Chapter 6
Nonlinear Systems

Up to now, we have considered linear systems. If for such a linear system the existence of a solution can be shown for a certain finite time interval, then the solution exists for all times provided that the control keeps its regularity. For *nonlinear systems*, the situation is completely different. In a nonlinear hyperbolic system, the solution can loose a part of its regularity after a finite time. For example, classical solutions typically break down after finite time since there is a blow up in certain partial derivatives.

6.1 The Korteweg-de Vries Equation (KdV)

JOHN SCOTT RUSSELL (1808–1882), a Scottish engineer, has made the following observations about waves (*Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp. 311–390, Plates XLVII-LVII*):

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two

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miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

As a model for the movement of water the Korteweg-de Vries equation (KdV equation)

$$\partial_t y + y\,\partial_x y + \partial_{xxx} y = 0 \tag{6.1}$$

is considered. An overview about the research about the KdV equation is given in [48]. In *Interaction of Solitons in a Colisionless Plasma and the Recurrence of Initial States* (Phy. Rev. Let. 15, 240–243, 1965) N.J. Zabusky and M.D. Kruskal describe certain traveling waves that solve the KdV equation, so-called **solitons**.

In this chapter we consider the KdV equation on a finite space interval $[0, L]$ with boundary control action. We consider the partial differential equation

$$\partial_t y + \partial_x y + \partial_{xxx} y + y\,\partial_x y = 0 \tag{6.2}$$

with the extra term $\partial_x y$ (see [6]). With the extra term the waves move in the positive direction. Equation (6.2) can be considered as a perturbed transport equation

$$\partial_t y + \partial_x y = 0.$$

The boundary conditions are

$$y(t, 0) = u_1(t),\ y(t, L) = u_2(t),\ y_x(t, L) = u_3(t).$$

The initial condition has the form

$$y(0, x) = y_0(x),\ x \in (0, L)$$

with

$$y_0 \in L^2(0, L).$$

In the sequel we consider the boundary control with $u_1 = u_2 = 0$ and $u(t) = u_3(t) \in L^2(0, T)$ (see [12]). We consider the initial boundary value problem for small initial data, that is with an assumption of the form

$$\|y_0\|_{L^2(0,L)} \le \delta$$

with a number $\delta > 0$ that is chosen sufficiently small.

6.1.1 Well-posedness of the linearized system

We start with a result about the well-posedness of the initial value problem for the linearized system. For this purpose we consider the linearized partial differential equation

$$\partial_t y + \partial_x y + \partial_{xxx} y = \tilde{h} \tag{6.3}$$

where

$$\tilde{h} \in L^1((0,T),\, L^2(0,L)).$$

We consider the spatial differential operator

$$A: \; y \mapsto -y_x - y_{xxx} \tag{6.4}$$

with the domain

$$D(A) = \{y \in H^3(0,\, L) : y(0) = y(L) = y'(L) = 0\}.$$

For all $w \in D(A)$ we have

$$\begin{aligned}
\int_0^L w\,A\,w\,dx &= \int_0^L w(-w' - w''') \\
&= \int_0^L (-\frac{1}{2}\, w^2)' + \int_0^L w'w'' \\
&= 0 + \int_0^L (\frac{1}{2}\, (w')^2)' \\
&= 0 + 0 - \frac{1}{2}(w'(0))^2 \\
&\leq 0.
\end{aligned}$$

If such an inequality $\int_0^L w\,Aw\,dx \leq 0$ holds whenever $w \in D(A)$, the operator A is called *dissipative* (see [9, 58]). The adjoint operator A^* is $A^*: \; w \mapsto w' + w'''$ with the domain

$$D(A^*) = \{y \in H^3(0,\, L) : y(0) = y(L) = y'(0) = 0\}.$$

This is shown using integration by parts. For all $w \in D(A^*)$ we have

$$\begin{aligned}
\int_0^L wA^*w\,dx &= \int_0^L w(w' + w''') \\
&= \int_0^L (\frac{1}{2}\, w^2)' - \int_0^L w'w'' \\
&= 0 + 0 - \frac{1}{2}(w'(L))^2 \leq 0.
\end{aligned}$$

Hence A^* is also dissipative. This implies that A generates what is called a strongly continuous semigroup of contractions (see for example Chapter 3 in [54], [4], [8]). Here is the corresponding definition:

Definition 6.1. Let X be a Banach space and $L(X)$ denote the set of linear operators on X. A strongly continuous semigroup of contractions is a map

$$\mathbb{T} : [0, \infty) \to L(X),$$

that we denote as a family of linear operators $\mathbb{T}_t$ that satisfies the following conditions:

1. $\mathbb{T}_0 = I$, that is $\mathbb{T}_0$ is the identity.
2. For all $t_1, t_2 \in [0, \infty)$ we have $\mathbb{T}_{t_1+t_2} = \mathbb{T}_{t_1}\ \mathbb{T}_{t_2}$
3. For all $\hat{x} \in X$ we have

$$\lim_{t \to 0+} \|\mathbb{T}_t \hat{x} - \hat{x}\|_X = 0.$$

4. For all $t \in [0, \infty)$ and all $\hat{x} \in X$ we have $\|\mathbb{T}_t \hat{x}\|_X \leq \|\hat{x}\|_X$.

The fact that the closure of A generates a strongly continuous semigroup of contractions follows from the LUMER–PHILLIPS THEOREM, that is stated in the language of functional analysis, see [58].

Lumer–Phillips Theorem [58] *Let X be a Banach space and $(A, D(A))$ a densely defined operator. If A and A^* are dissipative, the closure of A generates a strongly continuous semigroup of contractions.*

An excellent exposition of the use of semigroups in control is given in [54]. Using the semigroup $\mathbb{T}_t$ we can write the solutions of the initial value problem

$$\begin{cases} y(0, x) = y_0(x),\ x \in (0, L) \\ \partial_t y +\ \partial_x y + \partial_{xxx} y = 0 \\ y(t,\ 0) = y(t,\ L) = \partial_x y(t,\ L) = 0 \end{cases} \tag{6.5}$$

in the form

$$\mathbb{T}_t y_0.$$

In fact, the following theorem holds.

Theorem 6.1 ([9] Prop. 11, Prop. 13, [12], Lemma A1). *For all $y_0 \in L^2(0, L)$ the initial value problem (6.5) has a unique solution*

$$y \in C([0,\ T]; D(A)) \cap C^1([0, T]; L^2(0, L)).$$

With $u \in L^2(0, T)$ and

$$\tilde{h} \in L^1((0, T),\ L^2(0, L))$$

the initial boundary value problem

$$\begin{cases} y(0, x) = y_0(x),\ x \in (0, L) \\ \partial_t y = A\, y + \tilde{h} \\ y(t,\ 0) = y(t,\ L) = 0,\ \partial_x y(t,\ L) = u(t) \end{cases} \tag{6.6}$$

where the spatial operator A is as in (6.4) has a solution $y \in C([0,\, T]; L^2(0, L)) \cap L^2((0, T); H^1(0, L))$. There exists a constant $C > 0$ such that for all y_0, $\tilde{h}$, u the following inequality holds:

$$\begin{aligned} &\|y\|_{C([0,\,T],\,L^2(0,L))} + \|y\|_{L^2((0,\,T),\,H^1(0,L))} \\ &\leq C\left(\|y_0\|^2_{L^2(0,L)} + \|\tilde{h}\|^2_{L^1((0,T),\,L^2(0,L))} + \|u\|^2_{L^2(0,T)}\right)^{1/2}. \end{aligned} \tag{6.7}$$

Proof of Theorem 6.1. For the homogeneous case $\tilde{h} = 0$, $u = 0$ we get the solution of (6.5) from Proposition 2.1.5 from [54]. In order to obtain the solution for the inhomogeneous case, we consider the following reduction to the homogeneous case. We define the auxiliary function

$$\varphi(x) = -\frac{1}{L}x(L - x).$$

Then we have $\varphi(0) = \varphi(L) = 0$ and $\varphi'(L) = 1$. For

$$\psi(t, x) = \varphi(x)\, u(t)$$

this implies $\psi(t, 0) = \psi(t, L) = 0$ and $\psi_x(t,\, L) = u(t)$. We define

$$f = (-\psi_t + A\psi + \tilde{h})$$

and the initial value problem with an inhomogeneous differential equation and homogeneous boundary conditions

$$\begin{cases} z(0, x) = y_0(x),\ x \in (0, L) \\ \partial_t z = A\, z + f \\ z(t,\, 0) = z(t,\, L) = \partial_x z(t,\, L) = 0. \end{cases} \tag{6.8}$$

For a regular control $u \in C^2(0, T)$ with $u(0) = 0$, Duhamel's formula yields the solution

$$z(t, \cdot) = \mathbb{T}_t y_0 + \int_0^t \mathbb{T}_{t-s} f(s)\, ds.$$

We define

$$y = z + \psi.$$

Then we get

$$\begin{aligned} y_t &= z_t + \psi_t \\ &= Az + f + \psi_t \\ &= Az + A\psi - \psi_t + \tilde{h} + \psi_t \\ &= A(z + \psi) + \tilde{h} \\ &= A\, y + \tilde{h}. \end{aligned}$$

Moreover y satisfies the inhomogeneous boundary conditions. Thus y solves the initial value problem for $u \in C^2(0,T)$. By a density argument this implies the desired result for all $u \in L^2(0,T)$. Inequality (6.7) follows by multiplication of the pde with special test function and integration by parts (see [9]). Thus we have proved Theorem 6.1. □

6.1.2 A traveling waves solution for the linearized system

In order to get a better understanding of the linearized KdV-equation, we consider a traveling waves solution. In this case, the solution is the sum of three traveling waves.

