first proved by Nash [Nas56] and was later improved by Günther [Gun89a]. To prove Theorem 2.1.2, one needs to find a global solution of (2.1.1). We note that (2.1.1) is nonlinear. When iterations are used, a loss of differentiation occurs. Nash introduced an ingenuous iteration to handle this loss of differentiation. Such an iteration was later on improved by Moser, among many people, and is now called Nash-Moser iteration. Günther's argument is quite simple. He rewrote the first order system (2.1.1) as a second order elliptic differential system and then used the contraction mapping principle. Moreover, he improved the dimension of the ambient space. Specifically, he proved

$$N \geq \max\{s_n + 2n, s_n + n + 5\}.$$

If $n = 2$, then $N \geq 10$. Hence, any compact 2-dimensional smooth Riemannian manifold can be isometrically embedded in $\mathbb{R}^{10}$. A natural question is whether we can lower the dimension of the target Euclidean space.

For the local embedding, we are interested only in the case $N = s_n$ or when N is close to s_n . For the analytic case, we have the following optimal result.

Theorem 2.1.3 *Any analytic n -dimensional Riemannian manifold admits an analytic local isometric embedding in $\mathbb{R}^{s_n}$.*

Theorem 2.1.3 was proved by Janet [Jan26] for $n = 2$ and by Cartan [Car27] for $n \geq 3$. The proof is based on the Cauchy-Kowalewsky Theorem.

For the smooth case, we have the following result.

Theorem 2.1.4 *Any smooth n -dimensional Riemannian manifold admits a local smooth isometric embedding in $\mathbb{R}^{s_n+n}$.*

Theorem 2.1.4 was proved by Greene [Gre70] and by Gromov and Rokhlin [GR70] independently. Their proofs are based on the iteration scheme introduced by Nash. Günther [Gun89a] gave an alternative proof by using the contraction mapping principle.

For 2-dimensional Riemannian manifolds, a better result is available. Poznyak [Poz73] proved that any smooth 2-dimensional Riemannian manifold can be locally isometrically embedded in $\mathbb{R}^4$ smoothly.

Refer to [HH06] for proofs of these results and historical accounts.

In this note, we give a short survey on the global isometric embedding of surfaces (2-dimensional Riemannian manifolds) in $\mathbb{R}^3$ in the *smooth* or *sufficiently smooth* category.

2.2 Local Isometric Embedding

In this section, we briefly review the local isometric embedding. We begin with the following conjecture.

Conjecture 2.2.1 *Any smooth surface admits a smooth local isometric embedding in $\mathbb{R}^3$.*

This conjecture was raised by Schlaefli in 1873 and was given renewed attention by Yau in the 1980s and 1990s. It is still open.

Let g be a smooth metric in a neighborhood of $0 \in \mathbb{R}^2$. We are interested in whether g , restricted to a smaller neighborhood of 0, admits a smooth isometric embedding in $\mathbb{R}^3$. It turns out that the behavior of the Gauss curvature in a neighborhood of 0 plays an essential role. It is a classical result that g in a neighborhood of $0 \in \mathbb{R}^2$ admits a smooth isometric embedding in $\mathbb{R}^3$ if $K(0) \neq 0$. The general case when K assumes zero somewhere remains open in general.

We have the following result for the case of nonnegative Gauss curvature.

Theorem 2.2.2 *Suppose g is a C^r metric in a neighborhood of $0 \in \mathbb{R}^2$ with $K \geq 0$, for some integer $r \geq 14$. Then g admits a C^{r-10} isometric embedding in $\mathbb{R}^3$ locally in a neighborhood of 0.*

Theorem 2.2.2 was proved by Lin [Lin85]. We point out that the local isometric embedding established in Theorem 2.2.2 is not known to be smooth even if the metric g is smooth. The smoothness was proved in special cases by Hong and Zuilly [HZ87].

We have the following result when Gauss curvature changes its sign.

Theorem 2.2.3 *Suppose g is a C^r metric in a neighborhood of $0 \in \mathbb{R}^2$ with $K(0) = 0$ and $\nabla K(0) \neq 0$, for some integer $r \geq 9$. Then g admits a C^{r-6} isometric embedding in $\mathbb{R}^3$ locally in a neighborhood of 0.*

Theorem 2.2.3 was proved by Lin [Lin86]. An alternative proof was given by Han [Han05a]. For K in Theorem 2.2.3, the implicit function theorem implies the existence of a curve γ such that K changes sign across γ at order 1. Han [Han05b] proved a similar result if K changes sign across a curve γ at any order, or more general, if K changes sign monotonically across γ . See also [Khu07a].

Suppose a metric g defined in an open set $\Omega \subset \mathbb{R}^2$ is given by

$$g = \sum_{i,j=1}^2 g_{ij} dx_i dx_j.$$

To isometrically immerse g in $\mathbb{R}^3$, it is equivalent to finding a function $\mathbf{r} = (X_1, X_2, X_3) : \Omega \rightarrow \mathbb{R}^3$ such that $d\mathbf{r} \cdot d\mathbf{r} = g$, or

$$\sum_{k=1}^3 \partial_i X_k \cdot \partial_j X_k = g_{ij}, \quad i, j = 1, 2.$$

This is a first order differential system of three equations for three unknown functions. However, such a system is *not* covered by the general theory of first order differential systems. In order to study this system, we change it to an equivalent differential equation.

