Chapter 1
General View of Pole Solutions in Flame Front Propagation

Abstract We describe basic concepts and problems in flame front propagation; give a review of many relevant papers and books in this field.

Problems of interface growth have received much attention [1–11]. Such are, for example, the diffusion limited aggregation (DLA) [12], random sequential adsorption (RSA) [13], Laplacian growth [6, 14–23] or premixed flame front propagation [1, 3, 4, 7, 24–29]. We will mainly pay attention in this book to the numerical and analytical investigation of the last two problems.

We start to study the example of premixed flame propagation rather than Laplacian growth, simply because the former has an analytic description in terms of poles also in the experimentally relevant case of finite regularization term.

In addition to the fact that premixed flame front propagation is an interesting physical problem we feel that we can also explain experimental results on the basis of theoretical investigations.

The premixed flame—the self-sustaining wave of an exothermic chemical reaction—is one of the basic manifestations of gaseous combustion. It is well established, however, that the simplest imaginable flame configuration—unbounded planar flame freely propagating through initially motionless homogeneous combustible mixture—is intrinsically unstable and spontaneously assumes a characteristic two- or three-dimensional structure.

This book considers the very interesting problem of describing the nonlinear stage of development of hydrodynamic instability of the premixed flame. This problem can be considered for 1D (channel propagation), 2D (cylindrical propagation or rectangular channel cases), 3D (spherical case). The dimensionality of the problem is defined by the space dimensionality of the range of definition for the function, describing moving front. The direct numerical simulations can be made based on of the Navier-Stokes equations for expending premixed flames [30–33] and for channel geometry [34–36]. For 2D-cylindrical and 3D cases we can see experimentally observable effects [37–45] of the self-acceleration of the front of a divergent premixed flame, the formation of cellular structure, and other effects.

Much more simple equation for premixed flame can be obtained [29, 46]. It is Michelson-Sivashinsky approximation model for the channel case (and very similar

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Sivashinsky, Filyand and Frankel equation for cylindrical case). The Michelson-Sivashinsky model assumes very serious limitations, such as the smallness of the coefficient of gas expansion and, consequently, the potential flow in the combustion products and fresh mixture, weak nonlinearity, the assumptions of the stabilizing effect of the curvature of the premixed flame front and a linear dependence on the curvature of the normal speed, and others.

Interest of this model stems, firstly, from the fact that despite the serious limitations, this model can qualitatively describe the scenario of hydrodynamic instability and, in particular, the self-acceleration of the front of a divergent premixed flame, the formation of a cellular structure, and other effects.

Secondary, this nonlinear model has exact solutions which can be constructed on the basis of pole expansions. This pole expansion method got development in the following papers [47–50]. There was investigated relationships between pole solutions and partial decomposition in Fourier series [51], between the gas flow field and pole solutions [52]. The future development can be found in the following papers and the correspondent references inside of [24–27]. The Michelson-Sivashinsky model and pole solutions can be used for investigation of burning in Ia supernovae [53].

The aim of this book is to examine the role of random fluctuations on the dynamics of growing wrinkled interfaces which are governed by non-linear equations of motion. We are interested in those examples for which the growth of a flat or smooth interface are inherently unstable. A famous example of such growth phenomena is provided by Laplacian growth patterns [6, 8–10, 14, 15, 18–20]. The experimental realization of such patterns is seen for example in Hele-Shaw cells [6, 10, 16] in which air or another low viscosity fluid is displacing oil or some other high viscosity fluid. Under normal conditions the advancing fronts do not remain flat; in channel geometries they form in time a stable finger whose width is determined by delicate effects that arise from the existence of surface tension. In radial geometry, the growth the interface forms a contorted and ramified fractal shape. The same model of Laplacian growth can be used for description of Filtration Combustion, where no surface tension exists [14, 15, 18–20] (see also Appendix 2 in Chap. 5). A related phenomenon has been studied in a model equation for premixed flame propagation [29] which has the same linear stability properties as the Laplacian growth problem. The physical problem in this case is that of premixed flames which exist as self-sustaining fronts of exothermic chemical reactions in gaseous combustion. Experiments on Premixed flame propagation in radial (cylindrical) [37] and spherical [37–45] geometry show that the premixed flame front accelerates as time goes on, and roughens with characteristic exponents. Both observations must receive proper theoretical explanations. It is notable that the channel and radial growth are markedly different; the former leads to a single giant cusp in the moving front, whereas the latter exhibits infinitely many cusps that appear in a complex hierarchy as the premixed flame front develops ([28, 46] and Chap. 4).

