

Chapter 2

Topological Classification of Geodesic Flows on Revolution 2-Surfaces with Potential

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Abstract The paper is devoted to a short explanation of the topological classification (up to Liouville equivalence) of the integrable geodesic flows of two-dimensional surfaces of revolution with potential. The classification is given in the terms of so-called “marked molecules,” i.e., Fomenko–Zieschang invariants for integrable systems with two degrees of freedom on three-dimensional isoenergy surfaces.

2.1 “Atoms” and Morse Functions

The critical point is called *nondegenerate* if the second differential

$$d^2f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

is nondegenerate at this point. There are three possible types of nondegenerate critical points for functions on two-dimensional surfaces: maximum, minimum, and saddle (see Fig. 2.1).

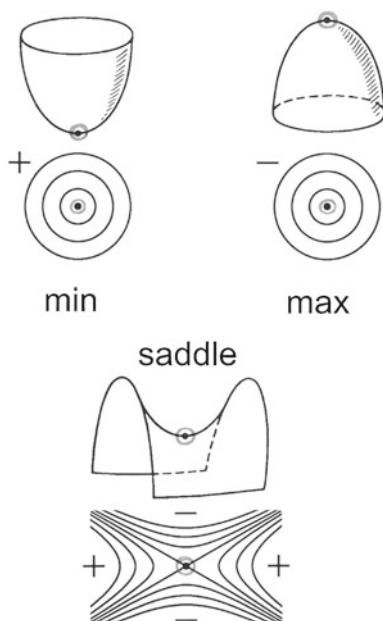
A smooth function is called a *Morse function* if all its critical points are nondegenerate. The Morse functions are everywhere dense in the space of all smooth functions on a smooth manifold.

By c we shall denote critical values of f , i.e., those in whose preimage there is at least one critical point. By arbitrary small perturbation, one can do so that, on every critical level c there is exactly one critical point. In other words, the critical points which occur in the same level can be moved close but in different levels. If each critical level $f^{-1}(c)$ contains exactly one critical point, then f is called *simple Morse function*.

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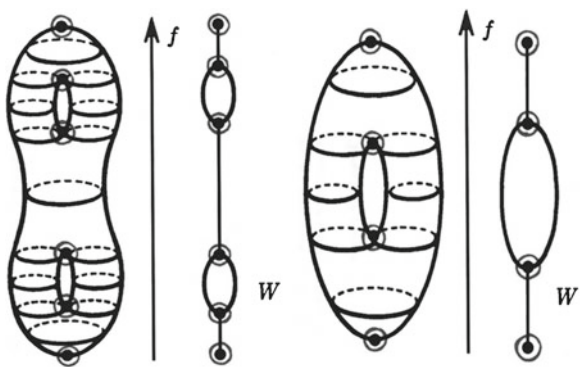
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Fig. 2.1 Types of critical points



Let f be a Morse function on a compact smooth manifold X^n . For any $a \in R$, consider the level surface $f^{-1}(a)$ and its connected components which can be called *fibers*. As a result, on the manifold there appears the structure of a foliation with singularities. By declaring each fiber to be a point and introducing the natural quotient topology in the space Γ of fibers, we obtain some quotient space. It can be considered as the base of the foliation. For Morse function, the space Γ is a finite graph. The graph is called the *Reeb graph* of the Morse function f on manifold X^n . Consider, for instance, the two-dimensional torus in R^3 embedded as shown in Fig. 2.2, and take the natural height function to be a Morse function on the torus.

Fig. 2.2 Reeb graphs for height functions



Then its Reeb graph has the form as shown in the same figure.

It is a natural problem to give the classification of Morse functions on two-dimensional surfaces up to the fiber equivalence. To solve it, first we need to study the local question, namely, to describe the local topological structure of singular fibers.

We begin with the informal definition. An *atom* is defined to be the topological type of a two-dimensional Morse singularity. In other words, this is the topological type of singular fiber of the foliation defined on a two-dimensional surface by a Morse function. More precisely, we can reformulate this as follows.

Definition 2.1 An *atom* is a neighborhood P^2 of a critical fiber (which is defined by inequality $c - \varepsilon \leq f(x) \leq c + \varepsilon$ for sufficiently small ε), foliated into level lines of f and considered up to the fiber equivalence. In other words, an atom is the germ of the foliation on a singular fiber.

The atom P^2 is called *simple*, if the Morse function f in the pair (P^2, f) is simple. The other atoms are called *complicated*. The *complexity* of an atom is a number of critical points on its critical level $f(x) = c$. The atom is called *orientable* (*oriented*) or *nonorientable* depending on whether the surface P^2 is orientable (*oriented*) or nonorientable.