We first note that $\mathbf{r}$ satisfies the following basic equations

$$\nabla_{ij}\mathbf{r} = h_{ij}\mathbf{n}, \quad i, j = 1, 2, \quad (2.2.1)$$

where ∇_{ij} denotes the covariant derivatives with respect to g , i.e.,

$$\nabla_{ij}\mathbf{r} = \partial_{ij}\mathbf{r} - \Gamma_{ij}^k \partial_k \mathbf{r},$$

and (h_{ij}) is the coefficient of the second fundamental form. Fix a unit vector $\mathbf{e}$ in $\mathbb{R}^3$ and consider $u = \mathbf{r} \cdot \mathbf{e}$. Taking the scalar product of $\mathbf{e}$ and (2.2.1) and then evaluating the determinant, we get

$$\det(\nabla_{ij}u) = K \det(g_{ij})(\mathbf{n} \cdot \mathbf{e})^2.$$

Note that

$$(\mathbf{n} \cdot \mathbf{e})^2 = 1 - \left(\frac{|(\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}) \times \mathbf{e}|}{|\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}|} \right)^2 = 1 - g^{ij} \partial_i u \partial_j u = 1 - |\nabla u|^2.$$

Then, we obtain

$$\det(\nabla^2 u) = K \det(g_{ij})(1 - |\nabla u|^2), \quad (2.2.2)$$

with a subsidiary condition $|\nabla u| < 1$. In local coordinates, (2.2.2) can be written as

$$\det(u_{ij} - \Gamma_{ij}^k u_k) = K \det(g_{ij})(1 - g^{ij} u_i u_j),$$

where Γ_{ij}^k is the Christoffel symbol and (g^{ij}) is the inverse of (g_{ij}) . This equation was derived by Darboux in 1894 and is referred to as the *Darboux equation*. Obviously, each component of $\mathbf{r}$ satisfies the Darboux equation.

It can be verified that isometrically embedding a given metric g in $\mathbb{R}^3$ is equivalent to finding a solution u to the Darboux equation (2.2.2).

The Eq. (2.2.2) is a fully nonlinear equation of the Monge-Ampère type. We are interested in a local solution in a neighborhood of any given point $p \in \Omega$. The type of the Eq. (2.2.2) is determined by the sign of the Gauss curvature K . If K is positive or negative, (2.2.2) is elliptic or hyperbolic. However, (2.2.2) is degenerate where K vanishes.

In the case that the Gauss curvature K does not vanish at $p \in \Omega$, (2.2.2) can be solved easily in a neighborhood of p . The difficulty arises if K vanishes at p .

To prove Theorems 2.2.2 and 2.2.3, we adopt a standard method to obtain local solutions of nonlinear differential equations. Basically, it consists of three steps.

Step 1. We choose an approximate solution and scale the equation appropriately. The purpose is to write the original equation as a perturbation of some standard equation.

Step 2. We derive *a priori* estimates for the linearized equation. This is the most difficult part.

Step 3. We obtain a solution by iterations, which may be an application of the contraction mapping principle or the complicated Nash-Moser iteration.

The crucial step here is to study the linearized equations and derive *a priori* estimates. The linearized equations of the Darboux equation are elliptic if the Gauss curvature is positive, hyperbolic if the Gauss curvature is negative, and of the mixed type if the Gauss curvature changes its sign. Moreover, the linearized equations are degenerate where the Gauss curvature vanishes.

2.3 Isometric Embedding of Closed Surfaces

In this section, we discuss the global isometric embedding of closed surfaces, 2-dimensional compact Riemannian manifold without boundary. Completely omitted is the isometric immersion of complete surfaces without boundary.

2.3.1 The Weyl Problem

The simplest closed surface is the sphere. We begin with

Question 2.3.1 Does any smooth metric on $\mathbb{S}^2$ with a pointwise positive Gauss curvature admit a smooth isometric embedding in $\mathbb{R}^3$?

The Question 2.3.1 is often referred to as the *Weyl Problem*, which was raised by Weyl [Wey16]. The first attempt to solve the problem was made by Weyl himself. He suggested the continuity method and obtained *a priori* estimates up to the second derivatives. Twenty years later, Lewy [Lew38a] solved the problem in the case of g being analytic. In the early 1950s, Nirenberg [Nir53] and Pogorelov [Pog52] independently solved the smooth case.

Theorem 2.3.2 Let g be a $C^{4,\alpha}$ metric on $\mathbb{S}^2$ with positive Gauss curvature, $\alpha \in (0, 1)$. Then there exists a $C^{4,\alpha}$ isometric embedding of g into $\mathbb{R}^3$.

The present form of Theorem 2.3.2 was proved by Nirenberg [Nir53] by the continuity method. The result was extended to the case of continuous third derivatives of the metric by Heinz [Hei62]. In a completely different approach to the problem, Alexandrov in 1942 obtained a generalized solution of the Weyl problem as a limit of polyhedra. The regularity of this generalized solution was proved by Pogorelov [Pog52]. Guan and Li [GL94], and Hong and Zuilu [HZ95], independently generalized Theorem 2.3.2 to metrics on $\mathbb{S}^2$ with nonnegative Gauss curvature.

Closely related to the global isometric embedding is the rigidity. The first rigidity result, proved by Cohn-Vossen [Coh27], states that any two closed isometric analytic convex surfaces are congruent (within a reflection) to each other. Herglotz [Her43] gave a very short proof of the rigidity, assuming that the surfaces are three times continuously differentiable. It was eventually extended to surfaces having merely two times continuously differentiable metrics by Sacksteder [Sac62].

