$\mathrm{Re}\ \nu = c$ with $c = -2a + 2 + \mathrm{Re}\ \nu$, the constraints on c being thus $\mathrm{Re}\ \nu < c < 2 \pm \mathrm{Re}\ \nu$: noting that $|x|^{\frac{\rho-\nu-2}{2}} |\xi|^{-\rho-\nu} = \hom^{(0)}_{\rho,-\nu}(x,\xi)$, one obtains

$$\mathfrak{S}_n = \frac{1}{8i\pi} \int_{\mathrm{Re}\ \rho=c} n^{\frac{\rho-\nu-2}{2}}\, B_0(\frac{\rho-\nu}{2}) \hom^{(0)}_{\rho,-\nu}\, d\rho, \quad \mathrm{Re}\ \nu < c < 2 \pm \mathrm{Re}\ \nu. \tag{1.1.45}$$

Theorem 1.1.8. *Assume that $-1 < \mathrm{Re}\ \nu < 1$, $\nu \neq 0$. Then, in the weak sense in $\mathcal{S}'(\mathbb{R}^2)$,*

$$\frac{1}{2}\, \mathfrak{E}_\nu(x,\xi) = \frac{1}{2}\, \zeta(-\nu) \left[\, |x|^{-\nu-1} + |\xi|^{-\nu-1} \right] + \frac{1}{2}\, \zeta(1-\nu) \left[\, |x|^{-\nu}\, \delta(\xi) + \delta(x)\, |\xi|^{-\nu} \right]$$
$$+ \frac{1}{8i\pi} \int_{\mathrm{Re}\ \rho=1} \zeta(\frac{2-\rho-\nu}{2})\, \zeta(\frac{\rho-\nu}{2}) \hom^{(0)}_{\rho,-\nu}(x,\xi)\, d\rho. \tag{1.1.46}$$

One can also, when desirable, replace the line $\mathrm{Re}\ \rho = 1$ by any line $\mathrm{Re}\ \rho = c$ with $c > 2 + |\mathrm{Re}\ \nu|$, provided that one deletes from the right-hand side the terms $\frac{1}{2}\, \zeta(-\nu)|\xi|^{-\nu-1}$ and $\frac{1}{2}\, \zeta(1-\nu)|x|^{-\nu}\delta(\xi)$.

Proof. We rewrite (1.1.38) as a decomposition

$$\frac{1}{2}\, \mathfrak{E}_\nu(x,\xi) = \frac{1}{2}\, \zeta(-\nu)\, |\xi|^{-\nu-1} + \frac{1}{2}\, \zeta(1-\nu)\, |x|^{-\nu}\, \delta(\xi) + \frac{1}{2}\, \mathfrak{E}^{\mathrm{main}}_\nu(x,\xi) \tag{1.1.47}$$

and from (1.1.45), assuming that $\mathrm{Re}\ \nu < c < 2 \pm \mathrm{Re}\ \nu$ (a condition certainly verified if $c = 1$),

$$\frac{1}{2}\, \mathfrak{E}^{\mathrm{main}}_\nu = \frac{1}{8i\pi} \sum_{n\geq 1} \sigma_\nu(n) \int_{\mathrm{Re}\ \rho=c} n^{\frac{\rho-\nu-2}{2}}\, B_0(\frac{\rho-\nu}{2}) \hom^{(0)}_{\rho,-\nu}\, d\rho. \tag{1.1.48}$$

We have shown in (1.1.14) in which sense this type of integral is always convergent: but we must still arrange for summability with respect to n. The product

$$B_0(\frac{\rho-\nu}{2}) \hom^{(0)}_{\rho,-\nu} = B_0(\frac{\rho-\nu}{2})\, |x|^{\frac{\rho-\nu-2}{2}} |\xi|^{\frac{-\rho-\nu}{2}}, \tag{1.1.49}$$

contrary to its second factor, is regular at $\rho = \nu$. This makes it possible, in (1.1.48), to move the line of integration to any line $\mathrm{Re}\ \rho = c$ with $c < \pm\mathrm{Re}\ \nu$. The right-hand side of (1.1.48) then becomes a convergent series of integrals, and one has for ρ on the new line of integration

$$\sum_{n\geq 1} \sigma_\nu(n)\, n^{\frac{\rho-\nu-2}{2}} = \sum_{d,k\geq 1} d^\nu (kd)^{\frac{\rho-\nu-2}{2}} = \zeta(\frac{2-\rho-\nu}{2})\, \zeta(\frac{2-\rho+\nu}{2}). \tag{1.1.50}$$

Using this identity together with the functional equation (1.1.3), one obtains the weak integral decomposition in $\mathcal{S}'(\mathbb{R}^2)$

$$\frac{1}{2}\, \mathfrak{E}^{\mathrm{main}}_\nu(x,\xi) = \frac{1}{8i\pi} \int_{\mathrm{Re}\ \rho=c} \zeta(\frac{2-\rho-\nu}{2})\, \zeta(\frac{\rho-\nu}{2}) \hom^{(0)}_{\rho,-\nu}(x,\xi)\, d\rho, \tag{1.1.51}$$

provided that Re $\rho < -|\text{Re } \nu|$. What has to be done finally is to move back the line of integration to the line Re $\rho = 1$, paying attention to the poles ρ of the integrand such that Re $\rho < 1$. The first zeta factor contributes a simple pole at $\rho = -\nu$, and the factor $\text{hom}^{(0)}_{\rho,-\nu}$ is singular when $\rho = \nu, \nu - 4, \dots$: but the simple poles $\nu - 4, \nu - 8, \dots$ can be discarded since they are (trivial) zeros of the second zeta factor. We shall have to add to the new integral obtained the product of the sum of residues by $-2i\pi$. The residue at $\rho = -\nu$ is

$$(-2)\,\zeta(-\nu)\,\text{hom}^{(0)}_{-\nu,-\nu}(x,\xi) = -2\,\zeta(-\nu)\,|x|^{-\nu-1} \tag{1.1.52}$$

while, as a consequence of (1.1.24) and of the equation $\zeta(0) = -\frac{1}{2}$, the residue of the integrand at $\rho = \nu$ is $-2\,\zeta(1-\nu)\,\delta(x)\,|\xi|^{-\nu}$. This leads to the decomposition (1.1.46).

If, as will be helpful later, one wishes to replace the line Re $\rho = 1$ by a line Re $\rho = c$ with c large, one must take into consideration the poles of the product $\zeta(\frac{\rho-\nu}{2})\,\text{hom}^{(0)}_{\rho,-\nu}$ with Re $\rho > 1$. There is one at $\rho = 2 + \nu$ because of the first factor, while the second factor has poles at $\rho = 2 - \nu, 6 - \nu, \dots$: but the poles $6 - \nu, 10 - \nu, \dots$ are killed by zeros of the other zeta factor $\zeta(\frac{2-\rho-\nu}{2})$. Finally, the residues at the only two remaining poles are obtained from an application of Lemma 1.1.4, together with the fact that $\zeta(0) = -\frac{1}{2}$. $\square$

Remark 1.1.2. Since $\text{hom}^{(0)}_{\rho,-\nu}(-\xi, x) = \text{hom}^{(0)}_{2-\rho,-\nu}(x,\xi)$, the fact that $\mathfrak{E}_\nu$ is invariant under the action of the map $(x,\xi) \mapsto (-\xi, x)$ remains apparent under the decomposition: not so the invariance under the action of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. It is the other way around when the Fourier expansion (1.1.38) is used instead.

Corollary 1.1.9. *Theorem* 1.1.8 *extends to the values of* ν *such that* Re $\nu \neq -1$, Re $\nu \neq 1, 5, \dots$ *and* $\nu \neq 0, 1, \dots$.

Proof. The left-hand side can be continued to values of ν distinct from ± 1. On the first line of the right-hand side, it suffices to assume that, moreover, $\nu \neq 0, 2, \dots$ and $\nu \neq 0$, $\nu \neq 1, 3, \dots$. So far as the integrand on the right-hand side is concerned, it suffices to manage so that one will always have $\rho + \nu \neq 0$, $\rho - \nu \neq 2$ (considering the zeta factors) and $\rho - \nu \neq 0, -4, \dots,\ 2 - \rho + \nu \neq 0, -4, \dots$ so that $\text{hom}^{(0)}_{\rho,-\nu}$ should be well-defined: this will be the case for every ρ on the line Re $\rho = 1$ provided that Re $\nu \neq -1$ and Re $\nu \neq 1, 5, \dots$. $\square$

Remarks 1.1.3. (i) If ν lies on one of the lines Re $\nu = -1$ or Re $\nu = 1, 5, \dots$ but $\nu \neq \pm 1$, a decomposition of $\mathfrak{E}_\nu(x,\xi)$ is still possible, only turning slightly around the point ρ on the line Re $\rho = 1$ responsible for the singularity, and computing a residue: this is trivial when the singularity originates from a zeta factor, and can be obtained, when the singularity originates from the factor $\text{hom}^{(0)}_{\rho,-\nu}(x,\xi)$, from an application of (1.1.14).

(ii) It is natural, imitating a definition given for Hecke (rather than Eisenstein) distributions later, in Theorem 2.1.2, to set, with $b_k = k^{-\frac{\nu}{2}}\sigma_\nu(k)$,

$$L\left(s, \frac{1}{2}\mathfrak{E}_\nu\right) = \sum_{k\geq 1} b_k\, k^{-s} = \zeta(s-\frac{\nu}{2})\,\zeta(s+\frac{\nu}{2}): \tag{1.1.53}$$

then, rewriting (1.1.46) in terms of this function, to wit

$$\begin{aligned}\frac{1}{2}\,\mathfrak{E}_\nu(x,\xi) = \frac{1}{2}\,\zeta(-\nu)\left[\,|x|^{-\nu-1} + |\xi|^{-\nu-1}\,\right] + \frac{1}{2}\,\zeta(1-\nu)\left[\,|x|^{-\nu}\,\delta(\xi) + \delta(x)\,|\xi|^{-\nu}\,\right]\\ + \frac{1}{8i\pi}\int_{\mathrm{Re}\ \rho=1} B_0(\frac{\rho-\nu}{2})\, L\left(\frac{2-\rho}{2}, \frac{1}{2}\mathfrak{E}_\nu\right)\ \mathrm{hom}^{(0)}_{\rho,-\nu}(x,\xi)\,d\rho,\end{aligned} \tag{1.1.54}$$

it will be seen in the next section that a fully analogous formula holds for Hecke distributions, except that there are no longer "cuspidal" terms (the analogues, in modular distribution theory, of the two terms, in the Fourier expansion of non-holomorphic Eisenstein series, not rapidly decreasing at the cusp of $\Gamma\backslash\Pi$). Note the functional equation

$$B_0(\frac{\rho-\nu}{2})\, L\left(\frac{2-\rho}{2}, \frac{1}{2}\mathfrak{E}_\nu\right) = B_0(\frac{2-\rho-\nu}{2})\, L\left(\frac{\rho}{2}, \frac{1}{2}\mathfrak{E}_\nu\right), \tag{1.1.55}$$

a consequence of (1.1.3).

1.2 Hecke distributions

We introduce here Hecke operators acting on automorphic distributions: these will be related later to the more traditional notion of Hecke operator acting on automorphic functions.

