Chapter 2
Linear Control Systems with Periodic Convex Integrands

We study the structure of approximate optimal trajectories of linear control systems with periodic convex integrands and show that these systems possess a turnpike property. To have this property means, roughly speaking, that the approximate optimal trajectories are determined mainly by the integrand, and are essentially independent of the choice of time interval and data, except in regions close to the endpoints of the time interval. We also show the stability of the turnpike phenomenon under small perturbations of integrands and study the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals.

2.1 Preliminaries and Turnpike Results

In this chapter we study the structure of approximate optimal trajectories of linear control systems

$$x'(t) = Ax(t) + Bu(t), \tag{2.1}$$

$$x(0) = x_0$$

with periodic convex integrands $f : [0, \infty) \times R^n \times R^m \to R^1$, where A and B are given matrices of dimensions $n \times n$ and $n \times m$, $x(t) \in R^n$, $u(t) \in R^m$ and the admissible controls are Lebesgue measurable functions.

We assume that the linear system (2.1) is controllable and that the integrand f is a Borel measurable function.

We denote by $|\cdot|$ the Euclidean norm and by $\langle \cdot, \cdot \rangle$ the inner product in the n-dimensional Euclidean space R^n. Denote by $\mathbf{Z}$ the set of all integers. For every

A.J. Zaslavski, *Turnpike Theory of Continuous-Time Linear Optimal Control Problems*, Springer Optimization and Its Applications 104,
DOI 10.1007/978-3-319-19141-6_2

$z \in R^1$ denote by $\lfloor z \rfloor$ the largest integer which does not exceed z: $\lfloor z \rfloor = \max\{i \in \mathbf{Z} : i \le z\}$.

The performance of the above control system is measured on any finite interval $[T_1, T_2] \subset [0, \infty)$ by the integral functional

$$I^f(T_1, T_2, x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt. \tag{2.2}$$

We suppose that the integrand $f : [0, \infty) \times R^n \times R^m \to R^1$ satisfies the following

Assumption (A)

(i) $f(t + \tau, x, u) = f(t, x, u)$ for all $t \in [0, \infty)$, all $x \in R^n$ and all $u \in R^m$ for some constant $\tau > 0$ depending only on f;
(ii) for any $t \in [0, \infty)$ the function $f(t, \cdot, \cdot) : R^n \times R^m \to R^1$ is strictly convex;
(iii) the function f is bounded on any bounded subset of $[0, \infty) \times R^n \times R^m$;
(iv) $f(t, x, u) \to \infty$ as $|x| \to \infty$ uniformly in $(t, u) \in [0, \infty) \times R^m$;
(v) $f(t, x, u)|u|^{-1} \to \infty$ as $|u| \to \infty$ uniformly in $(t, x) \in [0, \infty) \times R^n$.

Assumption (A) implies that f is bounded below on $[0, \infty) \times R^n \times R^m$.

Let $T_2 > T_1 \ge 0$. A pair of an absolutely continuous (a.c.) function $x : [T_1, T_2] \to R^n$ and a Lebesgue measurable function $u : [T_1, T_2] \to R^m$ is called an (A, B)-trajectory-control pair if for almost every (a. e.) $t \in [T_1, T_2]$ (2.1) holds. Denote by $X(A, B, T_1, T_2)$ the set of all (A, B)-trajectory-control pairs $x : [T_1, T_2] \to R^n$, $u : [T_1, T_2] \to R^m$.

Let $J = [a, \infty)$ be an infinite closed subinterval of $[0, \infty)$. A pair of functions $x : J \to R^n$ and $u : J \to R^m$ is called an (A, B)-trajectory-control pair if it is an (A, B)-trajectory-control pair on any bounded closed subinterval of J. Denote by $X(A, B, a, \infty)$ the set of all (A, B)-trajectory-control pairs $x : J \to R^n$, $u : J \to R^m$.

In this chapter we study the structure of approximate optimal trajectories of the linear control system (2.1) with the integrand f and show that the turnpike property holds. To have this property means, roughly speaking, that the approximate optimal trajectories on sufficiently large intervals are determined mainly by the integrand, and are essentially independent of the choice of time intervals and data, except in regions close to the endpoints of the time intervals. We also show the stability of the turnpike phenomenon under small perturbations of the integrand and study the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals.

More precisely, we consider the following optimal control problems

$$I^f(0, T, x, u) \to \min, \tag{P_1}$$

$$(x, u) \in X(A, B, 0, T) \text{ such that } x(0) = y,\ x(T) = z,$$

$$I^f(0, T, x, u) \to \min, \tag{P_2}$$

$$(x,u) \in X(A,B,0,T) \text{ such that } x(0) = y,$$

$$I^f(0,T,x,u) \to \min, \tag{P_3}$$

$$(x,u) \in X(A,B,0,T),$$

where $y, z \in R^n$ and $T > 0$. The study of these problems is based on the properties of solutions of the corresponding infinite horizon optimal control problem associated with the control system (2.1) and the integrand f.

In [57] we were interested in a turnpike property of the approximate solutions of problems (P_2). In this chapter we establish the turnpike property of the approximate solutions of problems (P_1) and (P_3). We show the stability of the turnpike phenomenon under small perturbations of the integrand f and study the structure of approximate optimal trajectories in regions close to the endpoints of the time intervals.

For the problems (P_2) and (P_3) we show that in regions close to the right endpoint T of the time interval these approximate solutions are determined only by the integrand, and are essentially independent of the choice of interval and the endpoint value y. For the problems (P_3), approximate solutions are determined only by the integrand also in regions close to the left endpoint 0 of the time interval.

The following result was obtained in [57] (see also Chap. 6 of [44]).

Proposition 2.1. *There exists* $(x_f, u_f) \in X(A,B,0,\tau)$ *which is the unique solution of the following minimization problem*

$$I^f(0,\tau,x,u) \to \min, \ \ (x,u) \in X(A,B,0,\tau) \textit{ such that } x(0) = x(\tau).$$

Let a trajectory-control pair $(x_f, u_f) \in X(A,B,0,\tau)$ be as guaranteed by Proposition 2.1. Put

$$\mu(f) = \tau^{-1} I^f(0,\tau,x_f,u_f). \tag{2.3}$$

The following results were obtained in [57] (see also Chap. 6 of [44]).

Theorem 2.2. *For any* $(x,u) \in X(A,B,0,\infty)$ *either*

$$\textit{(i) } I^f(0,T,x,u) - T\mu(f) \to \infty \textit{ as } T \to \infty$$

$$\textit{or (ii) } \sup\{|I^f(0,T,x,u) - T\mu(f)| : \ T > 0\} < \infty.$$

Moreover, if relation (ii) holds, then

$$\sup\{|x(i\tau + t) - x_f(t)| : \ t \in [0,\tau]\} \to 0 \textit{ as } i \to \infty, \ \textit{where } i \in \mathbf{Z}.$$

We say that $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-good [44, 53] if

$$\sup\{|I^f(0, T, x, u) - T\mu(f)| : \ T > 0\} < \infty.$$

The second statement of Theorem 2.2 describes the asymptotic behavior of (f, A, B)-good trajectory-control pairs, shows that the corresponding infinite horizon optimal control problem has the turnpike property, and that the function x_f is its turnpike.

We say that $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$ is (f, A, B)-overtaking optimal [44, 53] if for each $(x, u) \in X(A, B, 0, \infty)$ satisfying $x(0) = \tilde{x}(0)$,

$$\limsup_{T\to\infty}[I^f(0, T, \tilde{x}, \tilde{u}) - I^f(0, T, x, u)] \le 0.$$

Theorem 2.3. *Let $x_0 \in R^n$. Then there exists an (f, A, B)-overtaking optimal trajectory-control pair $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$ satisfying $\tilde{x}(0) = x_0$. Moreover, if $(x, u) \in X(A, B, 0, \infty) \setminus \{(\tilde{x}, \tilde{u})\}$ satisfies $x(0) = x_0$, then there are $T_0 > 0$ and $\epsilon > 0$ such that*

$$I^f(0, T, x, u) \ge I^f(0, T, \tilde{x}, \tilde{u}) + \epsilon \text{ for all } T \ge T_0.$$

The next result describes the limit behavior of overtaking optimal trajectories.

Theorem 2.4. *Let $M, \epsilon > 0$. Then there exists a natural number N such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ which satisfies $|x(0)| \le M$ the relation*

$$\sup\{|x(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} \le \epsilon \tag{2.4}$$

holds for all integers $i \ge N$. Moreover, there exists $\delta > 0$ such that for any (f, A, B)-overtaking optimal trajectory-control pair $(x, u) \in X(A, B, 0, \infty)$ satisfying $|x(0) - x_f(0)| \le \delta$, the relation (2.4) holds for all integers $i \ge 0$.

Let $T > 0$ and $y, z \in R^n$. Set

$$\begin{aligned} \sigma(f, y, z, T) = \inf\{I^f(0, T, x, u) : \\ (x, u) \in X(A, B, 0, T) \text{ and } x(0) = y, \ x(T) = z\}, \end{aligned} \tag{2.5}$$

$$\sigma(f, y, T) = \inf\{I^f(0, T, x, u) : \ (x, u) \in X(A, B, 0, T) \text{ and } x(0) = y\}, \tag{2.6}$$

$$\hat{\sigma}(f, z, T) = \inf\{I^f(0, T, x, u) : \ (x, u) \in X(A, B, 0, T) \text{ and } x(T) = z\}, \tag{2.7}$$

$$\sigma(f, T) = \inf\{I^f(0, T, x, u) : \ (x, u) \in X(A, B, 0, T)\}. \tag{2.8}$$

It follows from assumption (A) and Proposition 2.28 that

$$-\infty < \sigma(f, y, z, T), \sigma(f, y, T), \hat{\sigma}(f, z, T), \sigma(f, T) < \infty.$$

The next theorem establishes the turnpike property for approximate solutions of problems (P_2) with the turnpike $x_f(\cdot)$.

Theorem 2.5. *Let $M, \epsilon > 0$. Then there exist an integer $N \geq 1$ and $\delta > 0$ such that for each $T > 2N\tau$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$|x(0)| \leq M, \ I^f(0, T, x, u) \leq \sigma(f, x(0), T) + \delta$$

the inequality

$$\sup\{|x(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} \leq \epsilon \tag{2.9}$$

holds for all integers $i \in [N, \tau^{-1}T - N]$. Moreover if $|x(0) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [0, \tau^{-1}T - N]$.

Theorems 2.2–2.5 were obtained in [57] (see also Chap. 6 of [44]). Note that under assumptions of Theorem 2.5, if $|x(\lfloor \tau^{-1}T \rfloor \tau) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [N, \tau^{-1}T - 1]$.

The next two results establish the turnpike property for approximate solutions of problems (P_1) and (P_3) respectively with the turnpike $x_f(\cdot)$.

Theorem 2.6. *Let $M, \epsilon > 0$. Then there exist an integer $N \geq 1$ and $\delta > 0$ such that for each $T > 2N\tau$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$|x(0)|, \ |x(T)| \leq M, \ I^f(0, T, x, u) \leq \sigma(f, x(0), x(T), T) + \delta$$

inequality (2.9) holds for all integers $i \in [N, \tau^{-1}T - N]$. Moreover if $|x(0) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [0, \tau^{-1}T - N]$ and if $|x(\lfloor \tau^{-1}T \rfloor \tau) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [N, \tau^{-1}T - 1]$.

Theorem 2.7. *Let $\epsilon > 0$. Then there exist an integer $N \geq 1$ and $\delta > 0$ such that for each $T > 2N\tau$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies*

$$I^f(0, T, x, u) \leq \sigma(f, T) + \delta$$

inequality (2.9) holds for all integers $i \in [N, \tau^{-1}T - N]$. Moreover if $|x(0) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [0, \tau^{-1}T - N]$ and if $|x(\lfloor \tau^{-1}T \rfloor \tau) - x_f(0)| \leq \delta$, then inequality (2.9) holds for all integers $i \in [N, \tau^{-1}T - 1]$.

Theorems 2.5–2.7 are partial cases of Theorem 2.13 stated in Sect. 2.2 which is one of the main results of the chapter. The next theorem establishes a weak version of the turnpike property for approximate solutions of problems (P_1), (P_2), and (P_3) with the turnpike $x_f(\cdot)$.

Theorem 2.8. *Let $\epsilon, M_0, M_1 > 0$. Then there exist natural numbers Q, l such that for each $T > Ql\tau$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies at least one of the following conditions:*

$$|x(0)|,\ |x(T)| \le M_0,\ I^f(0,T,x,u) \le \sigma(f,x(0),x(T),T) + M_1;$$

$$|x(0)| \le M_0,\ I^f(0,T,x,u) \le \sigma(f,x(0),T) + M_1;$$

$$I^f(0,T,x,u) \le \sigma(f,T) + M_1$$

there exist strictly increasing sequences of nonnegative integers

$$\{a_i\}_{i=1}^q,\ \{b_i\}_{i=1}^q \subset [0, \tau^{-1}T]$$

such that $q \le Q$,

$$0 \le b_i - a_i \le l \textit{ for all } i = 1, \dots, q,$$

$b_i \le a_{i+1}$ *for all integers* i *satisfying* $1 \le i < q$ *and that for each integer* $i \in [0, \tau^{-1}T - 1] \setminus \cup_{j=1}^q [a_j, b_j]$,

$$|x(i\tau + t) - x_f(t)| \le \epsilon,\ t \in [0, \tau].$$

Theorem 2.8 is a partial case of Theorem 2.14, our stability result (see Sect. 2.2).

We say that $(x,u) \in X(A,B,0,\infty)$ is (f,A,B)-minimal [5, 53] if for each $T > 0$,

$$I^f(0,T,x,u) = \sigma(f,x(0),x(T),T). \tag{2.10}$$

The next result which is proved in Sect. 2.5 shows the equivalence of the optimality criterions introduced above.

Theorem 2.9. *Assume that* $(x,u) \in X(A,B,0,\infty)$. *Then the following conditions are equivalent:*

(i) (x,u) *is* (f,A,B)*-overtaking optimal; (ii)* (x,u) *is* (f,A,B)*-minimal and* (f,A,B)*-good; (iii)* (x,u) *is* (f,A,B)*-minimal and*

$$\max\{|x(i\tau + t) - x_f(t)| :\ t \in [0,\tau]\} \to 0 \textit{ as integers } i \to \infty;$$

(iv) (x,u) *is* (f,A,B)*-minimal and* $\liminf_{t\to\infty} |x(t)| < \infty$.

The following result is also proved in Sect. 2.5. It shows that if the integrand f does not depend on the variable t, then $x_f(\cdot)$ is a constant function.

Theorem 2.10. *Assume that for each* $x \in R^n$, *each* $u \in R^m$, *and each* $t_1, t_2 \ge 0$, $f(t_1,x,u) = f(t_2,x,u)$. *Then* $x_f(t) = x_f(0)$ *for all* $t \in [0,\tau]$ *and* $x_f(0)$ *does not depend of* τ.

Corollary 2.11. *Assume that for each* $x \in R^n$, *each* $u \in R^m$, *and each* $t_1, t_2 \ge 0$, $f(t_1,x,u) = f(t_2,x,u)$. *Then for all* $t \in [0,\tau]$, $x_f(t) = x_*$ *and* $u_f(t) = u_*$ *where* $(x_*, u_*) \in R^n \times R^m$ *is a unique solution of the minimization problem*

$$f(x,u) \to \min,\ (x,u) \in R^n \times R^m,\ Ax + Bu = 0.$$

2.2 Stability of the Turnpike Phenomenon

In this section we state Theorems 2.12–2.14 which show that the turnpike phenomenon is stable under small perturbations of the integrand f. We use the notation, definitions, and assumptions introduced in Sect. 2.1.

Recall that $f : [0,\infty) \times R^n \times R^m \to R^1$ is a Borel measurable function satisfying assumption (A). Let $a > 0$ and $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that

$$\lim_{t\to\infty} \psi(t) = \infty. \tag{2.11}$$

We suppose that for all $(t,x,u) \in [0,\infty) \times R^n \times R^m$,

$$f(t,x,u) \geq \max\{\psi(|x|),\ \psi(|u|)|u|\} - a. \tag{2.12}$$

Denote by $\mathcal{M}$ the set of all Borel measurable functions $g : [0,\infty) \times R^n \times R^m \to R^1$ which are bounded on all bounded subsets of $[0,\infty) \times R^n \times R^m$ and such that for all $(t,x,u) \in [0,\infty) \times R^n \times R^m$,

$$g(t,x,u) \geq \max\{\psi(|x|),\ \psi(|u|)|u|\} - a. \tag{2.13}$$

For the set $\mathcal{M}$ we consider the uniformity which is determined by the following base:

$$\begin{aligned} E(N,\epsilon,\lambda) = \{&(g_1,g_2) \in \mathcal{M}\times\mathcal{M} :\ |g_1(t,x,u) - g_2(t,x,u)| \leq \epsilon \text{ for each } t \geq 0, \\ &\text{each } x \in R^n \text{ satisfying } |x| \leq N \text{ and each } u \in R^m \text{ satisfying } |u| \leq N\} \\ &\cap\{(g_1,g_2) \in \mathcal{M}\times\mathcal{M} :\ (|g_1(t,x,u)|+1)(|g_2(t,x,u)|+1)^{-1} \in [\lambda^{-1},\lambda] \\ &\text{for each } t \geq 0,\ \text{each } x \in R^n \text{ satisfying } |x| \leq N \text{ and each } u \in R^m\}, \end{aligned} \tag{2.14}$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$. It is not difficult to see that the space $\mathcal{M}$ with this uniformity is metrizable and complete.

Let $T_2 > T_1 \geq 0$, $y,z \in R^n$, and $g \in \mathcal{M}$. For each pair of Lebesgue measurable functions $x : [T_1,T_2] \to R^n$, $u : [T_1,T_2] \to R^m$ set

$$I^g(T_1,T_2,x,u) = \int_{T_1}^{T_2} g(t,x(t),u(t))dt \tag{2.15}$$

and set

$$\begin{aligned} \sigma(g,y,z,T_1,T_2) = \inf\{&I^g(T_1,T_2,x,u) : \\ &(x,u) \in X(A,B,T_1,T_2) \text{ and } x(T_1) = y,\ x(T_2) = z\}, \end{aligned} \tag{2.16}$$

$$\sigma(g, y, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(A, B, T_1, T_2) \text{ and } x(T_1) = y\}, \tag{2.17}$$

$$\hat{\sigma}(g, z, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(A, B, T_1, T_2) \text{ and } x(T_2) = z\}, \tag{2.18}$$

$$\sigma(g, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) : \ (x, u) \in X(A, B, T_1, T_2)\}. \tag{2.19}$$

Since any $g \in \mathcal{M}$ is bounded on all the bounded subsets of $[0, \infty) \times R^n \times R^m$ it follows from Proposition 2.28 and (2.13) that all the values defined above are finite.

In this chapter we prove the following three stability results.

Theorem 2.12. *Let $\epsilon, M > 0$. Then there exist an integer $L_0 \geq 1$ and $\delta_0 > 0$ such that for each integer $L_1 \geq L_0$ there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following assertion holds.*

Assume that $T > 2L_1\tau$, $g \in \mathcal{U}$, $(x, u) \in X(A, B, 0, T)$ and that a finite sequence of integers $\{S_i\}_{i=0}^q$ satisfy

$$S_0 = 0, \ S_{i+1} - S_i \in [L_0, L_1], \ i = 0, \dots, q-1, \ S_q\tau \in (T - L_1\tau, T], \tag{2.20}$$

$$I^g(S_i\tau, S_{i+1}\tau, x, u) \leq (S_{i+1} - S_i)\tau\mu(f) + M$$

for each integer $i \in [0, q-1]$,

$$I^g(S_i\tau, S_{i+2}\tau, x, u) \leq \sigma(g, x(S_i\tau), x(S_{i+2}\tau), S_i\tau, S_{i+2}\tau) + \delta_0$$

for each nonnegative integer $i \leq q-2$ and

$$I^g(S_{q-2}\tau, T, x, u) \leq \sigma(g, x(S_{q-2}\tau), x(T), S_{q-2}\tau, T) + \delta_0.$$

Then there exist integers $p_1, p_2 \in [0, \tau^{-1}T]$ such that $p_1 \leq p_2$, $p_1 \leq 2L_0$, $p_2 > \tau^{-1}T - 2L_1$ and that for all integers $i = p_1, \dots, p_2 - 1$,

$$\max\{|x(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} \leq \epsilon.$$

Moreover if $|x(0) - x_f(0)| \leq \delta_0$, then $p_1 = 0$ and if $|x(\lfloor \tau^{-1}T \rfloor \tau) - x_f(0)| \leq \delta_0$, then $p_2 = [\tau^{-1}T]$.

Theorem 2.13. *Let $\epsilon \in (0, 1)$, $M_0, M_1 > 0$. Then there exist an integer $L \geq 1$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that for each $T > 2L\tau$, each $g \in \mathcal{U}$, and each $(x, u) \in X(A, B, 0, T)$ which satisfies for each $S \in [0, T - L\tau]$,*

$$I^g(S, S + L\tau, x, u) \leq \sigma(g, x(S), x(S + L\tau), S, S + L\tau) + \delta$$

and satisfies at least one of the following conditions:

$$\text{(a) } |x(0)|,\ |x(T)| \le M_0,\ I^g(0,T,x,u) \le \sigma(g,x(0),x(T),0,T) + M_1;$$

$$\text{(b) } |x(0)| \le M_0,\ I^g(0,T,x,u) \le \sigma(g,x(0),0,T) + M_1;$$

$$\text{(c) } I^g(0,T,x,u) \le \sigma(g,0,T) + M_1$$

there exist integers $p_1 \in [0,L]$, $p_2 \in [\lfloor \tau^{-1}T \rfloor - L, \tau^{-1}T]$ *such that for all integers* $i = p_1, \dots, p_2 - 1$,

$$|x(i\tau + t) - x_f(t)| \le \epsilon \text{ for all } t \in [0,\tau].$$

Moreover if $|x(0) - x_f(0)| \le \delta$, *then* $p_1 = 0$ *and if* $|x(\lfloor \tau^{-1}T \rfloor \tau) - x_f(0)| \le \delta$, *then* $p_2 = \lfloor \tau^{-1}T \rfloor$.