We now discuss Nirenberg's solution of the Weyl problem. It is based on the method of continuity and consists of three steps:

(a) The given $C^{4,\alpha}$ metric g on $\mathbb{S}^2$ with positive Gauss curvature is to be connected with the standard metric g_0 on $\mathbb{S}^2$ by a family of $C^{4,\alpha}$ metrics g_t , depending continuously on t , $0 \leq t \leq 1$, such that all metrics g_t have positive Gauss curvature.

For the next two steps, set

$$I = \{t \in [0, 1]; g_t \text{ can be isometrically embedded in } \mathbb{R}^3 \text{ in } C^{4,\alpha}\text{-category}\}.$$

(b) Show that I is open; that is, if g_{t_0} is isometrically embedded, then there exists a small neighborhood of t_0 , say $|t - t_0| < \varepsilon(t_0)$, such that g_t is isometrically embedded for all t in this neighborhood.

(c) I is closed.

Statements (a), (b) and (c) imply the set of values of t for which g_t is isometrically embedded in $C^{4,\alpha}$ is the whole segment $0 \leq t \leq 1$.

The statement (a) is proved with the aid of the uniformization theorem, which enables one to map conformally the Riemannian manifold defined by g globally onto the unit sphere—after which the construction of g_t is easily done.

The statement (b), which may be referred to as the statement of “openness”, requires one to solve a system of nonlinear partial differential equations which are degenerate in character. These are attacked by an iteration scheme. The key step is to solve a system of linear differential equations and to obtain estimates of its solutions.

The statement (c), which may be referred to as the statement of “closedness”, is based on *a priori* estimates for the second derivatives of the functions describing a convex surface with a given metric.

Now we derive the Darboux equation on the unit sphere. Let $\mathbf{r}(x_1, x_2)$ be a closed convex surface with positive Gauss curvature. The coefficients of the first and second fundamental forms of the surface are denoted by g_{ij} and h_{ij} . The g_{ij} 's are the components of the induced metric g given by

$$d\mathbf{r} \cdot d\mathbf{r} = (\partial_1 \mathbf{r} dx_1 + \partial_2 \mathbf{r} dx_2) \cdot (\partial_1 \mathbf{r} dx_1 + \partial_2 \mathbf{r} dx_2) = g_{ij} dx_i dx_j,$$

i.e., $g_{ij} = \partial_i \mathbf{r} \cdot \partial_j \mathbf{r}$. We set

$$|g| = g_{11}g_{22} - g_{12}^2.$$

The orientation is so chosen that the inner unit normal to the surface at any point is given by

$$\mathbf{n} = \frac{1}{\sqrt{|g|}} \partial_1 \mathbf{r} \times \partial_2 \mathbf{r}.$$

The Gauss curvature K of the surface, which is positive, is expressed by the formula

$$K = \frac{h_{11}h_{22} - h_{12}^2}{|g|}. \quad (2.3.1)$$

We also use the basic equation which takes the form

$$\partial_{ij}\mathbf{r} = \Gamma_{ij}^k \partial_k \mathbf{r} + h_{ij}\mathbf{n}, \quad i, j = 1, 2. \quad (2.3.2)$$

We now introduce one other function ρ . Choosing the origin as the center of the largest sphere which may be inscribed in $\mathbf{r}$, we define

$$\rho(x_1, x_2) = \frac{1}{2} \mathbf{r} \cdot \mathbf{r}. \quad (2.3.3)$$

This function ρ satisfies a second order differential equation, which is also called the Darboux equation. It can be easily derived by expressing K in terms of ρ and its derivatives as follows. Differentiating (2.3.3), we have

$$\partial_i \rho = \partial_i \mathbf{r} \cdot \mathbf{r}, \quad i = 1, 2, \quad (2.3.4)$$

and

$$\partial_{ij}\rho = \partial_{ij}\mathbf{r} \cdot \mathbf{r} + g_{ij} = \Gamma_{ij}^k \partial_k \rho + h_{ij}\mathbf{r} \cdot \mathbf{n} + g_{ij}. \quad (2.3.5)$$

In establishing (2.3.5), we have used (2.3.2) and (2.3.4). We may solve for h_{ij} in (2.3.5) and express K in terms of derivatives of ρ to obtain the equation

$$K(\mathbf{r} \cdot \mathbf{n})^2 = \frac{h_{11}h_{22} - h_{12}^2}{|g|}(\mathbf{r} \cdot \mathbf{n})^2 = \frac{1}{|g|} \det(\partial_{ij}\rho - \Gamma_{ij}^k \partial_k \rho - g_{ij}). \quad (2.3.6)$$

The expression $(\mathbf{r} \cdot \mathbf{n})^2$ represents the square of the distance from the origin to the plane tangent to the surface at the point (x_1, x_2) . It may in turn be expressed in terms of ρ and g_{ij} as follows

$$\begin{aligned} (\mathbf{r} \cdot \mathbf{n})^2 &= |\mathbf{r}|^2 - |\mathbf{r} \times \mathbf{n}|^2 = |\mathbf{r}|^2 - \left| \mathbf{r} \times \frac{\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}}{\sqrt{|g|}} \right|^2 \\ &= |\mathbf{r}|^2 - \frac{1}{|g|} |(\mathbf{r} \cdot \partial_1 \mathbf{r}) \partial_2 \mathbf{r} - (\mathbf{r} \cdot \partial_2 \mathbf{r}) \partial_1 \mathbf{r}|^2 \\ &= 2\rho - g^{ij} \partial_i \rho \partial_j \rho, \end{aligned} \quad (2.3.7)$$

as a consequence of (2.3.3) and (2.3.4). Substituting (2.3.7) into (2.3.6), we obtain the following nonlinear differential equation of Monge-Ampère type for the function ρ

$$\mathcal{F}(x, \rho, \partial \rho, \partial^2 \rho) = \frac{1}{|g|} \det(\partial_{ij}\rho - \Gamma_{ij}^k \partial_k \rho - g_{ij}) - K(2\rho - g^{ij} \partial_i \rho \partial_j \rho) = 0. \quad (2.3.8)$$

