

# Preface

The present book, just as some of its forerunners, grew out of the author's desire to find a different approach to analytic number theory, certainly one of the more aesthetically satisfying branches of mathematics: this is especially true so far as modular form and  $L$ -function theory are concerned. That we relied on pseudodifferential analysis to try such an approach was initially due to the fact that we had practiced the latter since its beginnings, half a century ago. But we discovered, on the way, that it has deep connections with many areas: quantization theory and mathematical Physics, the theory of symmetric spaces, representation theory and, our main interest here, modular form and  $L$ -function theory. Our aim in this preface is to convey to interested readers, not necessarily familiar with pseudodifferential analysis, the realization that some of their interests are closer to it than what they probably believe.

Pseudodifferential operators were first devised as a help in partial differential equations: the analysis of such problems required a constant use of auxiliary (non differential) operators, and pseudodifferential operators, of a more and more general type, soon provided the quite adaptable box of tools desired. At the same time, the structure of pseudodifferential analysis became a subject of interest of its own. It starts with a specific way (the Weyl calculus) of representing linear operators on functions of  $n$  variables by functions of  $(2n)$  variables, called their symbols. Such a correspondence is a linear isomorphism, but it cannot be an algebra isomorphism if, on the symbol side, the (commutative) pointwise product is considered. This is why much emphasis has always been put on the "sharp composition formula" expressing the symbol  $h_1 \# h_2$  of the composition of two operators with given symbols  $h_1$  and  $h_2$ . Such a formula is in general based on that valid for differential operators, which led many people to believe that asymptotic, no to say formal, expansions are a fixture of pseudodifferential analysis.

This is not the case and, in situations of interest (staying within the Weyl calculus or not), what is required instead is to combine the sharp product with the decomposition of symbols into irreducible parts provided by some relevant representation-theoretic notions. These come into play through the two properties of covariance of the Weyl calculus, which express that the symbol of an operator transforms by a translation or by a symplectic linear change of coordinates if the

operator undergoes a conjugation under an element of the Heisenberg representation or of the metaplectic representation. It is only the latter representation that will concern us here: actually, we shall specialize in this book in the one-dimensional case, so that the group of interest is simply  $SL(2, \mathbb{R})$ . As another major simplification, the only arithmetic subgroup of  $SL(2, \mathbb{R})$  considered here will be  $SL(2, \mathbb{Z})$ .

By definition, automorphic symbols will be symbols invariant under linear changes of coordinates associated to matrices in  $SL(2, \mathbb{Z})$ : they must be distributions, since no non-constant continuous function can qualify. Thanks to the covariance property, automorphic symbols can be expected to make up an algebra under the sharp product: there are difficulties, linked to the fact that two operators with automorphic symbols cannot be composed in the quite usual sense, but these are fully solved. Automorphic distributions which are at the same time homogeneous of some degree are to be called modular distributions: they are introduced, together with their associated  $L$ -functions, in an independent way. They can then be shown to be a notion only slightly more precise than that of non-holomorphic modular form (in the hyperbolic half-plane), and a complete two-to-one dictionary is provided. The main question solved in the book can then be formulated as follows: given two modular distributions  $\mathfrak{N}_1$  and  $\mathfrak{N}_2$ , decompose their sharp product  $\mathfrak{N}_1 \# \mathfrak{N}_2$  as a linear superposition (both integrals and series are needed) of modular distributions (Eisenstein distributions and Hecke distributions); then, express the coefficients of the decomposition in terms of  $L$ -functions, Rankin-Selberg product  $L$ -functions and related notions.

Developing this program necessitated some machinery, in particular a reinterpretation in terms of pseudodifferential analysis of the Radon transform from the homogeneous space  $G/MN$  of  $G = SL(2, \mathbb{R})$  (functions on this space can be identified with even functions on  $\mathbb{R}^2 \setminus \{0\}$ ) to the homogeneous space  $G/K$  (the hyperbolic half-plane). Of course, under this transformation, the non-commutative sharp product does not transfer to the pointwise product but if the two entries, as well as the output, of the sharp operation are reduced to homogeneous components, it becomes almost true: analyzing the terms of the sharp composition reduces ultimately to the analysis, in the hyperbolic half-plane, of the pair made up of the pointwise product and Poisson bracket, after functions in the hyperbolic half-plane have been decomposed into generalized eigenfunctions of the hyperbolic Laplacian. This is proved first in a non-automorphic environment, next in the automorphic case.

We hope, and believe, that several categories of readers may find useful developments in this volume. New methods had to be developed, of course, in pseudodifferential analysis, while new perspectives in classical (i.e., non-adelic) non-holomorphic modular form theory might be gained from the automorphic distribution point of view. So-called invariant triple kernels have been considered lately by several authors, with or without number-theoretic applications in mind:

the end of Section 3.3 will explain to what extent some of this this appears as a byproduct of pseudodifferential analysis. Connections with quantization (not only geometric quantization) theory, a subject which has been this author's central interest for a few years, are hinted at in a short last chapter. In view of the current interest in Rankin-Cohen brackets, we have also included a short review showing that series of such constitute the right-hand side of the sharp product formula in a genuine symbolic calculus, a counterpart of the Weyl calculus in which non-holomorphic modular forms are traded for modular forms of the holomorphic type: much remains to be done in this direction. One of our other projects would be to understand more about some easier cases of Langland's  $L$ -function theory: to this end, some generalization of the present theory to the  $n$ -dimensional case might prove manageable and helpful.



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