Let real numbers ω_1, ω_2, and ω_3 be given such that for all $i \in \{1,2,3\}$ we have

$$\omega_i^3 - \omega_i = \Lambda,$$

where $\Lambda = \omega_1\,\omega_2\,\omega_3$. We define the function

$$\begin{aligned} y(t,x) &= [\omega_3 - \omega_2]\,\cos(\omega_1 x + \Lambda\,t) \\ &\quad + [\omega_1 - \omega_3]\,\cos(\omega_2 x + \Lambda\,t) \\ &\quad + [\omega_2 - \omega_1]\,\cos(\omega_3 x + \Lambda\,t). \end{aligned}$$

Then we have the time-derivative

$$\begin{aligned} y_t(t,x) &= -\Lambda\,[\omega_3 - \omega_2]\,\sin(\omega_1 x + \Lambda\,t) \\ &\quad -\Lambda\,[\omega_1 - \omega_3]\,\sin(\omega_2 x + \Lambda\,t) \\ &\quad -\Lambda\,[\omega_2 - \omega_1]\,\sin(\omega_3 x + \Lambda\,t). \end{aligned}$$

Moreover, for the trigonometric function $\varphi_i(x) = \cos(\omega_i x + \Lambda\,t)$ we have the derivatives

$$\begin{aligned} \varphi_i'(x) &= -\omega_i \sin(\omega_i x + \Lambda\,t), \\ \varphi_i''(x) &= -\omega_i^2 \cos(\omega_i x + \Lambda\,t), \\ \varphi_i'''(x) &= \omega_i^3 \sin(\omega_i x + \Lambda\,t). \end{aligned}$$

Thus we get for all $i \in \{1,2,3\}$

$$\varphi_i'(x) + \varphi_i'''(x) = (-\omega_i + \omega_i^3)\sin(\omega_i x + \Lambda\,t) = \Lambda\sin(\Lambda\,t + \omega_i x).$$

This implies

$$\partial_t y(t,x) = -\partial_{xxx} y - \partial_x y,$$

hence y solves the linearized KdV equation. Moreover, for $x = 0$ we have

$$y(t,0) = [\omega_3 - \omega_2]\cos(\Lambda\, t) + [\omega_1 - \omega_3]\cos(\Lambda\, t) + [\omega_2 - \omega_1]\cos(\Lambda\, t) = 0.$$

Since for the partial derivative with respect to x we have

$$\begin{aligned} y_x(t,x) = &-\omega_1[\omega_3 - \omega_2]\sin(\omega_1 x + \Lambda\, t) \\ &-\omega_2[\omega_1 - \omega_3]\sin(\omega_2 x + \Lambda\, t) - \omega_3[\omega_2 - \omega_1]\sin(\omega_3 x + \Lambda\, t), \end{aligned}$$

we also have $y_x(t,0) = 0$.

6.1.3 Well-posedness for the nonlinear system

In this section we examine the well-posedness for the nonlinear system. In a first step we consider the nonlinear term $y\,\partial_x y$ and show that it can be considered as a source term in the linear equation.

Lemma 6.1 (See [47], Proposition 4.1). *For all $y \in L^2((0,T),H^1(0,L))$ we have $y\,y_x \in L^1((0,T),L^2(0,L))$ and the mapping $y \mapsto y\,y_x$ is continuous.*

Proof of Lemma 6.1. Let $y,\ z \in L^2((0,T),H^1(0,L))$ be given. Let K denote the norm of the embedding of $H^1(0,L)$ in $L^\infty(0,L)$. Then we have the inequality

$$\begin{aligned} \|y\,y_x - z\,z_x\|_{L^1((0,T),L^2(0,L))} &\le \int_0^T \|(y-z)y_x\|_{L^2(0,L)}\,dt \\ &+ \int_0^T \|z(y_x - z_x)\|_{L^2(0,L)}\,dt \\ &\le \int_0^T \|y-z\|_{L^\infty(0,L)}\,\|y_x\|_{L^2(0,L)}\,dt \\ &+ \int_0^T \|z\|_{L^\infty(0,L)}\,\|y_x - z_x\|_{L^2(0,L)}\,dt \\ &\le K\int_0^T \|y-z\|_{H^1(0,L)}\,\|y\|_{H^1(0,L)}\,dt \\ &+ K\int_0^T \|z\|_{H^1(0,L)}\,\|y_x - z_x\|_{L^2(0,L)}\,dt \\ &\le K\|y-z\|_{L^2((0,T),H^1(0,L))}\,\|y\|_{L^2((0,T),H^1(0,L))} \\ &+ K\|z\|_{L^2((0,T),H^1(0,L))}\,\|y-z\|_{L^2((0,T),H^1(0,L)}\,dt \\ &= K\left(\|y\|_{L^2((0,T),H^1(0,L))} + \|z\|_{L^2((0,T),H^1(0,L))}\right) \\ &\quad \|y-z\|_{L^2((0,T),H^1(0,L))}. \end{aligned}$$

If we insert $z = 0$, we see that we have $yy_x \in L^1((0,T), L^2(0,L))$. Moreover the inequality implies the continuity of the map $y \mapsto y y_x$. This finishes the proof of Lemma 6.1. □

Now we show the well-posedness of the initial boundary value problem

$$\begin{cases} y(0,x) = y_0(x),\ x \in (0,L) \\ \partial_t y = -\partial_x y - \partial_{xxx}\, y - y\,\partial_x y + \tilde{h} \\ y(t,\ 0) = y(t,\ L) = 0,\ \partial_x y(t,\ L) = u(t). \end{cases} \tag{6.9}$$

Lemma 6.2 ([11], Proposition 14). *Let $L > 0$ and $T > 0$ be given. Define the set*

$$\mathcal{B} = C([0,T], L^2(0,L)) \cap L^2((0,T), H^1(0,L)).$$

There exist numbers $\varepsilon > 0$ and $C > 0$ such that for all $\tilde{h} \in L^1((0,T), L^2(0,L))$, all $u \in L^2(0,T)$ and all $y_0 \in L^2(0,L)$ with

$$\|\tilde{h}\|_{L^1((0,T),L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} \le \varepsilon$$

the initial boundary value problem (6.9) has a solution y with

$$\|y\|_{\mathcal{B}} \le C\left(\|\tilde{h}\|_{L^1((0,T),L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}\right). \tag{6.10}$$

Proof of Lemma 6.2. For $z \in \mathcal{B}$ we define the map

$$M : \mathcal{B} \to \mathcal{B}$$

by $M(z) = \tilde{y}$ where $\tilde{y}$ denotes the solution of the initial value problem

$$\begin{cases} \tilde{y}(0,x) = y_0(x),\ x \in (0,L) \\ \partial_t \tilde{y} = -\partial_x \tilde{y} - \partial_{xxx}\, \tilde{y} - z\,\partial_x z + \tilde{h} \\ \tilde{y}(t,\ 0) = \tilde{y}(t,\ L) = 0,\ \partial_x \tilde{y}(t,\ L) = u(t). \end{cases} \tag{6.11}$$

Theorem 6.1 implies the existence of a solution $\tilde{y}$ that satisfies (6.7). We are looking for a fixed point y of M. For this purpose, we want to apply BANACH's fixed point theorem. Inequality (6.7) and Lemma 6.1 imply that there exists a number $D > 0$ such that

$$\|M(z)\|_{\mathcal{B}} \le D\left[\|\tilde{h}\|_{L^1((0,T),L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} + \|z\|_{\mathcal{B}}^2\right]. \tag{6.12}$$

Moreover we have

$$\|M(z_1) - M(z_2)\|_{\mathcal{B}} \le D\left(\|z_1\|_{\mathcal{B}} + \|z_2\|_{\mathcal{B}}\right)\ \|z_1 - z_2\|_{\mathcal{B}}.$$

Now we choose numbers $R > 0$ and $\varepsilon > 0$ sufficiently small such that

$$R < \frac{1}{2D} \text{ and } \varepsilon < \frac{R}{2D}.$$

We consider the set

$$Z = \{z \in \mathcal{B} : \ \|z\|_{\mathcal{B}} \le R\}.$$

Then $M(Z) \subset Z$ since for all $z \in Z$ we have the inequality

$$\|M(z)\|_{\mathcal{B}} \le D\,[\varepsilon + R^2] \le \frac{R}{2} + D\,R^2 = \frac{R}{2} + (D\,R)R$$
$$\le \frac{R}{2} + \frac{R}{2} = R.$$

The map M is a contraction in Z with the Lipschitz constant

$$L_M = 2\,D\,R \ < 1.$$

Now BANACH'S fixed point theorem implies the existence of a unique fixed point $z \in Z$ with $M(z) = z$. Since $D\,\|z\|_{\mathcal{B}} \le D\,R < 1/2$ inserting the fixed point in (6.12) yields (6.10) with $C = 2D$. Thus Lemma 6.2 is proved. □

6.1.4 A traveling wave solution for the nonlinear system

Also for the nonlinear system there is a traveling wave solution that satisfies the homogeneous KdV equation. We make the ansatz

$$y(t,\, x) = \frac{\alpha}{\cosh^2(a\,x + b\,t)}. \tag{6.13}$$

Then we get the partial derivatives

$$y_t(t,\, x) = -2\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} b, \tag{6.14}$$

$$y_x(t,\, x) = -2\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a, \tag{6.15}$$

$$y_{xx}(t,\, x) = 6\alpha \frac{\sinh^2(a\,x + b\,t)}{\cosh^4(a\,x + b\,t)} a^2 - 2\alpha \frac{1}{\cosh^2(a\,x + b\,t)} a^2, \tag{6.16}$$

$$y_{xxx}(t,\, x) = -24\alpha \frac{\sinh^3(a\,x + b\,t)}{\cosh^5(a\,x + b\,t)} a^3 \tag{6.17}$$

$$+\ 12\alpha \frac{\sinh(a\,x + b\,t)\ \cosh(a\,x + b\,t)}{\cosh^4(a\,x + b\,t)} a^3 \tag{6.18}$$

$$+\ 4\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a^3. \tag{6.19}$$

Since $\sinh^2(z) = \cosh^2(z) - 1$ this implies

$$y_{xxx}(t,\, x) = 24\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^5(a\,x + b\,t)} a^3 - 24\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a^3 \tag{6.20}$$

$$+\ 12\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a^3 + 4\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a^3 \tag{6.21}$$

$$= 24\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^5(a\,x + b\,t)} a^3 - 8\alpha \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} a^3. \tag{6.22}$$

This yields the product

$$y\,y_x = -2\alpha^2 \frac{\sinh(a\,x + b\,t)}{\cosh^5(a\,x + b\,t)} a.$$

Thus we get

$$y_x + y_{xxx} + y\,y_x - y_t \tag{6.23}$$

$$= \left[-2\alpha\, a - 8\alpha\, a^3 + 2\alpha\, b\right] \frac{\sinh(a\,x + b\,t)}{\cosh^3(a\,x + b\,t)} \tag{6.24}$$

$$+ \left[-2\alpha^2\, a + 24\alpha\, a^3\right] \frac{\sinh(a\,x + b\,t)}{\cosh^5(a\,x + b\,t)}. \tag{6.25}$$

With the choice

$$\alpha = 12\,a^2 \quad \text{und } b = a(1 + 4\,a^2)$$

we obtain the traveling waves solution

$$y(t,\, x) = \frac{12\,a^2}{\cosh^2(a\,(x + (1 + 4a^2)t))}$$

for (6.2). The speed $(1 + 4a^2)$ can become arbitrarily large and fixes the height of the wave.

Remark 6.1. For the viscous Burgers equation

$$y_t = y_x + y\,y_x + \nu\, y_{xx}$$

with viscosity $\nu > 0$ there also exists a traveling wave solution that has the form

$$y(t, x) = \alpha\, \tanh(a\,x + b\,t). \tag{6.26}$$

Exercise 6.1. Show that the ansatz (6.26) yields a traveling wave solution of the viscous Burgers equation with the choice $\alpha = 2\,a\,\nu$ und $b = a$. Thus for $\nu > 0$ and $a = 1/\nu$ we get the solution $y(t, x) = 2\tanh((x + t)/\nu)$. Note that the limit function for $\nu \to 0+$ is not continuous.