This equation is invariant under the change of coordinates. In addition, it is elliptic, since we have, in view of (2.3.7)

$$4\mathcal{F}_{\partial_{11}\rho}\mathcal{F}_{\partial_{22}\rho} - \mathcal{F}_{\partial_{12}\rho}^2 = \frac{4}{|g|}K(\mathbf{r} \cdot \mathbf{n})^2 > 0. \quad (2.3.9)$$

This is because the surface is convex ($K > 0$) and contains the origin in its interior.

In the Nirenberg's solution of Weyl problem, the Darboux equation (2.3.8) was used only in the proof of Part (c), the closedness. For the proof of Part (b), the openness, Nirenberg followed an idea of Weyl's by solving the first order system for the isometric embedding. It is an extremely complicated process.

In the rest of the subsection, we discuss the closedness in Nirenberg's solution of Weyl problem. The key result is the following theorem.

Theorem 2.3.3 *Let $\{g_{t_i}\}$ be a sequence of smooth metrics on $\mathbb{S}^2$ with positive Gauss curvature which can be isometrically embedded in $\mathbb{R}^3$ by a smooth embedding $\mathbf{r}_{t_i}$. Suppose g_{t_i} converges to g_t in C^4 for a smooth metric g_t on $\mathbb{S}^2$ with positive Gauss curvature. Then g_t can be isometrically embedded in $\mathbb{R}^3$ by a smooth embedding.*

In order to prove Theorem 2.3.3, we need to show that the $C^{3,\alpha}$ -norms of $\mathbf{r}_{t_i}$ can be estimated independent of t_i , for some $\alpha \in (0, 1)$. Then we simply apply the Ascoli theorem to prove that a subsequence of $\mathbf{r}_{t_i}$ converges in C^3 -norm to a $C^{3,\alpha}$ isometric embedding $\mathbf{r}_t$. Then the smoothness of $\mathbf{r}_t$ follows from the standard results in the theory of elliptic differential equations.

Let us now estimate the $C^{3,\alpha}$ -norm of $\mathbf{r}_{t_i}$. For convenience, we drop the dependence on t_i and prove a general result.

Theorem 2.3.4 *Let $\mathbf{r}$ be a closed smooth convex surface in $\mathbb{R}^3$ with a smooth first fundamental form g , with the center of the largest sphere inscribed in $\mathbf{r}$ taken as the origin. Then for any integer $m \geq 3$ and any $\alpha \in (0, 1)$*

$$|\mathbf{r}|_{C^{m,\alpha}} \leq C_{m,\alpha}, \quad (2.3.10)$$

where $C_{3,\alpha}$ is a positive constant depending only on α , $|g|_{C^4}$, $\min K$ and $\min |g|$; and $C_{m,\alpha}$ is a positive constant depending only on m , α , $|g|_{C^{m,\alpha}}$, $\min K$ and $\min |g|$ for $m \geq 4$.

The main part of the proof is to estimate the $C^{2,\beta}$ -norm of $\mathbf{r}$ for some $\beta \in (0, 1)$. Then the standard bootstrap argument yields the estimate for the $C^{m,\alpha}$ -norm.

We first need the following estimate of the mean curvature of convex surfaces in terms of the Gauss curvature. The proof is based on straightforward calculations.

Lemma 2.3.5 *For a compact surface (M, g) in $\mathbb{R}^3$ with positive curvature K , the mean curvature H satisfies*

$$\sup_M H^2 \leq \sup_M \left(K - \frac{\Delta_g K}{4K} \right). \quad (2.3.11)$$

We now sketch the proof of Theorem 2.3.4.

Proof of Theorem 2.3.4 The proof consists of several steps.

Step 1. First, by a comparison theorem, the intrinsic diameter of g is bounded in terms of $\min K$, and hence so also is the diameter of the closed convex surface $\mathbf{r}$. It then follows that the length of the vector $\mathbf{r}$ is bounded. Next, since $\partial_i \mathbf{r} \cdot \partial_i \mathbf{r} = g_{ii}$, the vector $\partial_i \mathbf{r}$ is also bounded in length, $i = 1, 2$. In conclusion, we have

$$|\mathbf{r}|_{C^1} \leq C_1, \quad (2.3.12)$$

where C_1 is a constant depending only on $|g|_{L^\infty}$ and $\min K$.

Step 2. The second derivatives of $\mathbf{r}$ may be bounded in terms of a bound for the mean curvature H of the surface $\mathbf{r}$ as follows. The expression of H implies

$$g_{11}H = \frac{1}{2|g|}(h_{22}g_{11}^2 - 2h_{12}g_{11}g_{12} + h_{11}g_{12}^2) + \frac{1}{2}h_{11}.$$

Since the surface is convex and the unit normal $\mathbf{n}$ was chosen to be an inner normal, the quadratic form on the right-hand side is positive definite so that we have

$$h_{11} < 2g_{11}H.$$

Similarly, we have

$$h_{22} < 2g_{22}H.$$

Then we get

$$|h_{12}| < 2H\sqrt{g_{11}g_{22}},$$

since (h_{ij}) is positive definite. By the Gauss equation, we obtain with (2.3.12)

$$|\partial_{ij}\mathbf{r}| \leq C(|\partial\mathbf{r}| + |h_{ij}|) \leq C(1 + H).$$

With Lemma 2.3.5, we obtain

$$|\mathbf{r}|_{C^2} \leq C_2, \quad (2.3.13)$$

where C_2 is a positive constant depending only on $|g|_{C^4}$, $\min K$ and $\min |g|$.

