CHAPTER 2

APPROXIMATION THEOREMS AND WHITNEY'S EMBEDDING

Perhaps the most important property of a manifold which opens up various developments of manifold theory is that a manifold can be embedded in a Euclidean space as a closed subspace. This is called Whitney's embedding theorem. Thus any manifold may be considered as a submanifold of a Euclidean space. Originally manifolds were introduced in this way, and Whitney's theorem reconciles this earlier concept of manifold with its modern abstract definition. We will prove the theorem by way of certain approximations. If we list the hierarchy of maps between manifolds as continuous, smooth, immersion, injective immersion, and embedding, then the approximation theorems say that each map in one of the above classes is approximable by a map in its immediate successor. Of course for a real analytic manifold, one can go one step further and prove that there is a real analytic embedding of a manifold in Euclidean space, but its proof lies much deeper and will not be considered here. Our approach to these approximation theorems will involve two important topological aspects of manifold, namely smooth partition of unity, and Sard's theorem. The first helps to construct smooth maps by piecing together local information, and the second leads to approximations.

In this chapter a manifold will always be a smooth manifold, and it may have boundary, unless it is stated explicitly otherwise.

2.1. Smooth partition unity

Recall that a covering $\mathcal{U}$ of a topological space X is called **locally finite** if each point of X has an open neighbourhood which intersects only finitely many members of $\mathcal{U}$. Another covering $\mathcal{V}$ of X is called a **refinement** of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A Hausdorff space X is **paracompact** if every open covering of X admits an open locally finite refinement.

Theorem 2.1.1. *Every manifold M is paracompact.*

Proof. Since M is locally homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$, it is locally compact (each of its points has a compact neighbourhood). Then each point $x \in M$ has an open neighbourhood V whose closure is compact. For, if U is an arbitrary open neighbourhood of x, and K a compact neighbourhood of x, then $V = U \cap \text{Int } K \subset K$, and so $\overline{V}$ is compact, being a closed subset of a compact set. It follows then, since M is second countable, that M admits a countable basis $\{V_j\}$ such that each $\overline{V}_j$ is compact.

Then, there is an increasing sequence $K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots$ of compact subsets whose union is M such that $K_j \subset \text{Int } K_{j+1}$, for each j. Indeed, we may take $K_1 = \overline{V}_1$, and, assuming inductively that K_j has been defined, if m is the smallest integer $> j$ such that $K_j \subset V_1 \cup \cdots \cup V_m$, then we may take

$$K_{j+1} = \overline{V}_1 \cup \cdots \cup \overline{V}_m = \overline{V_1 \cup \cdots \cup V_m}.$$

Now let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. Choose a locally finite refinement $\mathcal{V}$ as follows. Let $K_{-1} = K_0 = \emptyset$, and, for each $j \geq 0$, consider open sets

$$(\text{Int } K_{j+2} - K_{j-1}) \cap U_\alpha, \quad U_\alpha \in \mathcal{U}.$$

These open sets cover the compact set $K_{j+1} - \text{Int } K_j$. Therefore, we can find a finite subcover $\mathcal{V}_j = \{V_1^j, \ldots, V_{\alpha(j)}^j\}$ ($\alpha(j)$ an integer). The collection $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots$ covers M, since the sets $K_{j+1} - \text{Int } K_j$ cover M. The covering $\mathcal{V}$ is a refinement of $\mathcal{U}$ (each V_i^j is contained in some U_α). It is locally finite, because if $x \in K_j$, then $\text{Int } K_{j+1}$ is a neighbourhood of x which intersects no member of $\mathcal{V}_k$ for $k > j+1$. □

Remark 2.1.2. Actually we have constructed a locally finite refinement $\mathcal{V}$ which is countable.

Lemma 2.1.3. *Any open covering $\{U_\alpha\}$ of a manifold M has a locally finite countable refinement by coordinate neighbourhoods, each of which has compact closure.*

Proof. The proof follows the same line of arguments as that of the above theorem. One has only to choose the covering of each compact set $K_{j+1} - \text{Int } K_j$ suitably. Each point x of the open set

$$(\text{Int } K_{j+2} - K_{j-1}) \cap U_\alpha$$

has a coordinate neighbourhood V_x and a homeomorphism ϕ_x of V_x into $\mathbb{R}^n$ or $\mathbb{R}^n_+$ such that $V_x \subset (\text{Int } K_{j+2} - K_{j-1}) \cap U_\alpha$, and $\phi_x(V_x)$ contains a closed n-ball B^n with centre at $\phi_x(x)$, $n = \dim M$. Let $W_x = \phi_x^{-1}(\text{Int } B^n)$. With this choice, $\overline{W}_x$ will be compact. We may find a finite number of the W_x which cover $K_{j+1} - \text{Int } K_j$, and then proceed as before. □

A covering $\{V_i\}$ is called a **shrinking** of a covering $\{U_i\}$ if each $\overline{V}_i \subset U_i$.

Lemma 2.1.4 (Shrinking lemma). *Let $\mathcal{U} = \{U_i\}_{i \geq 1}$ be a countable locally finite open covering of M. Then there is another open covering $\{V_i\}$ of M such that $\overline{V}_i \subset U_i$ for every $i \geq 1$.*

PROOF. We may assume that M is connected. Now write $\mathcal{U}_k = \cup_{i\geq k} U_i$. and construct the open sets V_i inductively in the following way.

The closed set $A_1 = U_1 - \mathcal{U}_2$ is contained in U_1, and so $M = A_1 \cup \mathcal{U}_2$. Since M is paracompact, it is normal, and therefore we may choose an open set V_1 such that $A_1 \subset V_1 \subset \overline{V}_1 \subset U_1$. Then $M = V_1 \cup \mathcal{U}_2$. Next, suppose that the open sets $V_1, \ldots, V_{k-1}$ have been chosen so that $\overline{V}_i \subset U_i$ for $i = 1, \ldots, k-1$, and $M = V_1 \cup \cdots \cup V_{k-1} \cup \mathcal{U}_k$. Then the closed set $A_k = U_k - (V_1 \cup \cdots \cup V_{k-1} \cup \mathcal{U}_{k+1})$ is contained in U_k, and we have

$$M = V_1 \cup \cdots \cup V_{k-1} \cup A_k \cup \mathcal{U}_{k+1}.$$

Choose an open set V_k such that $A_k \subset V_k \subset \overline{V}_k \subset U_k$. Then we have

$$M = V_1 \cup \cdots \cup V_k \cup \mathcal{U}_{k+1}.$$

To see that the collection $\{V_i\}$ is a covering of M, take any point $x \in M$. Then, since the covering $\mathcal{U}$ is locally finite, there is a largest m such that $x \notin U_k$ for $k \geq m$, that is, $x \notin \mathcal{U}_m$. Since $M = V_1 \cup \cdots \cup V_{m-1} \cup \mathcal{U}_m$, it follows that $x \in V_1 \cup \cdots \cup V_{m-1}$. This completes the proof. □

Theorem 2.1.5. *Every manifold is metrisable.*

PROOF. Since every paracompact space is normal and a manifold is second countable, the proof follows trivially from Urysohn's metrisation theorem ([**15**], Theorem 2-46, p. 68). This theorem states that every second-countable normal space can be embedded topologically in infinite dimensional Hilbert coordinate space. This metric has nothing to do with the locally Euclidean structure on the manifold.

A natural metric on a manifold may be obtained from Smirnov's theorem ([**15**], Theorem 2-68, p. 81). This theorem states that a space is metrisable if and only if it is paracompact, and locally metrisable. Its proof uses paracompactness to pass from the local information to the global one.

Locally the topology of M is the same as the topology of $\mathbb{R}^n$, and therefore it is given by the standard metric in $\mathbb{R}^n$. If (U, ϕ) is a coordinate chart about a point $p \in M$ with coordinates $(x_1, \ldots, x_n)$ such that $\phi(U)$ is a convex set in $\mathbb{R}^n$, then we have a metric ρ on U such that for $x \in U$

$$\rho(x,p) = [(x_1 - x_1(p))^2 + \cdots + (x_n - x_n(p))^2]^{\frac{1}{2}},$$

or equivalently,

$$\rho(x,p) = \max\{|x_i - x_i(p)|\}.$$

□

We shall have occasions to use a bump function whose definition is as follows.

Definition 2.1.6. A **bump function** is a smooth function $\mathcal{B} : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\mathcal{B}(x) = 0 \text{ if } x \leq 0,\ 0 < \mathcal{B}(x) < 1 \text{ if } 0 < x < 1,\ \mathcal{B}(x) = 1 \text{ if } x \geq 1.$$

To construct a bump function $\mathcal{B}$, first define

$$\begin{aligned} \psi(x) &= \exp\left(\frac{1}{x(x-1)}\right) \text{ if } 0 < x < 1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then ψ is smooth, non-negative, and non-vanishing when $0 < x < 1$. Now define $\mathcal{B}$ by

$$\mathcal{B}(x) = \frac{\int_0^x \psi(t)dt}{\int_0^1 \psi(t)dt}.$$

Definition 2.1.7. The **support** of a function $f : M \longrightarrow \mathbb{R}$, denoted by $\text{supp}f$, is the closure of the set of points of M where f is non-zero.

Lemma 2.1.8. *If $K \subset U \subset M$, where K is compact and U is open, then there is a smooth function $\mu : M \longrightarrow [0, \infty)$ such that $\mu(x) > 0$ if $x \in K$, and* $\text{supp}\ \mu \subset U$.