Remark 6.2. A detailed discussion of interacting solitary waves that solve the KdV equation can be found in [56].

6.1.5 The linearized system with critical length: An example for a system that is not exactly controllable

In this section we study the linearized system for a special choice of the length L and show that the system is **not** exactly controllable. This result is due to L. Rosier, [47]. We are looking for a length L, for which the overdetermined system

$$\begin{cases} \partial_t y = -\partial_x y - \partial_{xxx} y \\ y(t,\, 0) = y(t,\, L) = 0,\ \partial_x y(t,\, 0) = \partial_x y(t,\, L) = 0 \end{cases} \tag{6.27}$$

has a nontrivial solution. For this purpose we consider the traveling wave solution from Section 6.1.2. As an example, let

$$\omega_1 = -\frac{1}{\sqrt{7}},\ \omega_2 = -\frac{2}{\sqrt{7}},\ \omega_3 = \frac{3}{\sqrt{7}}.$$

Then we have $\Lambda = \omega_1\, \omega_2\, \omega_2 = \frac{6}{7\sqrt{7}}$ and

$$\begin{aligned}
&(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3) \\
&\quad = \omega^3 - [\omega_1 + \omega_2 + \omega_3]\, \omega^2 + [\omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_3]\, \omega - \Lambda \\
&\quad = \omega^3 - \omega - \frac{6}{7\sqrt{7}}.
\end{aligned}$$

We define

$$L = 2\pi\sqrt{7}.$$

Then we have

$$\begin{aligned}
y(t, L) &= [\omega_3 - \omega_2]\ \cos(\omega_1 L + \Lambda\, t) \\
&\quad + [\omega_1 - \omega_3]\ \cos(\omega_2 L + \Lambda\, t) \\
&\quad + [\omega_2 - \omega_1]\ \cos(\omega_3 L + \Lambda\, t) \\
&= [\omega_3 - \omega_2]\ \cos(-2\pi + \Lambda\, t) \\
&\quad + [\omega_1 - \omega_3]\ \cos(-4\pi + \Lambda\, t) \\
&\quad + [\omega_2 - \omega_1]\ \cos(6\pi + \Lambda\, t) \\
&= y(t, 0) \\
&= 0.
\end{aligned}$$

Moreover we have

$$
\begin{aligned}
y_x(t,L) &= -\omega_1[\omega_3-\omega_2]\ \sin(\omega_1 L + \Lambda\, t)\\
&- \omega_2[\omega_1-\omega_3]\sin(\omega_2 L + \Lambda\, t)\\
&- \omega_3[\omega_2-\omega_1]\sin(\omega_3 L + \Lambda\, t)\\
&= -\omega_1[\omega_3-\omega_2]\ \sin(-2\pi + \Lambda\, t)\\
&- \omega_2[\omega_1-\omega_3]\sin(-4\pi + \Lambda\, t)\\
&- \omega_3[\omega_2-\omega_1]\sin(-6\pi + \Lambda\, t)\\
&= y_x(t,0) = 0.
\end{aligned}
$$

Hence for $L = 2\pi\sqrt{7}$ the function y solves our system with the overdetermined boundary conditions. Figure 6.1 shows the solution on the time interval $[0, 30]$.

Remark 6.3. A key fact for the generalization of this example is the equation

$$L = 2\pi\sqrt{7} = 2\pi\sqrt{\frac{1^2+4^2+1\cdot 4}{3}}.$$

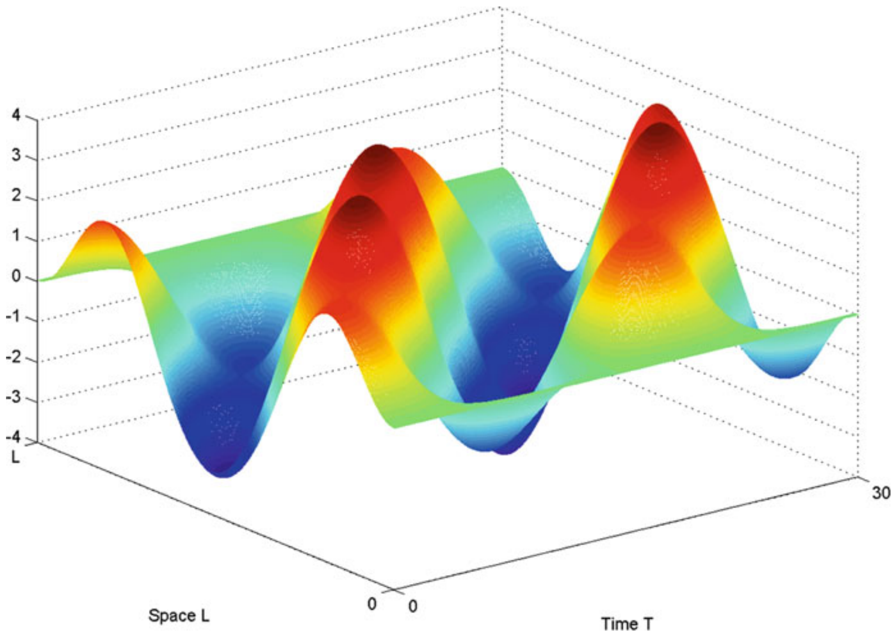

Fig. 6.1 A nontrivial traveling waves solution of the system (6.27) with overdetermined boundary conditions

In general nontrivial solutions of (6.27) exist for lengths

$$L \in \mathcal{N} = \left\{ 2\,\pi \sqrt{\frac{j^2 + l^2 + j\,l}{3}} \,:\, j, l \in \{1, 2, 3, \ldots\} \right\}$$

(see [47]). In particular, such a solution can only exist, if L is sufficiently large, that is if $L \geq 2\pi$!

The solutions of (6.27) are invisible for an observer in the following sense: If the value $y_x(t, 0)$ is observed, then the observation is always zero, no matter how long it is observed (with zero control).

Exercise 6.2. Consider the length $L = 2\sqrt{13}\,\pi$. Determine a nontrivial solution for the system (6.27) with overdetermined boundary conditions.

Solution of Exercise 6.2. *We put* $\omega_1 = -\frac{1}{\sqrt{13}}$, $\omega_2 = -\frac{3}{\sqrt{13}}$, $\omega_2 = \frac{4}{\sqrt{13}}$.

Exercise 6.3. Define the set of numbers

$$\mathcal{M} = \left\{ 2\,\pi \sqrt{\frac{k_2^2 - k_2\,k_3 + k_3^2}{3}},\ k_2, k_3 \in \mathbb{Z}, k_2 \neq k_3,\ k_2\,k_3 \neq 0 \right\}.$$

Show that for all $L \in \mathcal{M}$ there is a nontrivial solution for the homogeneous system (6.27) with overdetermined boundary conditions. Use solutions of the type defined in Section 6.1.2.

Solution of Exercise 6.3. *Let the numbers* $k_2, k_3 \in \mathbb{Z}, k_2 \neq k_3,\ k_2\,k_3 \neq 0$ *be given. We define*

$$\omega_1 = -\frac{1}{L}\frac{2\pi}{3}(k_2 + k_3),$$
$$\omega_2 = -\frac{1}{L}\frac{2\pi}{3}(-2k_2 + k_3),$$
$$\omega_3 = -\frac{1}{L}\frac{2\pi}{3}(k_2 - 2k_3).$$

Then we have $\omega_2 \neq \omega_3$, $\omega_1 \neq \omega_2$, *and* $\omega_1 \neq \omega_3$.

Hence the solution from Section 6.1.2 is nontrivial.

We have $\omega_1 + \omega_2 + \omega_3 = 0$. *Moreover*

$$\begin{aligned} \omega_1\,\omega_2 + \omega_1\,\omega_3 + \omega_2\,\omega_3 &= \frac{4\,\pi^2}{9\,L^2}\,\left[(k_2 + k_3)(-k_2 - k_3) + (-2k_2 + k_3)(k_2 - 2k_3)\right] \\ &= \frac{4\,\pi^2}{3\,L^2}\,\left[-k_2^2 + k_2\,k_3 - k_3^2\right]. \end{aligned}$$

Hence for all $L \in \mathcal{M}$ on account of

$$L^2 = 4\pi^2 \, \frac{k_2^2 - k_2\, k_3 + k_3^2}{3}.$$

we have the equation $\omega_1\, \omega_2 + \omega_1\, \omega_3 + \omega_2\, \omega_3 = -1$. *Thus we have*

$$(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3) = \omega^3 - \omega - \omega_1\, \omega_2\, \omega_3,$$

as required in Section 6.1.2. Thus we have $y(t,0) = 0 = y_x(t,0)$. *Moreover*

$$\begin{aligned} y(t,L) &= [\omega_3 - \omega_2]\cos(\omega_1 L + \Lambda\, t) \\ &+ [\omega_1 - \omega_3]\cos(\omega_2 L + \Lambda\, t) \\ &+ [\omega_2 - \omega_1]\cos(\omega_3 L + \Lambda\, t) \\ &= [\omega_3 - \omega_2]\cos(\omega_1 L + \Lambda\, t) \\ &+ [\omega_1 - \omega_3]\cos(\omega_1 L + \Lambda\, t + 2\,\pi\, k_2) \\ &+ [\omega_2 - \omega_1]\cos(\omega_1 L + \Lambda\, t + 2\,\pi\, k_3) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} y_x(t,L) &= -\omega_1[\omega_3 - \omega_2]\ \sin(\omega_1 L + \Lambda\, t) \\ &- \omega_2[\omega_1 - \omega_3]\sin(\omega_2 L + \Lambda\, t) \\ &- \omega_3[\omega_2 - \omega_1]\sin(\omega_3 L + \Lambda\, t) \\ &= -\omega_1[\omega_3 - \omega_2]\ \sin(\omega_1 L + \Lambda\, t) \\ &- \omega_2[\omega_1 - \omega_3]\sin(\omega_2 L + \Lambda\, t + 2\,\pi\, k_2) \\ &- \omega_3[\omega_2 - \omega_1]\sin(\omega_3 L + \Lambda\, t + 2\,\pi\, k_3) \\ &= 0. \end{aligned}$$

Therefore $y(t,x)$ *is a nontrivial solution for (6.27).*

Exercise 6.4. Show that for the set $\mathcal{M}$ that is defined in Exercise 6.3 we have

$$\mathcal{M} = \mathcal{N}.$$

Use the equation

$$k_2^2 - k_2\, k_3 + k_3^2 = (k_2 - k_3)^2 + (k_2 - k_3)k_3 + k_3^2.$$

In order to analyze the exact controllability, we consider the linear operator

$$\mathcal{F}_T : L^2(0,T) \to L^2(0,L),$$

that maps a control $u \in L^2(0,T)$ to the corresponding terminal state $y(T,\cdot) \in L^2(0,L)$ that is generated starting from the initial state $y_0 = 0$. The System (6.6) with $\tilde{h} = 0$ is exactly controllable if and only if $\mathcal{F}_T$ is surjective. To analyze the surjectivity of $\mathcal{F}_T$ we use a theorem from functional analysis (see [58]), where the adjoint map $\mathcal{F}^*$ plays a central role. Therefore, let us look at $\mathcal{F}_T{}^*$.