Analytic techniques to study such processes are available [47]. In the context of premixed flame propagation [24–28, 48–50], and in Laplacian growth in the zero surface tension limit [14, 16, 22, 54, 55] one can examine solutions that are described in terms of poles in the complex plane. This description is very useful in providing a set of ordinary differential equations for the positions of the poles, from which one can deduce the geometry of the developing front in an extremely economical and efficient way. Unfortunately this description is not available in the case of Laplacian growth with surface tension, and this makes the premixed flame propagation problem very attractive. However, it suffers from one fundamental drawback. For the noiseless equation the pole-dynamics always conserves the number of poles that existed in the initial conditions. As a result there is a final degree of ramification that is afforded by every set of initial conditions even in the radial geometry, and it is not obvious how to describe the continuing self-similar growth that is seen in experimental conditions or numerical simulations. Furthermore, as mentioned before, at least in the case of premixed flame propagation one observes [37–45] an *acceleration* of the premixed flame front with time. Such a phenomenon is impossible when the number of poles is conserved. It is therefore tempting to conjecture that noise may have an important role in affecting the actual growth phenomena that are observed in such systems. In fact, the effect of noise on unstable front dynamics must be adequately described. From the point of view of analytic techniques noise can certainly generate new poles even if the initial conditions had a finite number of poles. The subject of pole dynamics with the existence of random noise, and the interaction between random fluctuations and deterministic front propagation are the main issues of this book.

The consideration of the noise term is critically important for these processes. Indeed, in papers [56–60], the authors try to explain self-acceleration without involvement of the external forcing. However, no self-acceleration exist for the finite number of poles. So we can explain the self-acceleration and the appearance of new poles or by the noise or by the “rain” of poles from the “cloud” in infinity. Unfortunately, such rain can not “achieve” the “ground” for accelerating premixed flame as we will see in Chap. 4.

The authors of [56–60] suppose that some analytical solution without noise corresponds to the unsteady solution with increasing number of cusps in radial case. They base this conclusion on the weak dependence of numerical simulations on the noise reduction ([56], Fig. 2). However, we will see in Chap. 2 that for the big interval of the noise values (larger than some small value and up to some big value) in the regime III (corresponding to the increasing number of poles in radial case), the dependence on the noise is very slow: $f^{0.02}$. This result explains the weak dependence of numerical simulations on the noise reduction.

The calculations with a noise term was performed in both one-dimensional and two-dimensional formulations of the problem in the papers [24–27] and by Karlin and Sivashinsky [61, 62] for 1D, 2D and 3D cases.

The future development of these results is gotten in papers [63–65]. In the paper [63] the noise term was considered in the poles-like form. The numerical results for 1D case are in a good consistence with the theoretical results of this book. The acceleration value of premixed flames fronts changes in the interval 1.25–1.5 according experimental [37–45] and theoretical [24, 26, 27, 30, 33, 46, 61–63, 80–84] results. The dependence of the acceleration value on the other parameters can explain this variation. The dependence was investigated numerically [63] and experimentally

[40]. In [65], hydrodynamic instability of inward-propagating premixed flames was investigated.

Joulin et al. [66–69] use a vary similar to [24–28] approach for the channel and radial premixed flame growth. Main attention in the channel case in Joulin's work was made to the investigation of mean-spacing between cusps (crests). For the radial case only the linear dependence of radius on time (no self-acceleration) is considered in Joulin's work. The main attention in our work is made to the velocity of premixed flame front (self-acceleration for the radial case) and the premixed flame front width.