Proof. Define a smooth function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $\alpha(x) = \mathcal{B}(1 - |x|)$. Then $\alpha(x) > 0$ if $|x| < 1$, and $\alpha(x) = 0$ if $|x| \geq 1$. Now construct for each $p \in K \subset U$ a smooth function $\mu_p : M \longrightarrow [0, \infty)$ such that $\mu_p(p) > 0$ and $\text{supp}\ \mu_p \subset U$ in the following way. Choose local coordinates $(x_1, \ldots, x_n)$ about p with p corresponding to the origin such that $B_r = \{(x_1, \ldots, x_n) \mid |x_i| < r\} \subset U$, for a suitable $r > 0$. Define μ_p by

$$\begin{aligned} \mu_p(x) &= \alpha\left(\frac{x_1}{r}\right) \cdots \alpha\left(\frac{x_n}{r}\right) \text{ if } x \in B_r \\ &= 0 \text{ otherwise.} \end{aligned}$$

As p runs over K the open sets $\{x \in M \mid \mu_p(x) > 0\}$ cover K. By compactness, a finite number of them still cover K. Then the sum of the corresponding finite number of functions μ_p is the required function μ. □

Definition 2.1.9. Let M be a manifold with an open covering $\mathcal{U} = \{U_i\}_{i \in A}$. Then a **smooth partition of unity** subordinate to $\mathcal{U}$ is a family of smooth functions $\{\lambda_i : M \longrightarrow \mathbb{R}\}_{i \in A}$ satisfying the following conditions.

(i) $\text{supp}\lambda_i \subset U_i$ for all $i \in A$,
(ii) $0 \leq \lambda_i(x) \leq 1$ for all $x \in M$ and $i \in A$,
(iii) each $x \in M$ has a neighbourhood on which all but finitely many functions λ_i are identically zero,
(iv) $\sum_{i \in A} \lambda_i(x) = 1$ for all $x \in M$ (note that the sum is always finite by (iii)).

Lemma 2.1.10. *Let $\mathcal{U} = \{U_i\}_{i\in A}$ and $\mathcal{V} = \{V_j\}_{j\in B}$ be two open coverings of M such that $\mathcal{U}$ refines $\mathcal{V}$. Then, if $\mathcal{U}$ has a subordinate smooth partition of unity, so has $\mathcal{V}$.*

Proof. Let $\{\lambda_i\}_{i\in A}$ be a smooth partition of unity subordinate to $\mathcal{U}$. Let $f : A \longrightarrow B$ be a map of the index sets so that $U_i \subset V_{f(i)}$, $i \in A$. Define $\mu_j : M \longrightarrow \mathbb{R}$ by

$$\mu_j(x) = \sum_{i\in f^{-1}(j)} \lambda_i(x).$$

It is easily checked that the conditions (i)–(iv) hold for the family $\{\mu_j\}$, when U_i are replaced by V_j. □

Remark 2.1.11. Some people call $\{\lambda_i\}$ a partition of unity subordinate to $\mathcal{V}$, In this case the condition (i) has to be replaced by the following condition:

"for every $i \in A$ there is a $j \in B$ such that supp $\lambda_i \subset V_j$".

Theorem 2.1.12. *Any manifold M with an open covering $\{U_i\}$ admits a smooth partition of unity subordinate to $\{U_i\}$.*

Proof. We may assume that the given covering $\{U_i\}$ is countable and locally finite such that each of its members U_i is a coordinate neighbourhood with compact closure (Lemma 2.1.3). We may find another open covering $\{V_i\}$ of M such that $\overline{V}_i \subset U_i$ (Lemma 2.1.4). Now construct smooth functions $\mu_i : M \longrightarrow \mathbb{R}$ as described in Lemma 2.1.8 such that $\mu_i > 0$ on $\overline{V}_i$ and supp $\mu_i \subset U_i$. Then the function $\sum_i \mu_i$ is a well-defined positive smooth function, and the family of functions $\lambda_i = \mu_i / \sum_i \mu_i$ is the required smooth partition of unity. □

Lemma 2.1.13. *If $\{\lambda_i : U_i \longrightarrow \mathbb{R}\}$ is a smooth partition of unity on M, and $\{f_i : U_i \longrightarrow \mathbb{R}\}$ is a family of smooth functions, then the function $f : M \longrightarrow \mathbb{R}$ defined by $f(x) = \sum_i \lambda_i(x) f_i(x)$ is smooth.*

Proof. Since the function $\lambda_i f_i$ is smooth on U_i and vanishes on a neighbourhood of $M - U_i$, it can be extended over M using the zero function on $M - U_i$. Therefore the sum $f = \sum_i \lambda_i f_i$ is smooth. □

Lemma 2.1.14. *For a map $f : U \longrightarrow \mathbb{R}^m$, where U is open in $\mathbb{R}^n_+$, the following conditions are equivalent.*

(a) *f is smooth, as defined in Definition* 1.2.6 *using local extendability condition,*

(b) *there is an open set V in $\mathbb{R}^n$ and a smooth map $F : V \longrightarrow \mathbb{R}^m$ such that $V \cap \mathbb{R}^n_+ = U$ and $F|U = f$.*

Proof. The part (b) ⇒ (a) is trivial. Next, assume (a). Then, for each $x \in U$, there is an open neighbourhood V_x of x in $\mathbb{R}^n$ and a smooth map

$F_x : V_x \longrightarrow \mathbb{R}^m$ such that $f = F_x$ on $U \cap V_x$. Let $W = \cup_{x \in U} V_x$. Then W is open in $\mathbb{R}^n$, and $U \subset W$.

The manifold W admits a partition of unity $\{\lambda_x\}$ subordinate to the covering $\{V_x\}$. Then $G = \sum_{x \in U} \lambda_x F_x$ is a smooth map from W to $\mathbb{R}^m$, by Lemma 2.1.13. On the other hand, there exists an open set V' of $\mathbb{R}^n$ such that $U = V' \cap \mathbb{R}^n_+$. Then taking $V = W \cap V'$ and $F = G|V$, we get the condition (b). □

Lemma 2.1.15. *If $f : M \longrightarrow \mathbb{R}$ is a positive continuous function, then there is a smooth function $g : M \longrightarrow \mathbb{R}$ such that*

$$0 < g(x) < f(x) \text{ for all } x \in M.$$

When M is compact, g may be taken to be a constant function.

Proof. As in the proof of Theorem 2.1.12, consider a locally finite covering $\{U_i\}$ of M, and another open covering $\{V_i\}$ such that $\overline{V}_i$ is compact and $\overline{V}_i \subset U_i$. Take a smooth partition of unity $\{\lambda_i\}$ such that $\lambda_i > 0$ on $\overline{V}_i$ and supp $\lambda_i \subset U_i$. Choose $\delta_i > 0$ smaller than the infimum of f on the compact set $\overline{V}_i$, and define $g : M \longrightarrow \mathbb{R}$ by $g(x) = \sum_i \delta_i \lambda_i(x)$. Then g is smooth. Since the sum $\sum_i \lambda_i(x)$ is finite and equal to 1, and the maximum of the corresponding δ_i is less than $f(x)$, we have $g(x) < f(x)$. Also $g(x) > 0$, since all δ_i are so. □

Lemma 2.1.16. *If K is a closed subset of a manifold M, and $f : K \longrightarrow \mathbb{R}$ is a smooth function, then f extends to a smooth function $F : M \longrightarrow \mathbb{R}$.*

Proof. In view of Definition 1.2.6, cover K by open sets U_i such that there exist smooth functions $g_i : U_i \longrightarrow \mathbb{R}$ with $g_i = f$ on $K \cap U_i$. The sets U_i and $M - K$ form an open covering of M. Let $\{\lambda_i\}$ be a smooth partition of unity subordinate to this covering. Then the smooth extension F of f is given by

$$F(x) = \begin{cases} \sum_i \lambda_i(x) g_i(x), & \text{if } x \notin M - K, \\ 0, & \text{otherwise.} \end{cases}$$

□

Lemma 2.1.17 (Smooth Urysohn's lemma). *If $K \subset U \subset M$, K closed, U open, then there is a smooth function $f : M \longrightarrow \mathbb{R}$ such that $0 \leq f \leq 1$, $f|K = 1$ and supp $f \subset U$.*

Proof. The open sets $U_1 = U$ and $U_2 = M - K$ form a covering of M. Let $\lambda_1 : U_1 \longrightarrow \mathbb{R}$ and $\lambda_2 : U_2 \longrightarrow \mathbb{R}$ be a smooth partition of unity subordinate to this covering. Then λ_1 extended over M by the zero function outside U_1 is a solution f of the problem. □

Theorem 2.1.18 (Whitney's weak embedding theorem). *If M is a compact n-manifold, then there is an embedding $f : M \longrightarrow \mathbb{R}^m$, where $m = r(n+1)$ for some integer $r > 0$.*

PROOF. Find a finite covering of M by coordinate charts (U_i, ϕ_i), $i = 1, \ldots, r$, and open sets V_i also covering M such that $\overline{V_i} \subset U_i$ for all i. By Lemma 2.1.17, there are C^∞ functions $\lambda_i : M \longrightarrow \mathbb{R}$ such that $\lambda_i|\overline{V_i} = 1$ and supp $\lambda_i \subset U_i$. Let $\psi_i : M \longrightarrow \mathbb{R}^n$ be C^∞ maps given by

$$\begin{aligned} \psi_i(p) &= \lambda_i(p)\phi_i(p) \text{ if } p \in U_i, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Define $f : M \longrightarrow \mathbb{R}^{r(n+1)}$ by

$$f(p) = (\psi_1(p), \ldots, \psi_r(p), \lambda_1(p), \ldots, \lambda_r(p)), p \in M.$$

Then the Jacobian matrix $J(f)$ has rank n at every point $p \in M$. Because, $J(\phi_i)$ has rank n, and if $p \in M$, then

$$(p \in V_i \text{ for some } i) \Rightarrow (\lambda_i(p) = 1) \Rightarrow (\phi_i = \psi_i \text{ on } \overline{V}_i) \Rightarrow (J(\phi_i) = J(\psi_i) \text{ at } p),$$

so $J(\psi_i)$, and hence $J(f)$, has rank n at p.