Lemma 6.3 (Lemma 2.28 in [10]). *Let $z^T \in D(A^*)$ be given. Then we have*

$$\mathcal{F}_T{}^*(z^T) = z_x(\cdot, L),$$

where $z \in C([0,T],\, H^3(0,L))$ solves the following problem:

$$\begin{cases} z(T,x) = z^T(x),\ x \in (0,L) \\ \partial_t z = -A^* z, \\ z(t,\cdot) \in D(A^*). \end{cases} \tag{6.28}$$

Proof of Lemma 6.3. Let $u \in C^2([0,T])$ with $u(0) = 0$ and the solution y of the initial boundary value problem

$$\begin{cases} y(0,x) = 0,\ x \in (0,L) \\ \partial_t y = A\,y \\ y(t,\,0) = y(t,\,L) = 0,\ \partial_x y(t,\,L) = u(t) \end{cases} \tag{6.29}$$

be given. Then by the definition of $\mathcal{F}_T$ we have

$$\mathcal{F}_T(u) = y(T,\cdot).$$

Integration by parts implies

$$\begin{aligned} &\int_0^L z^T(x)\, \mathcal{F}_T(u)(x)\, dx \\ &= = \int_0^L z^T(x)\, y(T,x)\, dx - \int_0^L z(0,\,x)\, y(0,x)\, dx \\ &= \int_0^T \int_0^L \partial_t(zy)\, dx\, dt \\ &= \int_0^T \int_0^L (-A^* z)y + z(Ay)\, dx\, dt \end{aligned}$$

$$= \int_0^T \int_0^L -\partial_{xxx} z\, y - z\, \partial_{xxx} y - \partial_x z\, y - z\, \partial_x y\, dx\, dt$$

$$= \int_0^T \int_0^L -\partial_{xxx} z\, y - z\, \partial_{xxx} y\ dx\, dt$$

$$= \int_0^T \int_0^L \partial_{xx} z\, \partial_x y + \partial_x z\, \partial_{xx} y\ dx\, dt$$

$$- \int_0^T \partial_{xx} z\, y|_{x=0}^{L}\, dt - \int_0^T z\, \partial_{xx} y|_{x=0}^{L}\, dt$$

$$= \int_0^T \int_0^L -\partial_x z\, \partial_{xx} y + \partial_x z\, \partial_{xx} y\ dx\, dt$$

$$- \int_0^T \partial_{xx} z\, y|_{x=0}^{L}\, dt - \int_0^T z\, \partial_{xx} y|_{x=0}^{L}\, dt + \int_0^T \partial_x z\, \partial_x y|_{x=0}^{L}\, dt$$

With the boundary conditions this yields

$$\int_0^L z^T(x)\, \mathcal{F}_T(u)(x)\, dx$$

$$= \int_0^T \partial_x z\, \partial_x y|_{x=0}^{L}\, dt - \int_0^T \partial_{xx} z\, y|_{x=0}^{L}\, dt - \int_0^T z\, \partial_{xx} y|_{x=0}^{L}\, dt$$

$$= \int_0^T \partial_x z(t, L)\, u(t)\, dt$$

and the assertion follows. Thus we have proved Lemma 6.3. □

Now we can apply the following general result about the surjectivity of $\mathcal{F}$.

Theorem 6.2 (Closed Range Theorem). *Let H_1 and H_2 be Hilbert spaces and $\mathcal{F}$ a continuous linear mapping from H_1 to H_2. Then $\mathcal{F}$ is surjective if and only if there is a constant $\kappa > 0$ such that for all $x_2 \in H_2$ we have the inequality*

$$\|\mathcal{F}^*(x_2)\|_{H_1} \geq \kappa \|x_2\|_{H_2}. \tag{6.30}$$

Inequality (6.30) is called *observability inequality*. According to Lemma 6.3 for our KdV system it has the form

$$\|\partial_x z(t, L)\|_{L^2(0,T)} \geq \kappa \|z^T\|_{L^2(0,L)}, \tag{6.31}$$

where z is the solution of (6.28).

Now we consider the traveling waves solution y from Section 6.1.2. We define $z^T(x) = y(T, x)$. For the singular lengths $L \in \mathcal{N}$ the function y solves the system (6.27) with overdetermined boundary conditions. Therefore for the singular lengths

$L \in \mathcal{N}$ the function $z(t,x) = y(t,x)$ solves (6.28). Thus we have $\partial_x z(t,L) = 0$. Hence inequality (6.31) cannot hold, since the left-hand side is zero, but $z^T(x) \neq 0$. By Theorem 6.2 this implies that for the singular lengths from the set $\mathcal{N}$ the linearized KdV system is not exactly controllable.

Remark 6.4. We have seen in Section 6.1.1 that for all $w \in D(A)$ we have

$$\int_0^L w A w \, dx = -\frac{1}{2}(w'(0))^2.$$

Let y denote a nontrivial solution of the overdetermined system (6.27). Then $y \in D(A^*) \cap D(A)$. Now we consider the evolution of the L^2-norm

$$E(t) = \frac{1}{2}\int_0^L (y(t,\, x))^2 \, dx.$$

For the time-derivative we get

$$E'(t) = \int_0^L y(t,\, x)\, \partial_t y(t,\, x)\, dx = \int_0^L y(t,\, x)\, Ay(t,\, x)\, dx = 0.$$

Hence the function E is constant, that is the L^2-norm is a conserved quantity.

Remark 6.5. For $L \notin \mathcal{N}$ the linearized KdV system is exactly controllable (see [47]).

Remark 6.6. For $L \notin \mathcal{N}$ also the nonlinear system is locally exactly controllable. Locally means that the L^2-norm of the initial and the terminal state has to be sufficiently small. In fact, the nonlinear KdV system is also exactly controllable for $L \in \mathcal{N}$, see [9].

6.2 The isothermal Euler equations

As an example for a quasilinear 2×2 system we consider the *isothermal Euler equations* that can be used as a model for the flow of gas through pipelines:

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + (\frac{q^2}{\rho} + a^2\rho)_x = -\frac{1}{2}\theta \frac{q\,|q|}{\rho}. \end{cases} \tag{6.32}$$

Here ρ is the density and q the flow rate of the gas. The first equation guarantees the conservation of mass. In the second equation, a is the sound speed and $\theta = \frac{f_g}{\delta}$ where δ is the diameter of the pipe and f_g is a friction parameter. To see the connection to the wave equation, let us look at the velocity

$$v = \frac{q}{\rho}.$$

For the applications the *subcritical flows* are interesting, where we have

$$|v| = \left|\frac{q}{\rho}\right| < a$$

that is the velocity is less than the sound speed. For sufficiently regular solutions v solves a *quasilinear* wave equation, namely

$$v_{tt} = (a^2 - v^2)v_{xx} - 2v\,v_{tx} - 2v_t v_x - 2v(v_x)^2 - \theta|v|\,v_t - \frac{3}{2}\theta v\,|v|\,v_x. \tag{6.33}$$

Exercise 6.5. Show that the velocity v satisfies the quasilinear wave equation (6.33)!

Solution of Exercise 6.5. *Since $q = \rho\, v$, the equation $\rho_t + q_x = 0$ implies $\rho_t + \rho_x v + \rho\, v_x = 0$, hence we have*

$$\rho_t + v\rho_x = -\rho\, v_x. \tag{6.34}$$

Since we do not consider the vacuum case, for the density we have $\rho > 0$, therefore we can write the above equation in the form

$$\frac{\rho_t}{\rho} + v\frac{\rho_x}{\rho} = -v_x. \tag{6.35}$$

With the variable $\ln(\rho)$ this yields

$$\partial_t(\ln(\rho)) + v\,\partial_x(\ln(\rho)) = -v_x. \tag{6.36}$$

Hence we have

$$\partial_x(\ln(\rho)) = -\frac{v_x}{v} - \frac{1}{v}\,\partial_t(\ln(\rho)). \tag{6.37}$$

The second equation in (6.32) yields

$$v\,\rho_t + \rho v_t = -(\rho v^2 + a^2\,\rho)_x - \frac{1}{2}\theta\, v\,|v|\,\rho.$$

With (6.34) this implies

$$v(-v\rho_x - \rho\, v_x) + \rho v_t = -2\rho v\, v_x - \rho_x v^2 - a^2\rho_x - \frac{1}{2}\,\theta v\,|v|\,\rho.$$

Division by ρ yields

$$v_t + v\, v_x = -a^2\partial_x\ln(\rho) - \frac{1}{2}\,\theta\, v\,|v|. \tag{6.38}$$

Replacing the term $\partial_x \ln(\rho)$ *in (6.38) by the right-hand side of (6.37) yields*

$$v_t + v\, v_x = \frac{a^2}{v}\, \partial_t(\ln(\rho)) + a^2\, \frac{v_x}{v} - \frac{1}{2}\, \theta\, v\, |v|. \tag{6.39}$$

By multiplication with v *this implies*

$$v\, v_t + v^2\, v_x = a^2\, \partial_t(\ln(\rho)) + a^2\, v_x - \frac{1}{2}\, \theta\, v^2\, |v|. \tag{6.40}$$

By partial differentiation of (6.40) with respect to x *we get*

$$v_x\, v_t + v\, v_{tx} + v^2\, v_{xx} + 2v(v_x)^2 = a^2\, \partial_x\, \partial_t(\ln(\rho)) + a^2\, v_{xx} - \frac{3}{2}\, \theta\, v\, |v|\, v_x. \tag{6.41}$$

By partial differentiation of (6.38) with respect to t *we obtain*

$$v_{tt} + v_t\, v_x + v\, v_{tx} = -a^2 \partial_t\, \partial_x \ln(\rho) - \theta\, |v|\, v_t. \tag{6.42}$$

Adding (6.41) and (6.42) yields

$$v_{tt} + 2v_x\, v_t + 2v\, v_{tx} + v^2\, v_{xx} + 2v(v_x)^2 = a^2\, v_{xx} - \theta\, |v| \left(\frac{3}{2}\, v\, v_x + v_t\right). \tag{6.43}$$

Thus we get (6.33).

Let $\mathcal{I}$ denote the identity operator. We have the equation

$$\begin{aligned} & v_{tt} - (a^2 - v^2)v_{xx} + 2v v_{tx} + 2v_t v_x + 2v(v_x)^2 \\ = \quad & [\partial_t + (a+v)\partial_x + v_x \mathcal{I}]\, [\partial_t - (a-v)\partial_x]\, v \\ = \quad & [\partial_t - (a-v)\partial_x + v_x \mathcal{I}]\, [\partial_t + (a+v)\partial_x]\, v. \end{aligned} \tag{6.44}$$

Exercise 6.6. Show equation (6.44)!