First, consider a nonsingular level line which is close to a local maximum point. This line is a circle. As the regular value tends to the local maximum, the circle shrinks into a point (Fig. 2.3). Let us represent this evolution and the bifurcation in the following conventional, but in quite visual manner. Every regular level line (a circle) we represent as one point is located on the level a (Fig. 2.3). As a changes, this point moves running through a segment. At the moment, when the value of the function becomes critical (equal to c), a circle has shrunk into a point. Denote this event by the letter A with a segment going out of it. This segment is directed downwards. In the case of minimum, we proceed the similar way (Fig. 2.3).

If c is a critical saddle value, then the singular level line looks like a figure eight curve. As a tends to c , two circles are getting closer and, finally, touch at a point. After this, the level line bifurcation happens and, instead of two, we obtain just one circle, see Fig. 2.4.

Let f be a simple Morse function on a compact closed surface X^2 (orientable or nonorientable). Consider Reeb graph Γ . The vertices of Γ correspond to critical fibers of f . Let us replace these vertices by corresponding atoms (either A , or B , or \tilde{B} , which is nonorientable version of B). The graph obtained is called a *simple molecule* W . In fact, the notion of the simple molecule does not differ yet from that of the Reeb graph. However, for complicated Morse functions the molecule W will carry more information than the Reeb graph Γ (Fig. 2.5).

A minimal simple Morse function on the pretzel, i.e., on the sphere with two handles, is realized as the height function on the embedding of the pretzel, presented in Fig. 2.8. The corresponding simple molecule is also shown here.

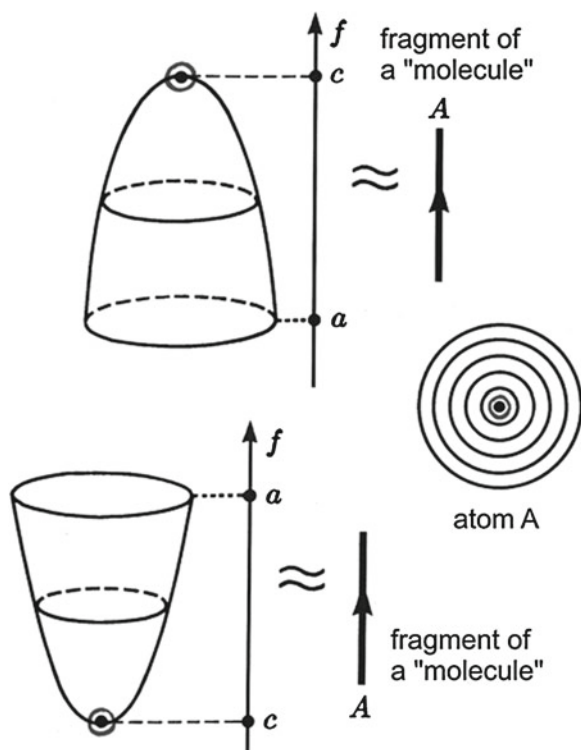
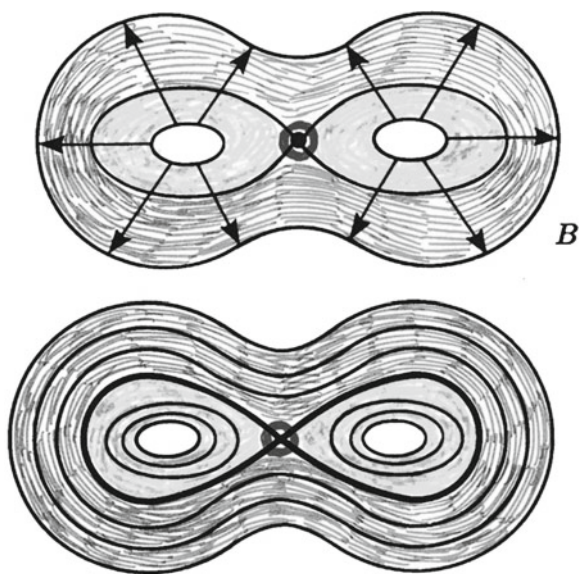
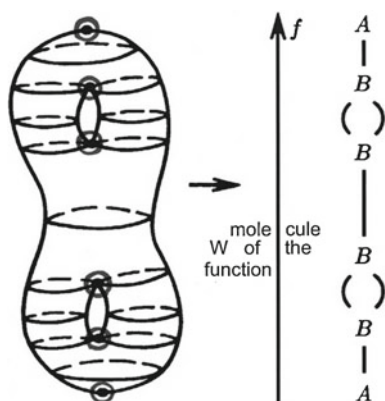
Fig. 2.3 Atom A**Fig. 2.4** Atom B. Level curves of a simple function f are shown

Fig. 2.5 Minimal simple Morse function on the pretzel and its simple molecule



2.2 Complicated Atoms and Molecules

Recall that an atom is complicated if critical connected level surface of function f contains several critical points. Such objects naturally arise in many problems in geometry and physics (see Fig. 2.6).