Also f is injective. Because, if $f(p) = f(q)$ then $\psi_i(p) = \psi_i(q)$ and $\lambda_i(p) = \lambda_i(q)$ for all i, and if $p \in V_j$ then $\lambda_j(p) = \lambda_j(q) = 1$, and so $\phi_j(p) = \phi_j(q)$, which implies $p = q$, ϕ_j being injective.

Thus f is an injective immersion. Since M is compact and $f(M)$ is Hausdorff, f is a homeomorphism onto its image, and hence it is an embedding. □

We remark that the theorem is unsatisfactory in that the value of m, which is the dimension of the Euclidean space, depends on the number r of coordinate neighbourhoods required to cover M, and therefore m may be much larger than we would like it. We will prove in Theorem 2.5.1 a stronger version of this theorem which removes the restriction of compactness and gives a much lower value for m, not depending on r.

◊ **Exercise 2.1.** Show that the projective space $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{(n+1)^2}$ by an embedding f where $f([x_0, \ldots, x_n])$ is a vector in $\mathbb{R}^{(n+1)^2}$ whose (i, j)-th coordinate in lexicographic order is $a_{ij} = x_i x_j / \sum_k x_k^2$, $i, j, k = 0, 1, \ldots, n$. The image of f consists of symmetric square matrices A of order $n + 1$ such that $A \cdot A = A$ and trace of A is 1.

In a similar way, the complex projective space $\mathbb{C}P^n$ embeds in $\mathbb{C}^{(n+1)^2}$.

Proposition 2.1.19. *Any metric on a manifold M compatible with the topology of M can be turned into a complete metric giving the same topology of M.*

PROOF. The proof will be clear from the next two lemmas. □

Recall that a continuous map between topological spaces is **proper** if the inverse image of any compact set is compact.

Lemma 2.1.20. *If X is a metric space with metric ρ and $f : X \longrightarrow \mathbb{R}$ is a continuous proper map, then the map $\rho' : X \times X \longrightarrow \mathbb{R}$ given by*

$$\rho'(x, y) = \rho(x, y) + |f(x) - f(y)|, x, y \in X$$

is a complete metric on X which is compatible with the topology of X.

PROOF. Clearly ρ' is a metric. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies on X induced by the metrics ρ and ρ' respectively. Then, since ρ' is continuous with respect to $\mathcal{T}$, we have $\mathcal{T}' \subset \mathcal{T}$. Conversely, take an open set U in $\mathcal{T}$ and a point x in U. Then we can find an $\epsilon > 0$ so that

$$B'(x, \epsilon) = \{y \in X \mid \rho'(x, y) < \epsilon\} \subset B(x, \epsilon) = \{y \in X \mid \rho(x, y) < \epsilon\} \subset U.$$

This means U is in $\mathcal{T}'$, and $\mathcal{T} = \mathcal{T}'$. Next, to see that ρ' is complete, take a Cauchy sequence $\{x_n\}$ in X with respect to the metric ρ'. Then there is a number $m > 0$ such that $\rho'(x_1, x_n) < m$ for all $n \geq 1$. Therefore

$$|f(x_1) - f(x_n)| < m$$

for all $n \geq 1$, and so

$$x_n \in f^{-1}([f(x_1) - m, f(x_1) + m]).$$

Since f is proper, the above set is compact, and the sequence $\{x_n\}$ converges to a limit in X. □

Lemma 2.1.21. *On a manifold M there always exists a proper smooth function $f : M \longrightarrow \mathbb{R}$.*

PROOF. Find an open covering of M by open sets with compact closure, and a smooth partition of unity $\{\lambda_i\}$ subordinate to a countable locally finite refinement of this covering. Since the refinement is countable, we may assume that the functions λ_i are indexed by integers $i > 0$. Then the function $f : M \longrightarrow [1, \infty)$ given by $f(x) = \sum_i i\lambda_i(x)$ is a well-defined, because all but a finite number of $\lambda_i(x)$ vanish. Now $f(x) \leq k$ implies at least one of the k functions $\lambda_1, \ldots, \lambda_k$ must not vanish at x (if all of them were zero, then $f(x)$ would be $\geq k + 1$). Therefore $f^{-1}([-k, k])$ is contained in the set

$$\cup_{i=1}^{k} \{x \in M : \lambda_i(x) \neq 0\}$$

which has compact closure. This implies that f is proper, because every compact subset of $\mathbb{R}$ is contained in some interval $[-k, k]$. □

We shall require some more facts about proper maps.

Recall that if X is a locally compact Hausdorff space, then its one-point compactification is a space $X^+ = X \cup \{\infty\}$ (∞ represents a point not in X) such that the topology of X^+ comprises all open sets in $X \subset X^+$ and all sets of the form $X^+ - K$, where K is a compact subset of X. This is the unique topology in X^+ which makes it a compact Hausdorff space with X as a subspace. If X is already compact, then ∞ is an isolated point in X^+.

Lemma 2.1.22. *Let $f : X \longrightarrow Y$ be a continuous map between locally compact Hausdorff spaces, and $f^+ : X^+ \longrightarrow Y^+$ be its extension obtained by setting $f^+(\infty) = \infty$. Then f is proper if and only if f^+ is continuous.*

PROOF. Suppose that a continuous map f is proper, and U is an open set in Y^+. Then, if $U \subset Y$, $(f^+)^{-1}(U) = f^{-1}(U)$ is open, and if $U = Y^+ - K$ with $K \subset Y$ compact, then $(f^+)^{-1}(U) = X^+ - f^{-1}(K)$ is also open, since $f^{-1}(K)$ is compact and hence closed. Therefore f^+ is continuous.

Conversely, suppose f^+ is continuous, and $K \subset Y$ is a compact set. Then K is compact, and hence closed in Y^+. Then $(f^+)^{-1}(K)$ is closed in X^+, and hence compact and contained in X. Thus f is proper. $\square$

Corollary 2.1.23. *Let X be locally compact Hausdorff, and Y Hausdorff. Then a continuous injective proper map $f : X \longrightarrow Y$ is a homeomorphism onto its image.*

PROOF. Since f is a continuous proper map from X onto $Z = f(X)$, it extends to a continuous map of the one-point compactifications

$$f^+ : X^+ \longrightarrow Z^+.$$

Since f^+ is a bijection from a compact space onto a Hausdorff space, it is a homeomorphism. Therefore f is a homeomorphism onto its image. $\square$

Corollary 2.1.24. *A proper injective immersion from a manifold M into a manifold N is an embedding.*

PROOF. An injective immersion which is a homeomorphism onto its image is an embedding. $\square$

Corollary 2.1.25. *Any continuous proper map $f : X \longrightarrow Y$ between locally compact Hausdorff spaces is a closed map.*

PROOF. Let C be any closed set in X, and $D = f(C)$. Then $C^+ = C \cup \{\infty\}$ is closed, and hence compact, in X^+. Since f^+ is continuous, $f^+(C^+)$ is compact, and hence closed, in Y^+. Therefore $f(C) = f^+(C^+) \cap Y$ is closed in Y. $\square$

2.2. Smooth approximations to continuous maps

Definition 2.2.1. Let M be a manifold, and N be a manifold with a metric ρ. Let $\delta : M \longrightarrow \mathbb{R}$ be a positive continuous function, and f, g be smooth maps from M to N. Then g is called a **δ-approximation** to f if

$$\rho(f(x), g(x)) < \delta(x),$$

for all $x \in M$.

The (fine) C^0**-topology** on the set $C^\infty(M,N)$ of smooth maps from M to N is a topology where the neighbourhood basis of $f \in C^\infty(M,N)$ comprises all sets of the form

$$B_0(f,\delta) = \{g \in C^\infty(M,N) \mid \rho(f(x),g(x)) < \delta(x)\}.$$

Thus g is a δ-approximation to f, if $g \in B_0(f,\delta)$.

The C^0 topology can be extended to the superset $C^0(M,N)$ of all continuous maps from M to N. This topology on $C^0(M,N)$ is larger than the compact open topology on $C^0(M,N)$. We shall prove this result in §8.2. We shall also see in there that the C^0 topology on $C^\infty(M,N)$ does not depend on the choice of the metric on N.

Lemma 2.2.2. *Let U be an open subset in $\mathbb{R}^n$ (or $\mathbb{R}^n_+$), and $f : U \longrightarrow \mathbb{R}$ be a continuous function such that f is smooth on an open set $V \subset U$. Let U' and V' be two other open sets in U such that $\overline{U}' \subset V'$, $\overline{V}' \subset U$, and $\overline{V}'$ is compact. Let $\delta : U \longrightarrow \mathbb{R}$ be a positive continuous function. Then there is a continuous function $g : U \longrightarrow \mathbb{R}$ such that g is smooth on $V \cup U'$, $g = f$ on $U - V'$, and $|g(x) - f(x)| < \delta(x)$ for all $x \in U$.*

The last condition means that g is a δ-approximation to f.