Definition 1.2.1. Given an automorphic distribution $\mathfrak{S}$, we set, for $N \geq 1$,

$$\left\langle T_N^{\mathrm{dist}}\mathfrak{S}\,,\, h\right\rangle = N^{-\frac{1}{2}} \sum_{\substack{ad=N,\, d>0\\ b \bmod d}} \left\langle \mathfrak{S}\,,\,(x,\,\xi)\mapsto h\left(\frac{dx-b\xi}{\sqrt{N}},\, \frac{a\xi}{\sqrt{N}}\right)\right\rangle \tag{1.2.1}$$

and

$$\left\langle T_{-1}^{\mathrm{dist}}\mathfrak{S}\,,\, h\right\rangle = \left\langle \mathfrak{S}\,,\,(x,\xi)\mapsto h(-x,\,\xi)\right\rangle. \tag{1.2.2}$$

Just as in the automorphic function environment, the linear span of the Hecke operators T_N^{dist} with $N \geq 1$ makes up an algebra, which is generated, as such, by the operators T_p^{dist} with p prime. Automorphic distributions which are left invariant, or change to their negatives, under T_{-1}^{dist}, are said to be of even or odd type: this answers the question whether such (globally even) distributions are separately even or odd with respect to each of the two variables x and ξ.

In what follows, we consider characters χ on $\mathbb{Q}^\times$, by which we mean homomorphisms from $\mathbb{Q}^\times$ to $\mathbb{C}^\times$: we do not assume these to be unitary, but tempered, in the sense that, for some $C \geq 0$, one has

$$\left|\chi\left(\frac{m}{n}\right)\right| \leq |mn|^C \text{ for every fraction } \frac{m}{n}. \tag{1.2.3}$$

In the usual way, an entire function f is said, below, to be polynomially bounded in vertical strips if, given a segment $[a,\, b] \subset \mathbb{R}$, one has $|f(s)| \leq C(1 + |\text{Im } s|)^N$ for some pair C, N and all s with $a \leq \text{Re } s \leq b$.

Theorem 1.2.2. *Given a tempered character χ on $\mathbb{Q}^\times$ and $\lambda \in \mathbb{R}$, the (even) distribution $\mathfrak{N} = \mathfrak{N}_{\chi,i\lambda} \in \mathcal{S}'(\mathbb{R}^2)$ defined by the equation*

$$\langle \mathfrak{N}\,,\, h\rangle = \frac{1}{4} \sum_{m,\, n \neq 0} \chi\left(\frac{m}{n}\right) \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{m}{t},\, nt\right) dt\,, \quad h \in \mathcal{S}(\mathbb{R}^2), \tag{1.2.4}$$

satisfies the identity $\langle \mathfrak{N}\,,\, h \circ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle = \langle \mathfrak{N}\,,\, h\rangle$ for every function $h \in \mathcal{S}(\mathbb{R}^2)$. Also, it is homogeneous of degree $-1 - i\lambda$. Set $\chi(-1) = (-1)^\varepsilon$ with $\varepsilon = 0$ or 1, and define

$$\psi_1(s) = \sum_{m \geq 1} \chi(m)\, m^{-s} = \prod_p \left(1 - \chi(p)\, p^{-s}\right)^{-1}, \quad \psi_2(s) = \sum_{n \geq 1} (\chi(n))^{-1} n^{-s}, \tag{1.2.5}$$

two convergent series for Re s *large enough. Also, define*

$$L(s, \mathfrak{N}) = \psi_1\left(s + \frac{i\lambda}{2}\right) \psi_2\left(s - \frac{i\lambda}{2}\right) \tag{1.2.6}$$

and assume that the function $s \mapsto L(s, \mathfrak{N})$ extends as an entire function of s, polynomially bounded in vertical strips. Then, the distribution $\mathfrak{N}$ admits a decomposition into bihomogeneous components, given as

$$\mathfrak{N} = \frac{1}{8i\pi} \int_{\text{Re } \rho = 1} B_\varepsilon\left(\frac{\rho - i\lambda}{2}\right) L\left(\frac{2 - \rho}{2}, \mathfrak{N}\right) \text{hom}^{(\varepsilon)}_{\rho, -i\lambda}\, d\rho : \tag{1.2.7}$$

one can also replace the line Re $\rho = 1$ *by any line* Re $\rho = c > 1$*. It is Γ-invariant, i.e., a modular distribution, if and only if the function*

$$L^\natural(s, \mathfrak{N}) = \frac{1}{2} B_\varepsilon\left(\frac{2 - i\lambda}{2} - s\right) L(s, \mathfrak{N}) \tag{1.2.8}$$

satisfies the functional equation

$$L^\natural(s, \mathfrak{N}) = (-1)^\varepsilon\, L^\natural(1 - s, \mathfrak{N}). \tag{1.2.9}$$

If such is the case, $\mathfrak{N}$ is of necessity a Hecke distribution, by which is meant that it is an eigendistribution of the operator T_N^{dist} for every $N = 1, 2, \dots$ and for $N = -1$.

Anticipating the relation, to be made explicit in Chapter 2, between the automorphic theories available in the plane and in the hyperbolic half-plane, let us make some points clear right now.

Remarks 1.2.1. (i) In Section 2.1, we shall define a linear map from automorphic distributions in the plane to automorphic functions in the hyperbolic half-plane: then, it will be shown that every Hecke eigenform $\mathcal{N}$, an eigenfunction of Δ for the eigenvalue $\frac{1+\lambda^2}{4}$, is the image of some Hecke distribution. The distribution $\mathfrak{N}$ contains more information than the Hecke eigenform $\mathcal{N}$, the knowledge of which only determines λ^2, not λ. The L-function of $\mathcal{N}$, as defined in a usual way, will be seen to coincide with the function $L(\,\bullet\,, \mathfrak{N})$ as defined in (1.2.6), Note, on the other hand, that while $L^{\natural}(\,\bullet\,, \mathfrak{N})$ is well-defined by (1.2.8), one could not substitute $\mathcal{N}$ for $\mathfrak{N}$ there, since this definition depends on λ, not only λ^2. Finally, the functional equations of the function $L^{\natural}(\,\bullet\,, \mathfrak{N})$ and of the more classical function

$$L^*(s,\, \mathfrak{N}) = \pi^{-s}\, \Gamma(\frac{s+\varepsilon}{2} + \frac{i\lambda}{4})\, \Gamma(\frac{s+\varepsilon}{2} - \frac{i\lambda}{4})\, L(s,\, \mathfrak{N}) \tag{1.2.10}$$

are identical, since

$$L^{\natural}(s, \mathfrak{N}) = \frac{(-i)^{\varepsilon} \pi^{\frac{1-i\lambda}{2}}}{2\, \Gamma\left(\frac{s+\varepsilon}{2} - \frac{i\lambda}{4}\right) \Gamma\left(\frac{1-s+\varepsilon}{2} - \frac{i\lambda}{4}\right)}\, L^*(s, \mathfrak{N}) \tag{1.2.11}$$

and the factor of proportionality is invariant under the change of s to $1-s$. Still, the function $L^{\natural}(s,\, \mathfrak{N})$ contains slightly more information than the function $L^*(s,\, \mathfrak{N})$ (the only one available in the modular form environment), and one may simplify (1.2.7) as

$$\mathfrak{N} = \frac{1}{4i\pi} \int_{\mathrm{Re}\ \rho=1} L^{\natural}(\frac{2-\rho}{2}, \mathfrak{N})\ \mathrm{hom}_{\rho,-i\lambda}^{(\varepsilon)}\, d\rho. \tag{1.2.12}$$

(ii) On globally even distributions, the partial Fourier transformation $\mathcal{F}_1$ relates to the symplectic Fourier transformation by the equation

$$\left(\mathcal{F}_1^{-1} \mathcal{F}^{\mathrm{symp}} h\right)(\xi, \eta) = \left(\mathcal{F}_1^{-1} h\right)(\eta, \xi):$$

it follows that

$$\mathcal{F}^{\mathrm{symp}}\, \mathfrak{N}_{\chi, i\lambda} = \mathfrak{N}_{\chi^{-1}, -i\lambda}. \tag{1.2.13}$$

As a consequence of this equation, together with (1.2.5), (1.2.6), one has $L(s, \mathcal{F}^{\mathrm{symp}} \mathfrak{N}) = L(s, \mathfrak{N})$ for every Hecke distribution $\mathfrak{N}$. Proposition 2.1.1 will give a better explanation of the fact.

(iii) The pair (λ, ε) is uniquely determined by $\mathfrak{N}$, but χ is not: indeed, as will be seen below, splitting the set of primes into two disjoint sets and changing the function $p \mapsto \chi(p)$ to $p \mapsto (\chi(p))^{-1}p^{i\lambda}$ on one of the two sets (then extending the modified version of χ as a character) does not change the distribution $\mathfrak{N}$.

(iv) As will be seen later as a consequence of (2.1.28) and of the fact that, in this equation, b_p is real, one has for every prime p either $|\chi(p)| = 1$ or $\bar{\chi}(p) = \chi(p)\, p^{-i\lambda}$ (whether the first condition always holds is the Ramanujan-Petersson conjecture): then, it follows from Remark (iii) that, if one sets $\chi_1(q) = (\bar{\chi}(q))^{-1}$ for $q \in \mathbb{Q}^\times$, one has $\mathfrak{N}_{\chi_1, i\lambda} = \mathfrak{N}_{\chi, i\lambda}$.

(v) From (1.1.38), the Eisenstein distribution $\frac{1}{2}\mathfrak{E}_{i\lambda}$ has, up to two extra terms, a totally similar Fourier decomposition, taking this time $\chi = 1$.

Proof of Theorem 1.2.2. One has

$$\left(\mathcal{F}_1^{-1}\left(h \circ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)\right)(s,\, t) = e^{-2i\pi st}\left(\mathcal{F}_1^{-1} h\right)(s,\, t), \tag{1.2.14}$$

from which the invariance of $\mathfrak{N}$ under $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ follows. However, invariance under $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ will necessitate a condition, similar to the ones occurring in so-called "converse theorems". Before analyzing it, let us observe, starting from the identity $\left(r^{2i\pi\mathcal{E}} h\right)(x,\, \xi) = r\, h(rx,\, r\xi)$, that, for $r > 0$, one has

$$\left(\mathcal{F}_1^{-1}\left(r^{2i\pi\mathcal{E}} h\right)\right)\left(\frac{m}{t},\, nt\right) = \left(\mathcal{F}_1^{-1} h\right)\left(\frac{m}{rt},\, rnt\right); \tag{1.2.15}$$

after a change of variable $t \mapsto r^{-1}t$ in the integral (1.2.4) defining $\langle \mathfrak{N}\,,\, r^{2i\pi\mathcal{E}} h\rangle$, one obtains that this coincides with $r^{i\lambda}\,\langle \mathfrak{N}\,,\, h\rangle$: in other words, $\mathfrak{N}$ is homogeneous of degree $-1 - i\lambda$. Another obvious point is the identity $\mathfrak{N}(-x, \xi) = \chi(-1)\,\mathfrak{N}(x, \xi)$.