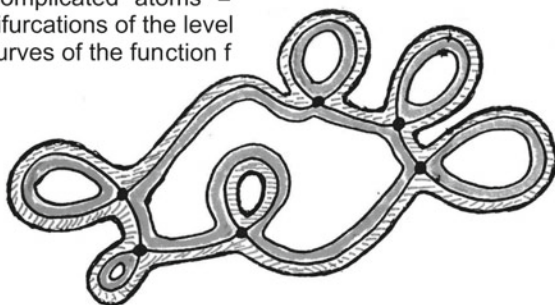
We now give a simple example. Suppose that a finite group G acts smoothly on a surface X^2 , and let f be a G -invariant Morse function; then, as a rule, such function will be complicated. Indeed, if, for instance, the orbit of a critical point x entirely belongs to a connected component of the level line $\{f(x) = \text{const}\}$, then this level contains several critical points.

Of course, a small perturbation can make the function simple by moving critical points into different levels. However, this destroys the symmetry. Thus, in the problems that require studying symmetries of different kinds, one has to investigate complicated Morse functions as an independent object. The molecules W classify Morse functions on M^2 up to fiber equivalence, see [1].

It is convenient to denote every atom (P^2, K) by some letter with number of incoming and outgoing edges. The end of each edge corresponds to a certain boundary

Fig. 2.6 Complicated Morse functions

Complicated "atoms" = bifurcations of the level curves of the function f



circle of the surface P . It is important to emphasize that, generally speaking, the ends of an atom (P, K) are not equivalent; because the boundary circles of the surface P are not equivalent in the sense that not every two of them can be matched by a homeomorphism of the pair (P, K) onto itself. Some atoms of low complexity (both orientable and nonorientable) are listed in Fig. 2.7. In the same table, one can see the corresponding pairs of f -graphs, as well as the surface \tilde{P} obtained from P by gluing disks to all of its boundary circles. See details in [1].

2.3 Topology of Integrable Hamiltonian Systems with Two Degrees of Freedom

Consider symplectic manifold M^{2n} with closed nondegenerate skew-symmetric two-form ω . It defines on the space of smooth functions on M^{2n} Poisson bracket

$$\{f, g\} = \omega^{-1}(df, dg).$$

The system of ordinary differential equations defined by vector field v is called *Hamiltonian system* if there exists such function H (called *Hamiltonian*) that for any function g on M^{2n} the equality holds $v(g) = \{g, H\}$. We denote $v = \text{sgrad}H$.

The system is integrable in Liouville sense, if there exists exactly n functionally independent commuting integrals f_1, \dots, f_n , and all vector fields $\text{sgrad}f_i$ are complete. They define the so-called Liouville foliation (see Fig. 2.8). The Liouville theorem states, that if the regular (i.e., f_i are functionally independent on it) common level surface of these functions is compact and connected, then it is a torus T^n . The solution (integral trajectory) in general case determines almost periodic motion on this torus.

This class of Hamiltonian systems contains many important examples from physics and classical mechanics: the different cases of motion of rigid body (Euler case, Lagrange top), geodesic flow on ellipsoid, interaction of the material points, located on the line or on the circle S^1 . The classification of such systems is a very difficult task. It turns out, however, that in case of $n = 2$ the full topological classification exists.

Consider a symplectic manifold M^4 with an integrable Hamiltonian system $v = \text{sgrad}H$; let Q_h^3 be a nonsingular compact connected isoenergy 3-surface in M^4 . Let f be an additional integral of the system v that is independent of H . We denote its restriction to Q_h^3 by the same letter f . It is assumed to be a Bott function (nondegenerate, see [1]) on Q_h^3 . Our aim is to investigate the topology of the Liouville foliation on Q_h^3 defined by the given integrable system. Its nonsingular leaves are Liouville tori, and the singular ones correspond to critical levels of the integral f on Q_h^3 .

Now consider a topologically stable (see [1]) integrable system with Bott integral f on an isoenergy 3-surface Q_h^3 and take some singular leaf L of the corresponding Liou-

No	ATOM	f -GRAPHS	CODE	GENUS
Complexity 1, orientable				
1			$-A$	S^2
2			$-B \leq$	S^2
Complexity 1, non-orientable				
$\tilde{1}$			$-\tilde{B}-$	$\mathbb{R}P^2$
Complexity 2, orientable				
1			$-C_1-$	T^2
2			$\geq C_2 \leq$	S^2
3			$-D_1 \equiv$	S^2