Denote by $\mathrm{Card}(A)$ the cardinality of the set A.

Theorem 2.14. *Let* $\epsilon \in (0,1)$, $M_0, M_1 > 0$. *Then there exist an integer* $L \ge 1$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{M}$ *such that for each* $T > L\tau$, *each* $g \in \mathcal{U}$, *and each* $(x,u) \in X(A,B,0,T)$ *which satisfies at least one of the following conditions:*

$$\text{(a) } |x(0)|,\ |x(T)| \le M_0,\ I^g(0,T,x,u) \le \sigma(g,x(0),x(T),0,T) + M_1;$$

$$\text{(b) } |x(0)| \le M_0,\ I^g(0,T,x,u) \le \sigma(g,x(0),0,T) + M_1;$$

$$\text{(c) } I^g(0,T,x,u) \le \sigma(g,0,T) + M_1$$

the following inequality holds:

$$\mathrm{Card}(\{i \in \{0,\dots,\lfloor \tau^{-1}T \rfloor - 1\} :\ \max\{|x(i\tau + t) - x_f(t)| :\ t \in [0,\tau]\} > \epsilon\}) \le L.$$

2.3 Structure of Solutions in the Regions Close to the End Points

In this section we state results which describe the structure of solutions of problems (P_1), (P_2) and (P_3) in the regions close to the end points. Combined with the turnpike results of Sect. 2.2 they provide the full description of the structure of their solutions. We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.2.

By Theorem 2.14 for each $z \in R^n$ there exists a unique (f,A,B)-overtaking optimal pair $(\xi^{(z)}, \eta^{(z)}) \in X(A,B,0,\infty)$ such that $\xi^{(z)}(0) = z$. Let $z \in R^n$. Set

$$\pi^f(z) = \liminf_{T \to \infty,\, T \in \mathbf{Z}} [I^f(0, T\tau, \xi^{(z)}, \eta^{(z)}) - T\tau\mu(f)]. \tag{2.21}$$

In view of Theorems 2.2, 2.3, and 2.9, $\pi^f(z)$ is a finite number. Definition (2.21) and the definition of (f, A, B)-overtaking optimal pairs imply the following result.

Proposition 2.15. *1. Let* $(x, u) \in X(A, B, 0, \infty)$ *be* (f, A, B)*-good. Then*

$$\pi^f(x(0)) \le \liminf_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, x, u) - T\tau\mu(f)]$$

and for each pair of integers $S > T \ge 0$*,*

$$\pi^f(x(T\tau)) \le I^f(T\tau, S\tau, x, u) - (S-T)\tau\mu(f) + \pi^f(x(S\tau)). \tag{2.22}$$

2. Let $S > T \ge 0$ *be integers and* $(x, u) \in X(A, B, T\tau, S\tau)$*. Then* (2.22) *holds.*

The next result follows from definition (2.21).

Proposition 2.16. *Let* $(x, u) \in X(A, B, 0, \infty)$ *be* (f, A, B)*-overtaking optimal. Then for each pair of integers* $S > T \ge 0$*,*

$$\pi^f(x(T\tau)) = I^f(T\tau, S\tau, x, u) - (S-T)\tau\mu(f) + \pi^f(x(S\tau)).$$

Theorems 2.3–2.5 and (2.21), (2.3) imply the following result.

Proposition 2.17. $\pi^f(x_f(0)) = 0$.

The following result is proved in Sect. 2.11.

Proposition 2.18. *The function* π^f *is continuous at* $x_f(0)$.

Proposition 2.19. *Let* $(x, u) \in X(A, B, 0, \infty)$ *be* (f, A, B)*-overtaking optimal. Then*

$$\pi^f(x(0)) = \lim_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, x, u) - T\tau\mu(f)].$$

Proof. It follows from Propositions 2.16–2.18 and Theorems 2.2 and 2.9 that

$$\begin{aligned}\pi^f(x(0)) &= \lim_{T\to\infty,\, T\in\mathbf{Z}} (\pi^f(x(0)) - \pi^f(x(T\tau))) \\ &= \lim_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, x, u) - T\tau\mu(f)].\end{aligned}$$

Proposition 2.19 is proved.

Proposition 2.20. *The function* π^f *is strictly convex and continuous.*

Proof. It is sufficient to show that the function π^f is strictly convex. Let $y, z \in R^n$, $y \ne z$ and $\alpha \in (0, 1)$. Consider $(\alpha\xi^{(y)} + (1-\alpha)\xi^{(z)}, \alpha\eta^{(y)} + (1-\alpha)\eta^{(z)}) \in X(A, B, 0, \infty)$ which satisfies $(\alpha\xi^{(y)} + (1-\alpha)\xi^{(z)})(0) = \alpha y + (1-\alpha)z$ and in view of (A) and Theorem 2.2, is (f, A, B)-good. By Propositions 2.15 and 2.19, assumption (A) and the relation $y \ne z$,

$$\begin{aligned}\pi^f(\alpha y + (1-\alpha)z) &= \liminf_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, \alpha\xi^{(y)} + (1-\alpha)\xi^{(z)}, \alpha\eta^{(y)} \\ &\quad + (1-\alpha)\eta^{(z)}) - T\tau\mu(f)] \\ &< \liminf_{T\to\infty,\, T\in\mathbf{Z}} [\alpha(I^f(0, T\tau, \xi^{(y)}, \eta^{(y)}) - T\tau\mu(f)) \\ &\quad + (1-\alpha)(I^f(0, T\tau, \xi^{(z)}, \eta^{(z)}) - T\tau\mu(f))] \\ &= \alpha \lim_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, \xi^{(y)}, \eta^{(y)}) - T\tau\mu(f)] \\ &\quad + (1-\alpha) \lim_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, \xi^{(z)}, \eta^{(z)}) - T\tau\mu(f)] \\ &= \alpha\pi^f(y) + (1-\alpha)\pi^f(z).\end{aligned}$$

Proposition 2.20 is proved.

The next result is proved in Sect. 2.11.

Proposition 2.21. *For each $M > 0$ the set $\{x \in R^n : \pi^f(x) \leq M\}$ is bounded.*

Set

$$\inf(\pi^f) = \inf\{\pi^f(z) : z \in R^n\}. \tag{2.23}$$

By Propositions 2.20 and 2.21, $\inf(\pi^f)$ is finite and there exists a unique $\theta_f \in R^n$ such that $\pi^f(\theta_f) = \inf(\pi^f)$.

Proposition 2.22. *Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-good such that for all integers $T > 0$,*

$$I^f(0, T\tau, x, u) - T\tau\mu(f) = \pi^f(x(0)) - \pi^f(x(T\tau)). \tag{2.24}$$

Then $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-overtaking optimal.

Proof. Theorem 2.3 implies that there exists an (f, A, B)-overtaking optimal pair $(x_1, u_1) \in X(A, B, 0, \infty)$ such that $x_1(0) = x(0)$. By Proposition 2.16, for each integer $T \geq 1$,

$$I^f(0, T\tau, x_1, u_1) - T\tau\mu(f) = \pi^f(x_1(0)) - \pi^f(x_1(T\tau)).$$

It follows from the equality above, (2.24), Theorems 2.2 and 2.9, and Propositions 2.17 and 2.18 that for all integers $T > 0$,

$$I^f(0, T\tau, x, u) - I^f(0, T\tau, x_1, u_1) = \pi^f(x_1(T\tau)) - \pi^f(x(T\tau)) \to 0 \text{ as } T \to \infty.$$

Thus

$$\lim_{T\to\infty,\, T\in\mathbf{Z}} [I^f(0, T\tau, x, u) - I^f(0, T\tau, x_1, u_1)] = 0$$

and in view of Theorem 2.3 this implies the pair (x, u) is (f, A, B)-overtaking optimal. Proposition 2.22 is proved.

Consider a linear control system

$$x'(t) = -Ax(t) - Bu(t),\ x(0) = x_0$$

which is also controllable. There exists a Borel measurable function $\bar{f} : [0, \infty) \times R^n \times R^m \to R^1$ such that for all $(x, u) \in R^n \times R^m$,

$$\bar{f}(t + \tau, x, u) = \bar{f}(t, x, u) \text{ for all } t \geq 0,$$

$$\bar{f}(t, x, u) = f(\tau - t, x, u) \text{ for all } t \in [0, \tau]. \tag{2.25}$$

Evidently, $\bar{f}$ satisfies assumption (A). For $\bar{f}$ we use all the notation and definitions introduced for f. It is clear that all the results obtained for the triplet (f, A, B) also hold for the triplet $(\bar{f}, -A, -B)$.

Assume that integers $S_2 > S_1 \geq 0$ and that $(x, u) \in X(A, B, S_1\tau, S_2\tau)$. For all $t \in [S_1\tau, S_2\tau]$ set

$$\bar{x}(t) = x(S_2\tau - t + S_1\tau),\ \bar{u}(t) = u(S_2\tau - t + S_1\tau). \tag{2.26}$$

In view of (2.26) for a. e. $t \in [S_1\tau, S_2\tau]$,

$$\begin{aligned}\bar{x}'(t) &= -x'(S_2\tau - t + S_1\tau) = -Ax(S_2\tau - t + S_1\tau) - Bu(S_2\tau - t + S_1\tau)\\ &= -A\bar{x}(t) - B\bar{u}(t)\end{aligned}$$

and $(\bar{x}, \bar{u}) \in X(-A, -B, S_1\tau, S_2\tau)$. By (2.25) and (2.26),

$$\begin{aligned}\int_{S_1\tau}^{S_2\tau} \bar{f}(t, \bar{x}(t), \bar{u}(t))dt &= \int_{S_1\tau}^{S_2\tau} \bar{f}(t, x(S_2\tau - t + S_1\tau), u(S_2\tau - t + S_1\tau))dt\\ &= \int_{S_1\tau}^{S_2\tau} f(S_2\tau - t + S_1\tau, x(S_2\tau - t + S_1\tau),\\ &\quad u(S_2\tau - t + S_1\tau))dt\\ &= \int_{S_1\tau}^{S_2\tau} f(t, x(t), u(t))dt.\end{aligned} \tag{2.27}$$

For each pair $T_2 > T_1 \geq 0$ and each $(x, u) \in X(-A, -B, T_1, T_2)$ set

$$I^{\bar{f}}(T_1, T_2, x, u) = \int_{T_1}^{T_2} \bar{f}(t, x(t), u(t))dt.$$

For each $y, z \in R^n$ and each $T > 0$ set

$$\sigma_-(\bar{f}, y, z, T) = \inf\{I^{\bar{f}}(0, T, x, u) :$$
$$(x, u) \in X(-A, -B, 0, T) \text{ and } x(0) = y,\ x(T) = z\},$$
$$\sigma_-(\bar{f}, y, T) = \inf\{I^{\bar{f}}(0, T, x, u) :\ (x, u) \in X(-A, -B, 0, T) \text{ and } x(0) = y\},$$
$$\hat{\sigma}_-(\bar{f}, z, T) = \inf\{I^{\bar{f}}(0, T, x, u) :\ (x, u) \in X(-A, -B, 0, T) \text{ and } x(T) = z\},$$
$$\sigma_-(\bar{f}, T) = \inf\{I^{\bar{f}}(0, T, x, u) :\ (x, u) \in X(-A, -B, 0, T)\}. \tag{2.28}$$

Relations (2.26) and (2.27) imply the following result.

Proposition 2.23. *Let $S_2 > S_1 \geq 0$ be integers, $M \geq 0$ and that $(x_i, u_i) \in X(A, B, S_1\tau, S_2\tau)$, $i = 1, 2$. Then*

$$I^f(S_1\tau, S_2\tau, x_1, u_1) \geq I^f(S_1\tau, S_2\tau, x_2, u_2) - M$$

if and only if $I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}_1, \bar{u}_1) \geq I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}_2, \bar{u}_2) - M$.

Proposition 2.23 implies the following result.

Proposition 2.24. *Let $S_2 > S_1 \geq 0$ be integers and*

$$(x, u) \in X(A, B, S_1\tau, S_2\tau).$$

Then the following assertion holds:

$$I^f(S_1\tau, S_2\tau, x, u) \leq \sigma(f, (S_2 - S_1)\tau) + M$$
$$\text{if and only if } I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \leq \sigma_-(\bar{f}, (S_2 - S_1)\tau) + M;$$
$$I^f(S_1\tau, S_2\tau, x, u) \leq \sigma(f, x(S_1\tau), x(S_2\tau), (S_2 - S_1)\tau) + M$$
$$\text{if and only if } I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \leq \sigma_-(\bar{f}, \bar{x}(S_1\tau), \bar{x}(S_2\tau), (S_2 - S_1)\tau) + M;$$
$$I^f(S_1\tau, S_2\tau, x, u) \leq \sigma(f, x(S_1\tau), (S_2 - S_1)\tau) + M$$
$$\text{if and only if } I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \leq \hat{\sigma}_-(\bar{f}, \bar{x}(S_2\tau), (S_2 - S_1)\tau) + M;$$
$$I^f(S_1\tau, S_2\tau, x, u) \leq \hat{\sigma}(f, x(S_2\tau), (S_2 - S_1)\tau) + M$$
$$\text{if and only if } I^{\bar{f}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \leq \sigma_-(\bar{f}, \bar{x}(S_1\tau), (S_2 - S_1)\tau) + M.$$

By Proposition 2.1, $(x_f, u_f) \in X(A, B, 0, \tau)$ is the unique solution of the minimization problem

$$I^f(0, \tau, x, u) \to \min, \ (x, u) \in X(A, B, 0, \tau) \text{ such that } x(0) = x(\tau).$$

Analogously there exists $(x_{\bar{f}}, u_{\bar{f}}) \in X(-A, -B, 0, \tau)$ which is the unique solution of the minimization problem

$$I^{\bar{f}}(0, \tau, x, u) \to \min, \ (x, u) \in X(-A, -B, 0, \tau) \text{ such that } x(0) = x(\tau).$$

In view of Proposition 2.23 and (2.27), for all $t \in [0, \tau]$,

$$x_{\bar{f}}(t) = x_f(\tau - t), \ u_{\bar{f}}(t) = u_f(\tau - t), \ \mu(\bar{f}) = \mu(f). \tag{2.29}$$

For each $z \in R^n$, set

$$\pi^{\bar{f}}(z) = \liminf_{T \to \infty, \, T \in \mathbf{Z}} [I^{\bar{f}}(0, T\tau, x, u) - T\tau\mu(f)], \tag{2.30}$$

where $(x, u) \in X(-A, -B, 0, \infty)$ is the unique $(\bar{f}, -A, -B)$-overtaking optimal pair such that $x(0) = z$. Let $(x_*, u_*) \in X(A, B, 0, \infty)$ be the unique (f, A, B)-overtaking optimal pair such that $\pi^f(x_*(0)) = \inf(\pi^f)$ and

$$(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$$

be the unique $(\bar{f}, -A, -B)$-overtaking optimal pair such that $\pi^{\bar{f}}(\bar{x}_*(0)) = \inf(\pi^{\bar{f}})$.

The following three theorems describe the structure of solutions of problems (P_1), (P_2), and (P_3) in the regions closed to the end points.

Theorem 2.25. *Let $L_0 > 0$ be an integer, $\epsilon \in (0, 1)$, $M > 0$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and an integer $L_1 > L_0$ such that for each integer $T \geq L_1$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T\tau)$ which satisfies*

$$|x(0)| \leq M, \ I^g(0, T\tau, x, u) \leq \sigma(g, x(0), 0, T\tau) + \delta$$

the following inequality holds:

$$|x(T\tau - t) - \bar{x}_*(t)| \leq \epsilon \text{ for all } t \in [0, L_0\tau].$$

Theorem 2.26. *Let $L_0 > 0$ be an integer, $\epsilon > 0$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and an integer $L_1 > L_0$ such that for each integer $T \geq L_1$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T\tau)$ which satisfies*

$$I^g(0, T\tau, x, u) \leq \sigma(g, 0, T\tau) + \delta$$

the following inequalities hold for all $t \in [0, L_0\tau]$*:*

$$|x(T\tau - t) - \bar{x}_*(t)| \le \epsilon,\ |x(t) - x_*(t)| \le \epsilon.$$

Theorem 2.27. *Let* $L_0 > 0$ *be an integer,* $\epsilon > 0,\ M_0 > 0$*. Then there exist* $\delta > 0$*, a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{M}$ *and an integer* $L_1 > L_0$ *such that for each integer* $T \ge L_1$*, each* $g \in \mathcal{U}$ *and each* $(x, u) \in X(A, B, 0, T\tau)$ *which satisfies*

$$|x(0)|,\ |x(T\tau)| \le M_0,\ I^g(0, T\tau, x, u) \le \sigma(g, x(0), x(T\tau), 0, T\tau) + \delta$$

the inequalities

$$|x(T\tau - t) - \bar{\xi}(t)| \le \epsilon,\ |x(t) - \xi(t)| \le \epsilon$$

hold for all $t \in [0, L_0\tau]$*, where* $(\xi, \eta) \in X(A, B, 0, \infty)$ *is the unique* (f, A, B)*-overtaking optimal pair such that* $\xi(0) = x(0)$ *and*

$$(\bar{\xi}, \bar{\eta}) \in X(-A, -B, 0, \infty)$$

is the unique $(\bar{f}, -A, -B)$*-overtaking optimal pair such that* $\bar{\xi}(0) = x(T\tau)$*.*

2.4 Auxiliary Results

In the sequel we use the following auxiliary results.

Proposition 2.28 (Proposition 6.2.1 of [44]). *For every* $\tilde{y}, \tilde{z} \in R^n$ *and every* $T > 0$ *there exists a solution* $x(\cdot)$*,* $y(\cdot)$ *of the system*

$$x' = Ax + BB^t y,\ y' = x - A^t y$$

with the boundary conditions $x(0) = \tilde{y}$*,* $x(T) = \tilde{z}$ *(where* B^t *denotes the transpose of* B*).*

Proposition 2.29 (Proposition 6.2.2 of [44]). *Let* $M_1 > 0$ *and* $0 < \tau_0 < \tau_1$*. Then there exists a positive number* M_2 *such that for each* $T \in [\tau_0, \tau_1]$ *and each* $(x, u) \in X(A, B, 0, T)$ *satisfying* $I^f(0, T, x, u) \le M_1$ *the inequality* $|x(t)| \le M_2$ *holds for all* $t \in [0, T]$*.*

Proposition 2.30 (Proposition 6.2.4 of [44]). *Let* M_1 *and* T *be positive numbers and let* $\mathcal{F}$ *be the set of all* $(x, u) \in X(A, B, 0, T)$ *satisfying* $I^f(0, T, x, u) \le M_1$*. Then for every sequence* $\{(x_i, u_i)\}_{i=1}^{\infty} \subset \mathcal{F}$ *there exist a subsequence* $\{(x_{i_k}, u_{i_k})\}_{k=1}^{\infty}$ *and* $(x, u) \in \mathcal{F}$ *such that* $x_{i_k}(t) \to x(t)$ *as* $k \to \infty$ *uniformly in* $[0, T]$*,* $x'_{i_k} \to x'$ *as* $k \to \infty$ *weakly in* $L^1(R^n; (0, T))$*, and* $u_{i_k} \to u$ *as* $k \to \infty$ *weakly in* $L^1(R^m; (0, T))$*.*

For each $y, z \in R^n$ define

$$v(y,z) = \inf\{I^f(0,\tau,x,u) : \ (x,u) \in X(A,B,0,\tau)$$

$$\text{such that } x(0) = y, \ x(\tau) = z\}. \tag{2.31}$$

It was shown in Sect. 6.2 of [44] that the function v is convex, satisfies

$$-\infty < v(y,z) < \infty \text{ for each } y, z \in R^n,$$

$$v(y,z) \to \infty \text{ as } |y| + |z| \to \infty \tag{2.32}$$

and that there exists $z_f \in R^n$ such that

$$v(z_f, z_f) < v(z,z) \text{ for all } z \in R^n \setminus \{z_f\}, \tag{2.33}$$

$$x_f(0) = z_f, \ \mu(f) = \tau^{-1} v(z_f, z_f). \tag{2.34}$$

Proposition 2.31 (Proposition 6.2.5 of [44]). *There exists $p_f \in R^n$ such that the function $\Theta_f : R^n \times R^n \to R^1$ defined by*

$$\Theta_f(y,z) = v(y,z) - v(z_f,z_f) - \langle p_f, y - z\rangle, \ y, z \in R^n$$

is strictly convex and

$$\Theta_f(z_f,z_f) = 0, \ \Theta_f(y,z) > 0 \textit{ for all } (y,z) \in R^n \times R^n \setminus \{(z_f,z_f)\}. \tag{2.35}$$

Define a function $f_0 : R^n \times R^m \to R^1$ by

$$f_0(x,u) = \sup\{f(t,x,u) : \ t \in [0,\infty)\}, \ (x,u) \in R^n \times R^m. \tag{2.36}$$

In view of assumption (A), the function f_0 is well defined, convex, and bounded on bounded subsets of $R^n \times R^m$. For all $y, z \in R^n$ set

$$v_0(y,z) = \inf\left\{ \int_0^\tau f_0(x(t),u(t))dt : \ (x,u) \in X(A,B,0,\tau)\right.$$

$$\left.\text{such that } x(0) = y, \ x(\tau) = z\right\}. \tag{2.37}$$

By (2.37), convexity of f_0, Proposition 2.28 and assumption (A), the function $v_0 : R^n \times R^n \to R^1$ is well defined, convex, and continuous.