We now turn to the question of decomposing $\mathfrak{N}$ into bihomogeneous components. One may rewrite (1.2.4) as

$$\langle \mathfrak{N}\,,\, h\rangle = \frac{1}{4}\sum_{k \in \mathbb{Z}^\times} \phi(k) \int_{-\infty}^{\infty} |t|^{-1-i\lambda}\, \left(\mathcal{F}_1^{-1} h\right)\left(\frac{k}{t},\, t\right) dt, \tag{1.2.16}$$

setting

$$\phi(k) = \sum_{mn = k} \chi\left(\frac{m}{n}\right) |n|^{i\lambda}, \quad k \in \mathbb{Z}^\times \tag{1.2.17}$$

(note that $\phi(1) = 2$). Permuting m and n, one sees that ϕ is unchanged if χ is changed to χ_1, with $\chi_1(s) = \chi\left(s^{-1}\right) |s|^{i\lambda}$: this justifies a remark made above, since we could also split the set of primes into two subsets and perform the change $\chi \mapsto \chi_1$ only on rational numbers which are products of powers of primes of the first category, leaving the other factor unchanged.

The case when $\varepsilon = 0$ can be treated in a way quite similar to the one used toward the decomposition into bihomogeneous components of the distribution $\frac{1}{2}\,\mathfrak{E}_\nu^{\mathrm{main}}$, as done in the proof of Theorem 1.1.8: only, we must replace ν by $i\lambda$ and

$\sigma_\nu(k)$ by $\frac{1}{2}\,\phi(k)$. Then, $\sum_{n\geq 1}\sigma_\nu(n)\,n^{\frac{\rho-\nu-2}{2}}$, as computed in (1.1.50), must be replaced by

$$\frac{1}{2}\sum_{k\geq 1}\phi(k)\,k^{\frac{\rho-i\lambda-2}{2}}=\psi_1\left(\frac{2-\rho+i\lambda}{2}\right)\psi_2\left(\frac{2-\rho-i\lambda}{2}\right) \tag{1.2.18}$$

(a convergent series when $-\mathrm{Re}\ \rho$ is large because the character χ is tempered), and (1.1.51) becomes now

$$\begin{aligned}
&\mathfrak{N}(x,\xi)\\
&=\frac{1}{8i\pi}\int_{\mathrm{Re}\ \rho=c}\pi^{\frac{\rho-i\lambda-1}{2}}\,\frac{\Gamma(\frac{2-\rho+i\lambda}{4})}{\Gamma(\frac{\rho-i\lambda}{4})}\,\psi_1(\frac{2-\rho+i\lambda}{2})\,\psi_2(\frac{2-\rho-i\lambda}{2})\ \mathrm{hom}^{(0)}_{\rho,-i\lambda}(x,\xi)\,d\rho\\
&=\frac{1}{8i\pi}\int_{\mathrm{Re}\ \rho=c}B_0\left(\frac{\rho-i\lambda}{2}\right)L(\frac{2-\rho}{2},\mathfrak{N})\ \mathrm{hom}^{(0)}_{\rho,-i\lambda}(x,\xi)\,d\rho.
\end{aligned} \tag{1.2.19}$$

The change of contour from $\mathrm{Re}\ \rho=c$ (with $-c$ large) to $\mathrm{Re}\ \rho=1$ does not require, this time, computing any residue, since the function $L(\,\textbf{.}\,,\mathfrak{N})$ is entire, while the factor $B_0(\frac{\rho-i\lambda}{2})$ kills the poles of the distribution $\mathrm{hom}^{(0)}_{\rho,-i\lambda}$ at $\rho=i\lambda,i\lambda-4,\dots$: we shall examine later the change of contour to a line $\mathrm{Re}\ \rho=c>1$.

Some changes are necessary when $\varepsilon=1$, replacing the study of the distribution $\mathfrak{S}_n$ in (1.1.41) by that of the distribution such that

$$\begin{aligned}
\langle\mathfrak{S}_n^-\,,\,h\rangle&=\frac{1}{2}\int_{-\infty}^{\infty}|t|^{-i\lambda-1}\left[\left(\mathcal{F}_1^{-1}h\right)\left(\frac{n}{t},\,t\right)-\left(\mathcal{F}_1^{-1}h\right)\left(-\frac{n}{t},\,t\right)\right]dt\\
&=i\int_{-\infty}^{\infty}|\xi|^{-i\lambda-1}\,d\xi\int_{-\infty}^{\infty}h(x,\xi)\,\sin\,\frac{2\pi nx}{\xi}\,dx,
\end{aligned} \tag{1.2.20}$$

in other words

$$\mathfrak{S}_n^-(x,\xi)=i\,|\xi|^{-1-i\lambda}\sin\,\frac{2\pi nx}{\xi}\ : \tag{1.2.21}$$

the only difference with the preceding case is that one must apply (1.1.18) in place of (1.1.17). The decomposition (1.2.7) is proved, whether $\varepsilon=0$ or 1.

Since

$$\mathrm{hom}^{(\varepsilon)}_{\rho,\nu}(-\xi,x)=(-1)^\varepsilon\,\mathrm{hom}^{(\varepsilon)}_{2-\rho,\nu}(x,\xi), \tag{1.2.22}$$

that the functional equation (1.2.9) is equivalent to the identity $\mathfrak{N}(-\xi,x)=\mathfrak{N}(x,\xi)$ follows from the decomposition.

Whether $\varepsilon=0$ or 1, we verify now that the line $\mathrm{Re}\ \rho=1$ can be changed to $\mathrm{Re}\ \rho=c>1$. The poles of $B_\varepsilon(\frac{\rho-i\lambda}{2})$ to be taken care of are simple, at the points $2+2\varepsilon+i\lambda,6+2\varepsilon+i\lambda,\dots$: but let us recall [4, p. 107] that not only the function $L(s,\mathfrak{N})$ is entire, but so is the function $L^*(s,\mathfrak{N})$ in (1.2.10), one of the two extra factors defining it being $\Gamma(\frac{s+\varepsilon}{2}+\frac{i\lambda}{4})$. Dividing by this factor, evaluated at $s=\frac{2-\rho}{2}$, precisely kills the poles of $B_\varepsilon(\frac{\rho-i\lambda}{2})$ under consideration. On the other hand, the poles of $\mathrm{hom}^{(\varepsilon)}_{2-\rho,-i\lambda}$ to be taken care of are at $\rho = 2+2\varepsilon-i\lambda, 6+2\varepsilon-i\lambda, \dots$: they are killed with the help of the other factor $\left(\Gamma(\frac{s+\varepsilon}{2}-\frac{i\lambda}{4})\right)^{-1}$ present if using the entire function $L^*(s,\mathfrak{N})$ in place of $L(s,\mathfrak{N})$.

What remains to be done is to show that $\mathfrak{N}$ changes, for every integer $N \geq 1$, to a multiple, under the operator T_N^{dist}. For $N \geq 1$, one has (with $d>0$)

$$\left(\mathcal{F}_1^{-1}\left((x,\,\xi)\mapsto N^{-\frac{1}{2}}\,h\left(\frac{dx-b\xi}{\sqrt{N}},\,\frac{a\xi}{\sqrt{N}}\right)\right)\right)(s,\,t)$$
$$= d^{-1}\,\left(\mathcal{F}_1^{-1}h\right)\left(\frac{\sqrt{N}}{d}s,\,\frac{at}{\sqrt{N}}\right)\exp\left(2i\pi\frac{bst}{d}\right), \quad (1.2.23)$$

so that

$$\langle T_N^{\mathrm{dist}}\,\mathfrak{N},\,h\rangle = \frac{1}{4}\sum_{m,\,n\neq 0}\chi\left(\frac{m}{n}\right)$$
$$\sum_{\substack{ad=N,\,d>0\\ b\bmod d}} d^{-1}\int_{-\infty}^{\infty}|t|^{-1-i\lambda}\,e^{2i\pi\frac{bmn}{d}}\,\left(\mathcal{F}_1^{-1}h\right)\left(\frac{\sqrt{N}}{d}\frac{m}{t},\,\frac{ant}{\sqrt{N}}\right)dt. \quad (1.2.24)$$

After a change of variable $t\mapsto\frac{\sqrt{N}}{an}\,t$, one finds

$$\langle T_N^{\mathrm{dist}}\,\mathfrak{N},\,h\rangle = \frac{1}{4}\sum_{m,\,n\neq 0}\chi\left(\frac{m}{n}\right)\sum_{\substack{ad=N,\,d>0\\ b\bmod d}} d^{-1}\left|\frac{an}{\sqrt{N}}\right|^{i\lambda}$$
$$e^{2i\pi\frac{bmn}{d}}\int_{-\infty}^{\infty}|t|^{-1-i\lambda}\,\left(\mathcal{F}_1^{-1}h\right)\left(\frac{a}{d}\frac{mn}{t},\,t\right)dt. \quad (1.2.25)$$

It is sufficient to examine further the case when N coincides with a prime number p: then, one can have $(a=1,\,d=p,\,b\bmod p)$ or $(a=p,\,d=1,\,b=0)$. In the first case, one has $\sum_b e^{2i\pi\frac{bmn}{d}}=0$ if p does not divide mn, and the same sum is p if $p|\,mn$. In the second case, this sum reduces to 1. Hence, (1.2.25) simplifies as

$$\langle T_p^{\mathrm{dist}}\,\mathfrak{N},\,h\rangle = \frac{1}{4}\sum_{k\in\mathbb{Z}^\times} a_k\int_{-\infty}^{\infty}|t|^{-1-i\lambda}\,\left(\mathcal{F}_1^{-1}h\right)\left(\frac{k}{t},\,t\right)dt, \quad (1.2.26)$$

with

$$a_k = \sum_{mn=pk}\left|\frac{n}{\sqrt{p}}\right|^{i\lambda}\chi\left(\frac{m}{n}\right)+\sum_{pmn=k}|n\sqrt{p}|^{i\lambda}\chi\left(\frac{m}{n}\right)$$
$$= p^{-\frac{i\lambda}{2}}\,\phi(pk)+p^{\frac{i\lambda}{2}}\,\phi\left(\frac{k}{p}\right), \quad (1.2.27)$$

with the convention that $\phi(k) = 0$ unless $k \in \mathbb{Z}^\times$. Given any integer $k \neq 0$, one has

$$\{(m,\, n)\colon mn = pk\} = \{(pm_1,\, n)\colon m_1 n = k\} \cup \{(m,\, pn_1)\colon mn_1 = k\}, \quad (1.2.28)$$

not a disjoint union if $p|\, k$: the two sets intersect along the set $\{(pm_1,\, pn_1)\colon m_1 n_1 = \frac{k}{p}\}$, which leads to the equation

$$\phi(pk) = \left[\, \chi(p) + p^{i\lambda}\, \chi(p^{-1}) \,\right]\, \phi(k) - p^{i\lambda}\, \phi\left(\frac{k}{p}\right), \quad k \in \mathbb{Z}^\times. \quad (1.2.29)$$

It follows that

$$p^{-\frac{i\lambda}{2}}\, \phi(pk) + p^{\frac{i\lambda}{2}}\, \phi\left(\frac{k}{p}\right) = \left[p^{-\frac{i\lambda}{2}}\, \chi(p) + p^{\frac{i\lambda}{2}}\, \chi(p^{-1}) \right]\, \phi(k). \quad (1.2.30)$$

Coupling this equation with the pair of equations (1.2.26), (1.2.27), one obtains the equation

$$T_p^{\mathrm{dist}}\, \mathfrak{N} = \left[\, \chi(p)\, p^{-\frac{i\lambda}{2}} + \chi(p^{-1})\, p^{\frac{i\lambda}{2}} \,\right] \mathfrak{N}, \quad (1.2.31)$$

proving that $\mathfrak{N}$ is indeed a Hecke eigenform if it is automorphic (i.e., invariant under $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$). This completes the proof of Theorem 1.2.2. $\square$