Proof. Let δ_0 be the positive minimum of the function δ on the compact set $\overline{V}'$. Then, by Weierstrass approximation theorem (see Dieudonné [**6**], (7.4.1), p. 139), there is a polynomial $p(x)$ such that

$$|p(x) - f(x)| < \delta_0 \text{ for } x \in \overline{V}'.$$

Let $h : U \longrightarrow \mathbb{R}$ be a smooth function such that $0 \le h \le 1$, $h = 1$ on $\overline{U}'$, and $h = 0$ on $U - V'$, as given by Lemma 2.1.17. Define $g : U \longrightarrow \mathbb{R}$ by

$$g(x) = h(x)p(x) + (1 - h(x))f(x),\ x \in U.$$

Then $g = f$ on $U - V'$, and $g = p$ on $\overline{U}'$. The last condition shows that g is smooth on U'. Also g is smooth on V, since p, h, f are smooth on it. Finally, on $\overline{V}'$ we have

$$|g(x) - f(x)| = |h(x)|\ |p(x) - f(x)| \ <\ \delta_0.$$

This completes the proof, since $g = f$ on $U - V'$. □

Theorem 2.2.3 (Smoothing theorem). *Let M and N be manifolds. Let K be a closed subset of M, and $f : M \longrightarrow N$ be a continuous map which is smooth on K. Then, there exist a positive continuous function $\delta : M \longrightarrow \mathbb{R}$, and a smooth map $g : M \longrightarrow N$ which agrees with f on K such that g is a δ-approximation to f.*

Remark 2.2.4. The possibility that $K = \emptyset$ is not ruled out.

PROOF. For each $x \in M$, let A_x be a coordinate neighbourhood of x in M, and B_x be a coordinate neighbourhood of $f(x)$ in N such that $f(A_x) \subset B_x$. Let $C_x \subset A_x$ be the compact closure of a neighbourhood of x. We shall show that it is possible to choose a countable collection of such C so that their interiors cover M, and any C intersects only a finite number of the other C's of the collection.

For this purpose, we construct, as in the proof of Theorem 2.1.1, a sequence of compact sets $\{K_j\}$ covering M such that $K_j \subset \text{Int } K_{j+1}$. Then the compact sets $L_j = K_j - \text{Int } K_{j-1}$ also cover M, and $L_j \cap L_m = \emptyset$ if $m \neq j-1, j$, or $j+1$. For each $x \in L_j$, we choose coordinate neighbourhoods A_x, B_x, and a compact neighbourhood $C_x \subset A_x$ as above. By shrinking C_x, if necessary, we may suppose that it does not intersect L_m for $m \neq j-1, j$, or $j+1$. Choose a finite number of such C's whose interiors cover L_j, and doing this for each j construct a sequence of sets $\{C_n\}$ such that the Int C_n cover M and any member of the sequence intersects only a finite number of other members. Let $\{A_n\}$ and $\{B_n\}$ be the corresponding sequences of A_x's and B_x's respectively.

Define a sequence of closed sets S_k inductively as follows. Take S_0 as the given closed set K, and then take $S_k = S_{k-1} \cup C_k$, for $k \geq 1$. Then M is the union of the interiors of the sets S_k. We shall construct inductively a sequence of maps $f_k : M \longrightarrow N$, $k \geq 0$, such that

(1) $f_k(x) = f_r(x)$ for $x \in S_r$, if $r < k$,

(2) f_k is smooth on S_k,

(3) $\rho(f_k(x), f(x)) < \delta(x)$, $x \in M$,

(4) f_k maps C_r into B_r for all k and r.

(Here ρ is a metric on N, and δ is a given positive continuous function on M which we shall adjust for completing the inductive step.)

Define $f_0 = f$, and suppose f_r has been defined for $r \leq k$ satisfying these conditions. Let us write $F = f_k$. Then, since F is smooth on S_k, it is smooth on an open neighbourhood V of S_k. Let $D = C_{k+1} - V \cap C_{k+1}$. Then by (4), $f_k = F$ maps D into B_{k+1}. Choose an open set W in C_{k+1} such that $D \subset W$, and $F(W)$ is contained in B_{k+1}. Since $S_k \subset V$, $D \cap S_k = \emptyset$. Therefore we can find open sets U', V', and U with $\overline{V}'$ compact such that

$$D \subset U',\ \overline{U}' \subset V',\ \overline{V}' \subset U,\ U \subset W,\ U \cap S_k = \emptyset,$$

and U intersects only a finite number of the sets C_r. Since B_{k+1} is a coordinate neighbourhood in N, the map $F|U : U \longrightarrow B_{k+1}$ is given by its components $F^{(i)} : U \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots, \dim N$. Then applying Lemma 2.2.2 to each component $F^{(i)}$, we get a map $F' : U \longrightarrow B_{k+1}$ such that $|F'^{(i)}(x) - F^{(i)}(x)| < \delta(x)$, $x \in U$, for each i, F' is smooth on $V \cup U'$, and $F' = F$ on $U - V'$. Define $f_{k+1} : M \longrightarrow N$ by

$$\begin{aligned} f_{k+1}(x) &= F(x) = f_k(x) \text{ if } x \notin U, \\ &= F'(x) \text{ if } x \in U. \end{aligned}$$

Then f_{k+1} satisfies (1), because if $x \in S_r \subset S_k$, $r \leq k$, then $x \notin U$ (as $U \cap S_k = \emptyset$), and so $f_{k+1}(x) = f_k(x) = f_r(x)$. Condition (2) holds, because $f_{k+1} = f_k$ is smooth on S_k, and F' is smooth on $V \cup U'$, which contains $(C_{k+1} \cap V) \cup D = C_{k+1}$. Condition (3) holds for f_k, and it will hold for f_{k+1} also, because

$$\rho(F'(x), F(x)) = \max\{|F'^{(i)}(x) - F^{(i)}(x)|\} < \delta(x),\ x \in U.$$

Condition (4) may be obtained by adjusting the size of δ; note that we need only to impose a finite number of restrictions on δ, since f_{k+1} differs from f_k only on V', and V' intersects only a finite number of the sets C_r.

Having constructed the sequence $\{f_k\}$, define $g : M \longrightarrow N$ by $g(x) = f_k(x)$ for $x \in S_k$. This gives g uniquely by (1), and g is smooth, since f_k is smooth on Int S_k and these sets cover M. Finally, $g(x) = f_0(x) = f(x)$ for $x \in S_0 = K$. □

2.3. Sard's theorem

Recall that the Lebesgue measure in $\mathbb{R}^n$ is given by a set function

$$\mu : \mathfrak{M} \longrightarrow [0, \infty]$$

satisfying certain axioms, where $\mathfrak{M}$ is a family of certain subsets of $\mathbb{R}^n$ that are called Lebesgue measurable sets. All open, closed, and compact subsets of $\mathbb{R}^n$ are Lebesgue measurable, so are all G_δ and F_σ subsets. We shall use the following properties of the Lebesgue measure: if S and T are Lebesgue measurable sets, then $\mu(S \cup T) \leq \mu(S) + \mu(T)$, and if $S \subset T$, then $\mu(S) \leq \mu(T)$.

An n-dimensional rectangle R in $\mathbb{R}^n$ is the Cartesian product of n intervals $I_1 \times \cdots \times I_n$; it is an n dimensional cube if all the intervals are of equal length. The Lebesgue measure of R is its volume vol(R) which is the product of the lengths of the n intervals. For an open set U in $\mathbb{R}^n$, $\mathrm{vol}(U) = \inf(\sum_i \mathrm{vol}(Q_i))$, where $\{Q_i\}$ is any sequence of n-dimensional cubes covering U.

A subset K of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$ if for any $\epsilon > 0$, K can be covered by a countable collection of n-dimensional cubes such that the sum of their volumes is less than ϵ. This definition may also be given in terms of rectangles, or even n-dimensional balls. We may say that K has measure zero if and only if for any $\epsilon > 0$ there is an open set U such that $K \subset U$ and $\mathrm{vol}(U) < \epsilon$. A countable union of sets of measure zero has measure zero. For, if $K = K_1 \cup K_2 \cup \cdots$, and $K_i \subset U_i$ where U_i is open and $\mathrm{vol}(U_i) < \epsilon/2^i$, then $K \subset U = \cup U_i$ and $\mathrm{vol}(U) \leq \sum_i \mathrm{vol}(U_i) < \sum \epsilon/2^i = \epsilon$.

The following lemma shows that the condition of being a set of measure zero is invariant under smooth map.

Lemma 2.3.1. *If a subset A of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$, and*

$$f : A \longrightarrow \mathbb{R}^m$$

is a smooth map, then $f(A)$ has measure zero in $\mathbb{R}^m$.

Proof. For each $p \in A$, f has a smooth extension on a neighbourhood of p in $\mathbb{R}^n$, which we still denote by f. By shrinking this neighbourhood, if necessary, we may suppose that f is smooth on a closed n-ball B centred at p. If $u_1, \ldots, u_n$ are the coordinate functions in $\mathbb{R}^n$, then the partial derivatives $\partial f_i / \partial u_j$ are bounded on the compact set B. Then by the fundamental theorem of calculus applied to each component f_i of f, together with the chain rule, we can find a constant c such that

$$\|f(x) - f(y)\| \leq c\|x - y\|$$

for all $x, y \in B$ (see Lemma 3.1.3 in Chapter 3). This is called the **Lipschitz estimate** for the smooth map f.