Proposition 2.32 (Corollary 6.2.1 of [44]). *Let $x_1, x_2 \in R^n$. Then there is a unique $(x,u) \in X(A,B,0,\tau)$ such that $x(0) = x_1$, $x(\tau) = x_2$ and $I^f(0,\tau,x,u) = v(x_1,x_2)$.*

Proposition 2.33. *Let $\epsilon > 0$. Then there exist $\delta > 0$ such that for each integer $k \geq 1$ and each $y, z \in R^n$ satisfying $|y - x_f(0)|,\ |z - x_f(0)| \leq \delta$,*

$$\sigma(f, y, z, k\tau) \leq k\tau\mu(f) + \epsilon.$$

Proof. Since the function v is continuous there exists $\delta \in (0, \epsilon)$ such that for each $y, z \in R^n$ satisfying

$$|y - x_f(0)|,\ |z - x_f(0)| \leq \delta, \tag{2.38}$$

we have

$$|v(y, z) - \tau\mu(f)| = |v(y, z) - v(x_f(0), x_f(0))| \leq \epsilon/4.$$

Let $k \geq 1$ be an integer and $y, z \in R^n$ satisfy (2.38). Assume that $k = 1$. By (2.31) and the choice of δ [see (2.38)],

$$\sigma(f, y, z, \tau) = v(y, z) \leq \tau\mu(f) + \epsilon.$$

Assume that $k > 1$. By Proposition 2.32, there exists $(x, u) \in X(A, B, 0, k\tau)$ such that

$$x(0) = y,\ x(\tau) = x_f(0),\ I^f(0, \tau, x, u) = v(y, x_f(0)),$$
$$x((k-1)\tau) = x_f(0),\ x(k\tau) = z,\ I^f((k-1)\tau, k\tau, x, u) = v(x_f(0), z),$$

and that for each integer i satisfying $1 \leq i < k - 1$,

$$x(i\tau + t) = x_f(t),\ u(i\tau + t) = u_f(t),\ t \in [0, \tau].$$

By the definition above, the choice of δ [see (2.38)],

$$\begin{aligned}\sigma(f, y, z, k\tau) &\leq I^f(0, k\tau, x, u) = v(x_f(0), z) + v(y, x_f(0)) + \tau\mu(f)(k-2)\\ &\leq 2(\mu(f)\tau + \epsilon/4) + \tau\mu(f)(k-2) \leq k\tau\mu(f) + \epsilon/2.\end{aligned}$$

Proposition 2.33 is proved.

Proposition 2.34. *There exists $M_* > 0$ such that for each $T > 0$ and each $(x, u) \in X(A, B, 0, T)$,*

$$I^f(0, T, x, u) \geq T\mu(f) - M_*.$$

Proof. By (A) there is $c_0 > 0$ such that

$$f(t, x, u) \geq -c_0 \text{ for all } (t, x, u) \in [0, \infty) \times R^n \times R^m. \tag{2.39}$$

In view of Proposition 2.29 there exists $M_0 > 0$ such that for each $T \in [\tau/2, 4\tau]$ and each $(x, u) \in X(A, B, 0, T)$ satisfying

$$I^f(0, T, x, u) \le 4\tau(|\mu(f)| + 1)$$

we have

$$|x(t)| \le M_0 \text{ for all } t \in [0, T]. \tag{2.40}$$

Let $p_f \in R^n$ be as guaranteed by Proposition 2.31. Choose

$$M_* > 4c_0\tau + 4\tau|\mu(f)| + 1 + 4|p_f|M_0. \tag{2.41}$$

Let $T > 0$ and $(x, u) \in X(A, B, 0, T)$. If $T \le 4\tau$, then by (2.39) and (2.41),

$$I^f(0, T, x, u) \ge -4c_0\tau \ge T\mu(f) - 4\tau|\mu(f)| - 4c_0\tau \ge T\mu(f) - M_*$$

and in this case Proposition 2.34 holds.

Assume that

$$T > 4\tau. \tag{2.42}$$

There are two cases:

$$|x(i\tau)| > M_0 \text{ for all integers } i \in [1, \tau^{-1}T - 1]; \tag{2.43}$$

$$\min\{|x(i\tau)| : \text{ an integer } i \in [1, \tau^{-1}T - 1]\} \le M_0. \tag{2.44}$$

Assume that (2.43) holds. Set

$$S_0 = 0,\ S_i = i\tau \text{ for all integers } i \in [0, \lfloor \tau^{-1}T \rfloor - 1],\ S_{\lfloor \tau^{-1}T \rfloor} = T. \tag{2.45}$$

By (2.45),

$$S_{\lfloor \tau^{-1}T \rfloor} - S_{\lfloor \tau^{-1}T \rfloor - 1} = T - \tau(\lfloor \tau^{-1}T \rfloor - 1) \in [\tau, 2\tau]. \tag{2.46}$$

By the choice of M_0 (see (2.40)), (2.42), (2.43), (2.45) and (2.46),

$$I^f(S_i, S_{i+1}, x, u) > 4\tau(|\mu(f)| + 1),\ i = 0, \dots, \lfloor T/\tau \rfloor - 1.$$

This implies that

$$I^f(0, T, x, u) \ge 4\tau(|\mu(f)| + 1)\lfloor T/\tau \rfloor > 2(|\mu(f)| + 1)T.$$

Thus in this case Proposition 2.34 holds.

Assume that (2.44) holds. Then there exist integers j_1, j_2 such that

$$1 \le j_1 \le j_2 \le \tau^{-1}T - 1, \tag{2.47}$$

$$|x(j_1\tau)|,\ |x(j_2\tau)| \le M_0, \tag{2.48}$$

$$|x(i\tau)| > M_0 \tag{2.49}$$

for each integer i satisfying $1 \le i < j_1$ and for each integer i satisfying $j_2 < i \le T/\tau - 1$. We will estimate $I^f(0, j_1\tau, x, u)$, $I^f(j_2\tau, T, x, u)$ and $I^f(j_1\tau, j_2\tau, x, u)$. By (2.39),

$$I^f(0, \tau, x, u) \ge -c_0\tau. \tag{2.50}$$

In view of (2.49) and the choice of j_1 and M_0 [see (2.40)], for each integer i satisfying $1 \le i < j_1$,

$$I^f(i\tau, (i+1)\tau, x, u) > 4\tau(|\mu(f)| + 1).$$

Together with (2.50) this implies that

$$I^f(0, j_1\tau, x, u) - j_1\tau\mu(f) \ge -c_0\tau - \tau|\mu(f)|. \tag{2.51}$$

By (2.39),

$$I^f(\tau\lfloor\tau^{-1}T\rfloor - \tau, T, x, u) \ge -2c_0\tau. \tag{2.52}$$

It follows from (2.49) and the choice of j_2 and M_0 [see (2.40)] that for each integer i satisfying $j_2 \le i < T/\tau - 1$,

$$I^f(i\tau, (i+1)\tau, x, u) \ge 4\tau(|\mu(f)| + 1).$$

Together with (2.52) this implies that

$$I^f(j_2\tau, T, x, u) - (T - j_2\tau)\mu(f) \ge -2c_0\tau - 2\tau|\mu(f)|. \tag{2.53}$$

If $j_1 = j_2$, then (2.51) and (2.53) imply that

$$I^f(0, T, x, u) - T\mu(f) \ge -3c_0\tau - 3\tau|\mu(f)| > -M_*.$$

Therefore we may assume without loss of generality that $j_1 < j_2$. We estimate $I^f(j_1\tau, j_2\tau, x, u)$. By (2.31), the choice of p_f, Proposition 2.31, (2.34), (2.42), and (2.48),

$$I^f(j_1\tau, j_2\tau, x, u) - (j_2 - j_1)\tau\mu(f) \geq \sum_{i=j_1}^{j_2-1} v(x(i\tau), x((i+1)\tau)) - (j_2 - j_1)v(z_f, z_f)$$
$$\geq \langle p_f, x(j_1\tau) - x(j_2\tau)\rangle \geq -2|p_f|M_0. \tag{2.54}$$

It follows from (2.41), (2.54), (2.51), and (2.53) that

$$\begin{aligned} I^f(0, T, x, u) - T\mu(f) &= I^f(0, j_1\tau, x, u) - j_1\tau\mu(f) + I^f(j_1\tau, j_2\tau, x, u) \\ &\quad - (j_2 - j_1)\tau\mu(f) + I^f(j_2\tau, T, x, u) - (T - j_2\tau)\mu(f) \\ &\geq -c_0\tau - \tau|\mu(f)| - 2|p_f|M_0 - 2c_0\tau - 2\tau|\mu(f)| \\ &\geq -3c_0\tau - 3\tau|\mu(f)| - 2|p_f|M_0 > -M_*. \end{aligned}$$

Proposition 2.34 is proved.

Proposition 2.35. *Let $M_0 > 0$. Then there exists $M > 0$ such that for each $T \geq 3\tau$ and each $y, z \in R^n$ satisfying $|y|, |z| \leq M_0$,*

$$\sigma(f, y, z, T) \leq T\mu(f) + M.$$

Proof. We may assume without loss of generality that

$$M_0 \geq |x_f(t)| \text{ for all } t \in [0, \tau]. \tag{2.55}$$

Since the function v_0 is continuous there exists $M_1 > 0$ such that

$$|v_0(y, z)| \leq M_1 \text{ for all } y, z \in R^n \text{ satisfying } |y|, |z| \leq M_0. \tag{2.56}$$

In view of assumption (A), there exists $c_0 > 0$ such that

$$f(t, x, u) \geq -c_0 \text{ for all } (t, x, u) \in [0, \infty) \times R^n \times R^m. \tag{2.57}$$

Choose

$$M > 2M_1 + 2c_0\tau + 2 + 2\tau|\mu(f)|. \tag{2.58}$$

Assume that

$$T \geq 3\tau, \;\; y, z \in R^n, \;\; |y|, |z| \leq M_0. \tag{2.59}$$

There exists

$$\xi \in [0, \tau) \tag{2.60}$$

such that

$$(T - \xi)\tau^{-1} \in \mathbf{Z}. \tag{2.61}$$

By (2.37), there exists $(x_1, u_1) \in X(A, B, 0, \tau)$ such that

$$x_1(0) = y, \; x_1(\tau) = x_f(0), \tag{2.62}$$

$$\int_0^\tau f_0(x_1(t), u_1(t))dt \le v_0(y, x_f(0)) + 1. \tag{2.63}$$

By (2.37), there exists $(x_2, u_2) \in X(A, B, T - \tau, T)$ such that

$$x_2(T - \tau) = x_f(\xi), \; x_2(T) = z, \tag{2.64}$$

$$\int_{T-\tau}^T f_0(x_2(t), u_2(t))dt \le v_0(x_f(\xi), z) + 1. \tag{2.65}$$

It follows from (2.36), (2.55), (2.56), (2.59), and (2.62)–(2.65) that

$$\int_0^\tau f(t, x_1(t), u_1(t))dt \le \int_0^\tau f_0(x_1(t), u_1(t))dt \le M_1 + 1,$$

$$\int_{T-\tau}^T f(t, x_2(t), u_2(t))dt \le \int_{T-\tau}^T f_0(x_2(t), u_2(t))dt \le M_1 + 1. \tag{2.66}$$

Define

$$x(t) = x_1(t), \; u(t) = u_1(t), \; t \in [0, \tau],$$

$$x(t) = x_f(t - \tau\lfloor \tau^{-1}t \rfloor), \; u(t) = u_f(t - \tau\lfloor \tau^{-1}t \rfloor), \; t \in (\tau, T - \tau],$$

$$x(t) = x_2(t), \; u(t) = u_2(t), \; t \in (T - \tau, T]. \tag{2.67}$$

By (2.61), (2.62), (2.64), (2.67),

$$(x, u) \in X(A, B, 0, T), \; x(0) = y, \; x(T) = z.$$

In view of (2.3), (2.47), (2.58), (2.66), and (2.67),

$$\begin{aligned} I^f(0, T, x, u) &= I^f(0, \tau, x_1, u_1) + I^f(T - \tau, T, x_2, u_2) + I^f(\tau, T - \tau, x, u) \\ &\le 2M_1 + 2 + (\lfloor \tau^{-1}T \rfloor - 1)\tau\mu(f) + c_0\tau < T\mu(f) + M. \end{aligned}$$

Proposition 2.35 is proved.

Proposition 2.36. *Let $M, \epsilon > 0$. Then there exists a natural number L such that for each $(x, u) \in X(A, B, 0, L\tau)$ satisfying*

$$I^f(0, L\tau, x, u) \le L\tau\mu(f) + M$$

there exists an integer $i \in [0, L-1]$ such that

$$\sup\{|x(\tau i + t) - x_f(t)| : t \in [0, \tau]\} \le \epsilon.$$

Proof. Assume that the proposition does not hold. Then there exist a strictly increasing sequence of natural numbers $\{L_k\}_{k=1}^{\infty}$ such that $L_k \ge k$ for all integers $k \ge 1$ and a sequence $(x_k, u_k) \in X(A, B, 0, L_k\tau)$, $k = 1, 2, \ldots$ such that for each integer $k \ge 1$ and each integer $i \in [0, L_k - 1]$,

$$I^f(0, L_k\tau, x_k, u_k) \le L_k\tau\mu(f) + M, \tag{2.68}$$

$$\sup\{|x_k(\tau i + t) - x_f(t)| : t \in [0, \tau]\} > \epsilon. \tag{2.69}$$

By Proposition 2.34, there exists $M_* > 0$ such that for each $T > 0$ and each $(x, u) \in X(A, B, 0, T)$,

$$I^f(0, T, x, u) \ge T\mu(f) - M_*. \tag{2.70}$$

Let $p \ge 1$ be an integer. It follows from (2.68) and (2.70) that for each integer $k > p$,

$$\begin{aligned} I^f(0, p\tau, x_k, u_k) &= I^f(0, L_k\tau, x_k, u_k) - I^f(p\tau, L_k\tau, x_k, u_k) \\ &\le L_k\tau\mu(f) + M - (L_k\tau - p\tau)\mu(f) + M_* \\ &\le p\tau\mu(f) + M + M_*. \end{aligned} \tag{2.71}$$

By (2.71) and Proposition 2.30, extracting a subsequence and re-indexing if necessary, we may assume without loss of generality that there exists $(x, u) \in X(A, B, 0, \infty)$ such that for each integer $p \ge 1$,

$$x_k(t) \to x(t) \text{ as } k \to \infty \text{ uniformly on } [0, p\tau], \tag{2.72}$$

$$x_k' \to x' \text{ as } k \to \infty \text{ weakly in } L^1(R^n; (0, p\tau)),$$

$$u_k \to u \text{ as } k \to \infty \text{ weakly in } L^1(R^m; (0, p\tau)),$$

$$I^f(0, p\tau, x, u) \le p\tau\mu(f) + M + M_*. \tag{2.73}$$

In view of (2.73) and Theorem 2.2, $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-good and there exists an integer $i_0 \ge 1$ such that for each integer $i \ge i_0$,

$$\sup\{|x(\tau i + t) - x_f(t)| : t \in [0, \tau]\} \le \epsilon/4. \tag{2.74}$$

Relation (2.72) implies that there exists an integer $k_0 > i_0 + 4$ such that for each integer $k \geq k_0$,

$$|x_k(t) - x(t)| \leq \epsilon/4 \text{ for all } t \in [0, (i_0 + 4)\tau]. \tag{2.75}$$

By (2.74) and (2.75), for each integer $k \geq k_0$ and each $t \in [0, \tau]$,

$$\begin{aligned} |x_f(t) - x_k(i_0\tau + t)| &\leq |x_f(t) - x(i_0\tau + t)| + |x(i_0\tau + t) - x_k(i_0\tau + t)| \\ &\leq \epsilon/2. \end{aligned}$$

This contradicts (2.69). The contradiction we have reached proves Proposition 2.36.

Proposition 2.37. *Any (f, A, B)-overtaking optimal $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-good.*

Proof. Let $(x, u) \in X(A, B, 0, \infty)$ be (f, A, B)-overtaking optimal. By Theorem 2.4, there is a number M_0 such that $M_0 > |x(t)|$ for all $t \geq 0$. Together with Proposition 2.35 this implies the existence of $M_1 > 0$ such that for each $T \geq 3\tau$, $I^f(0, T, x, u) \leq T\mu(f) + M_1$. In view of Theorem 2.2, the pair (x, u) is (f, A, B)-good.

Proposition 2.38. *Let $M, \epsilon > 0$. Then there exists a natural number L such that for each $T \geq L\tau$, each $(x, u) \in X(A, B, 0, T)$ satisfying*

$$I^f(0, T, x, u) \leq T\mu(f) + M \tag{2.76}$$

and each integer S satisfying

$$[S\tau, (S + L)\tau] \subset [0, T] \tag{2.77}$$

there exists an integer $i \in [S, S + L - 1]$ such that

$$|x(\tau i + t) - x_f(t)| \leq \epsilon \text{ for all } t \in [0, \tau]. \tag{2.78}$$

Proof. By Proposition 2.34, there exists $M_* > 0$ such that for each $T > 0$ and each $(x, u) \in X(A, B, 0, T)$,

$$I^f(0, T, x, u) \geq T\mu(f) - M_*. \tag{2.79}$$

By Proposition 2.36, there exists a natural number L such that the following property holds:

(i) for each $(x, u) \in X(A, B, 0, L\tau)$ satisfying

$$I^f(0, L\tau, x, u) \leq L\tau\mu(f) + M + 2M_*$$

there exists an integer $i \in [0, L-1]$ such that $|x(\tau i + t) - x_f(t)| \le \epsilon$ for all $t \in [0, \tau]$.

Assume that $T \ge L\tau$, an $(x, u) \in X(A, B, 0, T)$ satisfies (2.76) and an integer S satisfies (2.77). By the choice of M_* [see (2.79)],

$$I^f(0, S\tau, x, u) \ge S\tau\mu(f) - M_*, \ I^f(\tau(S+L), T, x, u) \ge (T - \tau(S+L))\mu(f) - M_*.$$

Together with (2.76) this implies that

$$\begin{aligned} I^f(\tau S, \tau(S+L), x, u) &= I^f(0, T, x, u) - I^f(0, S\tau, x, u) - I^f((S+L)\tau, T, x, u) \\ &\le T\mu(f) + M - S\mu(f) + M_* - (T - (S+L)\tau)\mu(f) \\ &\quad + M_* \le L\tau\mu(f) + 2M_*. \end{aligned}$$

By the inequality above and property (i), there exists an integer i such that $[i\tau, (i+1)\tau] \subset [S\tau, (S+L)\tau]$ and (2.78) holds. Proposition 2.38 is proved.

Proposition 2.39. *Let $\epsilon \in (0, 1)$. Then there exists $\delta > 0$ such that for each integer $p \ge 1$, each $(x, u) \in X(A, B, 0, p\tau)$ satisfying*

$$|x(0) - x_f(0)|, \ |x(p\tau) - x_f(0)| \le \delta, \ I^f(0, p\tau, x, u) \le \sigma(f, x(0), x(p\tau), p\tau) + \delta$$

and each integer $i \in [0, p-1]$, the inequality $|x(i\tau + t) - x_f(t)| \le \epsilon$ holds for all $t \in [0, \tau]$.

Proof. By the continuity of v, (2.34) and Proposition 2.33, for each integer $k \ge 1$, there is

$$\delta_k \in (0, 4^{-k}\epsilon) \tag{2.80}$$

such that the following properties hold:

(ii) for each $y, z \in R^n$ satisfying $|y - x_f(0)|, |z - x_f(0)| \le \delta_k$ we have $|v(y, z) - \mu(f)\tau| \le 4^{-k}$;

(iii) for each integer $p \ge 1$ and each $y, z \in R^n$ satisfying $|y - x_f(0)|, |z - x_f(0)| \le \delta_k$ we have $\sigma(f, y, z, p\tau) \le p\tau\mu(f) + 4^{-k}$.

We may assume without loss of generality that the sequence $\{\delta_k\}_{k=1}^{\infty}$ is decreasing. Assume that the proposition does not hold. Then for each natural number k there exist an integer $p_k \ge 1$ and $(x_k, u_k) \in X(A, B, 0, p_k\tau)$ satisfying

$$|x_k(0) - x_f(0)| \le \delta_k, \ |x_k(p_k\tau) - x_f(0)| \le \delta_k, \tag{2.81}$$

$$I^f(0, p_k\tau, x_k, u_k) \le \sigma(f, x_k(0), x_k(p_k\tau), p_k\tau) + \delta_k, \tag{2.82}$$

$$\sup\{\sup\{|x_k(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} : \ i = 0, \dots, p_k - 1\} > \epsilon. \tag{2.83}$$

By property (iii) and (2.80)–(2.82), for each integer $k \geq 1$,

$$I^f(0, p_k\tau, x_k, u_k) \leq p_k\tau\mu(f) + 2 \cdot 4^{-k}. \tag{2.84}$$

In view of Proposition 2.32 there exists $(x, u) \in X(A, B, 0, \infty)$ such that

$$x(t) = x_1(t),\ u(t) = u_1(t),\ t \in [0, p_1\tau],\ x((p_1+1)\tau) = x_2(0), \tag{2.85}$$

$$I^f(\tau p_1, (p_1+1)\tau, x, u) = v(x_1(p_1\tau), x_2(0)) \tag{2.86}$$

and for each integer $k \geq 1$,

$$x\left(\sum_{i=1}^{k}(p_i+1)\tau + t\right) = x_{k+1}(t), \tag{2.87}$$

$$u\left(\sum_{i=1}^{k}(p_i+1)\tau + t\right) = u_{k+1}(t), \tag{2.88}$$

$$x\left(\sum_{i=1}^{k+1}(p_i+1)\tau\right) = x_{k+2}(0), \tag{2.89}$$

$$I^f\left(\left(\sum_{i=1}^{k+1}(p_i+1) - 1\right)\tau, \sum_{i=1}^{k+1}(p_i+1)\tau, x, u\right) = v(x_{k+1}(p_{k+1}\tau), x_{k+2}(0)). \tag{2.90}$$

By (2.81), (2.84)–(2.90) and property (ii), for each integer $k \geq 2$,

$$\begin{aligned} I^f\left(0, \left(\sum_{i=1}^{k}(p_i+1)\right)\tau, x, u\right) &= \sum_{i=1}^{k}(I^f(0, p_i\tau, x_i, u_i) + v(x_i(p_i\tau), x_{i+1}(0))) \\ &\leq \sum_{i=1}^{k}[p_i\tau\mu(f) + 2 \cdot 4^{-i} + \mu(f)\tau + 4^{-i}] \\ &\leq \mu(f)\sum_{i=1}^{k}(p_i+1)\tau + 6. \end{aligned}$$

Since the relation above holds for any integer $k \geq 2$ it follows from Theorem 2.2 that the pair (x, u) is (f, A, B)-good and

$$\sup\{|x(i\tau + t) - x_f(t)| :\ t \in [0, \tau]\} \to 0 \text{ as } i \to \infty.$$

Thus there exists an integer $i_0 \geq 1$ such that for each integer $i \geq i_0$,

$$\epsilon \geq |x(i\tau + t) - x_f(t)|, \quad t \in [0, \tau].$$

In view of (2.85)–(2.90) this contradicts (2.83). The contradiction we have reached proves Proposition 2.39.