In the next papers of Joulin et al. [50, 70, 71], the pole solutions for the Zhdanov-Trubnikov equation (extensions of Michelson-Sivashinsky equation that incorporate higher orders of the coefficient of gas expansion) are developed.

In the paper of Joulin et al. [72] the pole solutions for steady forced premixed flames are considered.

In the paper of Joulin et al. [73], the steady pole solutions is analyzed for Neumann boundary conditions ("no-flux" boundary conditions suggested in [74, 75], which are more physical than periodical boundary conditions [48]) in the Michelson-Sivashinsky equation. For these boundary conditions, we get two cusps on boundaries with unequal fluctuating sizes (instead of one moving giant cusp for periodical boundary conditions).

Similar "no-flux" boundary conditions (instead periodical ones [22, 23]) was suggested previously for Laplacian growth in [76] and developed in [14, 21]. However, [76] repeats the conclusion about one finger asymptotic, which are made previously in [16].

In the paper of Joulin et al. [49, 77] approximate the giant cusp steady solution of the Michelson-Sivashinsky equation by parabola (geometrical stretch) and investigate its nonlinear stability: perturbations of this approximated solution can be described by poles.

In the paper of Joulin et al. [78], the authors discovered that pole positions in the giant cusp (for the number of poles $N > 3$) can be approximated with high precision by roots of polynomials, ruled by a discretized Burgers equation.

In the paper of Gostintsev, Istratov and Shulenin [37] an interesting survey of experimental studies on outward propagating spherical and cylindrical premixed flames in the regime of well developed Darrieus-Landau hydrodynamic instability [79] is presented. The available data clearly indicate that freely expanding wrinkled premixed flames possess two intrinsic features:

1. Multi-quasi-cusps structure of the premixed flame front. (The premixed flame front consists of a large number of quasi-cusps, i.e., cusps with rounded tips.)
2. Noticeable acceleration of the premixed flame front: the propagation of wrinkled premixed flames really self accelerates and the acceleration follows a self-similar law. In other words, we fit the velocity data with a power-law formula, the exponent not only is greater than unity but it is also a constant, at least within a certain stage of the propagation.

According [37], the temporal dependence of the premixed flame radius is nearly identical for all premixtures discussed and correlates well with the simple relation:

$$R_0(t) = At^{3/2} + B \tag{1.1}$$

Here $R_0(t)$ is the effective (average) radius of the wrinkled premixed flame and A, B are empirical constants.

In follow-on publications (both this authors and the other authors [37–45]) the value of acceleration 1.5 was lowered to 1.25–1.5, with the revised assessment that the value of 1.5 is only attained in limiting cases.

In this book we study the spatial and temporal behavior of a nonlinear continuum model (i.e., a model which possesses an infinite number of degrees of freedom) which embodies all the characteristics deemed essential to premixed flame systems; namely, dispersiveness, nonlinearity and linear instability. Sivashinsky, Filyand and Frankel [46] obtained an equation, denoted by SFF in what follows, to describe how two-dimensional wrinkles of the cylindrical premixed flame grow as a consequence of the well-known Darrieus-Landau hydrodynamic instability [79]. The formal derivation of (1.2) is quite similar to that Michelson-Sivashinsky (MS) (1.5) presented by Sivashmsky [29], where the hydrodynamically unstable planar premixed flames were studied. The SFF and MS equations were obtained under assumption that the Attwood number $A = (E-1)/(E+1)$ (based on the fresh-to-burnt density ratio $E > 1$) is small.