Fig. 2.7 Several atoms of low complexity

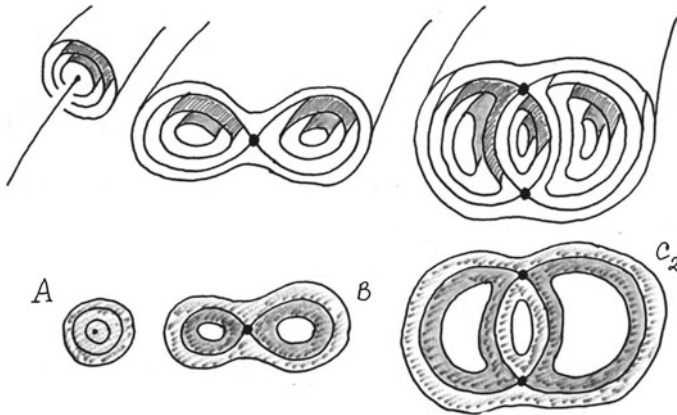


Fig. 2.8 The two-dimensional “cross-section” of Liouville foliation in three-dimensional invariant submanifold around singular leaf

ville foliation on Q_h^3 . Consider a neighborhood of this leaf, i.e., a three-dimensional manifold $U(L)$ with the Liouville foliation structure and fixed orientation. By analogy with the two-dimensional case, as neighborhood $U(L)$, we take the connected component of the set $c - \varepsilon \leq f(x) \leq c + \varepsilon$ that contains the singular leaf L (same as in previous section $f(L) = c$ is a critical value of f). Such an object is naturally called a *3-atom*. However, from the formal viewpoint, we have to be more careful. We shall assume two such 3-manifolds $U(L)$ and $U'(L)$ with the structure of the Liouville foliation to be fiberwise equivalent if

- (1) there exists a diffeomorphism between them that maps the leaves of the first Liouville foliation into those of the second one,
- (2) this diffeomorphism preserves both the orientation on 3-manifolds and the orientation on the critical circles defined by the Hamiltonian flows.

Definition 2.2 The equivalence class of the three-dimensional manifold $U(L)$ is called a *3-atom*. The number of critical circles in the 3-atom is called its *atomic weight* or *complexity* (for details, see [10, 11]).

Consider 3-atom $U(L)$ with the structure of a Seifert fibration on it. Let

$$\pi : U(L) \rightarrow P^2$$

denote its projection onto a two-dimensional base P^2 with the embedded graph $K = \pi(L)$. Let us mark those points on the base P^2 , into which the singular fibers of the Seifert fibration (i.e., the fibers of type $(2, 1)$) are projected. Recall that the base P^2 has a canonical orientation. The point is that an orientation is already fixed on $U(L)$, as well as on the fibers of the Seifert fibration. It is clear that, as a result, we obtain some oriented 2-atom (P^2, K) .

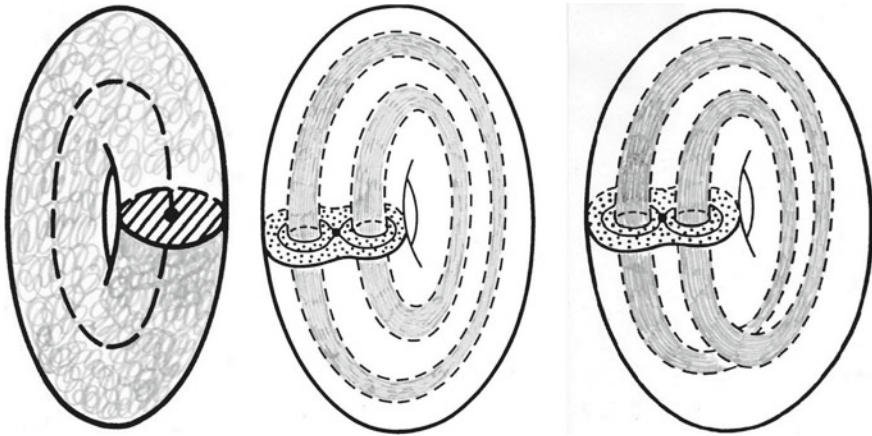


Fig. 2.9 3-atoms A , B and B^*

Theorem 2.1 (Fomenko). *Under the projection $\pi : U(L) \rightarrow P^2$, the 3-atom $U(L)$ turns into the 2-atom (P^2, K) ; and moreover, the singular fibers of the Seifert fibration on the 3-atom are in one-to-one correspondence with the star-vertices of the 2-atom. This correspondence between 2-atoms and 3-atoms is bijection.*

The example of 3-atom is shown in the Fig. 2.9.

Let us describe 3-atom A (see Fig. 2.9). This 3-atom is presented as a solid torus foliated into concentric tori, shrinking into the axis of the solid torus. In other words, the 3-atom A is the direct product of a circle and a disk foliated into concentric circles. From the viewpoint of the corresponding dynamical system, A is a neighborhood of a stable periodic orbit. The examples of saddle 3-atoms are presented in the Fig. 2.9.