2.5 Proofs of Theorems 2.9 and 2.10

Proof of Theorem 2.9. In view of Proposition 2.37, (i) implies (ii). By Theorem 2.2, (ii) implies (iii). Clearly, (iv) follows from (iii). Let us show that (iv) implies (i).

Assume that $(x, u) \in X(A, B, 0, \infty)$ is (f, A, B)-minimal and that

$$\liminf_{t \to \infty} |x(t)| < \infty. \tag{2.91}$$

We show that (x, u) is (f, A, B)-overtaking optimal. Since (x, u) is (f, A, B)-minimal it follows from (2.91), Proposition 2.35, and Theorem 2.2 that (x, u) is (f, A, B)-good and that

$$\lim_{i \to \infty,\, i \in \mathbf{Z}} \max\{|x(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} = 0. \tag{2.92}$$

Assume that (x, u) is not (f, A, B)-overtaking optimal. By Theorem 2.3, there exist an (f, A, B)-overtaking optimal $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$, $\epsilon > 0$ and $T_0 > 0$ such that

$$\tilde{x}(0) = x(0), \tag{2.93}$$

$$I^f(0, T, x, u) \geq I^f(0, T, \tilde{x}, \tilde{u}) + \epsilon \text{ for all } T \geq T_0. \tag{2.94}$$

In view of Theorem 2.2 and Proposition 2.37,

$$\lim_{i \to \infty,\, i \in \mathbf{Z}} \max\{|\tilde{x}(i\tau + t) - x_f(t)| : \ t \in [0, \tau]\} = 0. \tag{2.95}$$

Since the function v is continuous there exists $\delta > 0$ such that

$$|v(z_1, z_2) - v(x_f(0), x_f(0))| \leq \epsilon/4 \text{ for all } z_1, z_2 \in R^n$$

$$\text{satisfying } |z_i - x_f(0)| \leq \delta, \ i = 1, 2. \tag{2.96}$$

By (2.92) and (2.95), there exists a natural number $p > T_0/\tau$ such that for all integers $i \geq p$,

$$|\tilde{x}(i\tau) - x_f(0)|, \ |x(i\tau) - x_f(0)| \leq \delta. \tag{2.97}$$

Proposition 2.32 implies that there exists $(x_1, u_1) \in X(A, B, 0, (p+1)\tau)$ such that

$$x_1(t) = \tilde{x}(t),\ u_1(t) = \tilde{u}(t),\ t \in [0, p\tau],\ x_1((p+1)\tau) = x((p+1)\tau),$$
$$I^f(\tau p, \tau(p+1), x_1, u_1) = v(\tilde{x}(p\tau), x((p+1)\tau)). \tag{2.98}$$

By (2.94), (2.96), (2.97), (2.98), and (f, A, B)-minimality of (x, u),

$$\begin{aligned} I^f(0, (p+1)\tau, x_1, u_1) - I^f(0, (p+1)\tau, x, u) &= I^f(0, \tau p, \tilde{x}, \tilde{u}) - I^f(0, \tau p, x, u) \\ &\quad + I^f(\tau p, \tau(p+1), x_1, u_1) \\ &\quad - I^f(\tau p, \tau(p+1) x, u) \\ &\le -\epsilon + v(\tilde{x}(p\tau), x((p+1)\tau)) \\ &\quad - v(x(p\tau), x((p+1)\tau)) \\ &\le -\epsilon + \epsilon/2. \end{aligned}$$

This contradicts (f, A, B)-minimality of (x, u). The contradiction we have reached completes the proof of Theorem 2.9. □

Proof of Theorem 2.10. Since f satisfies assumption (A) with any $\tau > 0$ all our results hold for all $\tau > 0$. By Theorems 2.2, 2.3, and 2.9,

$$\mu(f) = \lim_{T \to \infty} T^{-1} I^f(0, T, x, u),$$

for any (f, A, B)-overtaking optimal $(x, u) \in X(A, B, 0, \infty)$. In view of Proposition 2.1, for any $\tau > 0$, there exists $(x_\tau, u_\tau) \in X(A, B, 0, \tau)$ which is a unique solution of the minimization problem

$$I^f(0, \tau, x, u) \to \min,\ (x, u) \in X(A, B, 0, \tau),\ x(0) = x(\tau), \tag{2.99}$$
$$\tau\mu(f) = I^f(0, \tau, x_\tau, u_\tau). \tag{2.100}$$

By (2.99) and (2.100), for any $\tau > 0$, $x_\tau(t) = x_{\tau/2}(t),\ t \in [0, \tau/2]$ and $x_\tau(\tau/2+t) = x_\tau(t),\ t \in [0, \tau/2]$. Since the relation above holds for any $\tau > 0$ we conclude that for any $\tau > 0$, $x_\tau(\cdot)$ is a constant function. In view of Theorem 2.2, $x_\tau(0)$ does not depend on τ. Theorem 2.10 is proved. □

2.6 Auxiliary Results for Theorem 2.12

Proposition 2.40. *Let $M_1 > 0$, $0 < \tau_0 < \tau_1$. Then there exists $M_2 > 0$ such that for each $g \in \mathcal{M}$, each $T_2 > T_1 \ge 0$ satisfying*

$$T_2 - T_1 \in [\tau_0, \tau_1] \tag{2.101}$$

and each $(x,u) \in X(A,B,T_1,T_2)$ *satisfying*

$$I^g(T_1,T_2,x,u) \le M_1 \tag{2.102}$$

the following inequality holds:

$$|x(t)| \le M_2 \textit{ for all } t \in [T_1,T_2]. \tag{2.103}$$

Proof. Fix

$$\delta \in (0, \min\{8^{-1}\tau_0, (2\|A\|+2)^{-1}\}). \tag{2.104}$$

By (2.11) and (2.13), there exists $c_0 > 1$ such that

$$g(t,x,u) \ge 8|u|(\|B\|+1) \tag{2.105}$$

for each $g \in \mathcal{M}$ and each $(t,x,u) \in [0,\infty) \times R^n \times R^m$ satisfying $|u| \ge c_0$ and $h_0 > 0$ such that

$$g(t,x,u) \ge 4M_1(\min\{1,\tau_0\})^{-1}\delta^{-1} + 2a\tau_1\delta^{-1} \tag{2.106}$$

for each $g \in \mathcal{M}$ and each $(t,x,u) \in [0,\infty) \times R^n \times R^m$ satisfying $|x| \ge h_0$. Fix

$$M_2 > 2 + 2M_1 + 2a\tau_1 + 2c_0(1+\tau_1)\|B\| + 2h_0. \tag{2.107}$$

Let $g \in \mathcal{M}$, $T_2 > T_1 \ge 0$ satisfy (2.101) and let $(x,u) \in X(A,B,T_1,T_2)$ satisfy (2.102). We show that (2.103) holds.

Assume the contrary. Then there exists $t_0 \in [T_1,T_2]$ such that

$$|x(t_0)| > M_2. \tag{2.108}$$

By the choice of h_0 [see (2.106)], (2.13), (2.102) and (2.104), there exists $t_1 \in [T_1,T_2]$ satisfying

$$|x(t_1)| \le h_0, \ |t_1 - t_0| \le \delta. \tag{2.109}$$

There exists a number t_2 such that

$$\min\{t_0,t_1\} \le t_2 \le \max\{t_0,t_1\},$$

$$|x(t_2)| \ge |x(t)|, \ t \in [\min\{t_0,t_1\}, \max\{t_0,t_1\}]. \tag{2.110}$$

It follows from (2.1), (2.109), and (2.110) that

$$|x(t_1) - x(t_2)| = \left| \int_{t_1}^{t_2} x'(t)dt \right| \leq \|A\| \left| \int_{t_1}^{t_2} |x(t)|dt \right| + \|B\| \left| \int_{t_1}^{t_2} |u(t)|dt \right|$$
$$\leq \|A\||x(t_2)|\delta + \|B\| \left| \int_{t_1}^{t_2} |u(t)|dt \right|. \tag{2.111}$$

By the choice of c_0 (see (2.105)), (2.13), (2.101), (2.102) and (2.109),

$$\left| \int_{t_1}^{t_2} |u(t)|dt \right| \leq \left| \int_{t_1}^{t_2} [8^{-1}g(t, x(t), u(t))(\|B\| + 1)^{-1} + c_0]dt \right|$$
$$\leq c_0|t_1 - t_2| + 8^{-1}a\tau_1(\|B\| + 1)^{-1} + 8^{-1}(\|B\| + 1)^{-1}I^g(T_1, T_2, x, u)$$
$$\leq c_0\delta + a\tau_1 8^{-1}(\|B\| + 1)^{-1} + 8^{-1}(\|B\| + 1)^{-1}M_1.$$

By this relation, (2.104) and (2.111),

$$|x(t_1) - x(t_2)| \leq 2^{-1}|x(t_2)| + \|B\|c_0\delta + a\tau_1 + M_1.$$

Combined with (2.108) and (2.109) this implies that $2^{-1}M_2 - h_0 \leq \|B\|c_0\delta + a\tau_1 + M_1$. This contradicts (2.107). The contradiction we have reached proves Proposition 2.40.

Proposition 2.41. *Let $0 < c_1 < c_2$, $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{M}$ such that for each $g \in V$, each $T_2 > T_1 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying*

$$\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \leq D \tag{2.112}$$

the inequality $|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \leq \epsilon$ holds.

Proof. By Proposition 2.40, there exists $S > 0$ such that

$$|x(t)| \leq S \text{ for all } t \in [T_1, T_2] \tag{2.113}$$

for each $g \in \mathcal{M}$, each $T_2 > T_1 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying $I^g(T_1, T_2, x, u) \leq D + 1$.

Choose $\delta \in (0, 1)$, $N > S$, $\Gamma > 1$ such that

$$\delta(c_2 + 1) \leq 4^{-1}\epsilon, \ \psi(N)N > 4a, \ (\Gamma - 1)(c_2 + D + a(c_2 + 1)) \leq \epsilon/4 \tag{2.114}$$

and set

$$V = \{g \in \mathcal{M} : \ (f, g) \in E(N, \delta, \Gamma)\}. \tag{2.115}$$

Assume that

$$g \in V,\ T_2 > T_1 \geq 0,\ T_2 - T_1 \in [c_1, c_2] \tag{2.116}$$

and that $(x, u) \in X(A, B, T_1, T_2)$ satisfies (2.112). By the choice of S, (2.112) and (2.116), (2.113) is true. Set

$$E_1 = \{t \in [T_1, T_2] :\ |u(t)| \leq N\},\ E_2 = [T_1, T_2] \setminus E_1. \tag{2.117}$$

By (2.113), (2.115), (2.116), (2.117), and the inequality $N > S$,

$$|f(t, x(t), u(t)) - g(t, x(t), u(t))| \leq \delta,\ t \in E_1. \tag{2.118}$$

Set

$$h(t) = \min\{f(t, x(t), u(t)),\ g(t, x(t), u(t))\},\ t \in [T_1, T_2]. \tag{2.119}$$

In view of (2.13), (2.113), (2.114), (2.115), (2.116), (2.117), (2.119), and the inequality $N > S$, for all $t \in E_2$,

$$(f(t, x(t), u(t)) + 1)(g(t, x(t), u(t)) + 1)^{-1} \in [\Gamma^{-1}, \Gamma]$$

and

$$|f(t, x(t), u(t)) - g(t, x(t), u(t))| \leq (\Gamma - 1)(h(t) + 1). \tag{2.120}$$

Relations (2.13), (2.112), (2.114), and (2.116)–(2.120) imply that

$$\begin{aligned}
|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \leq & \int_{E_1} |f(t, x(t), u(t)) - g(t, x(t), u(t))| dt \\
& + \int_{E_2} |f(t, x(t), u(t)) - g(t, x(t), u(t))| dt \\
& \leq \delta c_2 + (\Gamma - 1) \int_{E_2} (h(t) + 1) dt \leq \delta c_2 \\
& + (\Gamma - 1) c_2 + (\Gamma - 1)(D + a c_2) \leq \epsilon.
\end{aligned}$$

Proposition 2.41 is proved.

2.7 Proof of Theorem 2.12

By Proposition 2.39, there exists $\delta_0 \in (0, 1/8)$ such that the following property holds:

(P1) for each integer $p \geq 1$, each $(x, u) \in X(A, B, 0, p\tau)$ satisfying

$$|x(0) - x_f(0)|, \ |x(p\tau) - x_f(0)| \leq 4\delta_0,$$
$$I^f(0, p\tau, x, u) \leq \sigma(f, x(0), x(p\tau), p\tau) + 4\delta_0$$

and each integer $i \in [0, p-1]$, the inequality $|x(i\tau + t) - x_f(t)| \leq \epsilon$ holds for all $t \in [0, \tau]$.

By Proposition 2.38, there exists an integer $L_0 \geq 5$ such that the following property holds:

(P2) for each $T \geq (L_0 - 4)\tau$, each $(x, u) \in X(A, B, 0, T)$ satisfying

$$I^f(0, T, x, u) \leq T\mu(f) + M + 4$$

and each integer S satisfying $[S\tau, (S + L_0 - 4)\tau] \subset [0, T]$ there exists an integer $i \in [S, S + L_0 - 5]$ such that $|x(\tau i + t) - x_f(t)| \leq \delta_0$ for all $t \in [0, \tau]$.

Let an integer $L_1 \geq L_0$. By Proposition 2.35, there exists a number $M_0 > 0$ such that for each $S \geq 3\tau$ and each $y, z \in R^n$ satisfying $|y|, |z| \leq \max\{|x_f(t)| : \ t \in [0, \tau]\} + 4$,

$$\sigma(f, y, z, S) \leq S\mu(f) + M_0. \tag{2.121}$$

By Proposition 2.41, there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following property holds:

(P3) for each $g \in \mathcal{U}$, each $T_1 \geq 0$, each $T_2 \in [T_1 + 1, T_1 + 4L_1]$ and each $(x, u) \in X(A, B, T_1\tau, T_2\tau)$ satisfying

$$\min\{I^f(T_1\tau, T_2\tau, x, u), I^g(T_1\tau, T_2\tau, x, u)\} \leq (4L_1|\mu(f)| + M + M_0 + 1)(\tau + 1),$$
$$|I^f(T_1\tau, T_2\tau, x, u) - I^g(T_1\tau, T_2\tau, x, u)| \leq \delta_0.$$

Assume that $T > 2L_1\tau$, $g \in \mathcal{U}$, $(x, u) \in X(A, B, 0, T)$ and that a finite sequence of integers $\{S_i\}_{i=0}^q$ satisfy

$$S_0 = 0, \ S_{i+1} - S_i \in [L_0, L_1], \ i = 0, \ldots, q-1, \ S_q\tau \in (T - L_1\tau, T], \tag{2.122}$$
$$I^g(S_i\tau, S_{i+1}\tau, x, u) \leq (S_{i+1} - S_i)\tau\mu(f) + M \tag{2.123}$$

for each integer $i \in [0, q-1]$,

$$I^g(S_i\tau, S_{i+2}\tau, x, u) \leq \sigma(g, x(S_i\tau), x(S_{i+2}\tau), S_i\tau, S_{i+2}\tau) + \delta_0 \tag{2.124}$$

for each nonnegative integer $i \leq q-2$ and

$$I^g(S_{q-2}\tau, T, x, u) \leq \sigma(g, x(S_{q-2}\tau), x(T), S_{q-2}\tau, T) + \delta_0. \tag{2.125}$$

Let $i \in [0, q-1]$ be an integer. By (2.122), (2.123) and the choice of $\mathcal{U}$ (see property (P3)),

$$I^f(S_i\tau, S_{i+1}\tau, x, u) \le I^g(S_i\tau, S_{i+1}\tau, x, u) + \delta_0 \le (S_{i+1} - S_i)\tau\mu(f) + M + 1.$$

The inequality above, (2.122) and property (P2) imply that there exists an integer p_i such that

$$p_i \in [S_i + 3, S_i + L_0],\ |x(p_i\tau) - x_f(0)| \le \delta_0. \tag{2.126}$$

Let an integer $i \in [0, q-2]$. In view of (2.122) and (2.126),

$$p_i,\ p_{i+1} \in [S_i + 3, S_{i+2}],\ 3 \le p_{i+1} - p_i \le 2L_1. \tag{2.127}$$

It follows from (2.124) and (2.127) that

$$I^g(p_i\tau, p_{i+1}\tau, x, u) \le \sigma(g, x(p_i\tau), x(p_{i+1}\tau), p_i\tau, p_{i+1}\tau) + \delta_0 \tag{2.128}$$

Thus we have shown that there exists a strictly increasing sequence of nonnegative integers $\{p_i\}_{i=0}^k$ where k is a natural number such that

$$p_0 \le L_0,\ p_k\tau > T - 2\tau L_1,\ |x(p_i\tau) - x_f(0)| \le \delta_0,\ i = 0, \dots, k,$$

$$3 \le p_{i+1} - p_i \le 2L_1,\ i = 0, \dots, k-1 \tag{2.129}$$

and (2.128) holds for all $i = 0, \dots, k-1$. It is not difficult to see that if $|x(0) - x_f(0)| \le \delta_0$, then we may assume that $p_0 = 0$ and if $|x(\lfloor T/\tau \rfloor\tau) - x_f(0)| \le \delta_0$, then we may assume that $p_k = \lfloor T/\tau \rfloor$.

Let $i \in \{0, \dots, k-1\}$. By (2.129) and the choice of M_0 (see (2.121)),

$$\sigma(f, x(p_i\tau), x(p_{i+1}\tau), (p_{i+1} - p_i)\tau) \le \mu(f)(p_{i+1} - p_i)\tau + M_0. \tag{2.130}$$

Combined with (2.129) and the choice of $\mathcal{U}$ (see property (P3)) this implies that

$$|\sigma(f, x(p_i\tau), x(p_{i+1}\tau), (p_{i+1} - p_i)\tau) - \sigma(g, x(p_i\tau), x(p_{i+1}\tau), p_i\tau, p_{i+1}\tau)| \le \delta_0.$$

The inequality above and (2.128) imply that

$$\begin{aligned} I^g(p_i\tau, p_{i+1}\tau, x, u) &\le \sigma(f, x(p_i\tau), x(p_{i+1}\tau), (p_{i+1} - p_i)\tau) + 2\delta_0 \\ &\le \mu(f)(p_{i+1} - p_i)\tau + M_0 + 1. \end{aligned}$$

Together with (2.129) and the choice of $\mathcal{U}$ (see property (P3)) this implies that

$$\begin{aligned} I^f(p_i\tau, p_{i+1}\tau, x, u) &\le I^g(p_i\tau, p_{i+1}\tau, x, u) + \delta_0 \\ &\le \sigma(f, x(p_i\tau), x(p_{i+1}\tau), (p_{i+1} - p_i)\tau) + 3\delta_0. \end{aligned}$$

By the relation above, (2.129) and property (P1), for all $j \in \{p_i, \dots, p_{i+1} - 1\}$,

$$|x(\tau j + t) - x_f(t)| \le \epsilon,\ t \in [0, \tau].$$

Thus the relation above holds for all $j \in \{p_0, \dots, p_k - 1\}$ and Theorem 2.12 is proved. □

2.8 Basic Lemma for Theorem 2.13

Lemma 2.42. *Let $\epsilon \in (0, 1)$, $M_0, M_1 > 0$. Then there exist an integer $L \ge 1$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following assertion holds.*

Assume that $T > L\tau$, $g \in \mathcal{U}$, integers S_1, S_2 satisfy

$$0 \le S_1 \le S_2 - L,\ [S_1\tau, S_2\tau] \subset [0, T] \tag{2.131}$$

and $(x, u) \in X(A, B, 0, T)$ satisfies at least one of the following conditions:

$$\text{(a) } |x(0)|,\ |x(T)| \le M_0,\ I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$$

$$\text{(b) } |x(0)| \le M_0,\ I^g(0, T, x, u) \le \sigma(g, x(0), 0, T) + M_1;$$

$$\text{(c) } I^g(0, T, x, u) \le \sigma(g, 0, T) + M_1.$$

Then

$$\min\{|x(i\tau) - x_f(0)| :\ i = S_1, \dots, S_2\} \le \epsilon. \tag{2.132}$$

Proof. By Proposition 2.38 there exists a natural number L_0 such that the following property holds:

(P4) for each $T \ge L_0\tau$, each $(x, u) \in X(A, B, 0, T)$ satisfying

$$I^f(0, T, x, u) \le T\mu(f) + 16(1 + a)(\tau + 1)$$

and each integer S satisfying $[S\tau, (S + L_0)\tau] \subset [0, T]$,

$$\min\{|x(\tau i) - x_f(0)| :\ i = S, \dots, S + L_0 - 1\} \le \epsilon.$$

We may assume without loss of generality that

$$M_0 > \sup\{|x_f(t)| :\ t \in [0, \tau]\} + 4. \tag{2.133}$$

We use the functions f_0 and v_0 introduced in Sect. 2.4 [see (2.36) and (2.37)]. We may assume without loss of generality that

$$M_1 > \sup\{|v_0(z_1, z_2)| : \ z_1, z_2 \in R^n, \ |z_1|, |z_2| \le M_0\}. \tag{2.134}$$