Thus the solutions of the equations

$$v_t - (a - v)v_x = 0, \tag{6.45}$$

$$v_t + (a + v)v_x = 0 \tag{6.46}$$

also solve the quasilinear wave equation (6.33) with $\theta = 0$, that is

$$v_{tt} - (a^2 - v^2)v_{xx} + 2\left[v v_{tx} + v_t v_x + v(v_x)^2\right] = 0. \tag{6.47}$$

Equations (6.45), (6.46) are called (nonviscous) BURGERS equations. Often these equations appear with $a = 0$.

The solutions of these equations can be determined with the method of *characteristics*. The characteristic curves $\xi^v(s, x, t)$ are defined as the solutions of the initial value problems

$$\xi^v(t, x, t) = x, \ \partial_s \xi^v(s, x, t) = \pm a + v(s, \xi^v(s, x, t)). \tag{6.48}$$

Hence the $\xi^v(s, x, t)$ solve the integral equations

$$\xi^v(s, x, t) = x \pm a(s - t) + \int_t^s v(\tau, \xi^v(\tau, x, t))\, d\tau.$$

Now we consider the values of v along the characteristic curves. For the corresponding auxiliary function

$$h(s) = v(s, \xi^v(s, x, t))$$

we have

$$h'(s) = v_t + v_x \partial_s \xi^v = v_t + (v \pm a)\, v_x.$$

For a solution v of (6.45) ((6.46) respectively) this implies $h'(s) = 0$, hence v is constant along the characteristic curves. This implies that the characteristic curves have constant slopes, hence they are straight lines. Therefore in general different characteristic curves will intersect after finite time. In this case the solution in the sense of characteristics breaks down and a *shock* develops, namely a discontinuity of the solution. Before this happens, waves that appear in the solution become steeper and steeper. It can also happen that the characteristic curves diverge. In this case, a so-called *rarefaction fan* develops. Thus we see that *classical solutions of quasilinear equations can break down after finite time.*

On the other hand, for many quasilinear systems there exist so-called *semi-global* classical solutions that have the following property: *For a given time* $T > 0$ *there exists a classical solution on the time interval* $[0, T]$, *if the initial data and the boundary data are sufficiently small with respect to the* C^1*-norm and satisfy the* C^1*-compatibility conditions at the points where both the initial conditions and the boundary conditions hold at the initial time.*

A well-written introduction to the mathematical theory of waves is given in [37]. A detailed account of controllability and observability for quasilinear hyperbolic systems in the framework of semi-global classical solutions is given in [43]. In the next section, we present a result about semi-global Lipschitz-continuous solutions.

6.3 An initial boundary value problem for the Burgers equation

Theorem 6.3 states that for Lipschitz continuous initial data and boundary data that is Lipschitz-compatible with the initial state and sufficiently small (that is with sufficiently small maximum-norm and Lipschitz constant) the initial boundary value problem ($BARWP$) for the Burgers equation has a semi-global Lipschitz-continuous solution on the time-interval $[0, T]$ for a given time $T > 0$ in the sense of characteristics.

Theorem 6.3 (Quasilinear initial boundary value problem). *Let $T > 0$ and $a \in (0, \infty)$ be given. Assume that the function v_0 is Lipschitz continuous on $[0, \infty)$ and that u is Lipschitz continuous on $[0, T]$. We consider the system*

$$(BARWP) \begin{cases} v(0,x) &= v_0(x),\ x \in (0,\ \infty) \\ v(t,0) &= u(t),\ t \in (0,T) \\ \partial_t v &= -(a+v)\,\partial_x v. \end{cases}$$

Define the numbers

$$m = \inf_{(t,x)\in[0,T]\times[0,\infty)} \{u(t),\ v_0(x)\},\ M = \sup_{(t,x)\in[0,T]\times[0,\,\infty)} \{u(t),\ v_0(x)\}.$$

We assume that

$$m > -a, \tag{6.49}$$

and

$$M < a. \tag{6.50}$$

We define

$$\kappa = \max\{-m,\ M\} < a.$$

Assume that the C^0-compatibility conditions between v_0 and u hold, that is $v_0(0) = u(0)$. Let

$$\tilde{L}_R$$

denote a common Lipschitz constant for u and v_0 on $[0, T] \times \{0\} \cup \{0\} \times [0,\ \infty)$, such that we have

$$|u(t) - v_0(x)| \le \tilde{L}_R\ |at + x| \tag{6.51}$$

for all $t \in [0, T]$ and $x \ge 0$. Assume that

1. $T < \frac{1}{\tilde{L}_R}$,
2. $T < \frac{a-\kappa}{\tilde{L}_R}$
and
3. $T < \left(1 - \frac{\kappa}{a}\right)\ \frac{1}{\tilde{L}_R} = \frac{a-\kappa}{a\,\tilde{L}_R}$.

Then there exists a solution of ($BARWP$) on $[0, T]$ in the sense of characteristics.

Remark 6.7. To make sure that the solution exists on a given, possibly large time interval $[0,\ T]$, the Lipschitz constant $\tilde{L}_R$ must be sufficiently small.

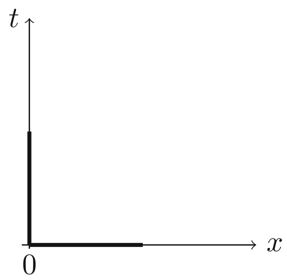

Fig. 6.2 We assume that the joint function that consists of u and v_0 glued together at the corner at 0 is Lipschitz continuous in the sense of (6.51) around the corner where the functions u and v_0 are glued together.

For a given value of $\tilde{L}_R$, Theorem 6.3 guarantees the existence of the solution only on a possibly short time interval $[0, T]$.

Remark 6.8. Condition (6.51) (see also (6.57) below) means that u and v_0 are Lipschitz continuous around the corner that is depicted in Figure 6.2.

Proof of Theorem 6.3. We consider a solution in the sense of characteristics. For the characteristic curves we have for $x \geq 0$ the explicit representation

$$\xi(s, x, 0) = x + (a + v_0(x))\, s \tag{6.52}$$

and for $t > 0$

$$\xi(s, 0, t) = (a + u(t))\,(s - t). \tag{6.53}$$

Due to the Lipschitz-continuity of u and v_0 and the compatibility condition $v_0(0) = u(0)$, the Picard-Lindelöf Theorem implies that the functions $\xi(\cdot, x, t)$ are uniquely defined as the solutions of the initial value problems (6.48) for sufficiently small $s > 0$, $t \geq 0$. Moreover, no regions in $(0, t) \times (0, \infty)$ occur, that do not contain characteristics, that is no rarefaction fans occur.

Now we determine a time interval, where it is impossible that the characteristic curves intersect.

In our case of the Burgers equation the *breaking time* can be estimated quite accurately. There are three possibilities how two characteristic curves can intersect:

1. Two characteristic curves of the type (6.52) intersect:
   The equation $\xi(s, x_1, 0) = \xi(s, x_2, 0)$ implies

$$s = -\frac{1}{\dfrac{v_0(x_2) - v_0(x_1)}{x_2 - x_1}}.$$

   Thus for a point s of intersection we have

$$|s| \geq \frac{1}{\tilde{L}_R}.$$

   Hence if $T < \frac{1}{\tilde{L}_R}$, an intersection of this kind cannot occur.

2. Two characteristic curves of the type (6.53) intersect:
   The equation $\xi(s,0,t_1) = \xi(s,0,t_2)$ implies

$$
\begin{aligned}
a &= \frac{u(t_2)-u(t_1)}{t_2-t_1}\, s + \frac{u(t_1)\, t_1 - u(t_2)\, t_2}{t_2-t_1} \\
&= s\,\frac{u(t_2)-u(t_1)}{t_2-t_1} + \frac{(u(t_1)-u(t_2))t_1 + u(t_2)(t_1-t_2)}{t_2-t_1} \\
&= s\,\frac{u(t_2)-u(t_1)}{t_2-t_1} - t_1\,\frac{u(t_2)-u(t_1)}{t_2-t_1} - u(t_2).
\end{aligned}
$$

   Hence we have

$$
\begin{aligned}
|s-t_1| &= \left| \frac{a+u(t_2)}{\frac{u(t_2)-u(t_1)}{t_2-t_1}} \right| \\
&\geq \frac{a-\kappa}{\tilde{L}_R}
\end{aligned}
$$

   Thus if $T < \frac{a-\kappa}{\tilde{L}_R}$, an intersection of this kind cannot occur.
3. A characteristic curves of the type (6.52) intersects a characteristic curves of the type (6.53): The equation $\xi(s,0,t) = \xi(s,x,0)$ implies

$$
\begin{aligned}
s &= \frac{-x-at-u(t)t}{v_0(x)-u(t)} \\
&= \frac{1}{-\frac{v_0(x)-u(t)}{x+at}} - u(t)t\frac{1}{v_0(x)-u(t)}
\end{aligned}
$$

   and thus we get the inequality

$$
\begin{aligned}
|s| &\geq \left| \frac{1}{\frac{v_0(x)-u(t)}{x+at}} \right| - \left| \frac{u(t)}{a}(at+x)\frac{1}{v_0(x)-u(t)} \right| \\
&\geq \frac{1}{\tilde{L}_R} - \frac{\kappa}{a}\frac{1}{\tilde{L}_R} \\
&= \left(1-\frac{\kappa}{a}\right)\,\frac{1}{\tilde{L}_R}.
\end{aligned}
$$

   If $T < \left(1-\frac{\kappa}{a}\right)\,\frac{1}{\tilde{L}_R}$, an intersection of this kind cannot occur.

Thus we have proved Theorem 6.3. □

Remark 6.9. In order to extend the solution if a shock has developed, only a solution in a weaker sense can be chosen. However, in general the weak solutions are not uniquely determined. It makes sense to choose the solution that is obtained as a limit of the solutions of the viscous Burgers equation for vanishing viscosity (see Exercise 6.1).