Step 3. We estimate the Hölder semi-norm of the second derivatives of $\mathbf{r}$. To do this, we study the function ρ introduced in (2.3.3),

$$\rho = \frac{1}{2}\mathbf{r} \cdot \mathbf{r}.$$

It suffices to estimate the Hölder semi-norm of the second derivatives of ρ . Recall that ρ satisfies the nonlinear differential equation of Monge-Ampère type $\mathcal{F}(x, \rho, \partial\rho, \partial^2\rho) = 0$ in (2.3.8). By (2.3.9) and a simple geometric argument, we have

$$4\mathcal{F}_{\partial_{11}\rho}\mathcal{F}_{\partial_{22}\rho} - \mathcal{F}_{\partial_{12}\rho}^2 = \frac{4}{|g|}K(\mathbf{r} \cdot \mathbf{n})^2 \geq c.$$

With (2.3.13), it follows

$$\lambda I \leq (\partial_{\rho_{ij}} \mathcal{F}) \leq \lambda^{-1} I,$$

where λ is a positive constant depending only on $|g|_4$, $\min K$ and $\min |g|$. Hence $\mathcal{F} = 0$ is uniformly elliptic. Therefore, standard results from the theory of fully nonlinear elliptic differential equations imply

$$|\partial^2 \rho|_{C^\beta} \leq C'_{2,\beta},$$

where $C'_{2,\beta}$ is a positive constant depending only on $|g|_4$, $\min K$ and $\min |g|$. By the relation between ρ and $\mathbf{r}$ discussed earlier in this section, we get

$$|\partial^2 \mathbf{r}|_{C^\beta} \leq C''_{2,\beta}.$$

With (2.3.13), we have

$$|\mathbf{r}|_{C^{2,\beta}} \leq C_{2,\beta},$$

where $C_{2,\beta}$ is a positive constant depending only on $|g|_4$, $\min K$ and $\min |g|$. Finally, the estimates for ρ , and hence for $\mathbf{r}$, can be extended to the $C^{m,\alpha}$ -norm for any integer $m \geq 3$ and any $\alpha \in (0, 1)$. $\square$

2.3.2 A Rigidity Result

In this subsection, we study the isometric embedding of general closed surfaces. We start with closed surfaces in $\mathbb{R}^3$. As is well-known, a closed surface M in $\mathbb{R}^3$ satisfies

$$\int_M K^+ dg \geq 4\pi,$$

where K is the Gauss curvature of M and K^+ is its positive part, i.e., $K^+ = \max\{0, K\}$. This simply says that the image of the Gauss map on $\{p \in M : K(p) > 0\}$ covers the unit sphere $\mathbb{S}^2$ at least once. Such an integral condition provides an obstruction for the existence of isometric embedding of metrics on closed surfaces.

To find sufficient conditions for the existence of isometric embedding, we first examine the rigidity, the uniqueness of the isometric embedding if it exists. For closed surfaces with Gauss curvature of the mixed sign, Alexandrov [Ale38] introduced a class of surfaces satisfying some integral condition for its Gauss curvature and proved that any compact analytic surfaces with this condition is rigid. Nirenberg [Nir63] partially generalized this result for smooth surfaces. To do this, he needed some extra conditions, one of which is not intrinsic.

Specifically, let (M, g) be a closed surface such that

$$\int_{\{K>0\}} K dg = 4\pi, \quad (2.3.14)$$

and

$$\nabla K \neq 0 \text{ whenever } K = 0. \quad (2.3.15)$$

The assumption (2.3.15) means the Gauss curvature changes sign cleanly and it implies that $\{p \in M; K(p) = 0\}$ consists of finitely many closed curves in M . Let $M_+ = \{p \in M : K(p) > 0\}$. It is proved in [Nir63] that $(M_+, g|_{M_+})$ is rigid in $\mathbb{R}^3$ and that (M, g) is rigid if there is at most one closed asymptotic curve in each component of $M_- = \{p \in M : K(p) < 0\}$. We need to point out that the extra assumption on asymptotic curves is not intrinsic. With this rigidity result, it seems reasonable to start with closed surfaces satisfying (2.3.14) and (2.3.15) in our discussion of the isometric embedding of (M, g) in $\mathbb{R}^3$.

Since (2.3.14) involves the part of the surface where the Gauss curvature is positive, we will focus on this part. Manifolds in the rest of this subsection are compact with nonempty boundary.

We now formulate the rigidity results by Alexandrov and Nirenberg as follows. Refer to [Nir63], or [HH06], for a proof.

Theorem 2.3.6 *Let Σ be an oriented and bounded C^4 -surface in $\mathbb{R}^3$ with nonempty boundary. Suppose*

$$\begin{aligned} K &> 0 \text{ in } \Sigma, \\ K &= 0 \text{ and } \nabla K \neq 0 \text{ on } \partial\Sigma, \\ \int_{\Sigma} K dg &= 4\pi. \end{aligned} \quad (2.3.16)$$

Then,

- (1) $\partial\Sigma$ consists of finitely many smooth planar convex curves σ_j , $j = 1, \dots, J$. Moreover, the plane containing σ_j is tangent to Σ along σ_j , for each $j = 1, \dots, J$;
- (2) the geodesic curvature k_g of σ_j is negative, for each $j = 1, \dots, J$;
- (3) $\Sigma \cup \partial\Sigma$ is rigid.