By Proposition 2.35, there exists a number $M_2 > M_1 + M_0$ such that for each $S \ge 3\tau$ and each $y, z \in R^n$ satisfying $|y|, |z| \le M_0$,

$$\sigma(f, y, z, S) \le S\mu(f) + M_2. \tag{2.135}$$

Choose a natural number l such that

$$\begin{aligned} \min\{1, \tau\}l > 4 + M_1 + 4(\tau + 1)\min\{1, \tau\}^{-1} + (\tau + 1)|\mu(f)|(2L_0 + M_2 + 4) \\ + (2L_0 + 18)(1 + a)(1 + \tau) + 2M_2 + 18 \\ + a(L_0\tau + \tau + 1) + \tau + a + 1 + \tau a \end{aligned} \tag{2.136}$$

and set

$$L = 2(L_0 + 1)l. \tag{2.137}$$

By Proposition 2.41, there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following property holds:

(P5) for each $g \in \mathcal{U}$, each $T_1 \ge 0$, each $T_2 \in [T_1 + \min\{1, \tau\}, T_1 + 4L\max\{1, \tau\}]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying

$$\begin{aligned} &\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \\ &\le ((4L + 2)|\mu(f)| + 4M_2 + 4 + 16(1 + a))(\tau + 1), \\ &|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le (4L)^{-1}(\tau + 1)^{-1}. \end{aligned}$$

Assume that

$$T > L\tau, \ g \in \mathcal{U}, \tag{2.138}$$

integers S_1, S_2 satisfy (2.131) and $(x, u) \in X(A, B, 0, T)$ satisfies at least one of the conditions (a), (b), (c). We show that (2.132) holds. Assume the contrary. Then

$$|x(i\tau) - x_f(0)| > \epsilon, \ i = S_1, \ldots, S_2. \tag{2.139}$$

We may assume without loss of generality that at least one of the following conditions hold:

$$S_1 = 0, \ S_2 = \lfloor T\tau^{-1} \rfloor; \tag{2.140}$$

$$S_1 \ge 1, \ |x((S_1 - 1)\tau) - x_f(0)| \le \epsilon, \ S_2 = \lfloor T\tau^{-1} \rfloor; \tag{2.141}$$

$$S_1 = 0,\ S_2 < \lfloor T\tau^{-1}\rfloor,\ |x((S_2+1)\tau) - x_f(0)| \le \epsilon \tag{2.142}$$

$$S_1 \ge 1,\ S_2 < \lfloor T\tau^{-1}\rfloor,\ |x(j\tau) - x_f(0)| \le \epsilon, j = S_1 - 1, S_2 + 1. \tag{2.143}$$

In view of (2.131) and (2.137),

$$\lfloor (S_2 - S_1)L_0^{-1}\rfloor \ge \lfloor LL_0^{-1}\rfloor \ge 2l. \tag{2.144}$$

It follows from (2.13) that

$$\begin{aligned} I^g(S_1\tau, S_2\tau, x, u) &= I^g(S_1\tau, S_1\tau + \lfloor (S_2 - S_1)L_0^{-1}\rfloor L_0\tau, x, u) \\ &\quad + I^g(S_1\tau + \lfloor (S_2 - S_1)L_0^{-1}\rfloor L_0\tau, S_2\tau, x, u) \\ &\ge \sum_{i=0}^{\lfloor (S_2 - S_1)L_0^{-1}\rfloor - 1} I^g((S_1 + iL_0)\tau, (S_1 + (i+1)L_0)\tau, x, u) - aL_0\tau. \end{aligned} \tag{2.145}$$

Let

$$j \in \{0, \dots, \lfloor (S_2 - S_1)L_0^{-1}\rfloor - 1\}. \tag{2.146}$$

By (2.139), (2.146) and property (P4),

$$I^f((S_1 + jL_0)\tau, (S_1 + (j+1)L_0)\tau, x, u) > L_0\tau\mu(f) + 16(1+a)(\tau+1). \tag{2.147}$$

We show that

$$I^g((S_1 + jL_0)\tau, (S_1 + (j+1)L_0)\tau, x, u) \ge L_0\tau\mu(f) + 16(1+a)(\tau+1) - 1. \tag{2.148}$$

Assume the contrary. Then

$$I^g((S_1 + jL_0)\tau, (S_1 + (j+1)L_0)\tau, x, u) < L_0\tau\mu(f) + 16(1+a)(\tau+1) - 1.$$

Combined with the choice of $\mathcal{U}$ [see property (P5)], (2.137) and (2.138) this implies that

$$\begin{aligned} &I^f((S_1 + jL_0)\tau, (S_1 + (j+1)L_0)\tau, x, u) \\ &\quad \le 1 + I^g((S_1 + jL_0)\tau, (S_1 + (j+1)L_0)\tau, x, u) \le L_0\tau\mu(f) + 16(1+a)(\tau+1). \end{aligned}$$

This contradicts (2.147). The contradiction we have reached proves (2.148). Thus (2.148) holds for all $j \in \{0, \dots, \lfloor (S_2 - S_1)L_0^{-1}\rfloor - 1\}$. Set

$$\begin{aligned} &z_0 = x(0) \text{ if } |x(0)| \le M_0,\ z_0 = 0 \text{ if } |x(0)| > M_0, \\ &z_1 = x(T) \text{ if } |x(T)| \le M_0,\ z_1 = 0 \text{ if } |x(T)| > M_0. \end{aligned} \tag{2.149}$$

It is not difficult to see that there exists $(x_1, u_1) \in X(A, B, 0, T)$ such that:
if (2.140) holds, then

$$x_1(0) = z_0,\ x_1(\tau) = x_f(0),\ I^f(0, \tau, x_1, u_1) \le v_0(z_0, x_f(0)) + 1,$$
$$x_1(T - \tau) = x_f(T - \lfloor \tau^{-1} T \rfloor \tau),\ x_1(T) = z_1,$$
$$I^f(T - \tau, T, x_1, u_1) \le v_0(x_f(T - \lfloor \tau^{-1} T \rfloor \tau), z_1) + 1,$$
$$x_1(t) = x_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ u_1(t) = u_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ t \in [\tau, T - \tau];$$

if (2.141) holds, then

$$x_1(t) = x(t),\ u_1(t) = u(t),\ t \in [0, \tau(S_1 - 1)],\ x_1(T - \tau) = x_f(T - \lfloor \tau^{-1} T \rfloor \tau),$$
$$x_1(T) = z_1,\ I^f(T - \tau, T, x_1, u_1) \le v_0(x_f(T - \lfloor \tau^{-1} T \rfloor \tau), z_1) + 1,$$
$$x_1(t) = x_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ u_1(t) = u_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ t \in [\tau S_1, T - \tau];$$
$$I^f(\tau(S_1 - 1), \tau S_1, x_1, u_1) \le v_0(x_1(\tau(S_1 - 1)), x_1(\tau S_1)) + 1;$$

if (2.142) holds, then

$$x_1(0) = z_0,\ x_1(\tau) = x_f(0),\ I^f(0, \tau, x_1, u_1) \le v_0(z_0, x_f(0)) + 1,$$
$$x_1(t) = x(t),\ u_1(t) = u(t),\ t \in [\tau(S_2 + 1), T],$$
$$x_1(t) = x_f(t - [\tau^{-1} t] \tau),\ u_1(t) = u_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ t \in [\tau, S_2 \tau],$$
$$I^f(\tau S_2, \tau(S_2 + 1), x_1, u_1) \le v_0(x_1(\tau S_2), x_1(\tau(S_2 + 1))) + 1;$$

if (2.143) holds, then

$$x_1(t) = x(t),\ u_1(t) = u(t),\ t \in [0, \tau(S_1 - 1)] \cup [\tau(S_2 + 1), T],$$
$$x_1(t) = x_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ u_1(t) = u_f(t - \lfloor \tau^{-1} t \rfloor \tau),\ t \in [\tau(S_1 + 1), \tau S_2],$$
$$I^f((S_1 - 1)\tau, S_1 \tau, x_1, u_1) \le v_0(x_1(S_1 - 1)\tau, x_1(S_1 \tau)) + 1,$$
$$I^f(\tau S_2, \tau(S_2 + 1), x_1, u_1) \le v_0(x_1(\tau S_2), x_1(\tau(S_2 + 1))) + 1.$$

In view of (2.149), conditions (a), (b), (c) and the choice of (x_1, u_1),

$$I^g(0, T, x, u) \le I^g(0, T, x_1, u_1) + M. \tag{2.150}$$

We consider the cases (2.140)–(2.143) separately and obtain a lower bound for $I^g(0, T, x, u) - I^g(0, T, x_1, u_1)$. Assume that (2.140) holds. By (2.13), (2.140) and (2.148),

$$
\begin{aligned}
I^g(0,T,x,u) &\geq I^g(0,\lfloor T\tau^{-1}\rfloor\tau,x,u)-\tau a\\
&\geq I^g(0,\lfloor\lfloor T\tau^{-1}\rfloor L_0^{-1}\rfloor L_0\tau,x,u)-\tau a(L_0+1)\\
&= \sum_{j=0}^{\lfloor\lfloor T\tau^{-1}\rfloor L_0^{-1}\rfloor-1} I^g((jL_0)\tau,(j+1)L_0\tau,x,u)-(L_0+1)a\tau\\
&\geq \lfloor L_0\tau\mu(f)+16(1+a)(1+\tau)-1\rfloor\lfloor\lfloor T\tau^{-1}\rfloor L_0^{-1}\rfloor-(L_0+1)\tau a\\
&\geq \lfloor L_0\tau\mu(f)+16(1+a)(1+\tau)-1\rfloor\lfloor T\tau^{-1}\rfloor L_0^{-1}\\
&\quad -L_0\tau\mu(f)-16(1+a)(1+\tau)-(L_0+1)\tau a\\
&\geq \lfloor L_0\tau\mu(f)+16(1+a)(1+\tau)-1\rfloor T\tau^{-1}L_0^{-1}\\
&\quad -2(L_0\tau\mu(f)+16(1+a)(1+\tau)+(L_0+1)\tau a)\\
&\geq T\mu(f)+TL_0^{-1}8(1+a)-2(L_0\tau\mu(f)+(L_0+17)(1+a)(1+\tau)).
\end{aligned}
\tag{2.151}
$$

Clearly,

$$
I^g(0,T,x_1,u_1)=I^g(0,\tau,x_1,u_1)+I^g(\tau,T-\tau,x_1,u_1)+I^g(T-\tau,T,x_1,u_1).
\tag{2.152}
$$

By (2.133), (2.134), (2.140), (2.149), and the choice of (x_1,u_1)

$$
\begin{aligned}
I^f(0,\tau,x_1,u_1) &\leq v_0(z_0,x_f(0))+1\leq M_1+1,\\
I^f(T-\tau,T,x_1,u_1) &\leq v_0(x_f(T-\lfloor\tau^{-1}T\rfloor\tau),z_1)+1\leq M_1+1.
\end{aligned}
$$

In view of these inequalities, (2.138) and property (P5),

$$
I^g(0,\tau,x_1,u_1),\ I^g(T-\tau,T,x_1,u_1)\leq M_1+5/4.
\tag{2.153}
$$

It follows from (2.3), (2.138), (2.140), and the choice of (x_1,u_1) that

$$
\begin{aligned}
I^g(\tau,T-\tau,x_1,u_1) &= I^g(\tau,(\lfloor T\tau^{-1}\rfloor-1)\tau,x_1,u_1)\\
&\quad + I^g((\lfloor T\tau^{-1}\rfloor-1)\tau,T-\tau,x_1,u_1)\\
&= \sum_{i=1}^{\lfloor T\tau^{-1}\rfloor-2} I^g(i\tau,(i+1)\tau,x_1,u_1)\\
&\quad + I^g((\lfloor T\tau^{-1}\rfloor-1)\tau,T-\tau,x_1,u_1),
\end{aligned}
\tag{2.154}
$$

for all integers $i=1,\dots,\lfloor\tau^{-1}T\rfloor-2$,

$$
I^f(i\tau,(i+1)\tau,x_1,u_1)=I^f(0,\tau,x_f,u_f)=\tau\mu(f)
$$

and in view of property (P5),

$$I^g(i\tau,(i+1)\tau,x_1,u_1)\le \tau\mu(f)+(\tau+1)^{-1}(4L)^{-1}.$$

This implies that

$$\sum_{i=1}^{\lfloor T\tau^{-1}\rfloor-2} I^g(i\tau,(i+1)\tau,x_1,u_1)\le T\mu(f)+(\tau+1)^{-1}(4L)^{-1}(T/\tau)+3\tau|\mu(f)|. \tag{2.155}$$

By (2.13), (2.138), (2.140), property (P5), and the choice of (x_1,u_1),

$$\begin{aligned}
&I^g((\lfloor T\tau^{-1}\rfloor-1)\tau,T-\tau,x_1,u_1)\\
&\quad=\int_{(\lfloor T\tau^{-1}\rfloor-1)\tau}^{T-\tau} g(t,x_f(t-(\lfloor\tau^{-1}T\rfloor-1)\tau),u_f(t-(\lfloor\tau^{-1}T\rfloor-1)\tau))dt\\
&\quad\le\int_{(\lfloor T\tau^{-1}\rfloor-1)\tau}^{\lfloor T/\tau\rfloor\tau} g(t,x_f(t-(\lfloor\tau^{-1}T\rfloor-1)\tau),u_f(t-(\lfloor\tau^{-1}T\rfloor-1)\tau))dt+\tau a\\
&\quad\le I^f(0,\tau,x_f,u_f)+1+\tau a.
\end{aligned}$$

Combined with (2.152)–(2.155) this implies that

$$\begin{aligned}
I^g(0,T,x_1,u_1)&\le 2(M_2+1)+I^g(\tau,T-\tau,x_1,u_1)\\
&\le 2(M_2+2)+T\mu(f)+(\tau+1)^{-1}(4L)^{-1}(T/\tau)\\
&\quad+3\tau|\mu(f)|+\tau\mu(f)+\tau a+1.
\end{aligned}$$

The relation above, (2.136)–(2.138), (2.150), and (2.151) imply that

$$\begin{aligned}
M_1&\ge I^g(0,T,x,u)-I^g(0,T,x_1,u_1)\\
&\ge T\mu(f)+8TL_0^{-1}(1+a)-2[L_0\tau\mu(f)+(L_0+17)(1+a)(\tau+1)]\\
&\quad-T\mu(f)-2M_2-4-(\tau+1)^{-1}(4L)^{-1}T\tau^{-1}-4\tau|\mu(f)|-1-\tau a\\
&=T[L_0^{-1}8(1+a)-(4L)^{-1}(\tau+1)^{-1}\tau^{-1}]\\
&\quad-2L_0\tau|\mu(f)|-2(L_0+17)(1+a)(\tau+1)-2M_2-4-4\tau|\mu(f)|-1-\tau a\\
&\ge 4TL_0^{-1}(1+a)-\tau|\mu(f)|(2L_0+4)\\
&\quad-2(L_0+17)(1+a)(\tau+1)-2M_2-5-\tau a\\
&\ge 4l\tau-\tau|\mu(f)|(2L_0+4)-2(L_0+17)(1+a)(\tau+1)-2M_2-5-\tau a.
\end{aligned}$$

This contradicts (2.137). Thus if (2.140) holds we have reached a contradiction.

Assume that (2.141) holds. By (2.141) and the choice of (x_1, u_1),

$$I^g(0, T, x, u) - I^g(0, T, x_1, u_1) = I^g(\tau(S_1 - 1), T, x, u) - I^g(\tau(S_1 - 1), T, x_1, u_1). \tag{2.156}$$

It follows from (2.13), (2.136)–(2.138), (2.141), and (2.148) that

$$\begin{aligned} I^g(\tau(S_1 - 1), T, x, u) &\geq -a + I^g(\tau S_1, \lfloor T\tau^{-1} \rfloor \tau, x, u) - \tau a \\ &\geq I^g(\tau S_1, \tau S_1 + \lfloor \lfloor T\tau^{-1} - S_1 \rfloor L_0^{-1} \rfloor L_0 \tau, x, u) \\ &\quad - L_0 \tau a - a(\tau + 1) = -a(L_0 \tau + \tau + 1) \\ &\quad + \sum_{j=0}^{\lfloor \lfloor T\tau^{-1} - S_1 \rfloor L_0^{-1} \rfloor - 1} I^g(S_1 \tau + jL_0 \tau, S_1 \tau + (j+1) L_0 \tau, x, u) \\ &\geq -a(L_0 \tau + \tau + 1) + \lfloor \lfloor T\tau^{-1} - S_1 \rfloor L_0^{-1} \rfloor (L_0 \tau \mu(f) \\ &\quad + 16(1 + a)(1 + \tau) - 1) \geq -a(L_0 \tau + \tau + 1) \\ &\quad - (L_0 |\mu(f)| + 16(1 + a)(1 + \tau)) + (\lfloor T\tau^{-1} \rfloor - S_1) \tau \mu(f) \\ &\quad + (\lfloor T\tau^{-1} \rfloor - S_1) L_0^{-1} (16(1 + a)(1 + \tau) - 1). \end{aligned} \tag{2.157}$$

It is clear that

$$I^g((S_1 - 1)\tau, T, x_1, u_1) = I^g(\tau(S_1 - 1), T - \tau, x_1, u_1) + I^g(T - \tau, T, x_1, u_1). \tag{2.158}$$

By the choice of (x_1, u_1), (2.133), (2.134), (2.141), and (2.149),

$$\begin{aligned} I^f(T - \tau, T, x_1, u_1) &\leq v_0(x_f(T - \lfloor \tau^{-1} T \rfloor \tau), z_1) + 1 \leq M_1 + 1, \\ I^f(\tau(S_1 - 1), \tau S_1, x_1, u_1) &\leq v_0(x_1(\tau(S_1 - 1)), x_1(\tau S_1)) + 1 \leq M_1 + 1. \end{aligned}$$

Together with (2.138) and property (P5) this implies that

$$I^g(T - \tau, T, x_1, u_1),\ I^g(\tau(S_1 - 1), \tau S_1, x_1, u_1) \leq M_1 + 2. \tag{2.159}$$

Clearly,

$$\begin{aligned} I^g((S_1 - 1)\tau, T - \tau, x_1, u_1) &= I^g(S_1 \tau, \lfloor T\tau^{-1} \rfloor \tau - \tau, x_1, u_1) \\ &\quad + I^g((S_1 - 1)\tau, S_1 \tau, x_1, u_1) \\ &\quad + I^g(\lfloor T\tau^{-1} \rfloor \tau - \tau, T - \tau, x_1, u_1). \end{aligned} \tag{2.160}$$

It follows from (2.141) and the choice of (x_1, u_1) that for each

$$j \in \{S_1, \dots, \lfloor \tau^{-1} T \rfloor - 2\},$$
$$I^f(j\tau, (j+1)\tau, x_1, u_1) = I^f(0, \tau, x_f, u_f) = \tau\mu(f)$$

and in view of property (P5),

$$I^g(j\tau, (j+1)\tau, x_1, u_1) \le \tau\mu(f) + (\tau+1)^{-1}(4L)^{-1}. \tag{2.161}$$

By (2.13), (2.138), (2.141), property (P5), and the choice of (x_1, u_1),

$$\begin{aligned}
&I^g((\lfloor T\tau^{-1} \rfloor - 1)\tau, T - \tau, x_1, u_1) \\
&\quad = \int_{(\lfloor T\tau^{-1} \rfloor - 1)\tau}^{T-\tau} g(t, x_f(t - (\lfloor \tau^{-1} T \rfloor - 1)\tau), u_f(t - (\lfloor \tau^{-1} T \rfloor - 1)\tau))dt \\
&\quad \le \int_{(\lfloor T\tau^{-1} \rfloor - 1)\tau}^{\lfloor T/\tau \rfloor \tau} g(t, x_f(t - (\lfloor \tau^{-1} T \rfloor - 1)\tau), u_f(t - (\lfloor \tau^{-1} T \rfloor - 1)\tau))dt + \tau a \\
&\quad \le I^f(0, \tau, x_f, u_f) + 1 + \tau a.
\end{aligned}$$

In view of the relation above and (2.159)–(2.161),

$$\begin{aligned}
I^g((S_1 - 1)\tau, T - \tau, x_1, u_1) &\le M_2 + 2 + \tau\mu(f) + 1 \\
&\quad + \tau a + I^g(S_1\tau, \lfloor T\tau^{-1} \rfloor \tau - \tau, x_1, u_1) \\
&\le M_2 + 4 + \mu(f)\tau + 1 + \tau a + (\lfloor T\tau^{-1} \rfloor - S_1)(\tau\mu(f) \\
&\quad + (\tau+1)^{-1}(4L)^{-1}) + 2\tau|\mu(f)|.
\end{aligned}$$

By the relation above, (2.131), (2.137), (2.141), (2.150), and (2.156)–(2.159),

$$\begin{aligned}
M_1 &\ge I^g(0, T, x, u) - I^g(0, T, x_1, u_1) \\
&= I^g(\tau(S_1 - 1), T, x, u) - I^g(\tau(S_1 - 1), T, x_1, u_1) \\
&\ge -a(L_0\tau + \tau + 1) - L_0|\mu(f)| - 16(a+1)(\tau+1) + (\lfloor T\tau^{-1} \rfloor - S_1)\tau\mu(f) \\
&\quad + (\lfloor T\tau^{-1} \rfloor - S_1)L_0^{-1}(16(1+a)(1+\tau) - 1) - 2M_2 - 8 - 3|\mu(f)|\tau - 1 \\
&\quad\quad - \tau a - (\lfloor T\tau^{-1} \rfloor - S_1)\tau\mu(f) - (\lfloor T\tau^{-1} \rfloor - S_1)(\tau+1)^{-1}(4L)^{-1} \\
&\ge -a(L_0\tau + \tau + 1) - L_0|\mu(f)| - 16(a+1)(\tau+1) - 2M_2 - 8 \\
&\quad\quad - 3|\mu(f)|\tau - 1 - \tau a + \lfloor \lfloor T\tau^{-1} \rfloor - S_1 \rfloor (L_0^{-1}(16(1+a)(1+\tau) - 1) \\
&\quad\quad - (4L)^{-1}(1+\tau)^{-1}) \ge -a(L_0\tau + \tau + 1) - L_0|\mu(f)| - 16(a+1)(\tau+1)
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2M_2 - 8 - 3|\mu(f)|\tau - 1 - \tau a + \lfloor \lfloor T\tau^{-1} \rfloor - S_1 \rfloor 4L_0^{-1}(1+a) \\
&\geq l - a(L_0\tau + \tau + 1) - L_0|\mu(f)| - 16(a+1)(\tau+1) \\
&\quad - 2M_2 - 8 - 3|\mu(f)|\tau - 1 - \tau a.
\end{aligned}
$$

This contradicts (2.136). Thus if (2.141) holds we have reached a contradiction.