6.4 The Burgers equation with source term

In the applications, often source terms appear in the balance laws, for example to model the effects of friction. Thus in this section we study the solutions of an initial boundary value problem for the Burgers equation with a source term. Let a continuous function

$$g : \mathbb{R} \to \mathbb{R}$$

with $g(0) = 0$ and $\theta, L_g \in [0, \infty)$ be given, such that for all $z \in [-a, a]$ we have:

$$|g(z)| \leq \theta |z|^2.$$

Assume that for all $z_1, z_2 \in [-a, a]$ we have the Lipschitz inequality

$$|g(z_1) - g(z_2)| \leq L_g |z_1 - z_2|.$$

Theorem 6.4 (Quasilinear initial boundary value problem with source term). *Let $T > 0$ and $a \in (0, \infty)$ be given. Assume that the function v_0 is Lipschitz continuous on $[0, \infty)$ and that u is Lipschitz continuous on $[0, T]$. We consider the system*

$$(QARWP) \begin{cases} v(0,x) &= v_0(x),\ x \in (0, \infty) \\ v(t,0) &= u(t),\ t \in (0, T) \\ \partial_t v &= -(a+v)\,\partial_x v + g(v). \end{cases}$$

Define the numbers

$$m = \inf_{(t,x)\in[0,T]\times[0,\infty)} \{u(t),\ v_0(x)\},\ M = \sup_{(t,x)\in[0,T]\times[0,\infty)} \{u(t),\ v_0(x)\}.$$

We assume that

$$m > -a, \tag{6.54}$$

and

$$M < a. \tag{6.55}$$

We define

$$\kappa = \max\{-m, M\} + \theta\, T\, a^2 \tag{6.56}$$

and assume that

$$\kappa < a.$$

Assume that the C^0-compatibility conditions between v_0 and u hold, that is $v_0(0) = u(0)$. Let

$$\tilde{L}_R$$

denote a common Lipschitz constant for u and v_0 on $[0, T] \times \{0\} \cup \{0\} \times [0, \infty)$, such that we have

$$|u(t) - v_0(x)| \leq \tilde{L}_R \; |at + x| \tag{6.57}$$

for all $t \in [0, T]$ and $x \geq 0$. Assume that there exists a number $L_M > \tilde{L}_R$, such that

$$L_{kontr} := T \; \exp(L_M \, T) \left[L_g \; (1 + L_M \; T) + \frac{\theta a^2}{a - \kappa} + \tilde{L}_R \; \left(1 + \frac{a}{a - \kappa}\right) \right] < 1$$

and

$$\exp(L_M \, T) \left[L_g \; L_M \; T \; + \frac{\theta a^2}{a - \kappa} + \tilde{L}_R \; \left(1 + \frac{a}{a - \kappa}\right) \right] \leq L_M.$$

Then there exists a solution of (QARWP) on $[0, T]$ in the sense of characteristics.

Remark 6.10. Due to the source term, in general now the characteristic curves are not given by straight lines. In contrast to the case $g = 0$ for nonvanishing source term in general the constant states are not stationary, since the solutions are not constant along the characteristic curves.

For the following proofs we use a fundamental Lemma of THOMAS HAKON GRÖNWALL, (1877–1932).

Lemma 6.4 (Gronwall's Lemma). *Let real numbers $L > 0$, $U_0 \geq 0$, $\varepsilon \geq 0$, and a continuous function U be given.*

Assume that for all $t \in [0, T]$ we have the integral inequality

$$0 \leq U(t) \leq U_0 + \int_0^t L \, U(\tau) + \varepsilon \, d\tau.$$

Then for all $t \in [0, T]$ for $U(t)$ we have the upper bound

$$U(t) \leq U_0 \; \mathrm{e}^{Lt} + \varepsilon \; \frac{\mathrm{e}^{Lt} - 1}{L}.$$

Proof of Lemma 6.4. We define the auxiliary function

$$F(t) = U_0 + \int_0^t LU(\tau) + \varepsilon \, d\tau.$$

Then we have $F'(t) = LU(t) + \varepsilon$ and $U(t) \leq F(t)$. Since $L > 0$ this implies the inequality $F'(t) \leq LF(t) + \varepsilon$.

We define $H(t) = \mathrm{e}^{-Lt}\, F(t)$. Then $H(0) = F(0) = U_0$. The product rule for differentiation implies

$$\begin{aligned} H'(t) &= -LH(t) + \mathrm{e}^{-Lt}\, F'(t) \\ &\leq -LH(t) + \mathrm{e}^{-Lt}\, (LF(t) + \varepsilon) \\ &= -LH(t) + LH(t) + \mathrm{e}^{-Lt}\, \varepsilon \\ &= \varepsilon \mathrm{e}^{-Lt}. \end{aligned}$$

By integration, the inequality $H'(\tau) \leq \varepsilon \mathrm{e}^{-L\tau}$ yields

$$\begin{aligned} H(t) - H(0) &= \int_0^t H'(\tau)\, d\tau \\ &\leq \int_0^t \varepsilon \mathrm{e}^{-L\tau}\, d\tau \\ &= \varepsilon \frac{1}{L}(1 - \mathrm{e}^{-Lt}). \end{aligned}$$

Hence we have

$$\begin{aligned} U(t) \leq F(t) &= \mathrm{e}^{Lt}\, H(t) \\ &\leq \mathrm{e}^{Lt} \left(H(0) + \varepsilon \frac{1}{L}(1 - \mathrm{e}^{-Lt}) \right) \\ &= \mathrm{e}^{Lt}\, U_0 + \varepsilon \frac{1}{L} \left(\mathrm{e}^{Lt} - 1 \right). \end{aligned}$$

Thus we have proved Lemma 6.4. □

For the proof of Theorem 6.4 we consider again the solution of our system in the sense of characteristics that is described by characteristic curves. For a given function v, the following lemma guarantees the existence of the characteristic curves without intersection on a given (possibly large) time interval $[0, T]$.

Lemma 6.5. *Let $T > 0$ be given. Let $v \in C([0, T] \times [0, \infty))$ be Lipschitz continuous with respect to x with the Lipschitz constant L_v. Assume that there is a real number $v_{\max}$ such that for all $(t, x) \in [0, T] \times [0, \infty)$*

$$|v(t, x)| \leq v_{\max} < a.$$

Then the characteristic curves $\xi^v(s, x, t)$ exist for all

$$(s, x, t) \in [0, T] \times [0, \infty) \times [0, T].$$

The functions $\xi^v(s,x,t)$ are continuously differentiable with respect to s with a Lipschitz continuous derivative (with the Lipschitz constant $a + v_{\max}$).

For all $w \in C([0,T] \times [0,\infty))$ (with the Lipschitz constant L_w) we have the inequality

$$|\xi^v(s,x,t) - \xi^w(s,x,t)| \leq \|v - w\|_{C([0,T]\times[0,\infty))} \frac{\exp(L_v s) - 1}{L_v} \\ \leq T\exp(L_v T)\,\|v - w\|_{C([0,T]\times[0,\infty))}. \tag{6.58}$$

If $(t,x) \in [0,T] \times [0,\infty)$ is such that for all $s \in [0,T]$ we have $\xi^v(s,x,t) > 0$, we define $t^v(x,t) = 0$. Else we define $t^v(x,t) \in [0,T]$ as the solution of the equation

$$\xi^v(t^v(x,t), x, t) = 0.$$

Then we have the inequality

$$|t^v(x,t) - t^w(x,t)| \leq \frac{1}{a - v_{\max}} T\exp(L_v T)\|v - w\|_{C([0,T]\times[0,\infty))}. \tag{6.59}$$

For all $x_1, x_2 \in [0,\infty)$, $s, t \in [0,T]$ we have

$$|\xi^v(s,x_1,t) - \xi^v(s,x_2,t)| \leq |x_1 - x_2|\exp(L_v s) \tag{6.60}$$

and

$$|t^v(x_1,t) - t^v(x_2,t)| \leq \frac{1}{a - v_{\max}} \exp(L_v T)\,|x_1 - x_2|. \tag{6.61}$$

Proof of Lemma 6.5. We define a fixed point iteration for $\xi(s,x,t) = \xi^v(s,x,t)$. For this purpose we extend v on the whole x-axis by defining $v(t,x) = v(t,0)$ if $x < 0$. Then the extension is continuous on $[0,T]\times\mathbb{R}$ and Lipschitz continuous with respect to x with the Lipschitz constant L_v. Now we consider the integral equation

$$\xi^v(s,x,t) = x + a(s-t) + \int_t^s v(\tau, \xi^v(\tau,x,t))\,d\tau \tag{6.62}$$

for $(s,x,t) \in [0,T] \times \mathbb{R} \times [0,T]$. In order to show the existence of a unique solution we consider the corresponding PICARD-LINDELÖF iteration with the starting point

$$\xi^{(1)}(s,x,t) = x + a(s-t)$$

and

$$\xi^{(k+1)}(s,x,t) = x + a(s-t) + \int_t^s v(\tau, \xi^{(k)}(\tau,x,t))\,d\tau$$

for $k \in \{1, 2, 3, \ldots\}$. For all $k \in \{2, 3, 4 \ldots\}$ we have the inequality

$$\left|\xi^{(k+1)}(s, x, t) - \xi^{(k)}(s, x, t)\right|$$

$$= \left|\int_t^s v(\tau,\ \xi^{(k)}(\tau,\ x,\ t)) - v(\tau,\ \xi^{(k-1)}(\tau,\ x, t))\, d\tau\right|$$
$$\leq L_v \left|\int_t^s |\xi^{(k)}(\tau,\ x,\ t) - \xi^{(k-1)}(\tau,\ x,\ t)|\, d\tau\right|.$$

We have

$$\left|\xi^{(2)}(s, x, t) - \xi^{(1)}(s, x, t)\right| = \left|\int_t^s v(\tau,\ \xi^{(1)}(\tau,\ x,\ t))\, d\tau\right|$$
$$\leq v_{\max}\ |t - s|.$$

By induction this implies

$$\left|\xi^{(k+1)}(s, x, t) - \xi^{(k)}(s, x, t)\right| \leq \frac{1}{k!}\ v_{\max}\ L_v^{k-1}\ |t - s|^k$$
$$= \frac{v_{\max}}{L_v}\ \frac{1}{k!}\ (|t - s|\ L_v)^k\,.$$

On account of

$$\sum_{k=0}^{\infty} \frac{1}{k!}\ (|t - s|\ L_v)^k = \exp\left(|t - s|\ L_v\right) < \infty$$

this implies: The sequence $(\xi^{(k)}(s, x, t))_k$ is a Cauchy sequence in the space

$$C([0, T] \times \mathbb{R} \times [0, T])$$

and hence convergent, since we have

$$\left|\xi^{(m)}(s, x, t) - \xi^{(n)}(s, x, t)\right| \leq \sum_{k=n}^{m-1} \left|\xi^{(k+1)}(s, x, t) - \xi^{(k)}(s, x, t)\right|$$
$$\leq \frac{v_{\max}}{L_v} \left|\sum_{k=n}^{m-1} \frac{1}{k!}\ (|t - s|\ L_v)^k\right|$$
$$\to 0 \text{ for } m,\ n \to \infty.$$

Hence there exists a limit function $\xi(s,x,t) \in C([0,T]\times\mathbb{R}\times[0,T])$ that satisfies the integral equation (6.62). Now let $\psi(s, x, t)$ be an arbitrary solution of the integral equation (6.62). Then we have

$$\begin{aligned}|\xi(s,x,t)-\psi(s,x,t)| &\le \left|\int_t^s v(\tau,\ \xi(s,x,\tau)) - v(\tau,\ \psi(s,x,\tau))\,d\tau\right| \\ &\le \left|\int_t^s 2\,v_{\max}\,d\tau\right| \\ &\le 2\ |t-s|\,v_{\max}.\end{aligned}$$

Hence for all $s,\ t \in [0,T]$, $x \in \mathbb{R}$ we have

$$|\xi(s,x,t)-\psi(s,x,t)| \le 2\,v_{\max}\ |t-s|.$$

By induction this implies

$$|\xi(s,x,t)-\psi(s,x,t)| \le 2\,\frac{v_{\max}}{L_v}\,\frac{1}{k!}\,L_v^k\,|t-s|^k$$

which yields

$$\sup_{s,\,t\in[0,T],\,x\in\mathbb{R}} |\xi(s,x,t)-\psi(s,x,t)| \le 2\,\frac{v_{\max}}{L_v}\,\frac{1}{k!}\,(L_v\,T)^k \to 0\ (k\to\infty).$$

Hence the solution of the integral equation (6.62) is uniquely determined.