Now given an $\epsilon > 0$, take a countable covering $\{U_j\}$ of $A \cap B$ by open n-balls such that

$$\sum_j \mathrm{vol}(U_j) < \epsilon.$$

Then, by the Lipschitz estimate, $f(B \cap U_j)$ is contained in an n-ball V_j whose radius is not greater than c times the radius of U_j. It follows that $f(B \cap U_j)$ is contained in some of the balls of the collection $\{V_j\}$ whose total volume is not greater than $\sum_j \mathrm{vol}(V_j)$, which is less than $c^n \epsilon$. Since this can be made as small as we like, $f(A \cap B)$ has measure zero. Since $f(A)$ is a union of countably many such sets, it has also measure zero. □

Remark 2.3.2. The lemma may be false if f is only assumed to be continuous. For example, the subset $A = [0, 1]$ has measure zero in $\mathbb{R}^2$, but there exists a continuous map $f : A \longrightarrow \mathbb{R}^2$ whose image fills up the entire square $[0, 1] \times [0, 1]$, which is not a set of measure zero in $\mathbb{R}^2$.

This is the Hahn-Mazurkiewicz theorem which says that a topological space is a Peano space (i.e. a space which is compact, connected, locally connected,and metric) if and only if it is the image of the unit interval under a continuous map into a Hausdorff space (see [**15**], p. 129).

Theorem 2.3.3 (**Fubini**). *If K is a compact set in $\mathbb{R}^n$ such that each subset $K \cap (t \times \mathbb{R}^{n-1})$ has measure zero in the hyperplane $\mathbb{R}^{n-1}$, then K has measure zero in $\mathbb{R}^n$.*

Proof. This simple and elegant proof is due to Bredon [**3**]. We may assume that K is contained in the cube I^n, where I is the unit interval $[0, 1]$. Define a function $f : I \longrightarrow \mathbb{R}$ by

$$f(t) = \mu(K \cap ([0, t] \times I^{n-1})), \qquad t \in I,$$

where μ is the Lebesgue measure on $\mathbb{R}^n$. It is required to show that $f(1) = 0$. By hypothesis, given $\epsilon > 0$, there is an open set U in I^{n-1} such that

$$K \cap (t \times I^{n-1}) \subset t \times U \qquad \text{with } \mathrm{vol}(U) < \epsilon.$$

By compactness of K, there is a $h_0 > 0$ such that

$$K \cap ([t-h_0, t+h_0] \times I^{n-1}) \subset [t-h_0, t+h_0] \times U.$$

Then, for any h, $0 \leq h < h_0$,

$$K \cap ([0, t+h] \times I^{n-1}) \subset (K \cap ([0,t] \times I^{n-1})) \cup ([t, t+h] \times U)$$

can be covered by an open set of volume $< f(t) + \epsilon h$. Therefore

$$f(t+h) \leq f(t) + \epsilon h \qquad \text{for } 0 \leq h < h_0.$$

Similarly, we have

$$K \cap ([0,t] \times I^{n-1}) \subset (K \cap ([0, t-h] \times I^{n-1})) \cup ([t-h, t] \times U)$$

so that

$$f(t) \leq f(t-h) + \epsilon h \qquad \text{for } 0 \leq h < h_0.$$

Therefore

$$\left|\frac{f(t+h) - f(t)}{h}\right| \leq \epsilon, \qquad \text{for all } |h| < h_0.$$

Therefore f is differentiable at t and its derivative is zero. Since $f(0) = 0$, we have $f(1) = 0$ also. □

Definition 2.3.4. A subset K of an n-manifold M is said to have **measure zero** if for each coordinate chart $\phi : U \longrightarrow \mathbb{R}^n$ (or $\mathbb{R}^n_+$) of M, the set $\phi(U \cap K)$ has measure zero in $\mathbb{R}^n$.

It is clear that if $K' \subset K \subset M$ and K has measure zero in M, then K' has also measure zero in M. It is also clear that if $\{K_n\}$ is a countable family of subsets of M such that each K_n has measure zero in M, then $\cup_n K_n$ has measure zero in M.

Let M and N be manifolds of dimension n and m respectively. Then, in view of Definition 1.5.5, a point $x \in M$ is a critical point of a smooth map $f : M \longrightarrow N$, and $f(x)$ is a critical value of f, provided the Jacobian matrix $Jf(x)$ has rank $< m$. A point $y \in N$ is a regular value of f, if it is not a critical value of f.

By convention, any point of N which is not in $f(M)$ is a regular value of f.

Thus, if $n < m$, then every point of M is a critical point of f, and if $n \geq m$ and $y \in f(M)$ is a regular value of f, then $Jf(x)$ has rank m at every point x of $f^{-1}(y)$.

Theorem 2.3.5 (Sard). *If $f : M \longrightarrow N$ is a smooth map of manifolds and C is the set of critical points of f in M, then $f(C)$ has measure zero.*

Remark 2.3.6. A more general version of Sard's theorem says that if

$$f : M \longrightarrow N$$

is a C^r map, where $\dim M = n$, $\dim N = m$, and $r > \max(0, n-m)$, then the set of critical values of f has measure zero. The smoothness condition is necessary, and a counter example (due to Whitney [**57**]) is available, if the inequality be refuted. Whitney constructed a C^1 function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ whose set of critical points C is homeomorphic to the open interval $(0, 1) \subset \mathbb{R}$, and $f(C)$ is not a set of measure zero in $\mathbb{R}$. Here $r = \max(0, n - m)$. We will skip the proof of the general version, because Theorem 2.3.5 is adequate for our purpose.

Lemma 2.3.7. *If Sard's theorem is true for every smooth map*

$$f : U \longrightarrow \mathbb{R}^m,$$

where U is an open subset of $\mathbb{R}^n$, then it is also true for every smooth map $g : V \longrightarrow \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n_+$.

Proof. Let $g : V \longrightarrow \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n_+$ be a smooth map, and C be the set of critical points of g. By Lemma 2.1.14(b), there is an open subset V' of $\mathbb{R}^n$ and a smooth map $g' : V' \longrightarrow \mathbb{R}^m$ such that $V = V' \cap \mathbb{R}^n_+$ and $g'|V = g$. If C' is the set of critical points of g', then $g'(C')$ is a set of measure zero in $\mathbb{R}^m$, by hypothesis. This implies $g(C)$ is a set of measure zero in $\mathbb{R}^m$, because $g(C) = g'(C) \subset g'(C')$. □

Proof of Theorem 2.3.5. The proof is by induction on n which is the dimension of M. The starting point is $n = 0$ which is trivial. Therefore suppose that the theorem has been proved for all manifolds of dimensions $\leq n-1$. Next note using the Second Axiom of Countability that it suffices to consider only the special case when $f : U \longrightarrow \mathbb{R}^m$, U being an open set of $\mathbb{R}^n_+$, and C is the critical set of f in U. In view of Lemma 2.3.7, we may suppose that U is an open set in the interior of $\mathbb{R}^n_+$, or in $\mathbb{R}^n$.

Let D be the set of points in C where the Jacobian matrix $J(f)$ vanishes. We shall show in the next two Lemmas 2.3.8 and 2.3.9 that both $f(D)$ and $f(C-D)$ have measure zero in $\mathbb{R}^m$. This will complete the proof of the theorem. □

Lemma 2.3.8. *The set $f(D)$ has measure zero in $\mathbb{R}^m$.*

Proof. Let $f_1 : U \longrightarrow \mathbb{R}$ be the first component of f. Then, if Jf vanishes at a point x, Jf_1 also vanishes at x, and if K is the set of points where Jf_1 vanishes (K is also the set of critical points of f_1), then $f(D) \subset f_1(K) \times \mathbb{R}^{m-1}$. Therefore if $f_1(K)$ has measure zero in $\mathbb{R}$, then $f_1(K) \times \mathbb{R}^{m-1}$, and hence $f(D)$, has measure zero in $\mathbb{R}^m$, because $\mathbb{R}^{m-1}$ has measure zero in $\mathbb{R}^m$. Hence it is sufficient to prove the lemma for the case $m = 1$.

Let D_i be the set of points of U at which all the partial derivatives of f of order $\leq i$ vanish. We have then a descending sequence of closed subsets of U :

$$D = D_1 \supset D_2 \supset \cdots \supset D_n \supset \cdots.$$

We shall show in the next two sublemmas 1 and 2 that each of the sets

$$f(D_i - D_{i+1}),\ 1 \leq i < n, \text{ and } \ f(D_n)$$

has measure zero. This will complete the proof of the lemma. □

Sublemma 1. *The set $f(D_i - D_{i+1})$, $1 \leq i < n$, has measure zero.*

Proof. It suffices to show that each point p of $D_i - D_{i+1}$, has a neighbourhood V in U such that $f(V \cap (D_i - D_{i+1}))$ has measure zero. This will prove that $f(D_i - D_{i+1})$ has measure zero, because $D_i - D_{i+1}$ can be covered by countably many of such neighbourhoods, by the Second Axiom of Countability.