Let us sum up the main results of the chapter just concluded as follows. Modular distributions, either of Eisenstein or of Hecke type, can be defined in the plane without appealing to a previously defined notion of non-holomorphic modular form: the relation between the two species of notions will be treated in the next chapter. Each modular distribution admits two types of expansions. First, the "Fourier series" expansion, to wit ((1.1.38) and (1.2.4))

$$\begin{aligned} \frac{1}{2}\, \mathfrak{E}_\nu(x,\xi) &= \frac{1}{2}\, \zeta(-\nu)\, |\xi|^{-\nu-1} \\ &\qquad + \frac{1}{2}\, \zeta(1-\nu)\, |x|^{-\nu} \delta(\xi) + \frac{1}{2} \sum_{k\neq 0} \sigma_\nu(|k|)\, |\xi|^{-1-\nu} \exp\left(2i\pi \frac{kx}{\xi}\right), \\ \mathfrak{N}(x,\xi) &= \frac{1}{4} \sum_{m,\, n\neq 0} |n|^{i\lambda} \chi\left(\frac{m}{n}\right)\, |\xi|^{-1-i\lambda} \exp\left(2i\pi \frac{mnx}{\xi}\right). \end{aligned} \quad (1.2.32)$$

On the other hand, according to Theorem 1.1.8 and Theorem 1.2.2 (or (1.2.12)), every modular distribution admits a continuous expansion (up to the addition of a few special terms in the case of Eisenstein distributions) into bihomogeneous functions, the coefficient of which is provided by the L-function relative to the given modular distribution.

Let us set $h_{\nu,q}(x,\xi) = |\xi|^{-1-\nu} e^{2i\pi \frac{qx}{\xi}}$, a symbol the analysis of which (with an emphasis on sharp products of such) will keep us busy throughout Chapter 4. The functions $h_{\nu,q}$ with $q \in \mathbb{Z}^\times$ are the basic terms of expansions (1.2.32) into Fourier series. They are linked to bihomogeneous functions by the pair of formulas

$$\mathrm{hom}_{\rho,\nu}^{(\varepsilon)} = B_\varepsilon\left(\frac{2-\rho-\nu}{2}\right)\int_{-\infty}^{\infty} |q|_\varepsilon^{\frac{-\rho-\nu}{2}} h_{\nu-1,q}dq, \quad \mathrm{Re}\ \nu > 0,\ 0 < \mathrm{Re}\ \frac{\rho+\nu}{2} < 1, \tag{1.2.33}$$

as it follows from (1.1.6), while, from Lemma 1.1.1, if $q \neq 0$, $a > 0$ and $x\xi \neq 0$,

$$h_{\nu,q}(x,\xi) = \frac{1}{4i\pi}\sum_{\varepsilon=0,1}(-1)^\varepsilon \int_{\mathrm{Re}\ \mu=a} B_\varepsilon(1-\mu)\,|q|_\varepsilon^{-\mu}\mathrm{hom}_{\nu-2\mu+2,-\nu}^{(\varepsilon)}(x,\xi)\,d\mu, \tag{1.2.34}$$

which can be rewritten in the following form, more immediately comparable to (1.1.46) and (1.2.7): if $q \neq 0$, $b < \mathrm{Re}\ \nu + 2$ and $x\xi \neq 0$,

$$h_{\nu,q}(x,\xi) = \frac{1}{8i\pi}\sum_{\varepsilon=0,1}(-1)^\varepsilon \int_{\mathrm{Re}\ \rho=b} B_\varepsilon(\frac{\rho-\nu}{2})\,|q|_\varepsilon^{\frac{\rho-\nu-2}{2}}\mathrm{hom}_{\rho,-\nu}^{(\varepsilon)}(x,\xi)\,d\rho. \tag{1.2.35}$$

Chapter 2

From the plane to the half-plane

This chapter provides a dictionary from automorphic distribution theory (in the plane) to automorphic function theory (in the hyperbolic half-plane). More precisely, one defines, with the help of the so-called dual Radon transformation, a linear operator $\Theta = (\Theta_0, \Theta_1)$ from automorphic distributions to pairs of automorphic functions: a two-component operator is needed because two distributions in the plane which are images of each other under the symplectic Fourier transformation have the same image under Θ_0. We show that the Θ_0-transforms of Eisenstein, or Hecke, distributions are Eisenstein, or Maass-Hecke modular forms, and that the notions of L-functions defined in the two environments are fully coherent.

We also transfer (non-automorphic) bihomogeneous functions, which leads to further decompositions of Eisenstein or Maass-Hecke modular forms. There, new functions $F_{\rho,\nu}^{(\varepsilon)}$ or, with a different normalization, $\Psi_{\rho,\nu}^{(\varepsilon)}$, show up: ν enters the (generalized) eigenvalue $\frac{1-\nu^2}{4}$ relative to the modular Laplacian Δ, while ρ is an eigenvalue associated to the operator

$$\mathrm{Eul}^{\Pi} = \frac{1}{i\pi}\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right). \tag{2.0.1}$$

The functions $F_{\rho,\nu}^{(\varepsilon)}$ are much more complicated than the distributions in the plane they originate from: the quite simple spectral-theoretic role of L-functions in automorphic distribution theory does not stay so simple in the automorphic function environment.

On the other hand, these functions led in [39, chap. 4] to the construction of a new class of automorphic functions in the hyperbolic half-plane with interesting singularities on the set of lines congruent to the line $(0, i\infty)$, a task briefly implemented in the last section of this chapter for the sake of completeness: only the case of functions invariant under the symmetry $z \mapsto -\bar{z}$ had been considered in the given reference. Disregarding completely Sections 2.2 and 2.3 would not harm understanding the rest of the book. But Section 2.1 and the notation, relating Hecke distributions and Hecke eigenforms, introduced there, is essential for the sequel.

2.1 Modular distributions and non-holomorphic modular forms

With the exception of Theorem 2.1.2 below (a converse of Theorem 1.2.2), all non classical facts in this section needed in this book have been detailed in [34] and, with a refreshed proof, in Sections 2.1 and 3.1 of [39]. Before dealing with the automorphic situation, we relate general analysis on the plane to that on the hyperbolic half-plane Π.

Recall that the standard Iwasawa decomposition NAK of $G = SL(2,\mathbb{R})$ involves the subgroup $K = SO(2)$ and the subgroups N and A consisting respectively of all matrices $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbb{R}$ and $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a > 0$; one considers also the group M consisting of the two matrices $\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The homogeneous space G/K can be identified with the half-plane Π with base-point i since K is the subgroup of G leaving this point fixed. The generic point (x,ξ) of $\mathbb{R}^2\backslash\{0\}$, regarded as the left column of the matrix $g = \left(\begin{smallmatrix} x & b \\ \xi & d \end{smallmatrix}\right)$, can be identified with the class gN: further dividing by M, we may regard G/MN as the quotient of the former space by the equivalence $(x,\xi) \sim (-x,-\xi)$, so that functions on G/MN become exactly even functions in $\mathbb{R}^2$.

The dual Radon transform V^* — a concept which can be defined and studied in considerable generality [11] — is the map from continuous even functions in $\mathbb{R}^2$ to functions on Π defined as

$$(V^* h)(g\,.\,i) = \int_K h((gk)\,.\,\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))\, dk \tag{2.1.1}$$

or, making the choice $g = \begin{pmatrix} y^{\frac12} & y^{-\frac12}x \\ 0 & y^{-\frac12} \end{pmatrix}$,

$$(V^* h)(x+iy) = \frac{1}{2\pi}\int_0^{2\pi} h\left(\pm\begin{pmatrix} y^{\frac12}\cos\frac{\theta}{2} - x\,y^{-\frac12}\sin\frac{\theta}{2} \\ -y^{-\frac12}\sin\frac{\theta}{2} \end{pmatrix}\right) d\theta. \tag{2.1.2}$$

In other words, $(V^*h)(i) = \langle d\sigma_i, h\rangle$ if $d\sigma_i$ is the rotation-invariant measure on the unit circle with total mass 1; more generally, $(V^*h)(z) = \langle d\sigma_z, h\rangle$ if $d\sigma_z$ is the measure supported in the ellipse $\{(x,\xi)\colon \frac{|x-z\xi|^2}{\text{Im } z} = 1\}$ (one irritant is that one cannot use simultaneously the coordinates x,ξ in $\mathbb{R}^2$ and $x+iy$ in Π), invariant under the group of linear transformations preserving this ellipse and with total mass 1.

The Radon transform V, which will be useful later as well, works in the other direction, from functions f on Π to even functions on $\mathbb{R}^2$: it is defined by the equation

$$(Vf)(g\,.\,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) = \int_N f((gn)\,.\,i)\,dn, \tag{2.1.3}$$

with $dn = db$ if $n = \left(\begin{smallmatrix}1 & b\\0 & 1\end{smallmatrix}\right)$. Recall that the hyperbolic distance d on Π is $SL(2,\mathbb{R})$-invariant and characterized as such by its special case $\cosh\, d(z,i) = \frac{1+|z|^2}{2\,\mathrm{Im}\; z}$. The integral is convergent, yielding a continuous function Vf if, say,

$$|f(z)| \leq C\,(\cosh\, d(i,z))^{-\frac{1}{2}-\varepsilon}$$

for some $\varepsilon > 0$: indeed, when $g = \left(\begin{smallmatrix}a & 0\\0 & a^{-1}\end{smallmatrix}\right)$, it is immediate that $2\,\cosh d(i,(gn)\,.\,i) = a^{-2} + a^2(1+b^2)$, a formula which remains true if g is replaced by kg with $k \in K$. Explicitly, completing if $x \neq 0$ the column $\left(\begin{smallmatrix}x\\\xi\end{smallmatrix}\right)$ into the matrix $\left(\begin{smallmatrix}x & 0\\\xi & x^{-1}\end{smallmatrix}\right)$, one has

$$(Vf)(\pm\left(\begin{smallmatrix}x\\\xi\end{smallmatrix}\right)) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\frac{x^2\,(i+b)}{x\xi\,(i+b)+1}\right) db\,, \quad x \neq 0. \tag{2.1.4}$$

One pair of transformations, working in the same direction as V^*, will be of considerable interest in this book. It is the pair (Θ_0, Θ_1) of maps from even functions, or even tempered distributions on $\mathbb{R}^2$, to functions on Π, defined by the equations

$$(\Theta_0\,\mathfrak{S})(z) = \left\langle \mathfrak{S}\,,\,(x,\xi) \mapsto 2\,\exp\left(-2\pi\,\frac{|x - z\,\xi|^2}{\mathrm{Im}\; z}\right)\right\rangle\,, \quad \Theta_1\,\mathfrak{S} = \Theta_0\,(2i\pi\mathcal{E}\mathfrak{S})\,. \tag{2.1.5}$$