By Theorem 2.3.6(1), the geodesic curvature k_g of each σ_j is simply the curvature of σ_j as a planar curve. As a consequence, we obtain

$$\int_{\sigma_i} k_g ds = -2\pi, \quad (2.3.17)$$

and

$$\int_0^{l_i} e^{\sqrt{-1} \int_0^s k_g(\tau) d\tau} ds = 0, \quad (2.3.18)$$

where σ_i is parametrized by $s \in [0, l_i]$.

We now formulate the following question.

Question 2.3.7 Let Ω be a smooth domain in $\mathbb{R}^2$ with nonempty boundary and g be a smooth metric in $\bar{\Omega}$. Suppose

$$\begin{aligned} K &> 0 \quad \text{in } \Omega, \\ K &= 0 \text{ and } \nabla K \neq 0 \quad \text{on } \partial\Omega, \\ \int_{\Omega} K dg &= 4\pi, \end{aligned} \tag{2.3.19}$$

and, for each connected component σ_i in $\partial\Omega$,

$$\begin{aligned} \int_{\sigma_i} k_g ds &= -2\pi, \\ \int_0^{l_i} e^{\sqrt{-1} \int_0^s k_g(\tau) d\tau} ds &= 0, \end{aligned} \tag{2.3.20}$$

where σ_i is parametrized by $s \in [0, l_i]$. Does (Ω, g) admit a smooth isometric embedding in $\mathbb{R}^3$?

2.3.3 Compactness of Alexandrov-Nirenberg Surfaces

Our main object in this subsection is the surfaces introduced by Alexandrov and Nirenberg, as in Theorem 2.3.6. For convenience, we introduce the following terminology.

Definition 2.3.8 We call Σ an Alexandrov-Nirenberg surface if it satisfies (2.3.16).

Our ultimate goal is to study the isometric embedding related to Alexandrov-Nirenberg surfaces. The rigidity result in Theorem 2.3.6(3) can be interpreted as the uniqueness of the isometric embedding. We are interested in the existence of the related isometric embedding. Following Nirenberg's solution of the Weyl problem, we plan to use the method of continuity to prove such an embedding. As discussed in Sect. 2.3.1, there are three steps in the method of continuity: connectedness, openness and closedness. The closedness often appears in the form of *a priori* estimates.

Now, we present a result by Han et al. [HHH14].

Theorem 2.3.9 For any integers $J \geq 1$ and $k \geq 2$ and any constant $\alpha \in (0, 1)$, let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $\mathbf{r} : \Omega \rightarrow \mathbb{R}^3$ be a $C^{k+3, \alpha}$ -mapping such that $\Sigma = \mathbf{r}(\Omega)$ is an Alexandrov-Nirenberg surface. Then,

$$|\mathbf{r}|_{C^{k, \alpha}(\bar{\Omega})} \leq C \left(|g|_{C^{k+2, \alpha}(\bar{\Sigma})}, \min_{\partial\Sigma} |\nabla K|, \max_{\partial\Sigma} |k_g| \right),$$

where g is the induced metric on Σ , K is the Gauss curvature of Σ and k_g is the geodesic curvature of $\partial\Sigma$.

We note that ∇K does not vanish on $\partial\Sigma$ by (2.3.16) and that k_g does not vanish on $\partial\Sigma$ by Theorem 2.3.6(2).

Difficulties in deriving the estimate in Theorem 2.3.9 arise from the condition $K = 0$ on $\partial\Sigma$. As discussed earlier, vanishing Gauss curvature results in degeneracy of the associated nonlinear elliptic equations. Hong [Hon99] studied the case where $\partial\Sigma$ consists of one connected component and the geodesic curvature k_g of $\partial\Sigma$ is positive everywhere. However, in the present case, $k_g < 0$ on $\partial\Sigma$ by Theorem 2.3.6(2). From an analytic point of view, the associated elliptic equation is non-characteristically degenerate on $\partial\Sigma$ if $k_g > 0$ on $\partial\Sigma$ and is characteristically degenerate if $k_g < 0$. The latter is presumably more difficult to study than the former.

To prove Theorem 2.3.9, we need to derive *a priori* estimates of the second fundamental forms. In the rest of this section, we describe the set up and major steps in proving Theorem 2.3.9.

Suppose Σ is an Alexandrov-Nirenberg surface as introduced in Definition 2.3.8. By Theorem 2.3.6, $\partial\Sigma$ consists of finitely many planar convex curves. Let σ be a connected component in $\partial\Sigma$. Without loss of generality, we assume that, in the geodesic coordinates with the base curve σ , the induced metric g is of the form

$$g = B^2 ds^2 + dt^2 \quad \text{for any } (s, t) \in [0, 2\pi] \times [0, 1] \quad (2.3.21)$$

where B is a positive function in $[0, 2\pi] \times [0, 1]$ satisfying

$$B(\cdot, 0) = 1, \quad B_t(\cdot, 0) = -k_g. \quad (2.3.22)$$

Here, $t = 0$ corresponds to the boundary curve σ and the negative sign in B_t indicates that the geodesic curvature of σ is calculated with respect to the anticlockwise orientation. Obviously, we have $B_t > 0$ on σ . Furthermore, we assume, by a scaling in t if necessary, that

$$B_t > 0 \text{ for all } t \in [0, 1].$$

Here and hereafter, we adopt the notion $(\partial_s, \partial_t) = (\partial_1, \partial_2)$. The Gauss-Codazzi equations are given by

$$L_t - M_s = \frac{B_t}{B}L - \frac{B_s}{B}M + BB_tN, \quad (2.3.23)$$

$$M_t - N_s = -\frac{B_t}{B}M, \quad (2.3.24)$$

and

$$NL - M^2 = KB^2. \quad (2.3.25)$$

The mean curvature H is given by

$$H = \frac{1}{2} \left(\frac{L}{B^2} + N \right). \quad (2.3.26)$$

We point out that by Definition 2.3.8, or by (2.3.16) specifically, we have

$$K(\cdot, 0) = 0, \quad K_t(\cdot, 0) > 0.$$

A simple calculation yields the following result.