Now we make more precise the definition of the Liouville equivalence for integrable Hamiltonian systems. From now on, we shall assume that two Liouville foliations are Liouville equivalent if and only if there exists a diffeomorphism that sends the leaves of the first foliation to those of the second one and satisfies two conditions related to the orientation. Namely, it preserves the orientation of 3-manifolds Q_h^3 and Q_h^{3*} ; and moreover, it also preserves the orientation on the critical circles given by the Hamiltonian flows. The molecule W contains a lot of essential information on the structure of the Liouville foliation on Q_h^3 . However, this information is not quite complete. Indeed, the molecule of the form $A-A$, for example, informs us that the manifold Q_h^3 is glued from two solid tori foliated into concentric tori in a natural way. However, it does not tell us how this gluing is made, and what three-dimensional manifold is obtained as a result. Therefore, we have to add some additional information to the molecule W , namely, the rules that clarify how to glue the isoenergy surface Q_h^3 from individual 3-atoms. As it was discovered by A.T. Fomenko and H. Zieschang, the molecule W , which corresponds to integrable Hamiltonian system, can be endowed by some numerical marks in such a way, that these *marked molecules* W^* will classify such a systems up to Liouville equivalence. In short, the

marked molecule W^* is molecule W , equipped with three sets of numbers r_i , ε_i and n_k called marks, see [1].

Theorem 2.2 (Fomenko, Zieschang). *Two integrable systems (v, Q_h^3) and $(v', Q_h'^3)$ are Liouville equivalent if and only if their marked molecule W^* and $W^{*'}$ coincide.*

The marks cannot be chosen arbitrary, as there are several important conditions on them. Any marked molecule W^* with marks, which satisfy these conditions, is called *abstract marked molecule*.

Theorem 2.3 *Any abstract marked molecule W^* is realized as a marked molecule of some integrable Hamiltonian system.*

Corollary 2.1 (1) *There exist a one-to-one correspondence between the Liouville equivalence classes of integrable systems and marked molecules. In particular, the set of Liouville equivalence classes of integrable systems is discrete (countable) and has no continuous parameters.*

(2) *There exists an enumeration algorithm for marked molecules (i.e., classes of integrable systems).*

(3) *There exists an algorithm for comparison of marked molecules, i.e., the algorithm that gives answer to the question whether two integrable systems corresponding to given molecule are Liouville equivalent or not.*

2.4 Geodesic Flows with Potential on the Surfaces of Revolution

Consider a manifold M , which is diffeomorphic to a cross product of finite interval (a, b) and circle S^1 . Let us define a metrics $ds^2 = dr^2 + f^2(r)d\varphi^2$ on M , where r is a natural parameter on $f(r)$. $f(r)$ is a smooth positive function on (a, b) , and $f(a) = f(b) = 0$; and, moreover, we assume that a surface of revolution obtained by rotation of this function around the axis Oz is a smooth manifold. φ is a conventional angle coordinate on a circle S^1 . Let $V(r)$ be a smooth function on the surface of revolution, obtained by the rotation of $f(r)$. It plays a role of a *potential* of the system.

Definition 2.3 We call a system, defined by pair $(f(r), V(r))$, a *system on the surface of revolution*.

The phase space of this system has a dimension of four: it has coordinates $(r, \varphi, p_r, p_\varphi)$.

Lemma 2.1 *The system given by a pair $(f(r), V(r))$ is integrable Hamiltonian system on T^*M (on the cotangent bundle of M) for any smooth potential function $V(r)$.*

This statement is clear since the investigated system has two integrals of motion: H — energy, and p_φ — linear first integral, the mechanical sense of the later being the projection of kinetic moment on axis of rotation. Hence, according to the definition of Liouville integrability (see [1–3, 12]), this system is integrable.

The Hamilton function of this system has a form

$$H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2f^2(r)} + V(r). \quad (2.1)$$

We can rewrite the function H in the form

$$H = g^{ij}(r)p_r p_\varphi + V(r). \quad (2.2)$$

According to a famous Maupertuis principle, there exists a value of energy $H = h_0 > \max(V(r))$, such that for any $h > h_0$ the integral trajectories of the vector field $v = \text{sgrad } H$ on the isoenergetic level $Q^3 = \{H(r, \varphi, p_r, p_\varphi) = h\}$ coincide with the integral trajectories of the vector field $\tilde{v} = \text{sgrad } \tilde{H}$, where the Hamiltonian \tilde{H} has a form

$$\tilde{H} = \frac{g^{ij}(r)}{h - V(r)} p_r p_\varphi. \quad (2.3)$$

It is clear that the vector field \tilde{v} defines a *geodesic flow* of a Riemannian metrics \tilde{g}^{ij} on the manifold T^*M , where $\tilde{g}^{ij}(r) = (h - V(r))g^{ij}(r)$.