If $p \notin D_{i+1}$, there is an i-th order derivative of f, say g, which vanishes on D_i, but its Jacobian matrix Jg is non-zero at p. Then some partial derivative of g say $\partial g/\partial x_1$, is non-zero at p. Define a map $h : U \longrightarrow \mathbb{R}^n$ by $h(x) = (g(x), x_2, \ldots, x_n)$. The Jacobian of h is non-singular at p, and so h maps a neighbourhood V of p diffeomorphically onto an open set W of $\mathbb{R}^n$, by the inverse function theorem. The critical set of $f : V \longrightarrow \mathbb{R}$ is $V \cap (D_i - D_{i+1})$, since $J(f)$ vanishes on D_i. Therefore, since h^{-1} is a diffeomorphism, the critical set of the composition

$$k = f \circ h^{-1} : W \longrightarrow \mathbb{R}$$

is $h(V \cap (D_i - D_{i+1}))$. But $h(V \cap (D_i - D_{i+1})) = (0 \times \mathbb{R}^{n-1})$, and this set is also the critical set of the restriction $k' = k|(0 \times \mathbb{R}^{n-1}) \cap W$. Therefore, by induction (Sard's theorem is true for $n-1$),

$$k'((0 \times \mathbb{R}^{n-1}) \cap W) = f \circ h^{-1}((0 \times \mathbb{R}^{n-1}) \cap W) = f(V \cap (D_i - D_{i+1}))$$

has measure zero. □

Sublemma 2. *The set $f(D_n)$ has measure zero.*

Proof. Again, it will be enough to show that $f(D_n \cap Q)$ has measure for any n-cube Q in U. Let r be the edge length of Q, and k be a positive integer. Subdivide Q into k^n subcubes of edge length r/k, and hence of diameter $r\sqrt{n}/k$. Let $p \in D_n \cap Q$, and Q_1 be one of the subcubes containing p. By Taylor's theorem of order n, if $p + h \in Q_1$, then

$$|f(p+h) - f(p)| \leq A \cdot ||h||^{n+1} \leq A \cdot (r\sqrt{n}/k)^{n+1},$$

where A is a constant independent of k obtained as a uniform estimate of partial derivatives of f of order $n+1$. Therefore $f(D_n \cap Q_1)$ is contained in an interval of length B/k^{n+1}, where B is a constant independent of k. Hence $f(D_n \cap Q)$ is contained in a union of intervals of total length $\leq B \cdot k^n/k^{n+1} = B/k$. Since $\lim_{k\to\infty} B/k = 0$, $f(D_n \cap Q)$ has measure zero. □

Lemma 2.3.9. *The set $f(C - D)$ has measure zero in $\mathbb{R}^m$.*

Proof. Let $p \notin D$. Then some first order partial derivative of some component of f, say $\partial f_1/\partial x_1$, fails to vanish at p. As in the proof of Sublemma 1, an application of the inverse function theorem asserts that the map $h : U \longrightarrow \mathbb{R}^n$, where $h(x) = (f_1(x), x_2, \ldots, x_n)$, sends a neighbourhood V of p diffeomorphically onto an open set W of $\mathbb{R}^n$. Then the set C_1 of critical points of $g = f \circ h^{-1} : W \longrightarrow \mathbb{R}^m$ is precisely $h(V \cap C)$, and $g(C_1) = f(V \cap C)$.

Now g carries each $(t, x_2, \ldots, x_n) \in W$ into the hyperplane $t \times \mathbb{R}^{m-1} \subset \mathbb{R}^m$. Let $g_t : W \cap (t \times \mathbb{R}^{n-1}) \longrightarrow t \times \mathbb{R}^{m-1}$ be the restriction of g. Since the Jacobian $J(g)$ of g is of the form

$$\begin{pmatrix} 1 & 0 \\ * & J(g_t) \end{pmatrix},$$

a point in $t \times \mathbb{R}^{n-1}$ is a critical point of g_t if and only if it is a critical point of g. By inductive hypothesis, the set of critical values of g_t has measure zero in $t \times \mathbb{R}^{m-1}$. Now Fubini's theorem implies that the set of critical values of g, that is, the set $f(V \cap C)$, is of measure zero. □

Corollary 2.3.10. *If* $\dim M < \dim N$*, then a smooth map* $f : M \longrightarrow N$ *cannot be surjective.*

Proof. The critical set of f is M. Therefore, if f is onto, then $N = f(M)$ will have measure zero, which is not possible. □

Corollary 2.3.11. *If* $f : M \longrightarrow N$ *is a smooth map with set of critical points* C*, then the set* $N - f(C)$ *is dense in* N*.*

Proof. A set of measure zero cannot contain a non-empty open set. □

Corollary 2.3.12. *If* $f_i : M \longrightarrow N$ *is a countable family of smooth maps, then the set of common regular values of all* f_i *is dense in* N*.*

Proof. Any countable union of sets of measure zero has measure zero. □

2.4. Approximations by immersions

Lemma 2.4.1. *Let* U *be an open set in* $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$*, and* $f : U \longrightarrow \mathbb{R}^m$ *a smooth map, where* $m \geq 2n$*. Then, given any* $\epsilon > 0$*, there is an* $m \times n$ *matrix* $A = (a_{ij})$ *with* $|a_{ij}| \leq \epsilon$ *such the map* $g : U \longrightarrow \mathbb{R}^m$ *given by* $g(x) = f(x) + A \cdot x$ *(*x *written as* $n \times 1$ *column matrix) is an immersion.*

Proof. The Jacobian matrix of f at $x \in U$ is $Jf(x)$, and that of g at x is $Jg(x) = Jf(x) + A$. The problem is to choose an A so that $Jg(x)$ has rank n at any x, or equivalently, to choose an A from the complement of the set

$$\{B - Jf(x) \mid B \in M(m,n),\ \text{rank}\ B = k < n, x \in U\},$$

where $M(m,n)$ is the space of $m \times n$ matrices.

Let $M_k(m,n)$ denote the space of $m \times n$ matrices of rank k. For any $k < n$, define a map $F_k : M_k(m,n) \times U \longrightarrow M(m,n)$ by $F_k(B,x) = B - Jf(x)$. Then F_k is smooth, and the domain of F_k has dimension $k(m+n-k)+n$ (see Example 1.2.7). Now the function $\alpha(k) = k(m+n-k)+n$ is monotone increasing with k for $k < n$, since its derivative $\alpha'(k) = m+n-2k > 0$, if $k < n < m$. Therefore

$$k(n+m-k)+n \leq (n-1)(n+m-n+1)+n = nm-(m-2n)-1 < nm,$$

if $m \geq 2n$. By Sard's theorem, Image F_k has measure zero in $M(m,n)$. Therefore it is possible to find an $A \in M(m,n)$ as close to the zero matrix as we please so that A does not lie in Image F_k for any $k < n$. $\square$

Theorem 2.4.2. *Let $f : M \longrightarrow \mathbb{R}^m$ be a smooth map,* $\dim M = n$ *and $m \geq 2n$. Let $\delta : M \longrightarrow \mathbb{R}$ be a positive continuous function. Then f can be δ-approximated by an immersion $g : M \longrightarrow \mathbb{R}^m$. Moreover, if f is an immersion on a closed subset $K \subset M$, then g can be chosen so that $g(x) = f(x)$ for $x \in K$.*

Proof. If $f|K$ is an immersion, it is an immersion on an open neighbourhood U of K in M, by the local immersion theorem (Theorem 1.4.9). The open covering $\{U, M - K\}$ has a countable locally finite refinement $\{U_i\}$ such that each $\overline{U}_i$ is compact, and each U_i is a coordinate neighbourhood of a chart (U_i, ϕ_i). Re-index the sets U_i by positive and negative integers so that $U_i \subset U$ if and only if $i \leq 0$. By applying the shrinking lemma (Lemma 2.1.4) twice, construct open sets V_i and W_i such that $\{W_i\}$ is a covering of M, and

$$\overline{W}_i \subset V_i, \quad \overline{V}_i \subset U_i.$$

Let $\epsilon_i = \min \delta(x)$ for $x \in \overline{U}_i$.

We shall construct a sequence of smooth maps $f_k : M \longrightarrow \mathbb{R}^m$, $k \geq 0$, such that

(1) $f_0 = f$,

(2) $f_k = f_{k-1}$ on $M - \overline{V}_k$,

(3) f_k has rank n on the set $S_k = \cup_{r \leq k} \overline{W}_r$,

(4) $\|f_k(x) - f_{k-1}(x)\| < \epsilon_k / 2^k$, $x \in M$.

The proof of the theorem will follow, once such a sequence is constructed. Indeed, since the covering $\{U_i\}$ is locally finite, and (2) holds, the f_k's become equal on any compact set when k is sufficiently large. Thus the sequence $\{f_k\}$ converges to a smooth map g. By (2), g agrees with f on K, because, by our indexing convention, $U_k \subset M - U$ if $k \geq 1$, and therefore $f_k = f_{k-1}$ on U for all $k \geq 1$. Also, g has rank n everywhere on M, by (3), and it is a δ-approximation of f by (4).

We now proceed to construct the sequence by induction. Take $f_0 = f$, and suppose f_{k-1} has been defined satisfying the conditions. Since Condition (2) determines f_k outside $\overline{V}_k$, the problem of defining f_k lies entirely within $\overline{V}_k$, and so we may transform the problem to Euclidean space.

For any $m \times n$ matrix A, consider the map $F_A : \phi_k(U_k) \longrightarrow \mathbb{R}^m$ given by

$$F_A(x) = f_{k-1}\phi_k^{-1}(x) + \alpha(x)A(x) \ (x \text{ is an } n \times 1 \text{ matrix}),$$

where α is a smooth map $\mathbb{R}^n \longrightarrow [0,1]$ with

$$\alpha|\phi_k(\overline{W}_k) = 1 \ \text{ and } \ \alpha(\mathbb{R}^n - \phi_k(V_k)) = 0,$$

obtained from smooth Urysohn's lemma (Lemma 2.1.17). The map F_A will be used in the construction of f_k. For this purpose, we need to choose A in three ways in order to achieve Conditions (3) and (4) for f_k.