Lemma 2.3.10 *Let Σ be an Alexandrov-Nirenberg surface in $\mathbb{R}^3$ of class C^4 and σ be a connected component in $\partial\Sigma$. Then, in the geodesic coordinates as in (2.3.21) and (2.3.22),*

$$L = M = 0, \quad N = \sqrt{\frac{K_t}{B_t}} \quad \text{on } t = 0,$$

and

$$L_t = \sqrt{K_t B_t} \quad \text{on } t = 0.$$

In other words, L , M , N and L_t are intrinsically determined on σ .

Next, for the Alexandrov-Nirenberg surface Σ in $\mathbb{R}^3$, we assume by Theorem 2.3.6(1) that $\partial\Sigma$ consists of J planar convex curves. Hence, Σ and the planar convex regions enclosed by these curves form a convex surface $\tilde{\Sigma}$ in $\mathbb{R}^3$. A simple geometric argument shows that *there exists a ball of radius R_0 inside $\tilde{\Sigma}$, where R_0 is a positive constant depending only on $1/\max K$ and the intrinsic diameter l of Σ* . In the following, we always take the origin as the center of this ball. We have the following upper bound of the mean curvature.

Lemma 2.3.11 *Let Σ be an Alexandrov-Nirenberg surface in $\mathbb{R}^3$ of class C^5 . Then,*

$$H \leq C \left\{ \max_{\partial\Sigma} \sqrt{\frac{|\nabla K|}{|k_g|}} + \max_{\Sigma} K + \max_{\Sigma} \sqrt{|\Delta K|} \right\},$$

where C is a positive constant depending only on the intrinsic diameter of Σ .

Lemma 2.3.11 extends Lemma 2.3.5 for closed surfaces without boundary to surfaces with boundary, where the Gauss curvature vanishes. Following steps outlined in the proof of Theorem 2.3.4, we can derive *interior* estimates of derivatives of the position vector $\mathbf{r}$. For estimates near the boundary, the crucial part is the estimate of the boundary Lipschitz norm. We achieve this in three successive steps:

- Step 1. Estimate the L^∞ -norm by the maximum principle;
- Step 2. Estimate the boundary Hölder norm by de Giorgi iteration;
- Step 3. Estimate the boundary Lipschitz norm by blow-up arguments.

After these 3 steps, we estimate the boundary higher order norm by results in [HH12] on L^p and Hölder boundary estimates for a class of characteristically degenerate elliptic equations.

2.3.4 Isometric Embedding Near Closed Curves

In this subsection, we describe a result due to Dong [Don93] concerning the isometric embedding near a closed curve where the Gauss curvature changes sign cleanly. Such a result can be considered as a semi-global version of Theorem 2.2.3.

Theorem 2.3.12 *Let ε_0 be a positive constant, m be a positive integer, and g be a C^m -metric in $\mathbb{S}^1 \times (-\varepsilon_0, \varepsilon_0)$ given by*

$$g = B^2(s, t)ds^2 + dt^2, \quad (2.3.27)$$

for some C^m -function B in $\mathbb{S}^1 \times (-\varepsilon_0, \varepsilon_0)$ with $B(\cdot, 0) = 1$. Assume $K = 0$ and $\nabla K \neq 0$ on $\{t = 0\}$. Suppose

$$K_t B_t > 0 \quad \text{on } \{t = 0\}, \quad (2.3.28)$$

and

$$\begin{aligned} \int_0^{2\pi} |B_t(s, 0)| ds &= 2\pi, \\ \int_0^{2\pi} \exp\left\{\sqrt{-1} \int_0^s |B_t(s, 0)| d\tau\right\} ds &= 0. \end{aligned} \quad (2.3.29)$$

Then for some $\varepsilon \in (0, \varepsilon_0)$, g restricted to $\mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ admits a C^{m-m_0} isometric embedding in $\mathbb{R}^3$, for some universal integer m_0 .

We point out that Theorem 2.3.12 will play an important role in answering Question 2.3.7. Let σ be a connected component in $\partial\Omega$. Assume that, in the geodesic coordinates with the base curve σ , g is given by (2.3.27), with $B(s, 0) = 1$. We assume $K_t(s, 0) > 0$. Then, (2.3.22) implies $B_t = -k_g$. By (2.3.28), we have $k_g < 0$. Then, (2.3.29) is equivalent to (2.3.20). Therefore, Theorem 2.3.12 asserts that the metric g in Question 2.3.7 restricted to a neighborhood of $\partial\Omega$ admits an isometric embedding in $\mathbb{R}^3$.

2.3.5 Torus-Like Surface

In this subsection, we briefly discuss the global isometric embedding of closed manifolds in $\mathbb{R}^3$. The torus $\mathbb{T}^2$ is our model.