So, we can consider the system on the surface of revolution as a geodesic flow with the potential on the surface of revolution.

2.5 The Case of Gravitational Potential: Topological Classification

Now let us examine the systems on the surfaces of revolution, which are defined by pairs $(f(r), V(r))$, where the function $f(r)$ gives us a smooth surface of revolution on $(0, \pi)$, and the potential function $V(r) = \cos r$, which means that we have the action of the gravitational field in such systems.

In this chapter, we want to give the topological classification of such systems. For this purpose, it is necessary to define some useful notions.

Definition 2.4 We will call the map

$$\Phi : M \rightarrow R^2 : (r, \varphi, p_r, p_\varphi) \mapsto (H(r, \varphi, p_r, p_\varphi), p_\varphi(r, \varphi, p_r, p_\varphi)) \quad (2.4)$$

the *momentum map*.

Definition 2.5 If the rank of $d\Phi(x)$ is less than 2, then x is a *critical point* of the momentum map, and $\Phi(x)$ is its *critical value*. A set of critical values $\Sigma = \{\xi = \Phi(x), x \text{ is a critical point}\}$ is called a *bifurcation diagram*.

For more details, see [7–9].

Let us define an *effective potential* function:

$$U_{\text{eff}}(p_\varphi, r) = \frac{p_\varphi^2}{2f^2(r)} + V(r). \quad (2.5)$$

Assume that the following conditions are satisfied:

1. In all points r_i , which are the solutions of the system

$$\begin{cases} \frac{\partial U_{\text{eff}}(r, p_\varphi)}{\partial r} = 0 \\ \frac{\partial^2 U_{\text{eff}}(r, p_\varphi)}{\partial r^2} = 0 \end{cases} \quad (2.6)$$

we have

$$\frac{\partial^3 U_{\text{eff}}(r_i, p_\varphi)}{\partial r^3} \neq 0 \quad (2.7)$$

(it is a condition of the existence of semicubical point of return on the bifurcation diagram);

2. In all points r_i where the function $U_{\text{eff}}(r, p_\varphi)$ has a local minima, all the values of $U_{\text{eff}}(r, p_\varphi)$ are different for the fixed p_φ . (this condition provides the existence of atoms of only two types A and B in the system).

Theorem 2.4 *If the system satisfies the conditions above, then the bifurcation diagram of this system is constructed of the curves of three types:*

- (i) The curve of a “parabole” type (see Fig. 2.10)
- (ii) Two points of the rank 0 with coordinates $(H, p_\varphi) = (\pm 1, 0)$. The point $(-1, 0)$ has a center–center type, the point $(1, 0)$ has a focus–focus type (see Fig. 2.11)
- (iii) The curve of a “beak” type (see Fig. 2.11: a, b, c particular cases, d the general case)

And besides, the bifurcation diagram may consist of only one curve of type (i) and of any number of the curves of type (iii).

Example 2.1 Let us consider a function $f(r) = \sin r$ (this system called a “spherical pendulum system” (this system was studied in [13])). In this case, the bifurcation diagram has a very simple form: it has no curves of type (iii). The function $f(r)$ and the bifurcation diagram are shown in the Fig. 2.12.

Example 2.2 Consider a function $f(r)$, which gives us a surface with two local maxima at the points a and b (see Fig. 2.13a). In this case, the bifurcation diagram also has the curves of type (iii) (see Fig. 2.13b)).

With the help of the bifurcation diagrams, we can construct the molecules. All the molecules consist of atoms of only two types A and B .