This pair of operators has a useful interpretation in terms of pseudodifferential analysis and of the canonical set of coherent states of the metaplectic representation, and the same is true of its adjoint: we shall come back to it in the next chapter. What we need to know about is the (immediate) covariance of this pair of maps, to wit the pair of relations

$$\Theta_\kappa\,(\mathfrak{S}\,\circ\,g) = (\Theta_\kappa\mathfrak{S})\,\circ\,g, \quad g \in G, \quad \kappa = 0, 1, \tag{2.1.6}$$

in which $g = \left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right) \in G = SL(2,\mathbb{R})$ acts on Π by means of the equation $\left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)\,.\,z = \frac{az+b}{cz+d}$, on $\mathbb{R}^2$ by means of the equation $\left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)\,.\,\left(\begin{smallmatrix}x\\\xi\end{smallmatrix}\right) = \left(\begin{smallmatrix}ax+b\xi\\cx+d\xi\end{smallmatrix}\right)$. Also, recalling that we have already defined, in (1.1.19), the Euler operator $2i\pi\mathcal{E}$ on $\mathbb{R}^2$, we need the fundamental transfer property expressed by the equation

$$\Theta_\kappa\left(\pi^2\mathcal{E}^2\mathfrak{S}\right) = \left(\Delta - \frac{1}{4}\right)\Theta_\kappa\,\mathfrak{S}, \tag{2.1.7}$$

with $\Delta = (z-\bar{z})^2\,\frac{\partial^2}{\partial z\partial\bar{z}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ if $z = x+iy$. Setting $\rho = \frac{|x-z\,\xi|^2}{\mathrm{Im}\; z}$, this identity can be written as

$$-\left(\rho\,\frac{d}{d\rho} + \frac{1}{2}\right)^2 k(\rho) = \left(\Delta - \frac{1}{4}\right)k(\rho) \tag{2.1.8}$$

for every C^2 function k: the calculation of the right-hand side presents no difficulty since, taking advantage of the invariance of both operators involved under the appropriate actions of G, one may assume that $(x, \xi) = (1, 0)$, so that $\rho = (\mathrm{Im}\ z)^{-1}$.

The operator $\mathcal{E}$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$ (i.e., it admits a unique self-adjoint extension) if given the initial domain $C_0^\infty(\mathbb{R}^2 \backslash \{0\})$. This makes it possible to define, in the spectral-theoretic sense, functions of $\mathcal{E}$. The map Θ_0 connects to the dual Radon transformation by the equation

$$\Theta_0 = V^* \, (2\pi)^{\frac{1}{2} - i\pi\mathcal{E}} \, \Gamma\left(\frac{1}{2} + i\pi\mathcal{E}\right). \tag{2.1.9}$$

To prove this, one first decomposes the function $h \in \mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$ the two sides of (2.1.9) are to be tested on into homogeneous components $h_{i\lambda}$, as will be done in (3.2.1), after which, performing a change of variable in Euler's integral formula for the Gamma function (details are given in [39, p.52] if so desired), one obtains

$$(\Theta_0 \, h_{i\lambda})(z) = (2\pi)^{\frac{i\lambda - 3}{2}} \, \Gamma(\frac{1 - i\lambda}{2}) \int_{\mathbb{R}^2} h(x, \, \xi) \, \left(\frac{|x - z\xi|^2}{\mathrm{Im}\ z}\right)^{\frac{i\lambda - 1}{2}} dx \, d\xi. \tag{2.1.10}$$

On the other hand, the operator $(2\pi)^{\frac{1}{2} - i\pi\mathcal{E}} \, \Gamma\left(\frac{1}{2} + i\pi\mathcal{E}\right)$ acts on $h_{i\lambda}$ as multiplication by the scalar $(2\pi)^{\frac{1+i\lambda}{2}} \Gamma(\frac{1-i\lambda}{2})$. Temporarily denoting z as $z = x' + iy$, one writes, with the help of (3.2.1) and (2.1.2),

$$\left(V^* h_{i\lambda}\right)(z) = \frac{1}{4\pi^2} \int_0^{2\pi} d\theta \int_0^\infty t^{i\lambda} h \left(\begin{pmatrix} t\left(y^{\frac{1}{2}} \cos\frac{\theta}{2} - x' y^{-\frac{1}{2}} \sin\frac{\theta}{2}\right) \\ t\left(-y^{-\frac{1}{2}} \sin\frac{\theta}{2}\right) \end{pmatrix} \right) dt : \tag{2.1.11}$$

then, one performs the change of variables defined by the equations

$$x = t\left(y^{\frac{1}{2}} \cos\frac{\theta}{2} - x' y^{-\frac{1}{2}} \sin\frac{\theta}{2}\right), \quad \xi = t\left(-y^{-\frac{1}{2}} \sin\frac{\theta}{2}\right), \tag{2.1.12}$$

so that $t^2 = \frac{|x - z\xi|^2}{\mathrm{Im}\ z}$ and $dx \, d\xi = t \, dt \, d\theta$. The identity (2.1.9) follows.

Let $\mathcal{G}$ be the rescaled version of the symplectic Fourier transformation (1.1.29) defined on $L^1(\mathbb{R}^2)$ (next on $\mathcal{S}'(\mathbb{R}^2)$) as $\mathcal{G} = 2^{2i\pi\mathcal{E}} \mathcal{F}^{\mathrm{symp}} = 2^{i\pi\mathcal{E}} \mathcal{F}^{\mathrm{symp}} 2^{-i\pi\mathcal{E}}$, i.e.,

$$\left(\mathcal{G} h\right)(x, \xi) = 2 \int_{\mathbb{R}^2} h(y, \eta) \, e^{4i\pi(x\eta - y\xi)} \, dy \, d\eta. \tag{2.1.13}$$

One then has the identities

$$\Theta_0 \left(\mathcal{G} \, \mathfrak{S}\right) = \Theta_0 \, \mathfrak{S}, \quad \Theta_1 \left(\mathcal{G} \, \mathfrak{S}\right) = -\Theta_1 \, \mathfrak{S}. \tag{2.1.14}$$

If $\mathfrak{S} \in \mathcal{S}'_{\mathrm{even}}(\mathbb{R}^2)$, the image of $\mathfrak{S}$ under Θ_0 (resp. Θ_1) characterizes the part of $\mathfrak{S}$ invariant (resp. changing to its negative) under $\mathcal{G}$. The proof of this fact, fundamental for our purposes, is to be postponed to the end of Section 3.1, after we have related the transforms Θ_0 and Θ_1 to pseudodifferential analysis:

We can now consider the automorphic situation, recalling that a tempered distribution $\mathfrak{S}$ is automorphic if it is invariant under the action of the group $\Gamma = SL(2,\mathbb{Z})$ by linear changes of coordinates. Because of the covariance formula (2.1.6), its Θ-transform will consist of a pair of automorphic functions in Π. A modular distribution is an automorphic distribution homogeneous of some degree $-1-\nu$. As a consequence of (2.1.7), its Θ_0-transform is a (possibly generalized) eigenfunction of Δ for the eigenvalue $\frac{1-\nu^2}{4}$, in other words a non-holomorphic modular form; so is its Θ_1-transform, but no novel information is carried by it if ν (not only ν^2) is known. Note, in view of the first equation (2.1.14) and of the identity $\mathcal{G}(2i\pi\mathcal{E}) = (-2i\pi\mathcal{E})\mathcal{G}$, that two modular distributions, one the image of the other under $\mathcal{G}$, have the same Θ_0-transform, and that their degrees of homogeneity are then $-1-\nu$ and $-1+\nu$ for some ν.

We make all this explicit, for which we need to give a crash course on automorphic function theory in Π, limiting ourselves to what is absolutely needed in the sequel. Very nice presentations of this theory (accessible to non-experts, including the present author) are to be found in [4, 14, 16] and elsewhere. The first thing to recall is that it is useful to complete the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ (a convergent series if Re $s > 1$) as the function

$$\zeta^*(s) = \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s), \tag{2.1.15}$$

which extends as a meromorphic function in the entire plane, the only poles of which lie at 1 and 0 and are simple: moreover, it satisfies the fundamental functional equation $\zeta^*(s) = \zeta^*(1-s)$. A great bulk of non-holomorphic modular form theory is made up of the so-called Eisenstein series. If Re $\nu < -1$, the series

$$E_{\frac{1-\nu}{2}}(z) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)=1}} \left(\frac{|mz-n|^2}{\text{Im } z}\right)^{\frac{\nu-1}{2}} \tag{2.1.16}$$

(where (m,n) denotes the g.c.d. of the pair m, n) is convergent, and its sum is a non-holomorphic modular form for the eigenvalue $\frac{1-\nu^2}{4}$. It is periodic of period 1, and the function

$$E^*_{\frac{1-\nu}{2}}(z) = \zeta^*(1-\nu)\,E_{\frac{1-\nu}{2}}(z) \tag{2.1.17}$$

admits the Fourier series expansion (with respect to $x = \text{Re } z$)

$$\begin{aligned} &E^*_{\frac{1-\nu}{2}}(x+iy) \\ &= \zeta^*(1-\nu)\,y^{\frac{1-\nu}{2}} + \zeta^*(1+\nu)\,y^{\frac{1+\nu}{2}} + 2\,y^{\frac{1}{2}} \sum_{k\neq 0} |k|^{-\frac{\nu}{2}}\,\sigma_\nu(|k|)\,K_{\frac{\nu}{2}}(2\pi\,|k|\,y)\,e^{2i\pi kx}, \end{aligned} \tag{2.1.18}$$

with $\sigma_\nu(|k|) = \sum_{1\leq d|\,k} d^\nu$: this provides its analytic continuation as a function of ν.

An easy separation of variables relative to the operator Δ shows that every generalized eigenfunction of Δ for the eigenvalue $\frac{1-\nu^2}{4}$, periodic of period 1 with respect to $x = \mathrm{Re}\, z$, admits a Fourier expansion of the kind

$$f(x+iy) = a_+\, y^{\frac{1-\nu}{2}} + a_-\, y^{\frac{1+\nu}{2}} + y^{\frac{1}{2}} \sum_{k\neq 0} b_k\, e^{2i\pi kx}\, K_{\frac{\nu}{2}}(2\pi\,|k|\,y), \tag{2.1.19}$$

unless it is far from being bounded, as $y \to \infty$, by some power of $1+y$ (in which case one would have to substitute for $K_{\frac{\nu}{2}}$ another linear combination of the functions $I_{\pm\frac{\nu}{2}}$).

We now introduce the *standard* fundamental domain D of Γ, consisting of all points $z \in \Pi$ with $-\frac{1}{2} < \mathrm{Re}\, z < \frac{1}{2}$ and $|z| > 1$: it satisfies the property that no two distinct points of D are congruent under Γ (i.e., the images of each other under some transformation in Γ), while every point of Π is congruent to at least one point in the topological closure of D. Outside a set of measure zero, an automorphic function is then characterized by its restriction to D: using in Π the invariant measure $dm(x+iy) = y^{-2}dx\,dy$, one may then introduce the Hilbert space, denoted as $L^2(\Gamma\backslash\Pi)$, which is just $L^2(D, dm)$ in terms of these restrictions. Standard Hilbert space techniques then show that there exists an at most — an adverb which can be dispensed with thanks to Selberg's trace formula — countable set of linearly independent modular forms (the so-called Maass forms), which satisfy the property (not shared by Eisenstein series) that they are rapidly decreasing as $y = \mathrm{Im}\, z \to \infty$, in a way uniform with respect to $x = \mathrm{Re}\, z$: in other words, the first two coefficients a_+ and a_- are zero. As can be seen, ν must be pure imaginary: one usually sets $\nu = i\lambda$ with, say, $\lambda > 0$, or $\nu = i\lambda_r$ since the possible λ's make up a sequence going to ∞. It is not known whether (in the case of Γ, the only discrete group under consideration here) there may exist linearly independent Maass forms corresponding to the same eigenvalue $\frac{1+\lambda_r^2}{4}$.