Firstly, we require rank F_A to be n on the set $R = \phi_k(S_{k-1} \cap \overline{V}_k)$. Now the Jacobian of F_A is

$$JF_A(x) = J(f_{k-1}\phi_k^{-1})(x) + A(x) \cdot J\alpha(x) + \alpha(x) \cdot A(x),$$

where $J\alpha$ is $1 \times n$ matrix. This gives a continuous map from $R \times M(m,n)$ to $M(m,n)$ sending (x, A) onto $JF_A(x)$. It maps $R \times (0)$ into the open subset $M_n(m,n)$ of $M(m,n)$. So if A is sufficiently small, this map will carry $R \times A$ into $M_n(m,n)$.

Secondly, by Lemma 2.4.1, we may choose A arbitrarily small so that

$$f_{k-1}\phi_k^{-1}(x) + \alpha(x)A(x)$$

has rank n on $\phi_k(U_k)$.

Finally, we choose A small enough so that $\|A(x)\| < \epsilon_k/2^k$ for all $x \in \phi_k(U_k)$.

Let A satisfy all these requirements. Then define $f_k : M \longrightarrow \mathbb{R}^m$ by

$$\begin{aligned} f_k(y) &= f_{k-1}(y) + \alpha(\phi_k(y)) \cdot A(\phi_k(y)) \text{ if } y \in U_k, \\ &= f_{k-1}(y) \text{ if } y \in M - \overline{V}_k. \end{aligned}$$

Two parts of the definition agree on the overlap $U_k - \overline{V}_k$, so f_k is smooth. Condition (2) follows from the definition. By the first choice of A, f_k has rank n on S_{k-1}, and by the second choice of A, f_k has rank n on $\overline{W}_k$, and so Condition (3) holds. Finally, the third choice of A ensures Condition (4) for f_k. This completes the proof. $\square$

Theorem 2.4.3. *If* $\dim M = n$*, and* $m > 2n$*, then any immersion*

$$f : M \longrightarrow \mathbb{R}^m$$

can be δ*-approximated by an injective immersion* $g : M \longrightarrow \mathbb{R}^m$ *for any positive continuous function* $\delta : M \longrightarrow \mathbb{R}$*. Moreover, if* f *is injective on an open neighbourhood* U *of a closed subset* K *of* M*, then we may choose* g *so that it agrees with* f *on* U*.*

Proof. For each $x \in M$, there is an open neighbourhood V_x such that $f|V_x$ is an embedding. The open covering $\{U \cap V_x, (M-K) \cap V_x\}_{x \in M}$ is a refinement of the covering $\{U, M-K\}$. Choose a partition of unity $\{\lambda_i\}$ subordinate to this refined covering, and re-index the λ_i by positive and negative integers so that $\text{supp}\lambda_i \subset U$ if and only if $i \leq 0$.

We shall construct $g : M \longrightarrow \mathbb{R}^m$ as the limit of an infinite series

$$f + v_1\lambda_1 + v_2\lambda_2 + \cdots + v_k\lambda_k + \cdots,$$

where $v_k \in \mathbb{R}^m$ will be chosen by induction. Suppose that $v_1, \ldots, v_k$ have been chosen so that

(1) the map $f_k = f + \sum_{i=1}^{k} v_i\lambda_i$ is an immersion,

(2) $\|f_r - f_{r-1}\| < \delta/2^r$ for $r \leq k$.

(The choice of v_1 will be clear from our arguments below, for which we must take $f_0 = f$.)

Since $\|f_{k+1} - f_k\| = \|v_{k+1}\lambda_{k+1}\| \leq \|v_{k+1}\|$, we may get (2) for $r = k+1$, simply by choosing $\|v_{k+1}\| \leq \delta/2^{k+1}$. The requirement that

$$f_{k+1} = f_k + v_{k+1}\lambda_{k+1}$$

be an immersion can also be met merely by taking $\|v_{k+1}\|$ sufficiently small by the arguments of Lemma 2.4.1. These choices of v_k show that the limit g is an immersion approximating f as required.

For the requirement that $g = f + \sum_{i=1}^{\infty} v_i\lambda_i$ be injective, we need to adjust the v_k still further. For this purpose, let W_{k+1} be the open set of $M \times M$ consisting of pairs (x_1, x_2) such that $\lambda_{k+1}(x_1) \neq \lambda_{k+1}(x_2)$. Let $\phi_{k+1} : W_{k+1} \longrightarrow \mathbb{R}^m$ be the map defined by

$$\phi_{k+1}(x_1, x_2) = \frac{f_k(x_2) - f_k(x_1)}{\lambda_{k+1}(x_1) - \lambda_{k+1}(x_2)}.$$

Since the map is smooth, and $\dim W_{k+1} = 2n < m$, the set $\phi_{k+1}(W_{k+1})$ has measure zero, by Sard's theorem. Therefore, it is possible to choose v_{k+1} arbitrarily small so that $v_{k+1} \notin \phi_{k+1}(W_{k+1})$. Suppose that v_{k+1} has been chosen in this way. Then

$$f_{k+1}(x_1) - f_{k+1}(x_2) = (f_k(x_1) - f_k(x_2)) + v_{k+1}(\lambda_{k+1}(x_1) - \lambda_{k+1}(x_2)).$$

Since $v_{k+1} \notin \phi_{k+1}(W_{k+1})$, it follows that $f_{k+1}(x_1) = f_{k+1}(x_2)$ if and only if $f_k(x_1) = f_k(x_2)$ and $\lambda_{k+1}(x_1) = \lambda_{k+1}(x_2)$.

Choosing all the v_k in this way, suppose $g(x_1) = g(x_2)$. Now, since $\{\lambda_i\}$ is a partition of unity, there is an r sufficient large such that $\lambda_k(x_1) = \lambda_k(x_2) = 0$ for $k > r$. Therefore

$$f_r(x_1) = g(x_1) = g(x_2) = f_r(x_2).$$

This implies by the above constructions of the v_k that $\lambda_r(x_1) = \lambda_r(x_2) = 0$, and $f_{r-1}(x_1) = f_{r-1}(x_2)$. Proceeding in this way in the direction of decreasing r, we arrive at $f(x_1) = f(x_2)$, and $\lambda_i(x_1) = \lambda_i(x_2) = 0$ for all $i > 0$. The last conditions imply that $x_1,\ x_2$ cannot belong to supp λ_i for all $i > 0$. But, by our indexing convention, supp $\lambda_i \subset M - U$ for all $i > 0$, Therefore $x_1,\ x_2$ must lie in U, on which f is injective. Therefore $x_1 = x_2$, showing that g is injective. □

2.5. Whitney's embedding theorem

We have now all the materials in hand to prove Whitney's theorem.

Theorem 2.5.1. *Any manifold M of dimension n can be embedded in $\mathbb{R}^{2n+1}$ as a closed subspace of $\mathbb{R}^{2n+1}$.*

PROOF. Consider the proper map $g = j \circ f : M \longrightarrow \mathbb{R}^{2n+1}$, where $f : M \longrightarrow \mathbb{R}$ is the proper function constructed in Lemma 2.1.21, and

$$j : \mathbb{R} \longrightarrow \mathbb{R}^{2n+1}$$

is the inclusion map. Recall that f is given in terms of a partition of unity $\{\lambda_i\}$ on M as $f(x) = \sum_{i=1}^{\infty} i\ \lambda_i(x)$. Then, taking $\delta(x) = 1/2$ and applying Theorems 2.4.2 and 2.4.3, we get an injective immersion $h : M \longrightarrow \mathbb{R}^{2n+1}$ such that

$$\|g(x) - h(x)\| < 1 \quad \text{for all } x \in M.$$

We will show that this h is also a proper map. This will imply that h is an embedding, by Corollary 2.1.24, and the proof will be complete.

It is sufficient to show that for each integer $k > 0$ the inverse image by h of the ball $B_k = \{u \in \mathbb{R}^{2n+1} | \|u\| \leq k\}$ is compact in M. By the above inequality, if $\|h(x)\| \leq k$, then

$$\|g(x)\| \leq \|g(x) - h(x)\| + \|h(x)\| < 1 + k,$$

which implies that $x \in \cup_{i=1}^{k+1} \mathrm{supp}\lambda_i$ (see the proof of Lemma 2.1.21). Thus $h^{-1}(B_k)$ is contained in a compact set, and so it is compact.

Finally, since h is proper, its image is a closed subset of $\mathbb{R}^{2n+1}$, by Corollary 2.1.25. □

Remark 2.5.2. The above proof by the method of approximation cannot be improved so as to get an embedding of an n-manifold M into an Euclidean space of dimension lower than $2n + 1$. For example, the immersion of S^1 into $\mathbb{R}^2$ for which the image crosses itself in the form of figure 8 cannot be approximated to any sufficiently close injective immersion whatsoever. In [**60**] Whitney employed a different method to prove that every n-manifold M can be embedded into $\mathbb{R}^{2n}$. He made deeper analysis on some homological conditions on M for removing double points. This result is best possible in the sense that there are manifolds of dimension n which cannot be embedded in $\mathbb{R}^{2n-1}$. For example, if $n = 2^r$, there is no embedding of the real projective space $\mathbb{R}P^n$ into $\mathbb{R}^{2n-1}$ (see Husemoller [**18**], Theorem 10.3, p. 262).

2.6. Homotopy of smooth maps

We will now extend the notion of homotopy to the smooth category. Two smooth maps are called smoothly homotopic if one can be deformed to the other through smooth maps. Here is the precise definition.

Definition 2.6.1. Two smooth maps $f, g : M \longrightarrow N$ are **smoothly homotopic** if there is a smooth map $H : M \times \mathbb{R} \longrightarrow N$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Thus we have a family of smooth maps $H_t : M \longrightarrow N$ given by $H_t(x) = H(x, t), t \in \mathbb{R}$. The smooth map H is called a **smooth homotopy** between f

and g. If H is just a continuous map, then f and g are continuously homotopic, or simply homotopic.