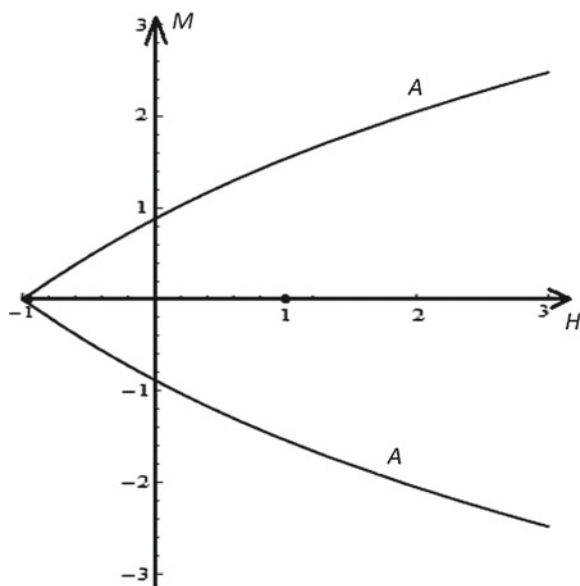


Fig. 2.10 The curve of a “parabole” type and critical points of rank 0

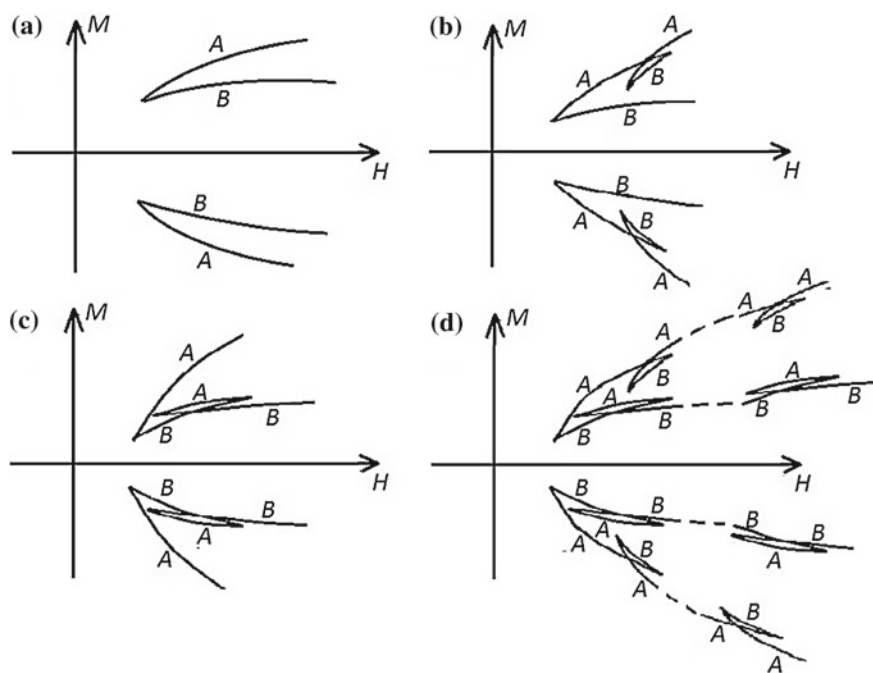


Fig. 2.11 Curves of a “beak” type

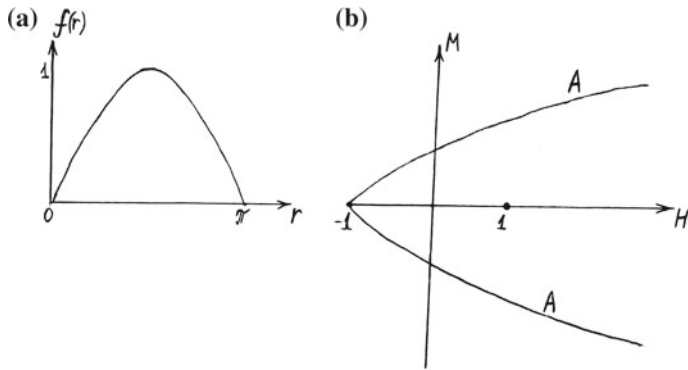


Fig. 2.12 The bifurcation diagram for “spherical pendulum” system

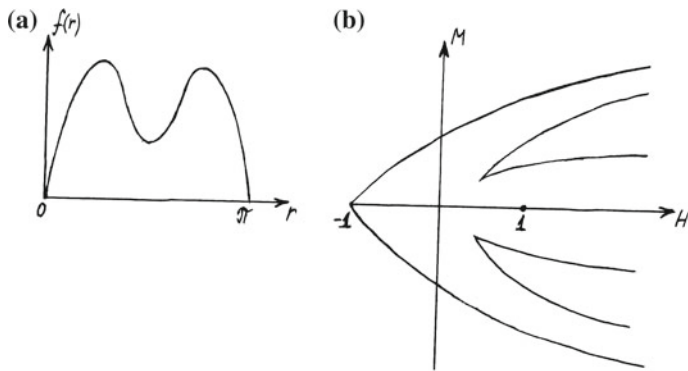


Fig. 2.13 The bifurcation diagram for the system with $f(r) = \sin r - a \cdot \sin^3 r$ and $V(r) = \cos r$

2.6 Topological Equivalence Between Different Integrable Systems

Now we continue to study the equivalence in the sense of Liouville. As soon as we have constructed the molecules, we have some information about the Liouville fibration of the system. But this information is not full (see Chap. 3). Let us consider an arbitrary rib of the molecule. In the preimage of any regular point from this rib, we have one regular torus in T^*M . So we can choose two regular tori, which correspond to the regular points of the rib near both edges of this rib.