The SFF equation reads as follows:

$$\frac{\partial R}{\partial t} = \frac{U_b}{2{R_0}^2(t)}\left(\frac{\partial R}{\partial \theta}\right)^2 + \frac{D_M}{{R_0}^2(t)}\frac{\partial^2 R}{\partial \theta^2} + \frac{\gamma U_b}{2R_0(t)} I\{R\} + U_b. \tag{1.2}$$

where $0 < \theta < 2\pi$ is an angle, $R(\theta, t)$ is the modulus of the radius-vector on the premixed flame interface (We can to increase the define area of the function $R(\theta, t)$ with periodical boundary condition to $-\infty < \theta < +\infty$ by the periodical continuation), U_b, D_M, γ are constants.

$$\begin{aligned} I(R) &= \frac{1}{\pi}\sum_{n=1}^{\infty} n \int_0^{2\pi} cos[n(\theta-\theta^*)]R(\theta^*, t)d\theta^* = \\ &= -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\frac{\partial R(\theta^*,t)}{\partial \theta^*}}{\theta^* - \theta} d\theta^* \end{aligned} \tag{1.3}$$

$$R_0(t) = \frac{1}{2\pi}\int_0^{2\pi} R(\theta, t)d\theta. \tag{1.4}$$

Sivashinsky, Filyand and Frankel [46] made a direct numerical simulation of this nonlinear evolution equation for the cylindrical premixed flame interface dynamics.

The result obtained shows that the two mentioned experimental effects take place. Moreover, the evaluated acceleration rate is not incompatible with the power law given by (1.1). For comparison, numerical simulations of freely expanding diffusively unstable premixed flames were presented as well. In this case no tendency towards acceleration has been observed.

The acceleration value of premixed flames fronts changes in the interval 1.25–1.5 according the other theoretical results [24, 26, 27, 30, 33, 61–63, 80–84].

To obtain results for radial premixed flame growth it is necessary [26, 30, 85] to investigate the channel case first. The channel version of equation for premixed flame front propagation is the so-called Michelson-Sivashinsky equation [29, 48, 86] (see also Appendix in Chap. 2) and looks like

$$\frac{\partial H}{\partial t} = \frac{1}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \nu\frac{\partial^2 H}{\partial x^2} + I\{H\}. \tag{1.5}$$

$$I(H) = -\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\frac{\partial H(x^*,t)}{\partial x^*}}{x^* - x}dx^*. \tag{1.6}$$

with periodic boundary condition on the interval x [0, L], where L is size of the system. ν is constant, $\nu > 0$. H is the height of the premixed flame front point, $P\int$ is the usual principal value integral.

There exists possibility to use methods found for the premixed flame front propagation, in different fields where similar problems appear such as the important model of Laplacian growth.

In the absence of surface tension, whose effect is to stabilize the short-wavelength perturbations of the interface, the problem of 2D Laplacian growth is described as follows

$$(\partial_x^2 + \partial_y^2)u = 0. \tag{1.7}$$

$$u\mid_{\Gamma(t)}= 0\ ,\ \partial_n u\mid_{\Sigma}= 1. \tag{1.8}$$

$$v_n = \partial_n u\mid_{\Gamma(t)}. \tag{1.9}$$

Here $u(x, y; t)$ is the scalar field mentioned, $\Gamma(t)$ is the moving interface, Σ is a fixed external boundary, ∂_n is a normal component of the gradient to the boundary (i.e. the normal derivative), and v_n is a normal component of the velocity of the front.

Equations for premixed flame front propagation and Laplacian growth with zero surface tension have remarkable property : these equations can be solved in terms of poles in the complex plane [22, 47–49, 69, 87]. So we obtain a set of ordinary differential equations for the coordinates of these poles. The number of the poles is constant value in the system, but to explain such effect as growth of the velocity premixed flame front we need to consider some noise that is a source of new poles. So we need to solve the problem of interaction of the random fluctuations and the pole motion.