Assume that (2.142) holds. By (2.142) and the choice of (x_1, u_1),

$$
I^g(0, T, x, u) - I^g(0, T, x_1, u_1) = I^g(0, \tau(S_2+1), x, u) - I^g(0, \tau(S_2+1), x_1, u_1). \tag{2.162}
$$

By (2.13),

$$
\begin{aligned}
I^g(0, \tau(S_2+1), x, u) &\geq I^g(0, S_2\tau, x, u) - \tau a \\
&= \sum_{j=0}^{\lfloor S_2 L_0^{-1} \rfloor - 1} I^g(jL_0\tau, (j+1)L_0\tau, x, u) - a(L_0\tau + \tau) \\
&\geq \lfloor S_2 L_0^{-1} \rfloor (L_0\tau\mu(f) + 16(1+a)(\tau+1) - 1) - a\tau(L_0+1).
\end{aligned} \tag{2.163}
$$

By the choice of (x_1, u_1), (8.133), (2.134), (2.138), (2.142), (2.149), and property (P5),

$$
\begin{aligned}
&I^f(0, \tau, x_1, u_1) \leq M_1 + 1, \; I^g(0, \tau, x_1, u_1) \leq M_1 + 2, \\
&I^f(\tau S_2, \tau(S_2+1), x_1, u_1) \leq M_1 + 1, \; I^g(\tau S_2, \tau(S_2+1), x_1, u_1) \leq M_1 + 2.
\end{aligned} \tag{2.164}
$$

By (2.164),

$$
\begin{aligned}
I^g(0, (S_2+1)\tau, x_1, u_1) &= I^g(\tau, (S_2+1)\tau, x_1, u_1) + I^g(0, \tau, x_1, u_1) \\
&\leq M_2 + 2 + I^g(\tau_1, (S_2+1)\tau, x_1, u_1).
\end{aligned} \tag{2.165}
$$

It follows from (2.142) and the choice of (x_1, u_1), (2.138) and property (P5) that for each $j \in \{1, \dots, S_2 - 1\}$,

$$
\begin{aligned}
&I^f(j\tau, (j+1)\tau, x_1, u_1) = I^f(0, \tau, x_f, u_f) = \tau\mu(f), \\
&I^g(j\tau, (j+1)\tau, x_1, u_1) \leq \tau\mu(f) + (\tau+1)^{-1}(4L)^{-1}.
\end{aligned}
$$

These relations imply that

$$
I^g(\tau, S_2\tau, x_1, u_1) \leq S_2\tau\mu(f) + S_2(4L)^{-1})(\tau+1)^{-1}. \tag{2.166}
$$

By (2.131), (2.137), (2.142), (2.150), and (2.162)–(2.166),

$$\begin{aligned}M_1 &\geq I^g(0,T,x,u) - I^g(0,T,x_1,u_1)\\&\geq \lfloor S_2 L_0^{-1}\rfloor (L_0\tau\mu(f) + 16(1+a)(\tau+1) - 1) - a\tau(L_0+1)\\&\quad - 2M_2 - 4 - S_2\tau\mu(f) - S_2(4L)^{-1}(\tau+1)^{-1}\\&\geq S_2(2L_0)^{-1}(16(1+a)(\tau+1)-1) - a\tau(L_0+1)\\&\quad - 2M_2 - 4 - S_2(4L)^{-1}(\tau+1)^{-1} - L_0\tau|\mu(f)|\\&\geq 4S_2L_0^{-1} - a\tau(L_0+1) - 2M_2 - 4 - L_0\tau|\mu(f)|\\&\geq 4l - a\tau(L_0+1) - L_0|\mu(f)|\tau - 2M_2 - 4.\end{aligned}$$

This contradicts (2.137). Thus if (2.142) holds we have reached a contradiction.

Assume that (2.143) holds. By (2.143) and the choice of (x_1, u_1),

$$\begin{aligned}I^g(0,T,x,u) - I^g(0,T,x_1,u_1) &= I^g(\tau(S_1-1), \tau(S_2+1), x, u)\\&\quad - I^g(\tau(S_1-1), \tau(S_2+1), x_1, u_1).\end{aligned} \tag{2.167}$$

By (2.13) and (2.148),

$$\begin{aligned}I^g(\tau(S_1-1), \tau(S_2+1), x, u) &\geq I^g(\tau S_1, S_2\tau, x, u) - 2\tau a\\&\geq -2a\tau + I^g(\tau S_1, S_1\tau\\&\quad + \lfloor (S_2-S_1)L_0^{-1}\rfloor L_0\tau, x, u) - L_0\tau a\\&\geq -\tau a(L_0+2) + \lfloor (S_2-S_1)L_0^{-1}\rfloor (L_0\tau\mu(f)\\&\quad + 16(1+a)(\tau+1)-1)).\end{aligned} \tag{2.168}$$

By the choice of (x_1, u_1), (2.133), (2.134), (2.143), and property (P5),

$$\begin{aligned}&I^f(\tau(S_1-1), \tau S_1, x_1, u_1) \leq M_1 + 1,\ I^f(\tau S_2, \tau(S_2+1), x_1, u_1) \leq M_1 + 1,\\&I^g(\tau(S_1-1), \tau S_1, x_1, u_1) \leq M_1 + 2,\ I^g(\tau S_2, \tau(S_2+1), x_1, u_1) \leq M_1 + 2.\end{aligned} \tag{2.169}$$

It follows from (2.141) and the choice of (x_1, u_1), (2.138) and property (P5) that for each $j \in \{S_1, \ldots, S_2 - 1\}$,

$$\begin{aligned}&I^f(j\tau, (j+1)\tau, x_1, u_1) = I^f(0, \tau, x_f, u_f) = \tau\mu(f),\\&I^g(j\tau, (j+1)\tau, x_1, u_1) \leq \tau\mu(f) + (\tau+1)^{-1}(4L)^{-1}.\end{aligned}$$

These relations and (2.169) imply that

$$I^g(\tau(S_1-1),\tau(S_2+1),x_1,u_1) \le (S_2-S_1)\tau\mu(f) + (S_2-S_1)(4L)^{-1})(\tau+1)^{-1} + 2M_1 + 4. \tag{2.170}$$

By (2.131), (2.137), (2.150), (2.167), (2.168), and (2.170),

$$\begin{aligned}
M_1 &\ge I^g(\tau(S_1-1),\tau(S_2+1),x,u) - I^g(\tau(S_1-1),\tau(S_2+1),x_1,u_1)\\
&\ge -\tau a(2+L_0) + \lfloor (S_2-S_1)L_0^{-1}\rfloor L_0\tau\mu(f)\\
&\quad + \lfloor (S_2-S_1)L_0^{-1}\rfloor (16(1+a)(\tau+1)-1)\\
&\quad - (S_2-S_1)\tau\mu(f) - (S_2-S_1)(4L)^{-1}(\tau+1)^{-1} - 2M_2 - 4\\
&\ge -(L_0+2)|\mu(f)|\tau - \tau a(2+L_0) - 2M_2 - 4 - 16(1+a)(\tau+1)\\
&\quad + (S_2-S_1)(L_0^{-1}(16(1+a)(1+\tau)-1) - (4L)^{-1})\\
&\ge -(L_0+2)\tau|\mu(f)| - \tau a(2+L_0) - 2M_2 - 16(1+a)(1+\tau) - 6 + 4l.
\end{aligned}$$

This contradicts (2.137). Thus in all the cases we have reached a contradiction which proves (2.132) and Lemma 2.42 itself.

2.9 Proof of Theorem 2.13

By Proposition 2.39, there exists $\delta_0 \in (0,\epsilon)$ such that the following property holds:

(P6) for each integer $p \ge 1$, each $(x,u) \in X(A,B,0,p\tau)$ satisfying

$$|x(0)-x_f(0)|,\ |x(p\tau)-x_f(0)| \le \delta_0,\ I^f(0,p\tau,x,u) \le \sigma(f,x(0),x(p\tau),p\tau)+\delta_0$$

and each integer $i \in [0,p-1]$, the inequality $|x(i\tau+t)-x_f(t)| \le \epsilon$ holds for all $t \in [0,\tau]$.

By Lemma 2.42, there exist an integer $L_0 \ge 1$ and a neighborhood $\mathcal{U}_0$ of f in $\mathcal{M}$ such that the following property holds:

(P7) for each $T > L_0\tau$, each $g \in \mathcal{U}_0$, each pair of integers S_1, S_2 satisfying $0 \le S_1 \le S_2 - L_0$, $S_2\tau \le T$ and each $(x,u) \in X(A,B,0,T)$ for which at least one of the conditions (a), (b), (c) holds,

$$\min\{|x(i\tau)-x_f(0)| :\ i = S_1,\dots,S_2\} \le \delta_0.$$

Fix an integer $L \ge 4(L_0+1)$ and $\delta \in (0, 4^{-1}\delta_0)$. By assumption (A), Proposition 2.35 and the boundedness of v_0 on bounded sets there is $M_2 > 0$ such that the following property holds:

(P8) for each $i \in \{1, \dots, L\}$ and each $y, z \in R^n$ satisfying $|y|, |z| < 1 + \sup\{|x_f(t)| : t \in [0, \tau]\}$ we have $|\sigma(f, y, z, i\tau)| \le M_2$.

By Proposition 2.41, there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{U}_0$ and that the following property holds:

(P9) for each $g \in \mathcal{U}$, each $T_1 \ge 0$, each $T_2 \in [T_1 + \min\{1, \tau\},\ T_1 + 4L \max\{1, \tau]\}$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying

$$\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \le M_2 + 4$$

the inequality $|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \le \delta$ holds.

Assume that

$$T > 2L\tau,\ g \in \mathcal{U}, \tag{2.171}$$

$(x, u) \in X(A, B, 0, T)$ satisfies for each $S \in [0, T - L\tau]$,

$$I^g(S, S + L\tau, x, u) \le \sigma(g, x(S), x(S + L\tau), S, S + L\tau) + \delta \tag{2.172}$$

and satisfies at least one of the following conditions:

$$\text{(a) } |x(0)|,\ |x(T)| \le M_0,\ I^g(0, T, x, u) \le \sigma(g, x(0), x(T), 0, T) + M_1;$$
$$\text{(b) } |x(0)| \le M_0,\ I^g(0, T, x, u) \le \sigma(g, x(0), 0, T) + M_1;$$
$$\text{(c) } I^g(0, T, x, u) \le \sigma(g, 0, T) + M_1.$$

By conditions (a)–(c) and property (P7), there exists a finite strictly increasing sequence of integers S_i, $i = 1, \dots, q$ such that

$$0 \le S_1 < L_0,\ T \ge S_q\tau \ge T - \tau(1 + L_0),\ S_{i+1} - S_i \le L_0 + 1,\ i = 1, \dots, q - 1, \tag{2.173}$$

$$|x(S_i\tau) - x_f(0)| \le \delta_0,\ i = 1, \dots, q. \tag{2.174}$$

We may assume without loss of generality that

$$\text{if } |x(0) - x_f(0)| \le \delta,\ \text{then } S_1 = 0$$

$$\text{and if } |x(\lfloor T\tau^{-1} \rfloor \tau) - x_f(0)| \le \delta,\ \text{then } S_q = \lfloor T\tau^{-1} \rfloor. \tag{2.175}$$

Assume that an integer $i \in [S_1, S_q)$. Then there exists a natural number $k \in \{1, \dots, q - 1\}$ such that

$$S_k \le i < S_{k+1}. \tag{2.176}$$

In view of (2.171), (2.173), and the inequality $L \geq 4(L_0+1)$, there is $S \in [0, T-L\tau]$ such that

$$[S_k\tau, S_{k+1}\tau] \subset [S, S+L\tau]. \tag{2.177}$$

It follows from (2.172) and (2.177) that

$$I^g(S_k\tau, S_{k+1}\tau, x, u) \leq \sigma(g, x(S_k\tau), x(S_{k+1}\tau), S_k\tau, S_{k+1}\tau) + \delta. \tag{2.178}$$

By (2.173), (2.174), and property (P8),

$$\sigma(f, x(S_k\tau), x(S_{k+1}\tau), (S_{k+1}-S_k)\tau) < M_2. \tag{2.179}$$

By (2.171), (2.173), (2.178), (2.179), and property (P9),

$$\begin{aligned}
&\sigma(g, x(S_k\tau), x(S_{k+1}\tau), S_k\tau, S_{k+1}\tau) \\
&\quad \leq \sigma(f, x(S_k\tau), x(S_{k+1}\tau), (S_{k+1}-S_k)\tau) + \delta
\end{aligned} \tag{2.180}$$

and

$$I^g(S_k\tau, S_{k+1}\tau, x, u) \leq M_2 + 2.$$

In view of the relation above, (2.171), (2.173), (2.178), (2.180), and property (P9),

$$\begin{aligned}
I^f(S_k\tau, S_{k+1}\tau, x, u) &\leq I^g(S_k\tau, S_{k+1}\tau, x, u) + \delta \\
&\leq \sigma(f, x(S_k\tau), x(S_{k+1}\tau), (S_{k+1}-S_k)\tau) + 3\delta.
\end{aligned}$$

Together with (2.174), (2.176), and property (P6), $|x(i\tau + t) - x_f(t)| \leq \epsilon$ for all $t \in [0, \tau]$. Theorem 2.13 is proved. □

2.10 Proof of Theorem 2.14

By Proposition 2.39, there exists $\delta_0 \in (0, \epsilon)$ such that the following property holds:

(P10) for each integer $p \geq 1$, each $(x, u) \in X(A, B, 0, p\tau)$ satisfying

$$|x(0) - x_f(0)|,\ |x(p\tau) - x_f(0)| \leq 4\delta_0,$$
$$I^f(0, p\tau, x, u) \leq \sigma(f, x(0), x(p\tau), p\tau) + 4\delta_0$$

and each integer $i \in [0, p-1]$, the inequality $|x(i\tau + t) - x_f(t)| \leq \epsilon$ holds for all $t \in [0, \tau]$.

By Lemma 2.42, there exist an integer $L_0 \geq 1$ and a neighborhood $\mathcal{U}_0$ of f in $\mathcal{M}$ such that the following property holds:

(P11) for each $T > L_0\tau$, each $g \in \mathcal{U}_0$, each pair of integers S_1, S_2 satisfying $0 \leq S_1 \leq S_2 - L_0$, $S_2\tau \leq T$ and each $(x, u) \in X(A, B, 0, T)$ which satisfies at least one of the conditions (a), (b) and (c),

$$\min\{|x(i\tau) - x_f(0)| : \ i = S_1, \dots, S_2\} \leq \delta_0.$$

Fix an integer

$$L \geq 4(L_0 + 1)(9 + 2\delta_0^{-1}M_1), \tag{2.181}$$

$$\delta \in (0, 4^{-1}\delta_0). \tag{2.182}$$

By Proposition 2.35 and the boundedness of v_0 on bounded sets there is $M_2 > 0$ such that for each $i \in \{1, \dots, L\}$ and each $y, z \in R^n$ satisfying $|y|, |z| \leq 1 + \sup\{|x_f(t)| : \ t \in [0, \tau]\}$ we have

$$|\sigma(f, y, z, i\tau)| \leq M_2. \tag{2.183}$$

By Proposition 2.41, there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{U}_0$ and that the following property holds:

(P12) for each $g \in \mathcal{U}$, each $T_1 \geq 0$, each $T_2 \in [T_1 + \min\{1, \tau\}, \ T_1 + 4L\max\{1, \tau\}]$ and each $(x, u) \in X(A, B, T_1, T_2)$ satisfying

$$\min\{I^f(T_1, T_2, x, u), I^g(T_1, T_2, x, u)\} \leq M_2 + 4$$

the inequality $|I^f(T_1, T_2, x, u) - I^g(T_1, T_2, x, u)| \leq \delta$ holds.

Assume that

$$T > L\tau, \ \ g \in \mathcal{U} \tag{2.184}$$

and that $(x, u) \in X(A, B, 0, T)$ satisfies at least one of the following conditions:

$$\text{(a) } |x(0)|, \ \ |x(T)| \leq M_0, \ \ I^g(0, T, x, u) \leq \sigma(g, x(0), x(T), 0, T) + M_1;$$

$$\text{(b) } |x(0)| \leq M_0, \ \ I^g(0, T, x, u) \leq \sigma(g, x(0), 0, T) + M_1;$$

$$\text{(c) } I^g(0, T, x, u) \leq \sigma(g, 0, T) + M_1.$$

By conditions (a)–(c), (2.184), and property (P11), there exists a finite strictly increasing sequence of integers S_i, $i = 1, \dots, q$ such that

$$0 \leq S_1 \leq L_0, \ \ T \geq S_q\tau \geq T - \tau(1 + L_0), \ \ S_{i+1} - S_i \leq L_0 + 1, \ \ i = 1, \dots, q - 1, \tag{2.185}$$

$$|x(S_i\tau) - x_f(0)| \le \delta_0, \; i = 1, \dots, q. \tag{2.186}$$

Define by induction a finite strictly increasing sequence of natural numbers $i_1, \dots, i_k \in \{1, \dots, q\}$. Set

$$i_1 = 1. \tag{2.187}$$

Assume that $p \ge 1$ is an integer and that we defined integers $i_1 < \dots < i_p$ belonging to $\{1, \dots, q\}$ such that for each natural number $m < p$ the following properties hold:

(i)

$$I^g(S_{i_m}\tau, S_{i_{m+1}}\tau, x, u) > \sigma(g, x(S_{i_m}\tau), x(S_{i_{m+1}}\tau), S_{i_m}\tau, S_{i_{m+1}}\tau) + \delta_0; \tag{2.188}$$

(ii) if $i_{m+1} > i_m + 1$, then

$$I^g(S_{i_m}\tau, S_{i_{m+1}-1}\tau, x, u) \le \sigma(g, x(S_{i_m}\tau), x(S_{i_{m+1}-1}\tau), S_{i_m}\tau, S_{i_{m+1}-1}\tau) + \delta_0. \tag{2.189}$$

(Note that by (2.187) our assumption holds for $p = 1$.) Let us define i_{p+1}. If $i_p = q$, then our construction is completed, $k = p$, $i_k = q$ and for each natural number $m < p = k$, properties (i) and (ii) hold.

Assume that $i_p < q$. There are two cases:

$$I^g(S_{i_p}\tau, S_q\tau, x, u) \le \sigma(g, x(S_{i_p}\tau), x(S_q\tau), S_{i_p}\tau, S_q\tau) + \delta_0; \tag{2.190}$$

$$I^g(S_{i_p}\tau, S_q\tau, x, u) > \sigma(g, x(S_{i_p}\tau), x(S_q\tau), S_{i_p}\tau, S_q\tau) + \delta_0. \tag{2.191}$$

Assume that (2.190) holds. Then we set $k = p + 1$, $i_k = q$ the construction is completed, for each natural number $m < k - 1$, (2.188) is true and for each natural number $m < k$, property (ii) holds.