Then, we choose a basis on each of these two tori, and we deform these bases toward each other. As a result, we obtain two bases on the torus, which correspond to the central (regular) point of the rib. We can write the transition matrix from one coordinate system to another, and this matrix gives us information about the way of gluing two tori in the ends of the rib (see Fig. 2.14).

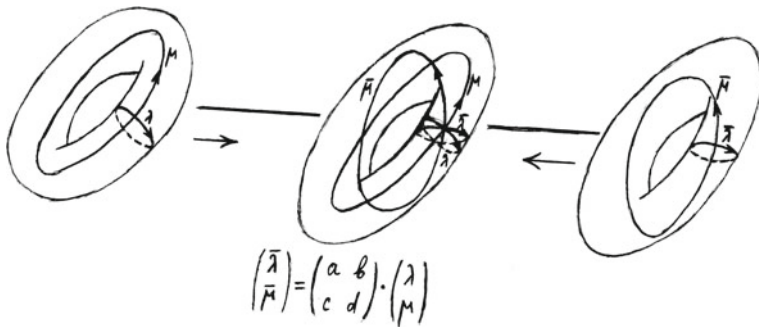


Fig. 2.14 The way of gluing two tori defined by matrix

We can choose the basis on a torus in different ways, therefore gluing matrix depends on the coordinate systems. But we can calculate some numbers given by this matrix, which do not depend on the coordinate system. These invariants are called the *marks* on the ribs of the molecule. The molecule with the marks is called a *Fomenko–Zieschang invariant* (or simply a *marked molecule*).

Theorem 2 says that the Fomenko–Zieschang invariant is the invariant of topological equivalence of the system. Namely, if two integrable Hamiltonian systems, defined by their Hamiltonian vector fields $v = \text{sgrad} H$ and $\tilde{v} = \text{sgrad} \tilde{H}$, have the same Fomenko–Zieschang invariants on the isoenergetic surfaces Q^3 and \tilde{Q}^3 , these systems are equivalent in the Liouville sense.

Hence, it is reasonable to calculate the marked molecules of integrable systems.

There exist three types of marks: r , ε and n . The mark r is defined by the rule

$$r = \begin{cases} \frac{a}{b} \bmod 1 \in Q/Z, & b \neq 0 \\ \infty, & b = 0 \end{cases} \quad (2.8)$$

The mark ε can be calculated in the following way:

$$\varepsilon_i = \begin{cases} \text{sign } b, & b \neq 0 \\ \text{sign } b, & b = 0 \end{cases} \quad (2.9)$$

The mark n has a more complicated definition (see [4–6, 14]).

Theorem 2.5 *If all the conditions of the Theorem 4 are satisfied, then the marked molecules of the systems with gravitational potential consist of the ribs of the following types: (a) the rib A–A with the mark $r = 0$, if $Q^3 = \{H = h < 1\}$; and with the mark $r = 1/2$, if $Q^3 = \{H = h > 1\}$. The mark $\varepsilon = +1$ in both cases;*

(b) the rib A–B with the mark $r = 0$ and the mark $\varepsilon = +1$;

(c) the rib B–B with the mark $r = \infty$ and the mark $\varepsilon = -1$, if the rib is symmetric relative to the axis OH; and the mark $\varepsilon = +1$ in other case;

(d) if the system admits the atoms of type B, then there exist marks of type n . If

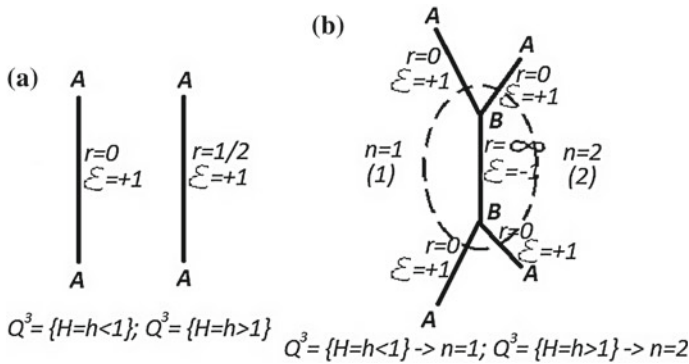


Fig. 2.15 The marked molecules for the systems from the examples 1 and 2

$Q^3 = \{H = h < 1\}$, then the mark $n = 1$, and if $Q^3 = \{H = h > 1\}$, the mark $n = 2$.

The marked molecules for the examples 1 and 2 are shown in Fig. 2.15.

Corollary 2.2 *The molecule of the example 2 (when the energy $H > 1$) coincides with a molecule in the classical integrable Hamiltonian system called Zhukovskii system. Almost all marks on the ribs of these molecules coincide except for one mark ε on the rib $B-B$. So, the systems are different in the Liouville sense, but their invariants almost coincide.*

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