A much clearer picture emerges after one has introduced the so-called Hecke operators T_N, $N \geq 1$, defined by the equation

$$(T_N f)(z) = N^{-\frac{1}{2}} \sum_{\substack{ad=N,\, d>0 \\ b \bmod d}} f\left(\frac{az+b}{d}\right): \tag{2.1.20}$$

they can be shown to commute pairwise, while commuting with Δ and with the parity operator T_{-1} defined by the equation $(T_{-1}f)(z) = f(-\bar{z})$. One has the fundamental formal relation between Dirichlet series

$$\sum_{N\geq 1} N^{-s} T_N = \prod_{p \text{ prime}} \left(1 - p^{-s}T_p + p^{-2s}\right)^{-1}, \tag{2.1.21}$$

a compact form of an infinite set of polynomial relations among the Hecke operators. Maass forms which are joint eigenfunctions of all Hecke operators (including T_{-1}) are called Hecke eigenforms. Just as Δ, the Hecke operators are self-adjoint in the space $L^2(\Gamma\backslash\Pi)$. Consider a true eigenvalue $\frac{1+\lambda_r^2}{4}$ of Δ, to wit one for which some Maass forms do exist. Then, standard Hilbert space methods (the theory of commuting families of compact self-adjoint operators) show that there exists a finite family $(\mathcal{M}_{r,\ell})_{1\leq\ell\leq\kappa}$, where κ is an r-dependent finite number, of Hecke eigenforms making up an orthonormal basis of the eigenspace of Δ corresponding to the given eigenvalue. Another normalization of Hecke eigenforms (to be referred to as Hecke's normalization) is very useful: it is the one for which one substitutes for $\mathcal{M}_{r,\ell}$ the proportional Hecke eigenform $\mathcal{N}_{r,\ell}$ such that the coefficient b_1 from its Fourier expansion (2.1.19) is 1: then, one has the collection of identities $T_N\mathcal{N}_{r,\ell} = b_N\,\mathcal{N}_{r,\ell}$. Again, a self-adjointness argument shows that all coefficients b_k of such a Hecke eigenform must be real numbers.

The spectral theorem relative to a certain natural self-adjoint realization of the operator Δ in $L^2(\Gamma\backslash\Pi)$, together with the collection of Hecke operators, makes it possible to show that every automorphic function $f\in L^2(\Gamma\backslash\Pi)$ admits a so-called Roelcke-Selberg expansion, to wit a decomposition of the kind

$$f(z) = \Phi^0 + \frac{1}{8\pi}\int_{-\infty}^{\infty}\Phi(\lambda)\,E_{\frac{1-i\lambda}{2}}(z)\,d\lambda + \sum_{r\geq 1}\sum_{\ell}\Phi^{r,\ell}\,\mathcal{M}_{r,\ell}(z). \tag{2.1.22}$$

In Chapter 5, we shall prove an analogous expansion for a large class of automorphic distributions. What is much harder to prove, at least in the case of automorphic functions invariant under the map $z\mapsto -\bar{z}$ (it requires using the so-called Selberg's trace formula), is that there does exist an infinite sequence $\left(\frac{1+\lambda_r^2}{4}\right)_{r\geq 1}$ of true eigenvalues of Δ. Note that Eisenstein series, as defined for general values of ν by their expansion (2.1.18), can never be *true* eigenfunctions of Δ, in that they never lie in $L^2(\Gamma\backslash\Delta)$ (as seen by an application of Hadamard's theorem that $\zeta(s)$ has no zero on the line $\mathrm{Re}\ s = 0$, while one trivially has $\zeta(s)\neq 0$ for $\mathrm{Re}\ s > 1$ in view of the Euler product expansion $(\zeta(s))^{-1} = \prod_{p\ \mathrm{prime}}(1-p^{-s})$).

The following proposition establishes the link between modular distribution theory and non-holomorphic modular form theory. Before stating it, we define the rescaled version of a tempered distribution $\mathfrak{S}$ as $\mathfrak{S}^{\mathrm{resc}} = 2^{-\frac{1}{2}+i\pi\mathcal{E}}\mathfrak{S}$: the operator $\mathcal{G}$ is the conjugate of the operator $\mathcal{F}^{\mathrm{symp}}$ under the rescaling operator. In particular, $\mathfrak{E}_\nu^{\mathrm{resc}} = 2^{\frac{-1-\nu}{2}}\mathfrak{E}_\nu$. The rescaling operator cannot be dispensed with when interested in the Weyl calculus, since the "most natural" one and two-dimensional Gaussian functions in this context are $x\mapsto 2^{\frac{1}{4}}e^{-\pi x^2}$ and $(x,\xi)\mapsto 2\,e^{-2\pi(x^2+\xi^2)}$: the second one (the symbol of the operator of orthogonal projection on the first one) is the rescaled version of $2\,e^{-\pi(x^2+\xi^2)}$. There are also reasons of elementary algebraic number theory leading to the same two types of normalization of Gaussian functions [20, p. 282].

Proposition 2.1.1. *For every $\nu \in \mathbb{C}$, $\nu \neq \pm 1$, one has $\Theta_0\left(\mathfrak{E}_\nu^{\mathrm{resc}}\right) = E^*_{\frac{1-\nu}{2}}$. Next, let $\mathcal{N}$ be a cusp-form with the Fourier expansion*

$$\mathcal{N}(x+iy) = y^{\frac{1}{2}} \sum_{k \neq 0} b_k \, K_{\frac{i\lambda}{2}}(2\pi\,|k|\,y)\, e^{2i\pi kx} : \tag{2.1.23}$$

this only defines the number λ^2 and, choosing $\lambda = \sqrt{\lambda^2}$, we define a pair $(\mathfrak{N}_\pm)$ of distributions in the plane by setting, for $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathfrak{N}_\pm\,,\,h\rangle = \frac{1}{2}\sum_{k\neq 0} |k|^{\frac{\pm i\lambda}{2}}\, b_k \int_{-\infty}^{\infty} |t|^{-1\mp i\lambda} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{k}{t},\,t\right) dt. \tag{2.1.24}$$

The distribution $\mathfrak{N}_\pm$ is a modular distribution, homogeneous of degree $-1 \mp i\lambda$. The two distributions are related by the identity $\mathcal{F}^{\mathrm{symp}}\,\mathfrak{N}_\pm = \mathfrak{N}_\mp$. The Θ-transform of the rescaled version $\mathfrak{N}_\pm^{\mathrm{resc}}$ is given by the equation

$$\left(\Theta_0\,\mathfrak{N}_\pm^{\mathrm{resc}}\right)(z) = \mathcal{N}(z). \tag{2.1.25}$$

Proof. In view of the (similar) Fourier expansions (1.2.32) and (1.2.4) of Eisenstein and Hecke distributions, the statement reduces to the results of computations involving the functions $h_{\nu,k}(x,\xi) = |\xi|^{-1-\nu} \exp\left(2i\pi \frac{k}{\xi}\right)$: these will be given (in a slightly more general version) in (4.1.2) and (4.6.3). $\square$

Remarks 2.1.1. (i) From (1.1.30), one has $\mathcal{G}\mathfrak{E}_\nu^{\mathrm{resc}} = \mathfrak{E}_{-\nu}^{\mathrm{resc}}$: the invariance of the function $E^*_{\frac{1-\nu}{2}}$ under the change of ν to $-\nu$, which follows then from (2.1.14), can also be seen from the Fourier expansion (2.1.18). In the same way, the modular distributions $\mathfrak{N}_\pm^{\mathrm{resc}}$ are $\mathcal{G}$-related and have the same Θ_0-transform.

(ii) While, as indicated above, we denote as $\left(\frac{1+\lambda_r^2}{4}\right)_{r\geq 1}$ the increasing sequence of true eigenvalues of the automorphic Laplacian and, for each r, we denote as $(\mathcal{N}_{r,\ell})_\ell$ the finite associated set (unique up to permutation) of Hecke eigenforms, normalized in Hecke's way, the following slight change is necessary when dealing with modular distributions: with the same convention about (r,ℓ), and given a Hecke eigenform $\mathcal{N} = \mathcal{N}_{r,\ell}$, we now denote as $\mathfrak{N}_{r,\ell}$ (resp. $\mathfrak{N}_{-r,\ell}$) the modular distribution (a Hecke distribution as will be seen presently) defined as $\mathfrak{N}_+$ (resp. $\mathfrak{N}_-$) by (2.1.24). In other words, a proper "total" set of Hecke distributions will then be the set $(\mathfrak{N}_{r,\ell})_{r,\ell}$ where, this time, the condition on r is $r \neq 0$. We shall always assume that, for $r \geq 1$, λ_r is the positive square root of λ_r^2, and it will be convenient to set $\lambda_{-r} = -\lambda_r$ so that, whether $r \geq 1$ or $r \leq -1$, one should always have

$$\langle \mathfrak{N}_{r,\ell}\,,\,h\rangle = \frac{1}{2}\sum_{k\neq 0} |k|^{\frac{i\lambda_r}{2}}\, b_k \int_{-\infty}^{\infty} |t|^{-1-i\lambda_r} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{k}{t},\,t\right) dt. \tag{2.1.26}$$

A clear understanding of the relation between the collection $(\mathcal{N}_{r,\ell})_{r\geq 1}$ of Hecke eigenforms and the collection $(\mathfrak{N}_{r,\ell})_{r\in\mathbb{Z}^\times}$ of Hecke distributions will be necessary in Chapter 5.

(iii) The coefficient b_1 in the series (2.1.23) for $\mathcal{N}$ is normalized (in Hecke's way, not in any Hilbert sense) to the value 1: making the same choice in the Eisenstein case leads (2.1.18) to considering $\frac{1}{2}E^*_{\frac{1-\nu}{2}}$ as the correctly normalized Eisenstein series. Finally, looking at the first sentence in Proposition 2.1.1, it is $\frac{1}{2}\mathfrak{E}_\nu$ that ought to be considered as normalized in Hecke's way.

We prove now a fact announced in Remark 1.2.1 (i) following Theorem 1.2.2, to some extent a converse of that theorem.

Theorem 2.1.2. *Every Hecke eigenform $\mathcal{N}$ with the Fourier expansion* (2.1.23)*, normalized so that the coefficient b_1 is* 1*, coincides, for some choice of χ, with the image under Θ_0 of the rescaled version of the Hecke distribution $\mathfrak{N} = \mathfrak{N}_{\chi,i\lambda}$ as defined in Theorem* 1.2.2*. Setting when* Re *s is large, as is usual, $L(s,\mathcal{N}) = \sum_{k\geq 1}\frac{b_k}{k^s}$, one has $L(s,\mathcal{N}) = L(s,\mathfrak{N})$. Recall that, with $\varepsilon = 0$ or* 1 *according to the parity of $\mathcal{N}$ under the map $z \mapsto -\bar{z}$, one sets*

$$L^*(s,\,\mathcal{N}) = \pi^{-s}\,\Gamma(\frac{s+\varepsilon}{2}+\frac{i\lambda}{4})\,\Gamma(\frac{s+\varepsilon}{2}-\frac{i\lambda}{4})\,L(s,\,\mathcal{N}), \tag{2.1.27}$$

obtaining as a result the identity $L^*(s,\,\mathcal{N}) = (-1)^\varepsilon L^*(1-s,\,\mathcal{N})$.