The simplest case is the channel geometry. Main results for this case is existence of the giant cusp solution [48] (Fig. 1.1), which is represented in configuration space

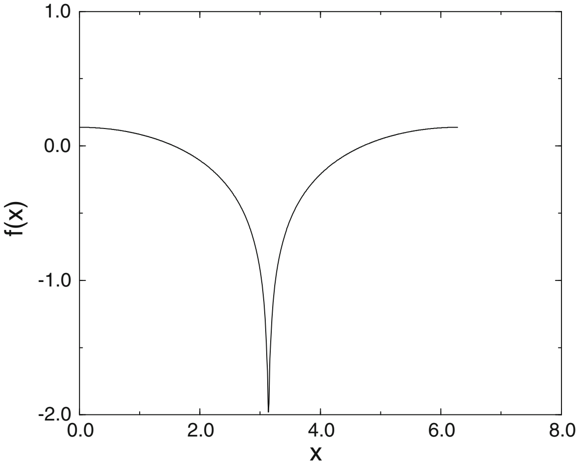

Fig. 1.1 Giant cusp solution

by poles which are organized on a line parallel to the imaginary axis. This pole solution is an attractor for pole dynamics.

A complete analysis of this steady-state solution was first presented in [48] and the main results are summarized as follows:

1. There is only one stable stationary solution which is geometrically represented by a giant cusp (or equivalently one finger) and analytically by $N(L)$ poles which are aligned on one line parallel to the imaginary axis. The existence of this solution is made clearer with the following remarks.
2. There exists an attraction between the poles along the real line. The resulting dynamics merges all the x positions of poles whose y-position remains finite.
3. The y positions are distinct, and the poles are aligned above each other in positions $y_{j-1} < y_j < y_{j+1}$ with the maximal being $y_{N(L)}$. This can be understood from equations for the poles motion in which the interaction is seen to be repulsive at short ranges, but changes sign at longer ranges.
4. If one adds an additional pole to such a solution, this pole (or another) will be pushed to infinity along the imaginary axis. If the system has less than $N(L)$ poles it is unstable to the addition of poles, and any noise will drive the system towards this unique state. The number $N(L)$ is

$$N(L) = \left[\frac{1}{2}\left(\frac{L}{\nu} + 1\right)\right], \tag{1.10}$$

where $\left[\ldots\right]$ is the integer part and $2\pi L$ is a system size. To see this consider a system with N poles and such that all the values of y_j satisfy the condition $0 < y_j < y_{max}$. Add now one additional pole whose coordinates are $z_a \equiv (x_a, y_a)$ with $y_a \gg y_{max}$. From the equation of motion for y_a, we see that the terms in the sum are all of the order of unity as is also the $\cot(y_a)$ term. Thus the equation of motion of y_a is approximately

$$\frac{dy_a}{dt} \approx \nu \frac{2N+1}{L^2} - \frac{1}{L}. \tag{1.11}$$

The fate of this pole depends on the number of other poles. If N is too large the pole will run to infinity, whereas if N is small the pole will be attracted towards the real axis. The condition for moving away to infinity is that $N > N(L)$ where $N(L)$ is given by (1.10). On the other hand the y coordinate of the poles cannot hit zero. Zero is a repulsive line, and poles are pushed away from zero with infinite velocity. To see this consider a pole whose y_j approaches zero. For any finite L the term $\coth(y_j)$ grows unboundedly whereas all the other terms in the equation for the poles motion remain bounded.

5. The height of the cusp is proportional to L. The distribution of positions of the poles along the line of constant x was worked out in [48].

We will refer to the solution with all these properties as the Thual-Frisch-Henon TFH-cusp solution.

The main results of our own work are as follow. Traditional linear analysis was made for this giant cusp solution. This analysis demonstrates the existence of negative eigenvalues that go to zero when the system size goes to infinity.

1. There exists an obvious Goldstone or translational mode with eigenvalue $\lambda_0 = 0$. This eigenmode stems from the Galilean invariance of the equation of motion.
2. The rescaled eigenvalues $(L^2\lambda_i)$ oscillate periodically between values that are L-independent in this presentation. In other words, up to the oscillatory behavior the eigenvalues depend on L like L^{-2}.
3. The eigenvalues λ_1 and λ_2 hit zero periodically. The functional dependence in this presentation appears almost piecewise linear.
4. The higher eigenvalues also exhibit similar qualitative behavior, but without reaching zero. We note that the solution becomes marginally stable for every value of L for which the eigenvalues λ_1 and λ_2 hit zero. The L^{-2} dependence of the spectrum indicates that the solution becomes more and more sensitive to noise as L increases.