Now we show that (6.58) holds. The integral equation (6.62) implies

$$\begin{aligned}&|\xi^v(s,x,t)-\xi^w(s,x,t)| \\ &= \left|\int_t^s v(\tau,\ \xi^v(\tau,\ x,\ t)) - w(\tau,\ \xi^w(\tau,\ x,\ t))\,d\tau\right| \\ &\le \left|\int_t^s v(\tau,\ \xi^v(\tau,\ x,\ t)) - v(\tau,\ \xi^w(\tau,\ x,\ t))\,d\tau\right| \\ &+ \left|\int_t^s v(\tau,\ \xi^w(\tau,\ x,\ t)) - w(\tau,\ \xi^w(\tau,\ x, t))\,d\tau\right| \\ &\le L_v\ \left|\int_t^s |\,\xi^v(\tau,\ x,\ t) - \xi^w(\tau,\ x,\ t)|\ d\tau\right| \\ &+\ |t-s|\ \max_{(\tau,\,z)\in[0,T]\times[0,\infty)} |v(\tau,\ z)-w(\tau,\ z)|\,.\end{aligned}$$

Now we can apply Lemma 6.4. We define

$$U(s) = |\xi^v(s,x,t)-\xi^w(s,x,t)|\,.$$

Then we have the integral inequality

$$U(s) \le \left| \int_t^s L_v \; U(\tau) \; + \|v - w\|_{C([0,T]\times[0,\infty))} \; d\tau \right| .$$

Gronwall's Lemma (Lemma 6.4) yields

$$U(s) \le \|v - w\|_{C([0,T]\times[0,\infty))} \; \frac{\exp(L_v s) - 1}{L_v}.$$

Hence we have shown (6.58). Now we show (6.59). Without restriction of generality we assume that

$$t^w(x,t) > t^v(x,t) \ge 0.$$

We have

$$\begin{aligned}
\xi^w(t^w(x,t),\, x,\, t) - \xi^w(t^v(x,t),\, x,\, t) &= \int_{t^v(x,t)}^{t^w(x,t)} \partial_s \xi^w(s,\, x,\, t)\, ds \\
&= \int_{t^v(x,t)}^{t^w(x,t)} a + w(s,\, \xi^w(s,\, x,\, t)\, ds \\
&\ge (a - w_{\max})\, [t^w(x,t) - t^v(x,t)] \, .
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\xi^w(t^w(x,t),\, x,\, t) - \xi^w(t^v(x,t),\, x,\, t) \\
&= 0 - \xi^w(t^v(x,t),\, x,\, t) \\
&\le \xi^v(t^v(x,t),\, x,\, t) - \xi^w(t^v(x,t),\, x,\, t) \\
&\le T \, \exp(L_w T) \, \|v - w\|_{C([0,T]\times[0,\infty))},
\end{aligned}$$

where the last inequality follows from (6.58). Putting the last two inequalities together yields (6.59). Now Gronwall's Lemma also yields (6.60).

Exercise 6.7. Show that (6.60) holds.

Now we show (6.61). Let $x_1, x_2 \in [0, \infty)$ be given. Without loss of generality we assume that $x_1 < x_2$. Then we have

$$t^v(x_1, t) \ge t^v(x_2, t).$$

Case 1: If $t^v(x_1, t) = 0$, then $t^v(x_2, t) = 0$ and thus $t^v(x_1, t) - t^v(x_2, t) = 0$.
Case 2: If $t^v(x_1, t) > 0$, we have

$$\xi^v(t^v(x_1, t),\, x_1,\, t) = 0.$$

This implies

$$\xi^v(t^v(x_1,t),\, x_1,\, t) - \xi^v(t^v(x_2,t),\, x_1,\, t) = \int_{t^v(x_2,t)}^{t^v(x_1,t)} \partial_s \xi^v(s,\, x_1,\, t)\, ds$$
$$\geq (a - v_{\max})\ \left[t^v(x_1,t) - t^v(x_2,t)\right].$$

On the other hand we have

$$\begin{aligned}
&\xi^v(t_+^u(x_1,t),\, x_1,\, t) - \xi^v(t^v(x_2,t),\, x_1,\, t)\\
&= 0 - \xi^v(t^v(x_2,t),\, x_1,\, t)\\
&\leq \xi^v(t^v(x_2,t),\, x_2,\, t) - \xi^v(t^v(x_2,t),\, x_1,\, t) \leq \exp(L_v\, T)\ |x_1 - x_2|
\end{aligned}$$

where the last inequality follows from (6.60). Thus (6.61) holds and we have shown Lemma 6.5. □

Proof of Theorem 6.4. We use a fixed point argument to show the existence of the solution v. Assume that L_M is as in Theorem 6.4 and define the set

$$M = \{v : v \text{ is continuous and Lipschitz continuous with respect to } x \text{ on } [0,T] \times [0,\infty)$$
$$\text{with a Lipschitz constant } L_v \leq L_M \text{ and } |v| \leq \kappa\}$$

where κ is as in (6.56). Now we consider a mapping Φ that is defined on the set M. For a given function $\nu \in M$ we have $\nu \in C([0,T] \times [0,\infty))$ with the Lipschitz constant $L_\nu \leq L_M$ and we have

$$|\nu(t,x)| \leq \kappa < a.$$

Now Lemma 6.5 implies the existence of characteristic curves ξ^ν. We use these characteristic curves for the definition of the map Φ. We define

$$\Phi(\nu)(t,x) = \nu(t^\nu(x,t),\, \xi^\nu(t^\nu(x,t),\, x,\, t))$$
$$+ \int_{t^\nu(x,t)}^{t} g\left(\nu(\tau,\, \xi^\nu(\tau,\, x,\, t)\right)\, d\tau.$$

Note that we have

$$\nu(t^\nu(x,t),\, \xi^\nu(t^\nu(x,t),\, x,\, t)) = \begin{cases} u(t^\nu(x,t)) \text{ if } t^\nu(x,t) > 0, \\ v_0(\xi^\nu(0,\, x,\, t)) \text{ if } t^\nu(x,t) = 0 \end{cases}$$

hence the corresponding values are determined by u and v_0.

Now we consider the fixed point iteration that is defined by the equation

$$\begin{aligned}
\nu_{k+1}(t,x) &= \nu_k(t^{\nu_k}(x,t),\ \xi^{\nu_k}(t^{\nu_k}(x,t),\ x,\ t)) \\
&\quad + \int_{t^{\nu_k}(x,t)}^{t} g\,(\nu_k(\tau,\ \xi^{\nu_k}(\tau,\ x,\ t)))\ d\tau \\
&= \Phi(\nu_k)(t,x).
\end{aligned}$$

Our aim is to apply Banach's fixed point theorem. We divide the corresponding analysis in 3 steps.

Step 1: The velocity remains subcritical For all $\nu_1 \in M$ we have

$$|\nu_1(t,x)| \leq \max\{-m,\ M\} + \ \theta\, T\, a^2 \ = \kappa < a.$$

By induction this implies that for all $k \in \{0, 1, 2, \ldots\}$ we have

$$|\nu_{k+1}(t,x)| \leq \kappa < a. \tag{6.63}$$

By (6.63) for all $k \in \{1, 2, 3, \ldots\}$ we have a subcritical flow. In particular Lemma 6.5 guarantees the existence of the characteristic curves, hence the fixed point iteration is well defined.

Let L_{ν_k} denote a Lipschitz constant of ν_k with respect to x.

Exercise 6.8. Show that the Lipschitz constants L_{ν_k} can be chosen in such a way that the sequence $(L_{\nu_k})_k$ is bounded by L_M.

Step 2: The Lipschitz constants are uniformly bounded.

Now we consider the Lipschitz constants of $\Phi(v)$ with respect to x.

For all $x_1, x_2 \in [0, \infty)$ we have the inequality

$$|\Phi(v)(t,\ x_1) - \Phi(v)(t,\ x_2)| \leq f_1(t,x) + f_2(t,x),$$

with

$$f_1(t,\ x) = |v(t^v(x_1,t),\ \xi^v(t^v(x_1,t),\ x_1,\ t)) - v(t^v(x_2,t),\ \xi^v(t^v(x_2,t),\ x_2,\ t))|$$

and

$$\begin{aligned}
&f_2(t,\ x) \\
&= \left| \int_{t^v(x_1,\,t)}^{t} g(v(\tau,\ \xi^v(\tau(x_1,\ t), x_1, t))\, d\tau - \int_{t^v(x_2,\,t)}^{t} g(v(\tau,\ \xi^v(\tau(x_2,\ t), x_2, t))\, d\tau \right| \\
&\leq \left| \int_{t^v(x_1,t)}^{t} g(v(\tau,\ \xi^v(\tau(x_1,t), x_1, t)) \ - g(v(\tau,\ \xi^v(\tau(x_2,t), x_2, t))\, d\tau \right|
\end{aligned}$$

$$+ \left| \int_{t^v(x_1,t)}^{t^v(x_2,t)} g(v(\tau,\, \xi^v(\tau(x_2,t), x_2, t))\, d\tau \right|$$

$$\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x_1,\, t)) - v(\tau,\, \xi^v(\tau,\, x_2,\, t))|\; d\tau$$

$$+ |t^v(x_1,t) - t^v(x_2,t)|\; \theta a^2$$

$$\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x_1,\, t)) - v(\tau,\, \xi^v(\tau,\, x_2,\, t))|\; d\tau$$

$$+ \;\theta a^2 \frac{1}{a-\kappa}\; \exp(L_v\, T)\, |x_1 - x_2|$$

where we have used (6.61) from Lemma 6.5. Inequality (6.60) implies

$$\begin{aligned} f_2(t,\, x) &\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x_1,\, t)) - v(\tau,\, \xi^v(\tau,\, x_2,\, t))|\; d\tau \\ &+ \theta a^2 \frac{1}{a-\kappa}\; \exp(L_v\, T)\, |x_1 - x_2| \\ &\le L_g\; T\; L_v\; \exp(L_v\, T)\; |x_1 - x_2| \\ &+ \theta a^2 \frac{1}{a-\kappa}\; \exp(L_v\, T)\, |x_1 - x_2| \\ &= [L_g\; T\; L_v \;+ \frac{\theta a^2}{a-\kappa}\,]\exp(L_v\, T)\, |x_1 - x_2|. \end{aligned}$$