It is natural to ask whether conditions (2.3.14) and (2.3.15) are sufficient for the isometric embedding of (M, g) in $\mathbb{R}^3$. We may even assume that M itself is already

an embedded closed surface in $\mathbb{R}^3$ and g is sufficiently close to the induced metric. It turns out that (2.3.14) and (2.3.15) are not sufficient even in this special case.

We now discuss torus $\mathbb{T}^2$. Suppose $\{\mathbb{T}^2, g_0\}$ is a standard torus with the standard metric in $\mathbb{R}^3$. It is easy to check that $\{\mathbb{T}^2, g_0\}$ satisfies (2.3.14) and (2.3.15). For metrics on $\mathbb{T}^2$, instead of (2.3.15), we assume

$$\{K = 0\} \text{ consists of two curves where } \nabla K \neq 0. \quad (2.3.30)$$

In the following, we identify $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and denote $(s, t) \in \mathbb{S}^1 \times \mathbb{S}^1$. Let g be a smooth metric on $\mathbb{T}^2$ given by

$$g = E(t)ds^2 + G(t)dt^2, \quad (2.3.31)$$

where E and G are smooth positive 2π -periodic functions.

The following result was proved by Han and Lin [HL08].

Theorem 2.3.13 *Suppose that g is a metric on $\mathbb{T}^2$ as in (2.3.31) and satisfies (2.3.14) and (2.3.30) with $K = 0$ on $\{t = t_1\}$ and $\{t = t_2\}$ for some $t_1, t_2 \in [0, 2\pi)$ with $t_1 < t_2$. Then g admits a smooth isometric embedding in $\mathbb{R}^3$ if and only if*

$$\int_{t_1}^{t_2} \sqrt{1 - \left(\frac{E'}{2\sqrt{EG}}\right)^2} \sqrt{G} dt = \int_{t_2}^{t_1+2\pi} \sqrt{1 - \left(\frac{E'}{2\sqrt{EG}}\right)^2} \sqrt{G} dt. \quad (2.3.32)$$

We point out that (2.3.32) is an additional assumption besides (2.3.14) and (2.3.30). It remains open to generalize to the general case.

2.4 Isometric Immersions of Complete Negatively Curved Surfaces

In this section, we discuss whether a complete negatively curved surface admits an isometric immersion in $\mathbb{R}^3$. Here a negatively curved surface is a surface with negative Gauss curvature. An example of such a surface is given by the hyperbolic surface whose Gauss curvature is -1 .

The study of negatively curved surfaces in $\mathbb{R}^3$ is closely related to the interpretation of non-Euclidean geometry. The first result concerning whether the entire hyperbolic plane can be realized globally in $\mathbb{R}^3$ is due to Hilbert [Hil01].

Theorem 2.4.1 *The hyperbolic plane does not admit any C^2 isometric immersion in $\mathbb{R}^3$.*

In fact, Hilbert originally proved that the hyperbolic plane does not admit any C^m isometric immersion in $\mathbb{R}^3$, for m sufficiently large. Here, the nonexistence of C^2 isometric immersion follows from a result of Efimov's.

During the 1960s, Efimov discussed various generalizations of Hilbert's result to complete negatively curved surfaces. He found different conditions on the Gauss curvature under which no isometric immersions in $\mathbb{R}^3$ exist. We now review two of his results. The first result due to Efimov [Efi63] is the following.

Theorem 2.4.2 *Any complete negatively curved smooth surface does not admit a C^2 isometric immersion in $\mathbb{R}^3$ if its Gauss curvature K is bounded away from zero, i.e., $K \leq \text{const} < 0$.*

Efimov's proof is very delicate and complicated. Readers can also refer to Klotz-Milnor [Klo72]. Based on his earlier results, Efimov [Efi68] made more progress in the study of nonexistence, proving the following result.

Theorem 2.4.3 *Any complete negatively curved smooth surface M has no C^2 isometric immersion in $\mathbb{R}^3$ if its Gauss curvature K satisfies*

$$\sup_M |K|, \sup_M |D(\frac{1}{\sqrt{|K|}})| < \infty.$$

Before the 1970s, most of the study on negatively curved surfaces involves nonexistence. As for affirmative answers, no result for complete negatively curved surfaces was known. Yau [Yau82] raised the following problem: *Find a sufficient condition for a complete negatively curved surface to be isometrically immersed in $\mathbb{R}^3$.* He also pointed out that a reasonable sufficient condition might be the decay rate of the Gauss curvature at infinity. In 1993, Hong [Hon93] gave an affirmative answer and showed that a correct sufficient condition is that the Gauss curvature decays at infinity faster than the inverse square of the geodesic distance.

Theorem 2.4.4 *Let (M, g) be a complete simply connected smooth surface with Gauss curvature $K < 0$ and (ρ, θ) be a (global) geodesic polar coordinate. Assume, for some constant $\delta > 0$,*

- (H₁) $\rho^{2+\delta}|K|$ is decreasing in ρ outside a compact set;
- (H₂) $\partial_\theta^i \ln |K|$, ($i = 1, 2$), $\rho \partial_\rho \partial_\theta \ln |K|$ are bounded.

Then (M, g) admits a smooth isometric immersion in $\mathbb{R}^3$.

Hong [Hon93] proved Theorem 2.4.4 by solving the Gauss-Codazzi system, which is equivalent to the Rozhdestvenskiĭ system for negatively curved surfaces.

In 2010, Chen et al. [CSW10], [CSW10] studied the Gauss-Codazzi system from another point of view. They established a connection between gas dynamics and differential geometry and showed how the fluid dynamics can be used to formulate a geometry problem.

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