It was proved that arbitrary initial conditions can be written in the term of poles in the complex plane. Inverse cascade process of giant cusp formation was investigated numerically and analytically. Dependencies of the premixed flame front width and mean velocity were found. The next step in investigation of the channel case was the influence of random noise on the pole dynamics. The main effect of the external noise is the appearance of new poles in the minima of the premixed flame front and the merging these poles with the giant cusp. The dependence of the mean premixed flame front velocity on the noise and the system size was found. The velocity is almost independent on the noise until the noise achieves some critical value. In the dependence of the velocity on the system size we see growth of the velocity with some exponent until the velocity achieves some saturation value.

Denoting v as the velocity of the premixed flame front and L the system size:

1. We can see two different regimes of behavior the average velocity v as a function of noise f for fixed system size L. For the noise f smaller then same fixed value f_{cr}
$$v \sim f^{\xi}. \tag{1.12}$$
For these values of f this dependence is very weak, and $\xi \approx 0.02$. For large values of f the dependence is much stronger
2. We can see growth of the average velocity v as a function of the system size L. After some values of L we can see saturation of the velocity. For regime $f < f_{cr}$ the growth of the velocity can be written as
$$v \sim L^{\mu}, \quad \mu \approx 0.35 \pm 0.03. \tag{1.13}$$
3. For $f > f_{cr}$ in Fig. 3.9 we will see qualitative change in the appearance of the premixed flame front: the noise introduces significant levels of small scales structure in addition to the cusps. The same observation was made in [88]

The dependence of the number of poles in the system and the number of the poles that appear in the system in unit time was investigated numerically as a function of the noise and the system parameters. The life time of a pole was found numerically. Theoretical discussion of the effect of noise on the pole dynamics and mean velocity was made [24, 27].

Pole dynamics can be used also to analyze small perturbation of the premixed flame front and make the full stability analysis of the giant cusp. Two kinds of modes were found. The first one is eigenoscillations of the poles in the giant cusp. The second one is modes connected to the appearance of the new poles in the system. The eigenvalues of these modes were found in [25, 89, 90]. The results of these papers are in good agreement with the traditional stability analysis ([89, 90] repeat many conclusions made previously in [25]).

The results found for the channel case can be used to analyze premixed flame front propagation in the radial case [26, 28]. Main feature of this case is a competition between attraction of the poles and expanding of the premixed flame front. So in this case we obtain not only one giant cusp but a set of cusps. New poles that appear in the system because of the noise form these cusps. On the basis of the equation of poles motion we can find connection between acceleration of the premixed flame front and the width of the interface. On the basis of the result for mean velocity in the channel case the acceleration of the premixed flame front can be found. So we obtain full picture of the premixed flame front propagation in the radial case.

The next step in the investigation of the problem is considering Laplacian growth with zero surface tension that also has pole solutions. In the case of Laplacian growth we obtain result that is analogous to the merging of the poles in the channel case of the premixed flame front propagation: all poles coalesce into one pole in the case of periodic boundary condition or two poles on the boundaries in the case of no-flux boundary conditions. This result can be proved theoretically [14, 16]. Moreover, it is possible to demonstrate that this solution is statistically stable: the width of the final finger will be oscillate near $1/2$ of the channel width in the presence of a noise [14].

The structure of this book is as follow.

In Chap. 2 we obtain main results for the channel case of the premixed flame front propagation. We give results about steady state solutions, present traditional linear analysis of the problem and investigate analytically and numerically the influence of noise on the mean velocity of the front and pole dynamics.

In Chap. 3 we obtain results of the linear stability analysis by the help of pole solutions.

In Chap. 4 we use the result obtained for the channel case for analysis of the premixed flame front propagation in the radial case

In Chap. 5 we investigate asymptotic behavior of the poles in the complex plane for the Laplacian growth with the zero surface tension in the case of periodic and no-flux boundary condition.

In Chap. 6 we give a short summary of the book ideas.

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