Proof. In this direction, we start from a Hecke eigenform $\mathcal{N}$, normalized in the way indicated and, defining $\mathfrak{N} = \mathfrak{N}_+$ according to Proposition 2.1.1, our problem is showing that $\mathfrak{N}$ coincides for some choice of $(\chi, i\lambda)$ with the Hecke distribution $\mathfrak{N}_{\chi,i\lambda}$ as defined by means of Theorem 1.2.2. First define $\varepsilon = 0$ or 1 according to the choice made in the statement of the theorem: then, $b_{-k} = (-1)^\varepsilon b_k$. On the other hand, for every prime p, let θ_p be any of the two roots of the equation

$$\theta_p^2 - b_p\,\theta_p + 1 = 0. \tag{2.1.28}$$

Denote as σ the collection of data

$$\sigma = \{\,\varepsilon\,,\, i\lambda\,,\,(\theta_p)_{p\,\text{prime}}\,\}, \tag{2.1.29}$$

where λ is any of the two square roots of λ^2. To each such set σ of spectral data, one associates in a one-to-one way the pair $(\chi, i\lambda)$, where the character χ on $\mathbb{Q}^\times$ is defined by the set of conditions

$$\chi(p) = p^{\frac{i\lambda}{2}}\,\theta_p, \quad \chi(\pm 1) = (-1)^\varepsilon\,: \tag{2.1.30}$$

it is quite well-known that χ, so defined, is a tempered character. The correspondence $\sigma \mapsto \mathfrak{N}_{\chi,i\lambda}$, or $(\chi, i\lambda) \mapsto \mathfrak{N}_{\chi,i\lambda}$, introduced in Theorem 1.2.2, is not one-to-one because, besides ε and λ, only the set of sums $\theta_p + \theta_p^{-1}$ is needed to define $\mathfrak{N}_{\chi,i\lambda}$, as it follows from (1.2.16), (1.2.17).

We now prove that, with $\mathfrak{N}$ as defined above, one has the identity (1.2.4), here recalled:

$$\langle \mathfrak{N},\, h\rangle = \frac{1}{4} \sum_{m,\,n\neq 0} \chi\left(\frac{m}{n}\right) \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left(\mathcal{F}_1^{-1} h\right) \left(\frac{m}{t},\, nt\right) dt, \quad h \in \mathcal{S}(\mathbb{R}^2). \tag{2.1.31}$$

To do so, let us rewrite the set of Fourier coefficients b_k in terms first of σ, next of the pair $(i\lambda,\, \chi)$, relying on Hecke's theory: the computation reduces to that of b_k for $k \geq 1$. One has $T_k\, \mathcal{N} = b_k\, \mathcal{N}$. On the other hand, one has the formal identity (2.1.21) between Dirichlet series: applying the operator there to $\mathcal{N}$ and using (2.1.28), one has

$$\begin{aligned}\sum_{k\geq 1} k^{-s}\, b_k &= \prod_p \left(1 - p^{-s}\, \theta_p\right)^{-1} \left(1 - p^{-s}\, \theta_p^{-1}\right)^{-1} \\ &= \prod_p \left(1 + p^{-s}\, \theta_p + p^{-2s}\, \theta_p^2 + \dots\right) \left(1 + p^{-s}\, \theta_p^{-1} + p^{-2s}\, \theta_p^{-2} + \dots\right).\end{aligned} \tag{2.1.32}$$

The number b_k, which is the coefficient of k^{-s} in the right-hand side, can thus be written, if $k = \prod_p p^{j_p}$, as

$$b_k = \prod_p \Bigg(\sum_{\substack{r_p,\, s_p \geq 0 \\ r_p + s_p = j_p}} \theta_p^{r_p - s_p} \Bigg) = \sum_{r+s=j} \prod_p \theta_p^{r_p - s_p}, \quad k \geq 1 \tag{2.1.33}$$

if, in the last expression, one sets $j = (j_2,\, j_3,\, j_5,\, \dots)$ and one considers similarly defined vectors r and s with non-negative coordinates, indexed by the set of primes. To each pair $(r,\, s)$, associate the pair $(m,\, n)$ of positive integers $m = \prod_p p^{r_p},\ n = \prod_p p^{s_p}$, so that $k = mn$, an arbitrary decomposition of $k \geq 1$ as a product of two integers ≥ 1. Defining the character θ on $\mathbb{Q}_+^\times$ by the equation $\theta(p) = \theta_p$ for all p, one can thus write (2.1.33) as

$$b_k = \sum_{\substack{m,\, n \geq 1 \\ mn = k}} \theta(m)\, \theta(n)^{-1} = \sum_{\substack{m,\, n \geq 1 \\ mn = k}} \theta\left(\frac{m}{n}\right), \quad k \geq 1. \tag{2.1.34}$$

Then, for any $k \in \mathbb{Z}^\times$,

$$\begin{aligned}|k|^{\frac{i\lambda}{2}}\, b_k = |k|_\varepsilon^{\frac{i\lambda}{2}}\, b_{|k|} &= \sum_{\substack{m,\, n \geq 1 \\ mn = |k|}} (\operatorname{sign} k)^\varepsilon\, |mn|^{\frac{i\lambda}{2}}\, \theta\left(\left|\frac{m}{n}\right|\right) \\ &= \frac{1}{4} \sum_{\substack{m,\, n \neq 0 \\ mn = k}} |m|_\varepsilon^{\frac{i\lambda}{2}}\, \theta(|m|) \times |n|^{i\lambda} \left[|n|_\varepsilon^{\frac{i\lambda}{2}}\, \theta(|n|) \right]^{-1}\end{aligned}$$

$$= \frac{1}{4} \sum_{\substack{m,\, n\neq 0 \\ mn=k}} |n|^{i\lambda}\, \chi\left(\frac{m}{n}\right). \tag{2.1.35}$$

Hence,

$$\langle \mathfrak{N},\, h\rangle = \frac{1}{4} \sum_{m,\, n\neq 0} \chi\left(\frac{m}{n}\right) |n|^{i\lambda} \int_{-\infty}^{\infty} |t|^{-1-i\lambda} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{mn}{t},\, t\right) dt : \tag{2.1.36}$$

performing the change of variable $t \mapsto nt$, we are done. $\square$

Remarks 2.1.2. (i) Recall that the Hecke eigenform $\mathcal{N}$ is said to satisfy the Ramanujan-Petersson conjecture if $|\theta_p| = 1$ for every prime p, in other words if χ is unitary.

(ii) That the Hecke operator T_N (acting on non-holomorphic modular forms), as defined in (2.1.20), is the transfer under Θ_0 of the operator T_N^{dist} (acting on automorphic distributions) in (1.2.1) is easy: nothing more than relating the two actions (by linear or fractional-linear transformations) of the group G is needed.

(iii) Eisenstein series are of course not cusp-forms: however, Eisenstein distributions can be recovered in the same way, with the exception of the first two terms of its decomposition (1.1.38). We start from the Fourier series expansion (2.1.18), both sides of which have been multiplied by $\frac{1}{2}$ for proper normalization, so that the coefficient b_1 taken from this expansion should be 1. Following the construction in the proof which precedes, one sees that $\varepsilon = 0$ and that, for $k \geq 1$, one has $b_k = k^{-\frac{\nu}{2}}\, \sigma_\nu(k)$. In particular, for any prime p, one has $b_p = p^{-\frac{\nu}{2}}\, (1 + p^\nu)$, so that $\theta_p = p^{-\frac{\nu}{2}}$ is one solution of equation (2.1.28). Any corresponding character χ is trivial on p. This leads to the main part of the expansion (1.1.38) of the Eisenstein distribution $\frac{1}{2}\, \mathfrak{E}_\nu$.

Even though it is natural to put less emphasis on Hilbert space methods in the automorphic distribution environment than in the automorphic function (in Π) environment, there is a perfectly natural Hilbert space $L^2(\Gamma\backslash\mathbb{R}^2)$, despite the fact that there is no fundamental domain for the action by linear changes of coordinates of Γ in $\mathbb{R}^2$ (most orbits are dense). This will be recalled at the end of Section 3.2. The decomposition of rather general automorphic distributions into their homogeneous components (the analogue of the Roelcke-Selberg theory) will be treated in Section 5.1.

2.2 Bihomogeneous functions and joint eigenfunctions of $(\Delta,\, \mathrm{Eul}^{\Pi})$

N.B. The present section and the one which follows are not required for further reading.

However, the question of how bihomogeneous functions in the plane transfer to the half-plane is a natural one. Theorems 2.2.4 and 2.2.5 below are the analogues, in the half-plane, of the decomposition formulas (1.1.46) and (1.2.7) of Eisenstein or Hecke distributions into bihomogeneous components. On the other hand, Theorem 2.2.3 can be stated as the fact that $\Psi_{\rho,\nu}^{(\varepsilon)}$ is *almost* a generalized eigenfunction of Δ, in that it satisfies the required differential equation outside a one-dimensional set.

We first compute the dual Radon transform of $\mathrm{hom}_{\rho,\nu}^{(\varepsilon)}$, a task already performed in [39, section 2.3] in the case when $\varepsilon = 0$. To do so, we need to introduce for $\nu \notin \mathbb{Z}$ and $\rho \pm \nu \notin 2\mathbb{Z}$ the function on $\mathbb{R}\backslash\{0\}$

$$\chi_{\rho,\,\nu}(t) = 2^{\nu-1}\,\pi^{-\frac{1}{2}}\,\frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})} \times \left(\frac{-1-i\,t}{2}\right)_+^{\frac{\rho+\nu-2}{2}}\;{}_2F_1\left(\frac{1-\nu}{2}\,,\,\frac{2-\rho-\nu}{2}\,;\,1-\nu\,;\,\frac{2}{1+it}\right), \tag{2.2.1}$$

where the power z_+^{α}, for $z \notin]-\infty,\,0]$, is defined as $e^{i\alpha\theta}$, θ being the argument of z lying in $]-\pi,\,\pi[$. It is undefined at 0 since, for a proper definition of the hypergeometric function, we must exclude values of the argument lying on the half-line $[1,\,\infty[$. Its main property is that the function

$$z \mapsto (\mathrm{Im}\ z)^{\frac{\rho-1}{2}}\,\chi_{\rho,\,\nu}\left(\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\right) \tag{2.2.2}$$

is in the complement, in the hyperbolic half-plane, of the line $\mathrm{Re}\ z = 0$, a generalized eigenvalue of Δ for the eigenvalue $\frac{1-\nu^2}{4}$: this will be detailed below in (2.2.30), (2.2.31). We also set

$$F_{\rho,\nu}^{(\varepsilon)}(z) = \begin{cases} (\mathrm{Im}\ z)^{\frac{\rho-1}{2}}\,\chi_{\rho,\nu}^{\mathrm{even}}\left(\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\right) & \text{if } \varepsilon = 0,\\ (\mathrm{Im}\ z)^{\frac{\rho-1}{2}}\,\chi_{\rho,\nu}^{\mathrm{odd}}\left(\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\right) & \text{if } \varepsilon = 1,\end{cases} \tag{2.2.3}$$

and, with another normalization,

$$\Psi_{\rho,\nu}^{(\varepsilon)}(z) = \pi^{-\frac{1}{2}}\,\Gamma\left(\frac{4-2\varepsilon-\rho-\nu}{4}\right)\Gamma\left(\frac{4-2\varepsilon-\rho+\nu}{4}\right)\,F_{\rho,\nu}^{(\varepsilon)}(z). \tag{2.2.4}$$

These functions are obviously generalized eigenfunctions of the operator Eul^{Π} introduced in (2.0.1).