Hence

$$L_2 = \left[L_g\; T\; L_v \;+ \frac{\theta a^2}{a-\kappa} \right] \exp(L_v\, T)$$

is a Lipschitz constant of f_2.
To get a Lipschitz constant for f_1 we distinguish three cases.
Without loss of generality we assume that

$$t^v(x_1,t) \le t^v(x_2,t).$$

Case 1.: If $t^v(x_1,t) > 0$, we have $\xi^v(t^v(x_1,t),\, x_1,\, t){=}0$ and $\xi^v(t^v(x_2,t),\, x_2,\, t){=}0$. This implies

$$\begin{aligned} f_1(t,x) &\le |u(t^v(x_1,t)) - u(t^v(x_2,t))| \\ &\le \tilde{L}_R\, |t^v(x_1,t) - t^v(x_2,t)| \\ &\le \tilde{L}_R\; \frac{1}{a-\kappa}\; \exp(L_v T)\; |x_1 - x_2|. \end{aligned}$$

Case 2.: If $t^v(x_2, t) = 0$, we have $t^v(x_1, t) = 0$ and $\xi^v(t^v(x_1, t), x_1, t) > 0$, $\xi^v(t^v(x_2, t), x_2, t) > 0$. This implies

$$\begin{aligned}
f_1(t, x) &\leq |v_0(\xi^v(0, x_1, t)) - v_0(\xi^v(0, x_2, t))| \\
&\leq \tilde{L}_R \, |\xi^v(0, x_1, t) - \xi^v(0, x_2, t)| \\
&\leq \tilde{L}_R \, \exp(L_v T) \, |x_1 - x_2|.
\end{aligned}$$

Case 3.: If $t^v(x_2, t) > 0$ and $t^v(x_1, t) = 0$, we have

$$\begin{aligned}
f_1(t, x) &= |v_0(\xi^v(0, x_1, t)) - u(t^v(x_2, t))| \\
&\leq \tilde{L}_R \, |a \, t^v(x_2, t) + \xi^v(0, x_1, t)| \\
&\leq \tilde{L}_R \, a \, | \, t^v(x_2, t) - 0| + \tilde{L}_R \, |\xi^v(0, x_1, t) - 0| \\
&\leq \tilde{L}_R \, a \, | \, t^v(x_2, t) - t^v(x_1, t)| + \tilde{L}_R \, |\xi^v(0, x_1, t) - \xi^v(0, \tilde{x}, t)| \\
&\leq \tilde{L}_R \, a \, \frac{1}{a - \kappa} \, \exp(L_v T) \, |x_1 - x_2| \\
&+ \tilde{L}_R \, \exp(L_v T) \, |x_2 - x_2|.
\end{aligned}$$

Here $\tilde{x}$ is the point between x_1 and x_2 with $\xi_v(0, \tilde{x}, t) = 0$.

Case 1–Case 3 yield the Lipschitz constant L_1 for f_1 that is given by the equation

$$L_1 = \tilde{L}_R \, \exp(L_v T) \left[1 + \frac{a}{a - \kappa} \right].$$

Thus we get the Lipschitz constant L_Φ for $\Phi(v)$ that is given by

$$\begin{aligned}
L_\Phi &= L_1 + L_2 \\
&= \exp(L_v \, T) \left[L_g \, T \, L_v \, + \frac{\theta a^2}{a - \kappa} + \tilde{L}_R \, \left(1 + \frac{a}{a - \kappa} \right) \right] \\
&\leq L_M
\end{aligned}$$

where the last inequality follows from our assumptions in Theorem 6.4. Therefore the Lipschitz constants are uniformly bounded by L_M during the fixed point iteration.

In Step 1 we have shown that the solution remains subcritical during the iteration. Hence for all $v \in M$ we have $\Phi(v) \in M$, that is

$$\Phi(M) \subset M.$$

Step 3: Contractivity Now we show that Φ is a contraction. For all $v, w \in M$ we have the inequality

$$|\Phi(v) - \Phi(w)| \leq A + I$$

with

$$
\begin{aligned}
A &= |v(t^v(x,t),\, \xi^v(t^v(x,t),\, x,\, t)) - w(t^w(x,t),\, \xi^w(t^w(x,t),\, x,\, t))|\,, \\
I &= \left| \int_{t^v(x,t)}^{t} g(v(\tau,\, \xi^v(\tau(x,t)))\, d\tau - \int_{t^w(x,t)}^{t} g(w(\tau,\, \xi^w(\tau(x,t)))\, d\tau \right| \\
&\le \left| \int_{t^v(x,t)}^{t} g(v(\tau,\, \xi^v(\tau(x,t))) \, - g(w(\tau,\, \xi^w(\tau(x,t)))\, d\tau \right| \\
&+ \left| \int_{t^w(x,t)}^{t^v(x,t)} g(w(\tau,\, \xi^w(\tau(x,t)))\, d\tau \right| \\
&\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x,\, t)) - w(\tau,\, \xi^w(\tau,\, x,\, t))|\; d\tau \\
&+ |t^w(x,t) - t^v(x,t)|\; \theta a^2 \\
&\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x,\, t)) - w(\tau,\, \xi^w(\tau,\, x,\, t))|\; d\tau \\
&+ \; \theta a^2 \frac{1}{a-\kappa} T \,\exp(L_v\, T) \|v - w\|_{C([0,\,T]\times[0,\infty))}.
\end{aligned}
$$

We have

$$
\begin{aligned}
I &\le L_g \int_0^t |v(\tau,\, \xi^v(\tau,\, x,\, t)) - v(\tau,\, \xi^w(\tau,\, x,\, t))|\; d\tau \\
&+ L_g \int_0^t |v(\tau,\, \xi^w(\tau,\, x,\, t)) - w(\tau,\, \xi^w(\tau,\, x,\, t))|\; d\tau \\
&+ \; \frac{\theta a^2}{a-\kappa} T \,\exp(L_v\, T) \|v - w\|_{C([0,\,T]\times[0,\infty))} \\
&\le L_g\, T\, L_v\, T \exp(L_v\, T)\; \|v - w\|_{C([0,T]\times[0,\infty))} \\
&+ L_g\, T\; \|v - w\|_{C([0,T]\times[0,\infty))} \\
&+ \; \frac{\theta a^2}{a-\kappa} T \,\exp(L_v\, T) \|v - w\|_{C([0,\,T]\times[0,\infty))} \\
&= L_g\, T\, [1 + L_v\, T \exp(L_v\, T)]\; \|v - w\|_{C([0,T]\times[0,\infty))} \\
&+ \; \frac{\theta a^2}{a-\kappa} T \,\exp(L_v\, T) \|v - w\|_{C([0,\,T]\times[0,\infty))} \\
&= T \left[L_g\, (1 + L_v\, T \exp(L_v\, T)) + \frac{\theta a^2}{a-\kappa}\, \exp(L_v\, T) \right] \|v - w\|_{C([0,\,T]\times[0,\infty))}.
\end{aligned}
$$

Now we consider again three cases. Without loss of generality we assume that

$$t^v(x,t) \le t^w(x,t).$$

Case 1.: If $t^w(x,t) > 0$, we have $\xi^v(t^v(x,t),\, x,\, t) = 0$ and $\xi^w(t^w(x,t),\, x,\, t) = 0$. Hence we get

$$\begin{aligned} A &\le |u(t^v(x,t)) - u(t^w(x,t))| \\ &\le \tilde{L}_R\, |t^v(x,t) - t^w(x,t)| \\ &\le \tilde{L}_R\, \frac{1}{a-\kappa}\, T \exp(L_v T)\ \|v - w\|_{C([0,\,T]\times[0,\infty))}. \end{aligned}$$

Case 2.: If $t^w(x,t) = 0$, we have $t^v(x,t) = 0$ and $\xi^v(t^v(x,t),\, x,\, t) > 0$, $\xi^w(t^w(x,t),\, x,\, t) > 0$. Thus we get

$$\begin{aligned} A &\le |v_0(\xi^v(0,\, x,\, t)) - v_0(\xi^w(0,\, x,\, t))| \\ &\le \tilde{L}_R\, |\xi^v(0,\, x,\, t) - \xi^w(0,\, x,\, t)| \\ &\le \tilde{L}_R\, T \exp(L_v T)\ \|v - w\|_{C([0,\,T]\times[0,\infty))}. \end{aligned}$$

Case 3.: If $t^w(x,t) > 0$ and $t^v(x,t) = 0$, we have

$$\begin{aligned} A &= |v_0(\xi^v(0,\, x,\, t)) - u(t^w(x,t))| \\ &\le \tilde{L}_R\ |a\, t^w(x,t) + \xi^v(0,\, x,\, t)| \\ &\le \tilde{L}_R\, a\, |\, t^w(x,t) - 0| + \tilde{L}_R\ |\xi^v(0,\, x,\, t) - 0| \\ &\le \tilde{L}_R\, a\, |\, t^w(x,t) - t^v(x,t)| + \tilde{L}_R\ |\xi^v(0,\, x,\, t) - \xi^w(0,\, x,\, t)| \\ &\le \tilde{L}_R\, a\, \frac{1}{a-\kappa}\, T \exp(L_v T)\ \|v - w\|_{C([0,\,T]\times[0,\infty))} \\ &\quad + \tilde{L}_R\, T \exp(L_v T)\ \|v - w\|_{C([0,\,T]\times[0,\infty))}. \end{aligned}$$

Here $\xi^w(0,\, x,\, t) < 0$ is defined by the extension of the characteristic curves for $(s,\, x,\, t) \in [0,T] \times (-\infty,\, \infty) \times [0,T]$. For this purpose w is extended on $(-\infty, 0)\times[0,T]$ by $w(x,t) = w(0,t)$ $(x < 0)$. Note that this extension is Lipschitz continuous.

From Case 1–Case 3 we get

$$\begin{aligned} &\|\Phi(v) - \Phi(w)\|_{C([0,T]\times[0,\infty))} \\ &\le L_{kontr}\ \|v - w\|_{C([0,T]\times[0,\infty))} \end{aligned}$$

with the Lipschitz constant

$$L_{kontr} = T\left[L_g\,(1 + L_v\,T\exp(L_v\,T)) + \frac{\theta a^2}{a-\kappa}\,\exp(L_v\,T)\right]$$
$$+\,\tilde{L}_R\,\frac{\max\{a,\,1\}}{a-\kappa}\,T\exp(L_v T) + \tilde{L}_R\,T\exp(L_v T).$$

Thus we have shown that if $T\,[\tilde{L}_R + L_g + \theta]$ is sufficiently small the map Φ is a contraction. Now Banach's fixed point theorem implies the existence of a unique fixed point of Φ in M, which is the solution of the quasilinear initial boundary value problem (*QARWP*). Thus Theorem 6.4 is proved. □

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