The smooth homotopy is defined for all $t \in \mathbb{R}$, rather than on the interval $I = [0,1]$, because we want to avoid a technical difficulty, namely, $M \times I$ is not a smooth manifold when M has boundary. In §7.5, we shall show that $M \times I$ can be given a unique smooth structure. Then we will have no problem in replacing $\mathbb{R}$ by I in the above definition.

The portion of $\mathbb{R}$ outside I does not play any important role. Given H as above, we can always find a smooth map $\overline{H} : M \times \mathbb{R} \longrightarrow N$ such that $\overline{H}(x,t) = f(x)$ if $t \leq 0$ and $\overline{H}(x,t) = g(x)$ if $t \geq 1$. Just define $\overline{H}(x,t) = H(x, \mathcal{B}(t))$, where $\mathcal{B}(t)$ is a bump function (Definition 2.1.6). The smooth map $\overline{H}$ is called the **normalised homotopy** corresponding to the homotopy H. Note that H and $\overline{H}$ are smoothly homotopic by $F : M \times \mathbb{R} \times \mathbb{R} \longrightarrow N$, where $F(x,s,t) = H(x, (1-s)\cdot t + s \cdot \mathcal{B}(t))$.

Lemma 2.6.2. *Smooth homotopy is an equivalence relation.*

Proof. That the relation is reflexive and symmetric are obvious. To see that it is transitive, take smooth maps f, g, and h from M to N, and let H and F be normalised smooth homotopies between f and g and between g and h respectively. Define $K : M \times \mathbb{R} \longrightarrow N$ by

$$\begin{aligned} K(x,t) &= H(x,3t) && \text{if } t \leq 1/2, \\ &= F(x,3t-2) && \text{if } t \geq 1/2. \end{aligned}$$

This is a smooth map, since H and F are smooth maps and $K(x,t) = g(x)$ for $1/3 \leq t \leq 2/3$ so that two parts of the definition match together smoothly. Clearly K is a normalised homotopy between f and h. $\square$

Lemma 2.6.3. *If two smooth maps $f, g : M \longrightarrow N$ are continuously homotopic, then they are smoothly homotopic.*

Proof. Let $H : M \times \mathbb{R} \longrightarrow N$ be a normalised continuous homotopy between f and g. Then H is smooth on the closed set $M \times J$, where $J = (-\infty, 0] \cup [1, \infty)$, since $H \mid M \times (-\infty, 0] = f$ and $H \mid M \times [1,\infty) = g$. By the smoothing theorem (Theorem 2.2.3), there is a positive continuous function δ on M such that H can be δ-approximated by a smooth map $F : M \times \mathbb{R} \longrightarrow N$ which agrees with H on $M \times J$. $\square$

$\Diamond$ **Exercise 2.2.** Show that if m is sufficiently large, then any smooth map

$$f : M \longrightarrow \mathbb{R}^m$$

is δ-approximable by an embedding $g : M \longrightarrow \mathbb{R}^m$ which is homotopic to f by a smooth homotopy $H_t : M \longrightarrow \mathbb{R}^m$ so that each H_t is a δ-approximation to f.

Hint. $H_t(x) = (1-t)f(x) + tg(x)$.

Definition 2.6.4. Two embeddings $f, g : M \longrightarrow N$ are **isotopic** if there exists a smooth homotopy $H : M \times \mathbb{R} \longrightarrow N$ such that for each $t \in \mathbb{R}$, the map

$$H_t : M \longrightarrow N$$

is an embedding.

Remark 2.6.5. If $H_t : M \longrightarrow N$ is an isotopy, and

$$\alpha : M_1 \longrightarrow M, \quad \beta : N \longrightarrow N_1$$

are embeddings, then $\beta \circ H_t \circ \alpha : M_1 \longrightarrow N_1$ is an embedding.

Proposition 2.6.6. *Any two embeddings $f, g : M \longrightarrow \mathbb{R}^m$ are isotopic, provided m is sufficiently large (in fact, $m \geq 2n + 2$, where $n = \dim M$).*

Proof. Since $\mathbb{R}^m$ is contractible to a point, the embeddings f and g are continuously homotopic, and hence homotopic by a smooth homotopy $H : M \times \mathbb{R} \longrightarrow \mathbb{R}^m$. If m is sufficiently large, H may be deformed to an embedding $F : M \times \mathbb{R} \longrightarrow \mathbb{R}^m$ which agrees with H on $(-\infty, 0] \cup [1, \infty)$. This F serves as the required isotopy between f and g. $\square$

2.7. Stability of smooth maps

We now consider a different set of problems, whose elegant formulation and presentation are influenced by Guillemin and Pollack [**12**]. Suppose, for example, an embedding f is deformed slightly to a map g; then we would like to pose the question whether g also an embedding.

Definition 2.7.1. Let $\mathcal{C}$ be a class of smooth maps from M to N defined by a property. Then $\mathcal{C}$ is called a **stable class** with respect to the property if for any $f \in \mathcal{C}$ and any smooth homotopy $f_t : M \longrightarrow N$ of f, there is an $\epsilon > 0$ such that $f_t \in \mathcal{C}$ for all $t < \epsilon$.

Theorem 2.7.2. *Each of the following classes of smooth maps from M to N, where M is compact and $\partial M = \partial N = \emptyset$, is a stable class:*

(1) *local diffeomorphisms,*
(2) *immersions,*
(3) *submersions,*
(4) *embeddings,*
(5) *diffeomorphisms.*

To this list of classes of maps, we may add one more class, namely, the class of maps transversal to a given submanifold A of N. We will read about this class of maps in Chapter 6, and show that locally the transversality condition is the same as the submersion condition (3). Therefore this class will be stable.

Proof. We shall prove only (2) and (4). Because, (1) is a special case of (2) when $\dim M = \dim N$, and the proof of (3) is essentially identical with the proof of (2). The proof of (5) will follow from (4) and the fact that a local diffeomorphism maps open sets into open sets.

Proof of (2). Let f_t be a smooth homotopy of an immersion f_0. Then the problem is to find an $\epsilon > 0$ so that $d(f_t)_x$ is injective for all points

$$(x, t) \in M \times [0, \epsilon) \subset M \times I.$$

Since M is compact, any open neighbourhood of $M \times \{0\}$ in $M \times I$ contains $M \times [0, \epsilon)$ if ϵ is small enough. Therefore, it is sufficient only to show that each point $(x_0, 0) \in M \times \{0\}$ has an open neighbourhood U in $M \times I$ such that $d(f_t)_x$ is injective for $(x, t) \in U$. Since this assertion is local, it is enough to consider only the case when M is an open subset of $\mathbb{R}^n$, and N is an open subset of $\mathbb{R}^m$.

Since $d(f_0)_{x_0}$ is injective, the Jacobian matrix $Jf_0(x_0)$ of f_0 at x_0 has a minor $R(x_0, 0)$ of order n whose determinant is non-zero. The function

$$M \times I \longrightarrow \mathbb{R},$$

which sends (x, t) to the determinant of the minor $R(x, t)$ of the Jacobian matrix $Jf_t(x)$ (formed by the same rows and columns as $R(x_0, 0)$) is continuous, since each entry of $R(x, t)$ is continuous on $M \times I$, and the determinant function is continuous. Therefore there is an open neighbourhood U of $(x_0, 0)$ in $M \times I$ such that $R(x, t)$ is non-singular for all $(x, t) \in U$. This completes the proof of (2).

Proof of (4). As shown in the above proof, f_0 is an immersion implies that f_t is an immersion for small values of t. We shall show that if f_0 is injective, then so is f_t for sufficiently small t. This will complete the proof of (4), because any injective immersion on a compact manifold is an embedding.

Suppose our assertion is false. Take a sequence of real numbers $\{t_k\}$ which converges to zero. For each k, we can find a pair of distinct points (x_k, y_k) of M such that $f_{t_k}(x_k) = f_{t_k}(y_k)$. Since M is compact, each of the sequences $\{x_k\}$ and $\{y_k\}$ has convergent subsequences. Denoting them by the same notations, let $\lim x_k = x_0$ and $\lim y_k = y_0$. Then

$$f_0(x_0) = \lim f_{t_k}(x_k) = \lim f_{t_k}(y_k) = f_0(y_0).$$

This implies that $x_0 = y_0$, since f_0 is injective.

Define a smooth map $G : M \times I \longrightarrow N \times I$ by $G(x, t) = (f_t(x), t)$ A simple computation shows that the Jacobian matrix $JG(x_0, 0)$ is

$$\begin{pmatrix} \boxed{Jf_0(x_0)} & * \\ 0, \cdots, 0 & 1 \end{pmatrix},$$

which is non-singular, since $Jf_0(x_0)$ is so. Then, by the inverse function theorem (Theorem 1.4.3), G is injective in a neighbourhood of $(x_0, 0)$. But for large

k, both (x_k, t_k) and (y_k, t_k) belong to this neighbourhood, and so $x_k = y_k$, which is a contradiction. Therefore we may conclude that f_t is injective when t is sufficiently small. $\square$

$\Diamond$ **Exercise 2.3.** Show that Theorem 2.7.2 is false if M is not compact, by constructing counterexamples to all the classes in the following way:

Define $f_t : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_t(s) = s\lambda(ts)$, where $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $\lambda(s) = 1$ if $|s| < 1$, and $\lambda(s) = 0$ if $|s| > 2$.

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