Lemma 2.2.1. *One has*

$$\begin{aligned}(1+t^2)^{\frac{1-\rho}{2}}\,\chi_{\rho,\nu}^{\mathrm{even}}(t) &= \frac{\Gamma(\frac{2+\rho-\nu}{4})\Gamma(\frac{2+\rho+\nu}{4})}{\Gamma(\frac{4-\rho-\nu}{4})\Gamma(\frac{4-\rho+\nu}{4})}\,\chi_{2-\rho,\nu}^{\mathrm{even}}(t),\\ (1+t^2)^{\frac{1-\rho}{2}}\,\chi_{\rho,\nu}^{\mathrm{odd}}(t) &= \frac{\Gamma(\frac{\rho-\nu}{4})\Gamma(\frac{\rho+\nu}{4})}{\Gamma(\frac{2-\rho-\nu}{4})\Gamma(\frac{2-\rho+\nu}{4})}\,\chi_{2-\rho,\nu}^{\mathrm{odd}}(t).\end{aligned} \tag{2.2.5}$$

Proof. The even case is just equation (2.3.52) in [39]: we (wrongly) omitted the superscript "even" there: the proof of Lemma 2.3.4, from a certain point on, only considered the even part of $\chi_{\rho,\nu}$ (the odd part of this function played no part in that book). Equation (2.3.55) in the given reference can be written as

$$\chi_{2-\rho,\nu}(t) = e^{i\pi\frac{\rho-1}{2}\operatorname{sign} t}\,\frac{\Gamma(\frac{2-\rho+\nu}{2})}{\Gamma(\frac{\rho+\nu}{2})}\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}}\chi_{\rho,\nu}(t). \tag{2.2.6}$$

Then,

$$\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}}\chi^{\mathrm{odd}}_{\rho,\nu}(t) = \frac{1}{2}\,\frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})}\left[e^{i\pi\frac{1-\rho}{2}}\chi_{2-\rho,\nu}(t) - e^{i\pi\frac{\rho-1}{2}}\chi_{2-\rho,\nu}(-t)\right]. \tag{2.2.7}$$

Using twice the equation [39, (2.3.29)]

$$\chi_{\rho,\nu}(t) = e^{\frac{i\pi(2-\nu-\rho)}{2}}\,\chi_{\rho,\nu}(-t), \quad t>0 \tag{2.2.8}$$

and the equation

$$\frac{1}{2}\left(e^{i\pi\frac{1-\rho}{2}} - e^{i\pi\frac{\nu-1}{2}}\right) = \frac{i\pi\, e^{\frac{i\pi(\nu-\rho)}{4}}}{\Gamma(\frac{2+\rho+\nu}{4})\Gamma(\frac{2-\rho-\nu}{4})}, \tag{2.2.9}$$

one arrives at the second equation (2.2.5). □

It follows that

$$|z|^{1-\rho}\,F^{(\varepsilon)}_{\rho,\nu}(z) = \frac{\Gamma(\frac{2-2\varepsilon+\rho-\nu}{4})\Gamma(\frac{2-2\varepsilon+\rho+\nu}{4})}{\Gamma(\frac{4-2\varepsilon-\rho-\nu}{4})\Gamma(\frac{4-2\varepsilon-\rho+\nu}{4})}\,F^{(\varepsilon)}_{2-\rho,\nu}(z), \tag{2.2.10}$$

hence

$$F^{(\varepsilon)}_{\rho,\nu}(z) = (-1)^{\varepsilon}\,\frac{\Gamma(\frac{2-2\varepsilon+\rho-\nu}{4})\Gamma(\frac{2-2\varepsilon+\rho+\nu}{4})}{\Gamma(\frac{4-2\varepsilon-\rho-\nu}{4})\Gamma(\frac{4-2\varepsilon-\rho+\nu}{4})}\,F^{(\varepsilon)}_{2-\rho,\nu}(-z^{-1}); \tag{2.2.11}$$

in other words

$$\Psi^{(\varepsilon)}_{\rho,\nu}(z) = (-1)^{\varepsilon}\,\Psi^{(\varepsilon)}_{2-\rho,\nu}(-z^{-1}). \tag{2.2.12}$$

Another symmetry worth mentioning concerns the function $hom^{(\varepsilon)}_{\rho,\,\nu}$: one has

$$\mathcal{F}^{\mathrm{symp}}\,\mathrm{hom}^{(\varepsilon)}_{\rho,\nu} = \pi^{-\nu}\,\frac{\Gamma(\frac{2-\rho+\nu+2\varepsilon}{4})\Gamma(\frac{\rho+\nu+2\varepsilon}{4})}{\Gamma(\frac{\rho-\nu+2\varepsilon}{4})\Gamma(\frac{2-\rho-\nu+2\varepsilon}{4})}\,\mathrm{hom}^{(\varepsilon)}_{\rho,-\nu}, \tag{2.2.13}$$

or

$$\mathcal{F}^{\mathrm{symp}}\left(B_{\varepsilon}(\frac{\rho+\nu}{2})\,\mathrm{hom}^{(\varepsilon)}_{\rho,\nu}\right) = B_{\varepsilon}(\frac{\rho-\nu}{2})\,\mathrm{hom}^{(\varepsilon)}_{\rho,-\nu}. \tag{2.2.14}$$

Theorem 2.2.2. *Assume* Re $\nu > -1 + |\mathrm{Re}\ \rho - 1|$ *and* $\nu \notin \mathbb{Z},\ \rho \pm \nu \notin 2\mathbb{Z}$. *Then, one has, denoting also the even and odd parts of* $\chi_{\rho,\nu}$ *as* $\chi_{\rho,\nu}^{(\varepsilon)}$ *with* $\varepsilon = 0$ *or* 1,

$$\begin{aligned}\left(V^*\, hom^{(\varepsilon)}_{\rho,\,\nu}\right)(z) = i^{\varepsilon}\, (\mathrm{Im}\ z)^{\frac{\rho-1}{2}} & \\ \times\ 2^{\frac{\rho-\nu}{2}}\, \pi^{-1}\, & \frac{\Gamma(\frac{2-\rho+\nu}{2})\,\Gamma(\frac{2\varepsilon+\rho+\nu}{4})\,\Gamma(\frac{4-2\varepsilon-\rho-\nu}{4})}{\Gamma(\frac{\nu+1}{2})} \\ \times & \left[\chi^{(\varepsilon)}_{\rho,\,-\nu}\left(\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\right) + \chi^{(\varepsilon)}_{\rho,\,\nu}\left(\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\right)\right],\end{aligned} \tag{2.2.15}$$

or

$$V^*\, hom^{(\varepsilon)}_{\rho,\,\nu} = i^{\varepsilon}\, 2^{\frac{\rho-\nu}{2}}\, \pi^{-1}\, \frac{\Gamma(\frac{2-\rho+\nu}{2})\,\Gamma(\frac{2\varepsilon+\rho+\nu}{4})\,\Gamma(\frac{4-2\varepsilon-\rho-\nu}{4})}{\Gamma(\frac{\nu+1}{2})}\, \left[F^{(\varepsilon)}_{\rho,\,-\nu} + F^{(\varepsilon)}_{\rho,\,\nu}\right]. \tag{2.2.16}$$

Proof. In the case when $\varepsilon = 0$, this is [39, p. 70], and we consider now the case when $\varepsilon = 1$. One has, if Re $\nu > \max(\mathrm{Re}\ \rho - 2, -\mathrm{Re}\ \rho) = -1 + |\mathrm{Re}\ \rho - 1|$,

$$\begin{aligned}(V^*\, hom^{(1)}_{\rho,\,\nu})(x+iy) &= \frac{1}{2\pi}\int_0^{2\pi} \left|y^{-\frac12}\ \sin\frac{\theta}{2}\right|_1^{\frac{\nu-1}{2}}\, \left|y^{\frac12}\ \cos\frac{\theta}{2} - x\,y^{-\frac12}\ \sin\frac{\theta}{2}\right|_1^{\frac{\rho+\nu-2}{2}} d\theta \\ &= y^{\frac{\rho-1}{2}} \times \frac{1}{2\pi}\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right|_1^{\frac{\nu-1}{2}}\, \left|\cos\frac{\theta}{2} - \frac{x}{y}\ \sin\frac{\theta}{2}\right|_1^{\frac{\rho+\nu-2}{2}} d\theta.\end{aligned} \tag{2.2.17}$$

It is immediate (compare [39, p. 71]) that, as $\frac{x}{y} \to +\infty$, one has the equivalent

$$(V^*\, hom^{(1)}_{\rho,\,\nu})(x+iy) \sim -\pi^{-\frac12}\, \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}\, y^{\frac{\rho-1}{2}}\, \left|\frac{x}{y}\right|^{\frac{\rho+\nu-2}{2}}. \tag{2.2.18}$$

On the other hand, $y^{\frac{1-\rho}{2}}(V^*\, hom^{(1)}_{\rho,\,\nu})(x+iy)$ is an odd function of x, or of $t = \frac{x}{y}$. To see this, starting from the definition

$$(V^*h)(g.i) = \int_K h\left((gk).\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)\, dk, \tag{2.2.19}$$

only note that $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \begin{pmatrix} y^{\frac12} & xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix} \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) = \begin{pmatrix} y^{\frac12} & -xy^{-\frac12} \\ 0 & y^{-\frac12}\end{pmatrix}$ and $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) K \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) = K$.

To prove Theorem 2.2.2, one may assume, using analytic continuation, that Re $\nu > 0$, in which case an equivalent of $\chi^{\mathrm{odd}}_{\rho,\,\nu}(t) + \chi^{\mathrm{odd}}_{\rho,\,-\nu}(t)$ as $t \to \infty$ reduces to an equivalent of the first term. With $C = 2^{\nu-1}\, \pi^{-\frac12}\, \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})}$, one has

$$\begin{aligned}\chi^{\mathrm{odd}}_{\rho,\,\nu}(t) &\sim C\, 2^{\frac{-\rho-\nu}{2}}\, t^{\frac{\rho+\nu-2}{2}}\, [e^{-\frac{i\pi}{4}(\rho+\nu-2)} - e^{\frac{i\pi}{4}(\rho+\nu-2)}] \\ &= i\, 2^{\frac{\nu-\rho}{2}}\, \pi^{\frac12}\, \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})\Gamma(\frac{2+\rho+\nu}{4})\Gamma(\frac{2-\rho-\nu}{4})}\, t^{\frac{\rho+\nu-2}{2}}.\end{aligned} \tag{2.2.20}$$

http://www.springer.com/978-3-319-18656-6

Pseudodifferential Operators with Automorphic Symbols
Unterberger, A.
2015, X, 202 p., Softcover
ISBN: 978-3-319-18656-6
A product of Birkhäuser Basel