Assume that (2.191) holds. Then we set

$$\begin{aligned} i_{p+1} = \min\{&j > S_{i_p} : \; j \text{ is an integer and} \\ &I^g(S_{i_p}\tau, S_j\tau, x, u) > \sigma(g, x(S_{i_p}\tau), x(S_j\tau), S_{i_p}\tau, S_j\tau) + \delta_0\}. \end{aligned} \tag{2.192}$$

It is easy to see that the assumption made for p also holds for $p + 1$. As a result we obtain a finite strictly increasing sequence of integers $i_1, \dots, i_k \in \{1, \dots, q\}$ such that $i_k = q$, for all integers m satisfying $1 \le m < k - 1$, (2.188) holds and for each integer m satisfying $1 \le m < k$, (ii) holds. By conditions (a)–(c) and (2.188),

$$\begin{aligned} M_1 &\ge I^g(0, T, x, u) - \sigma(g, x(0), x(T), 0, T) \\ &\ge \sum \{ I^g(S_{i_j}\tau, S_{i_{j+1}}\tau, x, u) - \sigma(g, x(S_{i_j}\tau), x(S_{i_{j+1}}\tau), S_{i_j}\tau, S_{i_{j+1}}\tau) : \end{aligned}$$

$$j \text{ is an integer satisfying } 1 \le j < k-1\} \ge \delta_0(k-2),$$

$$k \le \delta_0^{-1} M_1 + 2. \tag{2.193}$$

Set

$$A = \{j \in \{1, \dots, k\} : \ j < k \text{ and } S_{i_{j+1}} - S_{i_j} \ge 4(L_0 + 1)\}. \tag{2.194}$$

Let

$$j \in A. \tag{2.195}$$

By (2.185), (2.189), (2.194), (2.195), and property (ii),

$$I^g(S_{i_j}\tau, S_{i_{j+1}-1}\tau, x, u) \le \sigma(g, x(S_{i_j}\tau), x(S_{i_{j+1}-1}\tau), S_{i_j}\tau, S_{i_{j+1}-1}\tau) + \delta_0, \ \ i_{j+1} > i_j + 3. \tag{2.196}$$

Let

$$p \in \{i_j, \dots, i_{j+1} - 2\}.$$

This implies that $\{S_p, S_{p+1}\} \subset \{S_{i_j}, \dots, S_{i_{j+1}-1}\}$ and in view of (2.196),

$$I^g(S_p\tau, S_{p+1}\tau, x, u) \le \sigma(g, x(S_p\tau), x(S_{p+1}\tau), S_p\tau, S_{p+1}\tau) + \delta_0. \tag{2.197}$$

By the choice of M_2 [see (2.183)], (2.185), and (2.186),

$$\sigma(f, x(S_p\tau), x(S_{p+1}\tau), S_{p+1}\tau - S_p\tau) \le M_2. \tag{2.198}$$

Together with (2.181), (2.184), (2.185), and property (P12) this implies that

$$\sigma(g, x(S_p\tau), x(S_{p+1}\tau), S_p\tau, S_{p+1}\tau) \le \sigma(f, x(S_p\tau), x(S_{p+1}\tau), (S_{p+1} - S_p)\tau) + \delta_0.$$

By (2.181), (2.184), (2.185), (2.197), (2.198), the inequality above and property (P12),

$$I^f(S_p\tau, S_{p+1}\tau, x, u) \le \sigma(f, x(S_p\tau), x(S_{p+1}\tau), (S_{p+1} - S_p)\tau) + 3\delta_0. \tag{2.199}$$

It follows from (2.186), (2.199), and property (P10) that for all integers $m \in \{S_p, \dots, S_{p+1} - 1\}$,

$$|x(m\tau + t) - x_f(t)| \le \epsilon, \ \ t \in [0, \tau]. \tag{2.200}$$

Thus (2.200) holds for all integers $m \in \{S_{i_j}, \dots, S_{i_{j+1}-1} - 1\}$. Since j is any integer belonging to A we conclude that (2.200) holds for all

$$m \in \cup\{\{S_{i_j}, \dots, S_{i_{j+1}-1} - 1\} : j \in A\}$$

and that

$$\{m \in \{0, \dots, \lfloor T\tau^{-1}\rfloor - 1\} : \text{(2.200) does not hold}\}$$
$$\subset \{0, \dots, S_1\} \cup \{S_q, \dots, \lfloor T\tau^{-1}\rfloor\} \cup \{\{S_{i_j}, \dots, S_{i_{j+1}}\} : j \in \{1, \dots, k\} \setminus A\}$$
$$\cup\{\{S_{i_{j+1}-1}, \dots, S_{i_{j+1}}\} : j \in A\}$$

and in view of (2.181), (2.185), (2.193), and (2.194), the cardinality of the right-hand side of the inclusion above does not exceed

$$4(L_0 + 1)(2 + 2k) \le 4(L_0 + 1)(8 + 2\delta_0^{-1}M_1) < L.$$

Theorem 2.14 is proved. □

2.11 Proofs of Propositions 2.18 and 2.21

Proof of Proposition 2.18. Let $\epsilon > 0$. Since the function v is continuous there exists $\delta > 0$ such that

$$|v(z_1, z_2) - \mu(f)\tau| \le \epsilon/4 \tag{2.201}$$

$$\text{for all } z_1, z_2 \in R^n \text{ satisfying } |z_i - x_f(0)| \le \delta,\ i = 1, 2.$$

Let

$$z_1, z_2 \in R^n,\ |z_i - x_f(0)| \le \delta,\ i = 1, 2. \tag{2.202}$$

By Proposition 2.32, there exists $(x, u) \in X(A, B, 0, \infty)$

$$x(0) = z_2,\ x(\tau) = z_1,\ I^f(0, \tau, x, u) = v(z_2, z_1), \tag{2.203}$$

$$x(t) = \xi^{(z_1)}(t - \tau),\ u(t) = \eta^{(z_1)}(t - \tau) \text{ for all } t \ge \tau. \tag{2.204}$$

By (2.21), (2.202)–(2.204), and Proposition 2.15,

$$\pi^f(z_2) = \pi^f(x(0)) \le \liminf_{T \in \mathbf{Z}, T \to \infty} [I^f(0, \tau, x, u) - T\mu(f)]$$

$$= I^f(0,\tau,x,u) - \tau\mu(f) + \liminf_{T\in\mathbf{Z},T\to\infty}[I^f(0,T,\xi^{(z_1)},\eta^{(z_1)}) - T\mu(f)]$$

$$\leq v(z_2,z_1) + \pi^f(z_1) - \mu(f)\tau \leq \pi^f(z_1) + \epsilon/2.$$

Proposition 2.18 is proved. □

Proof of Proposition 2.21. Let $M > 0$. by Proposition 2.18, there exists $\delta > 0$ such that for each $x \in R^n$ satisfying $|x - x_f(0)| \leq \delta$,

$$|\pi^f(x)| \leq 1. \tag{2.205}$$

By Proposition 2.38, there exists a natural number L_0 such that the following property holds:

(i) for each $T \geq L_0\tau$, each $(x,u) \in X(A,B,0,T)$ satisfying

$$I^f(0,T,x,u) \leq T\mu(f) + M + 1$$

and each integer S satisfying $[S\tau,(S+L_0)\tau] \subset [0,T]$ there exists an integer $i \in [S, S+L_0-1]$ such that

$$|x(\tau i + t) - x_f(t)| \leq \delta \text{ for all } t \in [0,\tau].$$

By Proposition 2.40, there exists $M_1 > 0$ such that the following property holds:

(ii) for each $T \in [\tau,(L_0+1)\tau]$ and each $(x,u) \in X(A,B,0,T)$ satisfying

$$I^f(0,T,x,u) \leq \tau|\mu(f)|(L_0+1) + M + 1$$

we have

$$|x(t)| \leq M_1 \text{ for all } t \in [0,T].$$

Assume that $x \in R^n$ satisfies

$$\pi^f(x) \leq M. \tag{2.206}$$

By (2.21) and (2.206),

$$\pi^f(x) = \liminf_{T\in\mathbf{Z},T\to\infty}[I^f(0,T,\xi^{(x)},\eta^{(x)}) - T\mu(f)] \leq M. \tag{2.207}$$

In view of (2.207) and property (i), there exists an integer $t_0 \in [1, L_0+1]$ such that

$$|\xi^{(x)}(t_0\tau) - x_f(0)| \leq \delta. \tag{2.208}$$

It follows from (2.208) and the choice of δ [see (2.205)] that

$$|\pi^f(\xi^{(x)}(t_0\tau))| \leq 1. \tag{2.209}$$

Proposition 2.16, (2.206) and (2.209) imply that

$$I^f(0, t_0\tau, \xi^{(x)}, \eta^{(x)}) - t_0\tau\mu(f) = \pi^f(x) - \pi^f(\xi^{(x)}(t_0\tau)) \leq M + 1. \tag{2.210}$$

By (2.210) and property (ii), $|\xi^{(x)}(t)| \leq M_1$ for all $t \in [0, t_0\tau]$. Thus $|x| \leq M_1$. Proposition 2.21 is proved. □

2.12 Auxiliary Results for Theorem 2.25

We continue to use the notation, definitions, and assumptions introduced in Sects. 2.1–2.3.

Assume that $S_2 > S_1 \geq 0$ are integers and $g \in \mathcal{M}$. There exists a unique function $\mathcal{L}_{S_1,S_2}(g)(t, x, u)$, $(t, x, u) \in [0, \infty) \times R^n \times R^m$ such that for each $x \in R^n$ and each $u \in R^m$,

$$\mathcal{L}_{S_1,S_2}(g)(t, x, u) = g(S_2\tau - t + S_1\tau, x, u) \text{ for each } t \in [S_1\tau, S_2\tau], \tag{2.211}$$

$$\mathcal{L}_{S_1,S_2}(g)(t + (S_2 - S_1)\tau, x, u) = \mathcal{L}_{S_1,S_2}(g)(t, x, u) \text{ for each } t \geq 0. \tag{2.212}$$

Clearly, $\mathcal{L}_{S_1,S_2}(g) \in \mathcal{M}$ and $\mathcal{L}_{S_1,S_2}$ is a self-mapping of $\mathcal{M}$. It is easy to see that the following proposition holds.

Proposition 2.43. *1. Let V be a neighborhood of $\bar{f}$ in $\mathcal{M}$. Then there exists a neighborhood U of f in $\mathcal{M}$ such that $\mathcal{L}_{S_1,S_2}(g) \in V$ for all $g \in U$ and all integers $S_2 > S_1 \geq 0$.*

2. Let V be a neighborhood of f in $\mathcal{M}$. Then there exists a neighborhood U of $\bar{f}$ in $\mathcal{M}$ such that $\mathcal{L}_{S_1,S_2}(g) \in V$ for all $g \in U$ and all integers $S_2 > S_1 \geq 0$.

Let $S_2 > S_1 \geq 0$ be integers, $g \in \mathcal{M}$ and $(x, u) \in X(A, B, S_1\tau, S_2\tau)$ $(X(-A, -B, S_1\tau, S_2\tau)$ respectively). Then in view of (2.26) and (2.211),

$$\begin{aligned}
&\int_{S_1\tau}^{S_2\tau} \mathcal{L}_{S_1,S_2}(g)(t, \bar{x}(t), \bar{u}(t))dt \\
&\quad = \int_{S_1\tau}^{S_2\tau} g(S_2\tau - t + S_1\tau, x(S_2\tau - t + S_1\tau), u(S_2\tau - t + S_1\tau))dt \\
&\quad = \int_{S_1\tau}^{S_2\tau} g(t, x(t), u(t))dt.
\end{aligned} \tag{2.213}$$

Let $T_2 > T_1 \geq 0$, $y, z \in R^n$ and $g \in \mathcal{M}$. For each $(x, u) \in X(-A, -B, T_1, T_2)$, put

$$I^g(T_1, T_2, x, u) = \int_{T_1}^{T_2} g(t, x(t), u(t))dt \tag{2.214}$$

and set

$$\sigma_-(g, y, z, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(-A, -B, T_1, T_2) \text{ and } x(T_1) = y,\ x(T_2) = z\}, \tag{2.215}$$

$$\sigma_-(g, y, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(-A, -B, T_1, T_2) \text{ and } x(T_1) = y\}, \tag{2.216}$$

$$\hat{\sigma}_-(g, z, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :$$

$$(x, u) \in X(-A, -B, T_1, T_2) \text{ and } x(T_2) = z\}, \tag{2.217}$$

$$\sigma_-(g, T_1, T_2) = \inf\{I^g(T_1, T_2, x, u) :\ (x, u) \in X(-A, -B, T_1, T_2)\}. \tag{2.218}$$

Relation (2.213) implies the following result.

Proposition 2.44. *Let $S_2 > S_1 \geq 0$ be integers, $g \in \mathcal{M}$ and $(x_i, u_i) \in X(A, B, S_1\tau, S_2\tau)$, $i = 1, 2$. Then*

$$I^g(S_1\tau, S_2\tau, x_1, u_1) \geq I^g(S_1\tau, S_2\tau, x_2, u_2) - M \tag{2.219}$$

if and only if

$$I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}_1, \bar{u}_1) \geq I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}_2, \bar{u}_2) - M, \tag{2.220}$$

where $\bar{g} = \mathcal{L}_{S_1,S_2}(g)$.

Proposition 2.44 [see (2.219) and (2.220)] implies the following result.

Proposition 2.45. *Let $S_2 > S_1 \geq 0$ be integers, $M \geq 0$, $g \in \mathcal{M}$, $\bar{g} = \mathcal{L}_{S_1,S_2}(g)$ and $(x, u) \in X(A, B, S_1\tau, S_2\tau)$. Then the following assertions are equivalent:*

$$I^g(S_1\tau, S_2\tau, x, u) \leq \sigma(g, S_1\tau, S_2\tau) + M$$

if and only if

$$I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \leq \sigma_-(\bar{g}, S_1\tau, S_2\tau) + M;$$

$$I^g(S_1\tau, S_2\tau, x, u) \leq \sigma(g, x(S_1\tau), x(S_2\tau), S_1\tau, S_2\tau) + M$$

if and only if

$$I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \le \sigma_-(\bar{g}, \bar{x}(S_1\tau), \bar{x}(S_2\tau), S_1\tau, S_2\tau) + M;$$
$$I^{g}(S_1\tau, S_2\tau, x, u) \le \hat{\sigma}(g, x(S_2\tau), S_1\tau, S_2\tau) + M$$

if and only if

$$I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \le \sigma_-(\bar{g}, \bar{x}(S_1\tau), S_1\tau, S_2\tau) + M;$$
$$I^{g}(S_1\tau, S_2\tau, x, u) \le \sigma(g, x(S_1\tau), S_1\tau, S_2\tau) + M$$

if and only if

$$I^{\bar{g}}(S_1\tau, S_2\tau, \bar{x}, \bar{u}) \le \hat{\sigma}_-(\bar{g}, \bar{x}(S_2\tau), S_1\tau, S_2\tau) + M.$$

2.13 The Basic Lemma for Theorem 2.25

Let $\theta_f \in R^n$ satisfy

$$\pi^f(\theta_f) = \inf(\pi^f). \tag{2.221}$$

Lemma 2.46. *Let* $S_0 \ge 1$ *be an integer,* $\epsilon \in (0, 1)$ *and*

$$(x_*, u_*) \in X(A, B, 0, \infty)$$

be an (f, A, B)*-overtaking optimal pair satisfying*

$$x_*(0) = \theta_f. \tag{2.222}$$

Then there exists $\delta \in (0, \epsilon)$ *such that for each* $(x, u) \in X(A, B, 0, S_0\tau)$ *which satisfies*

$$\pi^f(x(0)) \le \inf(\pi^f) + \delta,$$
$$I^f(0, S_0\tau, x, u) - S_0\tau\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0\tau)) \le \delta$$

the inequality $|x(t) - x_*(t)| \le \epsilon$ *holds for all* $t \in [0, S_0\tau]$.

Proof. Assume that the lemma does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0, 1]$ and a sequence $\{(x_k, u_k)\}_{k=1}^{\infty} \subset X(A, B, 0, S_0\tau)$ such that

$$\lim_{k\to\infty} \delta_k = 0$$

and that for all integers $k \geq 1$,

$$\pi^f(x_k(0)) \leq \inf(\pi^f) + \delta_k, \tag{2.223}$$

$$I^f(0, S_0\tau, x_k, u_k) - S_0\tau\mu(f) - \pi^f(x_k(0)) + \pi^f(x_k(S_0\tau)) \leq \delta_k, \tag{2.224}$$

$$\sup\{|x_k(t) - x_*(t)| : t \in [0, S_0\tau]\} > \epsilon. \tag{2.225}$$

In view of (2.223) and Proposition 2.21, the sequence $\{x_k(0)\}_{k=1}^{\infty}$ is bounded. By (2.224), the continuity and the boundedness from below of the function π^f (see Proposition 2.22) and boundedness of the sequence $\{x_k(0)\}_{k=1}^{\infty}$, the sequence $\{I^f(0, S_0\tau, x_k, u_k)\}_{k=1}^{\infty}$ is bounded. By Proposition 2.30, we may assume without loss of generality that there exists $(x, u) \in X(A, B, 0, S_0\tau)$ such that

$$x_k(t) \to x(t) \text{ as } k \to \infty \text{ uniformly on } [0, S_0\tau], \tag{2.226}$$

$$I^f(0, S_0\tau, x, u) \leq \liminf_{k\to\infty} I^f(0, S_0\tau, x_k, u_k), \tag{2.227}$$

$$u_k \to u \text{ as } k \to \infty \text{ weakly in } L^1(R^m; (0, S_0\tau)).$$

It follows from (2.221), (2.223), (2.226) and the continuity and strict convexity of π^f (see Proposition 2.20) that

$$\pi^f(x(0)) = \lim_{k\to\infty} \pi^f(x_k(0)) = \inf(\pi^f),\ x(0) = \theta_f.$$

By (2.224), (2.226), (2.227), and the continuity of π^f (see Proposition 2.20),

$$\pi^f(x(S_0\tau)) = \lim_{k\to\infty} \pi^f(x_k(S_0\tau)),$$

$$I^f(0, S_0\tau, x, u) - S_0\tau\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0\tau))$$

$$\leq \liminf_{k\to\infty}[I^f(0, S_0\tau, x_k, u_k) - S_0\tau\mu(f) - \pi^f(x_k(0)) + \pi^f(x_k(S_0\tau))] \leq 0.$$

In view of the inequality above and Proposition 2.15,

$$I^f(0, S_0\tau, x, u) - S_0\tau\mu(f) - \pi^f(x(0)) + \pi^f(x(S_0\tau)) = 0. \tag{2.228}$$

Theorem 2.3 implies that there exists an (f, A, B)-overtaking optimal pair $(\tilde{x}, \tilde{u}) \in X(A, B, 0, \infty)$ such that

$$\tilde{x}(0) = x(S_0\tau). \tag{2.229}$$

For all $t > S_0\tau$ set

$$x(t) = \tilde{x}(t - S_0\tau),\ u(t) = \tilde{u}(t - S_0\tau). \tag{2.230}$$

It is not difficult to see that the pair $(x, u) \in X(A, B, 0, \infty)$ is an (f, A, B)-good pair. By its definition, (2.228)–(2.230) and Propositions 2.15 and 2.16,

$$I^f(0, S\tau, x, u) - S\tau\mu(f) - \pi^f(x(0)) + \pi^f(x(S\tau)) = 0 \text{ for all integers } S \geq 1.$$

Combined with Proposition 2.23 this implies that (x, u) is an (f, A, B)-overtaking optimal pair satisfying $x(0) = \theta_f$. By Theorem 2.3 and (2.222), $x(t) = x_*(t)$ and $u(t) = u_*(t)$ for all $t \geq 0$. Together with (2.226) this implies that for all sufficiently large natural numbers k,

$$|x_k(t) - x_*(t)| \leq \epsilon/2 \text{ for all } t \in [0, S_0\tau].$$

This contradicts (2.225). The contradiction we have reached proves Lemma 2.46.

Note that Lemma 2.46 can also be applied for the triplet $(\bar{f}, -A, -B)$.

2.14 Proof of Theorem 2.25

By Lemma 2.46 applied to the triplet $(\bar{f}, -A - B)$ there exist

$$\delta_1 \in (0, \epsilon/4)$$

such that the following property holds:

(P13) for each $(x, u) \in X(-A, -B, 0, L_0\tau)$ which satisfies

$$\pi^{\bar{f}}(x(0)) \leq \inf(\pi^{\bar{f}}) + \delta_1,$$

$$I^{\bar{f}}(0, L_0\tau, x, u) - L_0\tau\mu(f) - \pi^{\bar{f}}(x(0)) + \pi^{\bar{f}}(x(L_0\tau)) \leq \delta_1$$

we have

$$|x(t) - \bar{x}_*(t)| \leq \epsilon \text{ holds for all } t \in [0, L_0\tau].$$

In view of the continuity of π^f, Proposition 2.17, and (2.34), there exists $\delta_2 \in (0, \delta_1)$ such that for each $z \in R^n$ satisfying $|z - x_f(0)| \leq 2\delta_2$,

$$|\pi^{\bar{f}}(z)| = |\pi^{\bar{f}}(z) - \pi^{\bar{f}}(x_f(0))| \leq \delta_1/8; \tag{2.231}$$

for each $y, z \in R^n$ satisfying $|y - x_f(0)| \leq 2\delta_2$, $|z - x_f(0)| \leq 2\delta_2$,

$$|v(y, z) - \tau\mu(f)| \leq \delta_1/8. \tag{2.232}$$

By Theorem 2.13, there exist an integer $l_0 \geq 1$, $\delta_3 \in (0, \delta_2/8)$ and a neighborhood $\mathcal{U}_1$ of f in $\mathcal{M}$ such that the following property holds:

(P14) for each integer $T > 2l_0$, each $g \in \mathcal{U}_1$ and each

$$(x, u) \in X(A, B, 0, T\tau)$$

such that

$$|x(0)| \leq M, \ I^g(0, T\tau, x, u) \leq \sigma(g, x(0), 0, T\tau) + \delta_3$$

we have

$$|x(i\tau) - x_f(0)| \leq \delta_2 \text{ for all } i = l_0, \dots, T - l_0. \tag{2.233}$$

Since the pair $(\bar{x}_*, \bar{u}_*) \in X(-A, -B, 0, \infty)$ is $(\bar{f}, -A, -B)$-good it follows from Theorem 2.2 and (2.29) that there exists an integer $l_1 \geq 1$ such that

$$|\bar{x}_*(i\tau) - x_f(0)| \leq \delta_2 \text{ for all integers } i \geq l_1. \tag{2.234}$$

By Proposition 2.41, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f in $\mathcal{M}$ such that the following property holds:

(P15) for each $g \in \mathcal{U}$, each integer $j \in \{1, \dots, 2L_0 + 2l_0 + 2l_1 + 4\}$ and each $(x, u) \in X(A, B, 0, j\tau)$ satisfying

$$\begin{aligned} &\min\{I^f(0, j\tau, x, u), I^g(0, j\tau, x, u)\} \\ &\quad \leq (|\mu(f)| + 2)(2L_0 + 2l_0 + 2l_1 + 4)\tau + |\pi^{\bar{f}}(\bar{x}_*(0))| + 2 \end{aligned}$$

we have $|I^f(0, j\tau, x, u) - I^g(0, j\tau, x, u)| \leq \delta_3/8$.

Choose $\delta > 0$ and an integer L_1 such that

$$\delta \leq \delta_3/4, \tag{2.235}$$

$$L_1 > 2L_0 + 2l_0 + 2l_1 + 4. \tag{2.236}$$

Assume that an integer

$$T \geq L_1, \ g \in \mathcal{U} \tag{2.237}$$

and $(x, u) \in X(A, B, 0, T\tau)$ satisfies

$$|x(0)| \leq M, \ I^g(0, T\tau, x, u) \leq \sigma(g, x(0), 0, T\tau) + \delta. \tag{2.238}$$

By property (P14) and (2.235)–(2.238), (2.233) holds. In view of (2.236) and (2.237),

$$[T - l_0 - l_1 - L_0 - 4, T - l_0 - l_1 - L_0] \subset [l_0, T - l_0 - l_1 - L_0]. \tag{2.239}$$

In view of (2.233) and (2.239),

$$|x(i\tau) - x_f(0)| \leq \delta_2 \text{ for all } i \in \{T - l_0 - l_1 - L_0 - 4, \dots, T - l_0 - l_1 - L_0\}. \tag{2.240}$$

Proposition 2.32 implies that there exists $(x_1, u_1) \in X(A, B, 0, T\tau)$ such that

$$x_1(t) = x(t),\ u_1(t) = u(t),\ t \in [0, \tau(T - l_0 - l_1 - L_0 - 4)], \tag{2.241}$$

$$x_1(t) = \bar{x}_*(T\tau - t),\ u_1(t) = \bar{u}_*(T\tau - t),\ t \in [\tau(T - l_0 - l_1 - L_0 - 3), \tau T], \tag{2.242}$$

$$\begin{aligned} &I^f(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x_1, u_1) \\ &\quad = v(x(\tau(T - l_0 - l_1 - L_0 - 4)), \bar{x}_*(\tau(l_0 + l_1 + L_0 + 3))). \end{aligned} \tag{2.243}$$

By (2.238) and (2.241),

$$\begin{aligned} -\delta &\leq I^g(0, T\tau, x_1, u_1) - I^g(0, T\tau, x, u) \\ &= I^g(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x_1, u_1) \\ &\quad + I^g(\tau(T - l_0 - l_1 - L_0 - 3), \tau T, x_1, u_1) \\ &\quad - I^g(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x, u) \\ &\quad - I^g(\tau(T - l_0 - l_1 - L_0 - 3), \tau T, x, u). \end{aligned} \tag{2.244}$$

We show that

$$\begin{aligned} &I^g(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x_1, u_1) \\ &- I^g(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x, u) \\ &\leq \delta_1/8 + \delta_3/8 + \delta_1/2. \end{aligned} \tag{2.245}$$

In view of (2.234), (2.240), (2.243), and the choice of δ_2 [see (2.232)],

$$I^f(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x_1, u_1) \leq \tau\mu(f) + \delta_1/8.$$

Combined with (2.237) and property (P15) this implies that

$$I^g(\tau(T - l_0 - l_1 - L_0 - 4), \tau(T - l_0 - l_1 - L_0 - 3), x_1, u_1) \leq \tau\mu(f) + \delta_1/8 + \delta_3/8. \tag{2.246}$$

It follows from (2.240) and the choice of δ_2 [see (2.232)] that

$$I^f(\tau(T-l_0-l_1-L_0-4), \tau(T-l_0-l_1-L_0-3), x, u) \geq \tau\mu(f)-\delta_1/8. \tag{2.247}$$

If

$$I^g(\tau(T-l_0-l_1-L_0-4), \tau(T-l_0-l_1-L_0-3), x, u) < \tau\mu(f)-\delta_1/2,$$

then by property (P15) and (2.237),

$$\begin{aligned}&I^f(\tau(T-l_0-l_1-L_0-4), \tau(T-l_0-l_1-L_0-3), x, u)\\ &\quad < \tau\mu(f)-\delta_1/2+\delta_3/8 < \tau\mu(f)-3\delta_1/8\end{aligned}$$

and this contradicts (2.247). Thus

$$I^g(\tau(T-l_0-l_1-L_0-4), \tau(T-l_0-l_1-L_0-3), x, u) \geq \tau\mu(f)-\delta_1/2. \tag{2.248}$$

It follows from (2.246) and (2.248) that (2.245) holds. By (2.244) and (2.245),

$$\begin{aligned}&I^g(\tau(T-l_0-l_1-L_0-3), \tau T, x_1, u_1) - I^g(\tau(T-l_0-l_1-L_0-3), \tau T, x, u)\\ &\quad \geq -\delta-\delta_1/8-\delta_3/8-\delta_1/2.\end{aligned} \tag{2.249}$$

Since $(\bar{x}_*, \bar{u}_*)$ is an $(\bar{f}, -A, -B)$-overtaking optimal pair it follows from (2.27), (2.242), and Proposition 2.16 that

$$\begin{aligned}I^f(\tau(T-l_0-l_1-L_0-3), \tau T, x_1, u_1) &= I^{\bar{f}}(0, \tau(l_0+l_1+L_0+3), \bar{x}_*, \bar{u}_*)\\ &= \mu(f)\tau(l_0+l_1+L_0+3)+\pi^{\bar{f}}(\bar{x}_*(0))\\ &\quad -\pi^{\bar{f}}(\bar{x}_*(\tau(l_0+l_1+L_0+3))).\end{aligned} \tag{2.250}$$

By (2.234) and the choice of δ_2 (see (2.231)),

$$|\pi^{\bar{f}}(\bar{x}_*(\tau(l_0+l_1+L_0+3)))| \leq \delta_1/8.$$

Together with (2.250) this implies that

$$I^f(\tau(T-l_0-l_1-L_0-3), \tau T, x_1, u_1) \leq \pi^{\bar{f}}(\bar{x}_*(0))+\mu(f)\tau(l_0+l_1+L_0+3)+\delta_1/8. \tag{2.251}$$

Property (P15), (2.237), and (2.251) imply that

$$\begin{aligned}&I^g(\tau(T-l_0-l_1-L_0-3), \tau T, x_1, u_1)\\ &\quad \leq \pi^{\bar{f}}(\bar{x}_*(0))+\mu(f)\tau(l_0+l_1+L_0+3)+\delta_1/8+\delta_3/8.\end{aligned} \tag{2.252}$$

In view of (2.249) and (2.252),

$$\begin{aligned} &I^g(\tau(T-l_0-l_1-L_0-3),\tau T,x,u) \\ &\quad \le \pi^{\bar{f}}(\bar{x}_*(0)) + \mu(f)\tau(l_0+l_1+L_0+3) + \delta + 3\delta_1/4 + \delta_3/4. \end{aligned} \tag{2.253}$$

Property (P15) and (2.253) imply that

$$\begin{aligned} &I^f(\tau(T-l_0-l_1-L_0-3),\tau T,x,u) \\ &\quad \le \pi^{\bar{f}}(\bar{x}_*(0)) + \mu(f)\tau(l_0+l_1+L_0+3) + \delta + 3\delta_1/4 + 3\delta_3/8. \end{aligned} \tag{2.254}$$

Set

$$\tilde{x}(t) = x(T\tau - t),\ \tilde{u}(t) = u(T\tau - t),\ t \in [0,\tau T]. \tag{2.255}$$

Clearly, $(\tilde{x},\tilde{u}) \in X(-A,-B,0,T\tau)$ and in view of (2.27), (2.254), and (2.255),

$$\begin{aligned} &I^{\bar{f}}(0,\tau(l_0+l_1+L_0+3),\tilde{x},\tilde{u}) = I^f(\tau(T-l_0-l_1-L_0-3),\tau T,x,u) \\ &\quad \le \pi^{\bar{f}}(\bar{x}_*(0)) + \mu(f)\tau(l_0+l_1+L_0+3) + \delta + 3\delta_1/4 + 3\delta_3/8. \end{aligned} \tag{2.256}$$

It follows from (2.240) and (2.255) that

$$|\tilde{x}(\tau(l_0+l_1+L_0+3)) - x_f(0)| \le \delta_2. \tag{2.257}$$

By (2.257) and the choice of δ_2 (see (2.231)),

$$|\pi^{\bar{f}}(\tilde{x}(\tau(l_0+l_1+L_0+3)))| \le \delta_1/8. \tag{2.258}$$

By (2.238), (2.256), and Proposition 2.15,

$$\begin{aligned} &\pi^{\bar{f}}(\tilde{x}(0)) - \pi^{\bar{f}}(\bar{x}_*(0)) + I^{\bar{f}}(0,L_0\tau,\tilde{x},\tilde{u}) - L_0\tau\mu(f) - \pi^{\bar{f}}(\tilde{x}(0)) + \pi^{\bar{f}}(\tilde{x}(L_0\tau)) \\ &\quad \le \pi^{\bar{f}}(\tilde{x}(0)) - \pi^{\bar{f}}(\bar{x}_*(0)) + I^{\bar{f}}(0,\tau(l_0+l_1+L_0+3),\tilde{x},\tilde{u}) \\ &\qquad - \mu(f)\tau(l_0+l_1+L_0+3) - \pi^{\bar{f}}(\tilde{x}(0)) + \pi^{\bar{f}}(\tilde{x}(\tau(l_0+l_1+L_0+3))) \\ &\quad \le \pi^{\bar{f}}(\tilde{x}(0)) - \pi^{\bar{f}}(\bar{x}_*(0)) + \mu(f)\tau(l_0+l_1+L_0+3) + \delta + 3\delta_3/8 + 3\delta_1/4 \\ &\qquad + \pi^{\bar{f}}(\bar{x}_*(0)) - \mu(f)\tau(l_0+l_1+L_0+3) - \pi^{\bar{f}}(\tilde{x}(0)) + \delta_1/8 \\ &\quad \le \delta + 3\delta_3/8 + 3\delta_1/4 + \delta_1/8 \le \delta_1. \end{aligned}$$

By the relation above, Proposition 2.15 and the relation $\pi^{\bar{f}}(\bar{x}_*(0)) = \inf(\pi^{\bar{f}})$,

$$\pi^{\bar{f}}(\tilde{x}(0)) \le \pi^{\bar{f}}(\bar{x}_*(0)) + \delta_1, \tag{2.259}$$

$$I^{\bar{f}}(0, L_0\tau, \tilde{x}, \tilde{u}) - L_0\tau\mu(f) + \pi^{\bar{f}}(\tilde{x}(0)) + \pi^{\bar{f}}(\tilde{x}(L_0\tau)) \le \delta_1. \tag{2.260}$$

It follows from (2.259), (2.260), and property (P13) that

$$|\tilde{x}(t) - \bar{x}_*(t)| \le \epsilon \text{ holds for all } t \in [0, L_0\tau].$$

Together with (2.245) this implies that

$$|x(T\tau - t) - \bar{x}_*(t)| \le \epsilon \text{ holds for all } t \in [0, L_0\tau].$$

Theorem 2.25 is proved. □

2.15 Proof of Theorem 2.26

Theorems 2.13 and 2.25 imply the following result.

Theorem 2.47. *Let $L_0 > 0$ be an integer, $\epsilon > 0$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and an integer $L_1 > L_0$ such that for each integer $T \ge L_1$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T\tau)$ which satisfies*

$$I^g(0, T\tau, x, u) \le \sigma(g, 0, T\tau) + \delta$$

the following inequality holds for all $t \in [0, L_0\tau]$:

$$|x(T\tau - t) - \bar{x}_*(t)| \le \epsilon.$$

Theorem 2.47 and Propositions 2.43 and 2.45 imply Theorem 2.26.

2.16 Proof of Theorem 2.27

Theorem 2.27 follows from Propositions 2.43 and 2.45 and the next result.

Theorem 2.48. *Let $L_0 > 0$ be an integer, $\epsilon > 0$, $M_0 > 0$. Then there exist $\delta > 0$, a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and an integer $L_1 > L_0$ such that for each integer $T \ge L_1$, each $g \in \mathcal{U}$ and each $(x, u) \in X(A, B, 0, T\tau)$ which satisfies*

$$|x(0)|,\ |x(T\tau)| \le M_0,\ I^g(0, T\tau, x, u) \le \sigma(g, x(0), x(T\tau), 0, T\tau) + \delta$$

for all $t \in [0, L_0\tau]$,

$$|x(t) - \xi(t)| \le \epsilon,$$

where $(\xi, \eta) \in X(A, B, 0, \infty)$ *is the unique* (f, A, B)*-overtaking optimal pair such that* $\xi(0) = x(0)$.

Proof. Denote by d the metric of the space $\mathcal{M}$. Assume that Theorem 2.48 does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0, 1)$ such that

$$\delta_k < 4^{-k},\ k = 1, 2, \ldots, \tag{2.261}$$

a sequence of integers

$$T_k \ge L_0 + 2k,\ k = 1, 2, \ldots \tag{2.262}$$

a sequence $\{g_k\}_{k=1}^{\infty} \subset \mathcal{M}$ such that

$$d(g_k, f) \le k^{-1},\ k = 1, 2, \ldots \tag{2.263}$$

and a sequence $(x_k, u_k) \in X(A, B, 0, T_k\tau)$, $k = 1, 2, \ldots$ such that for each natural number k,

$$|x_k(0)|,\ |x_k(T_k\tau)| \le M_0, \tag{2.264}$$

$$I^{g_k}(0, T_k\tau, x_k, u_k) \le \sigma(g_k, x_k(0), x_k(T_k\tau), 0, T_k\tau) + \delta_k, \tag{2.265}$$

$$\max\{|x_k(t) - \xi_k(t)| :\ t \in [0, L_0\tau]\} > \epsilon, \tag{2.266}$$

where the pair $(\xi_k, \eta_k) \in X(A, B, 0, \infty)$ is (f, A, B)-overtaking optimal and

$$x_k(0) = \xi_k(0). \tag{2.267}$$

In view of (2.264), (2.267) and Theorem 2.4, the following property holds:

(P16) for each $\gamma > 0$ there exists a natural number $m(\gamma)$ such that for each integer $k \ge 1$ and each integer $p \ge m(\gamma)$,

$$|\xi_k(p\tau) - x_f(0)| \le \gamma.$$

Proposition 2.16 implies that for each pair of natural number p, k,

$$I^f(0, p\tau, \xi_k, \eta_k) - p\tau\mu(f) + \pi^f(\xi_k(0)) + \pi^f(\xi_k(p\tau)) = 0. \tag{2.268}$$

It follows from (2.264), (2.267), (2.268), and the continuity and boundedness from below of the function π^f that for each integer $p \ge 1$ the sequence $\{I^f(0, p\tau, \xi_k, \eta_k)\}_{k=1}^{\infty}$ is bounded. By Proposition 2.30, extracting subsequences and re-indexing we may assume without loss of generality that there exists $(\xi, \eta) \in X(A, B, 0, \infty)$ such that for each integer $p \geq 1$,

$$\xi_k(t) \to \xi(t) \text{ as } k \to \infty \text{ uniformly on } [0, p\tau], \tag{2.269}$$

$$I^f(0, p\tau, \xi, \eta) \leq \liminf_{k\to\infty} I^f(0, p\tau, \xi_k, \eta_k). \tag{2.270}$$

In view of (2.268)–(2.270) and the continuity of π^f, for each integer $p \geq 1$,

$$I^f(0, p\tau, \xi, \eta) - p\tau\mu(f) + \pi^f(\xi(0)) + \pi^f(\xi(p\tau)) \leq 0.$$

Together with Proposition 2.15 this implies that for each integer $p \geq 1$,

$$I^f(0, p\tau, \xi, \eta) - p\tau\mu(f) - \pi^f(\xi(0)) + \pi^f(\xi(p\tau)) = 0. \tag{2.271}$$

By (2.271), the boundedness from below of π^f and Theorem 2.2, the pair $(\xi, \eta) \in X(A, B, 0, \infty)$ is (f, A, B)-good. Together with (2.271) and Proposition 2.22 this implies that the pair (ξ, η) is (f, A, B)-overtaking optimal. In view of (2.267) and (2.269),

$$\xi(0) = \lim_{k\to\infty} \xi_k(0) = \lim_{k\to\infty} x_k(0). \tag{2.272}$$

Let $\Delta > 0$. Proposition 2.17, (2.34), and the continuity of π^f imply that there exists $\delta > 0$ such that for each $y, z \in R^n$ satisfying $|y - x_f(0)|,\ |z - x_f(0)| \leq \delta$,

$$|\pi^f(y)| \leq \Delta/8,\ |v(y, z) - \tau\mu(f)| \leq \Delta/8. \tag{2.273}$$

By (2.262)–(2.265), (2.267), and Theorems 2.13 and 2.4, there exists an integer $k(\Delta) \geq 1$ such that for each integer $k \geq 4k(\Delta) + 8$,

$$|x_k(i\tau) - x_f(0)| \leq \delta,\ i = k(\Delta) + 1, \dots, T_k - k(\Delta), \tag{2.274}$$

$$|\xi_k(i\tau) - x_f(0)| \leq \delta \text{ for all integers } i \geq k(\Delta) + 1. \tag{2.275}$$

Let $S \geq 1$ and $k \geq 4k(\Delta) + S + 8$ be integers. By Proposition 2.32 and (2.262), there exists $(\tilde{x}_k, \tilde{u}_k) \in X(A, B, 0, (S + k(\Delta) + 2)\tau)$ such that

$$\tilde{x}_k(t) = \xi_k(t),\ \tilde{u}_k(t) = \eta_k(t),\ t \in [0, (S + k(\Delta) + 1)\tau], \tag{2.276}$$

$$\tilde{x}_k((S + k(\Delta) + 2)\tau) = x_k((S + k(\Delta) + 2)\tau), \tag{2.277}$$

$$\begin{aligned} &I^f((S + k(\Delta) + 1)\tau, (S + k(\Delta) + 2)\tau, \tilde{x}_k, \tilde{u}_k) \\ &= v(\xi_k((S + k(\Delta) + 1)\tau), x_k((S + k(\Delta) + 2)\tau)). \end{aligned} \tag{2.278}$$

In view of (2.267), (2.268), and (2.273)–(2.278), for each integer $k \geq 4k(\Delta)+S+8$,

$$\begin{aligned} I^f(0,(S+k(\Delta)+2)\tau,\tilde{x}_k,\tilde{u}_k) &= I^f(0,(S+k(\Delta)+1)\tau,\xi_k,\eta_k) \\ &\quad + v(\xi_k((S+k(\Delta)+1)\tau),x_k((S+k(\Delta)+2)\tau)) \\ &\leq \mu(f)\tau(S+k(\Delta)+1)+\pi^f(x_k(0)) \\ &\quad - \pi^f(\xi_k((S+k(\Delta)+1)\tau))+\tau\mu(f)+\Delta/8 \\ &\leq \mu(f)\tau(S+k(\Delta)+2)+\pi^f(x_k(0))+\Delta/4. \end{aligned} \tag{2.279}$$

Proposition 2.41 and (2.263) imply that there exists an integer $k_1 \geq 4k(\Delta)+S+8$ such that for all integers $k \geq k_1$,

$$I^{g_k}(0,(S+k(\Delta)+2)\tau,\tilde{x}_k,\tilde{u}_k) \leq \mu(f)\tau(S+k(\Delta)+2)+\pi^f(x_k(0))+3\Delta/8. \tag{2.280}$$

By (2.261), (2.276), and (2.267), there exists an integer $k_2 \geq k_1$ such that for all integers $k \geq k_2$,

$$I^{g_k}(0,(S+k(\Delta)+2)\tau,x_k,u_k) \leq \mu(f)\tau(S+k(\Delta)+2)+\pi^f(x_k(0))+\Delta/2. \tag{2.281}$$

Proposition 2.41, (2.263), and (2.281) imply that there exists an integer $k_3 \geq k_2$ such that for all integers $k \geq k_3$,

$$I^f(0,(S+k(\Delta)+2)\tau,x_k,u_k) \leq \mu(f)\tau(S+k(\Delta)+2)+\pi^f(x_k(0))+5\Delta/8. \tag{2.282}$$

Since S is any natural number we conclude that for any integer $p \geq 1$, the sequence $\{I^f(0,p\tau,x_k,u_k)\}_{k=p}^{\infty}$ is bounded. By Proposition 2.30, extracting subsequences and re-indexing we may assume without loss of generality that there exists $(x,u) \in X(A,B,0,\infty)$ such that for each integer $p \geq 1$,

$$x_k(t) \to x(t) \text{ as } k \to \infty \text{ uniformly on } [0,p\tau], \tag{2.283}$$

$$I^f(0,p\tau,x,u) \leq \liminf_{k\to\infty} I^f(0,p\tau,x_k,u_k). \tag{2.284}$$

In view of (2.274), (2.282)–(2.284), and the continuity of π^f,

$$I^f(0,(S+k(\Delta)+2)\tau,x,u) \leq \mu(f)(S+k(\Delta)+2)\tau+\pi^f(x(0))+5\Delta/8, \tag{2.285}$$

$$|x((S+k(\Delta)+2)\tau)-x_f(0)| \leq \delta. \tag{2.286}$$

By (2.285), (2.286), and the choice of δ [see (2.273)],

$$I^f(0,(S+k(\Delta)+2)\tau,x,u)-\mu(f)(S+k(\Delta)+2)\tau$$
$$-\pi^f(x(0))+\pi^f(x((S+k(\Delta)+2)\tau))\leq\Delta. \tag{2.287}$$

Since S is any natural number it follows from (2.287) and Proposition 2.15 that for any integer $p\geq 1$,

$$I^f(0,p\tau,x,u)-\mu(f)\tau p-\pi^f(x(0))+\pi^f(x(p\tau))\leq\Delta. \tag{2.288}$$

In view of (2.288), the boundedness from below of the function π^f and Theorem 2.2, $(x,u)\in X(A,B,0,\infty)$ is an (f,A,B)-good pair. Since Δ is any positive number Proposition 2.15 and (2.288) imply that for any integer $p\geq 1$,

$$I^f(0,p\tau,x,u)-\mu(f)\tau p-\pi^f(x(0))+\pi^f(x(p\tau))=0. \tag{2.289}$$

By Proposition 2.22 and (2.289), (x,u) is an (f,A,B)-overtaking optimal pair. Since (x,u) and (ξ,η) are (f,A,B)-overtaking optimal pairs it follows from (2.272), (2.283), and Theorem 2.3 that

$$x(t)=\xi(t),\ u(t)=\eta(t),\ t\in[0,\infty). \tag{2.290}$$

By (2.269), (2.283), and (2.290), for all sufficiently large natural numbers k,

$$|x_k(t)-\xi_k(t)|\leq\epsilon/4,\ t\in[0,L_0\tau].$$

This contradicts (2.266). The contradiction we have reached proves Theorem 2.48.

http://www.springer.com/978-3-319-19140-9

Turnpike Theory of Continuous-Time Linear Optimal
Control Problems
Zaslavski, A.J.
2015, IX, 296 p., Hardcover
ISBN: 978-3-319-19140-9