

Chapter 2

Introduction and Theory

This chapter aims to convey the underlying theoretical concepts and the current state of research in the field of turbulent multi-particle dispersion. Some of these concepts are well known in the turbulence community and are described in detail in e.g. Monin and Yaglom (2007), Frisch (1995), Argyris et al. (2010) and Pope (2000). Wherever possible, however, I will rely on the excellent and historically important texts by Richardson (1922), Kolmogorov (1941a, b) and Batchelor (1950) in order to present these ideas.

Section 2.1 describes the governing equations of a turbulent flow and explains the necessity of a statistical description. Section 2.2 focuses on the famous theory of Kolmogorov (1941a) and the underlying picture of the turbulence energy cascade. In Sect. 2.3, different aspects of turbulent dispersion for two or more fluid particles are described and current theoretical and experimental findings are presented.

2.1 The Governing Equations

The first fundamental assumption when dealing with fluid dynamics is the *continuum approximation*. This assumption implies that the fluid fills space continuously and that its composition of molecules and ions can be neglected. This approximation is valid as long as the smallest scales of the flow are much larger than the mean free path in the fluid, a condition satisfied by the majority of flows. In a strongly turbulent cumulus cloud, e.g., the smallest scales of the flow are of the order of $\eta \approx 10^{-3}$ m and thus some orders of magnitude larger than the typical mean free path.¹ Using the continuum approximation, the equations for fluid motion can be derived from first principles.

¹The smallest turbulent scale is estimated from measurements in atmospheric clouds at an altitude of approx. 2000 m (Siebert et al. 2006). The atmospheric pressure at this height is about 65 kPa for a sea-surface temperature of 288 K and an average molecular mass of the atmosphere of 0.02897 kg/mole, leading to a mean free path of $\lambda \approx 10^{-7}$ m.

2.1.1 The Navier-Stokes Equations

The evolution of a fluid element is governed by two conservation laws: the conservation of mass and the conservation of momentum. Mass conservation leads to the continuity equation

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla \cdot (\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) = 0, \quad (2.1)$$

with $\rho(\mathbf{x}, t)$ being the density of the fluid element and $\mathbf{u}(\mathbf{x}, t)$ being its velocity. Momentum conservation, together with the second Newtonian theorem, results in

$$\begin{aligned} \frac{d}{dt} (\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) &= \left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \right) (\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) \\ &= \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \rho(\mathbf{x}, t) \mathbf{f}(\mathbf{x}, t), \end{aligned} \quad (2.2)$$

with the stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ and $\mathbf{f}(\mathbf{x}, t)$ being the body force (e.g. gravity or the Coriolis force).

Most natural flows exhibit incompressibility, meaning that the density remains constant along fluid trajectories, according to $\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \rho(\mathbf{x}, t) = 0$. In the case of low-intensity water flows, where only very small Mach numbers are reached, it is justified to even assume a constant density, $\rho(\mathbf{x}, t) \equiv \rho$, throughout the flow. For the rest of this thesis, the density is assumed to be constant if not stated otherwise. In this case, the above equations simplify to

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0, \quad (2.3)$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \right) \mathbf{u}(\mathbf{x}, t) = \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \rho \mathbf{f}(\mathbf{x}, t). \quad (2.4)$$

For an incompressible viscous fluid, the stress tensor is given by

$$\sigma_{ij}(\mathbf{x}, t) = -p(\mathbf{x}, t) \delta_{ij} + \mu \left(\frac{\partial u_i(\mathbf{x}, t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x}, t)}{\partial x_i} \right), \quad (2.5)$$

where $p(\mathbf{x}, t)$ is the scalar pressure field, μ is the dynamic viscosity, and δ_{ij} is the Kronecker delta. With this stress tensor and Eq. (2.4), one obtains the *Navier-Stokes equations* (Navier 1827; Stokes 1845) for a fluid with constant density:

$$\rho \left(\frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) \right) = -\nabla p(\mathbf{x}, t) + \mu \Delta \mathbf{u}(\mathbf{x}, t) + \rho \mathbf{f}(\mathbf{x}, t). \quad (2.6)$$

Together with the continuity Eq. (2.3) and initial as well as boundary conditions, the Navier-Stokes equations fully describe a fluid flow. It is informative to non-dimensionalize Eq. (2.6) by introducing the new variables

$$\mathbf{x}' = \mathbf{x}/L, \quad \mathbf{u}' = \mathbf{u}/U, \quad t' = t U/L, \quad p' = p/(\rho U^2) \quad \text{and} \quad \mathbf{f}' = \mathbf{f} L/U^2, \quad (2.7)$$

with L and U being the characteristic length scale and velocity of the flow. Inserting the new variables into Eq.(2.6), dividing by U^2/L and omitting the primes, one obtains

$$\frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}, t) + (\mathbf{u}(\mathbf{x}, t) \cdot \nabla) \mathbf{u}(\mathbf{x}, t) = -\nabla p(\mathbf{x}, t) + \frac{1}{\text{Re}} \Delta \mathbf{u}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t). \quad (2.8)$$

The only free parameter in Eq.(2.8) is the *Reynolds number*, $\text{Re} = \frac{UL}{\nu}$, with the kinematic viscosity $\nu = \mu/\rho$ (Reynolds 1883). For very small Reynolds numbers, $\text{Re} \lesssim 1$, the viscous term is dominating and the flow is purely laminar, i.e. all stream lines are in parallel. For $\text{Re} \gtrsim 10^3$, on the other hand, the viscous damping term is very weak and turbulence can evolve (Reynolds 1883; Avila et al. 2011). For turbulent, atmospheric flows one can find Reynolds numbers as high as $\text{Re} \approx 10^8 - 10^9$.

The Navier-Stokes equations are nonlinear and, due to the pressure term, also non-local. For turbulent flows, for which the damping term, $\frac{1}{\text{Re}} \Delta \mathbf{u}(\mathbf{x}, t)$, becomes relatively weak due to a high Reynolds number, this leads to a phenomenon called *deterministic chaos*: Tiny differences in the initial conditions lead to very different outcomes, making the flow unpredictable (see e.g. Lorenz 1963). Only averaged quantities can be reproduced reliably, showing the necessity of a statistical approach to turbulence.

2.1.2 Statistical Description of Turbulence

A natural first attempt for a statistical description of turbulent flows is the averaging of the Navier-Stokes equations in order to obtain an expression for the mean flow. This was first done by Reynolds (1895) using a time average. In this section, I will use an ensemble average instead, but the concept remains the same.

Let us assume several independent statistical events, e.g. repetitions of the same experiment, where for each event a random variable x is measured. The ensemble average of x is then defined as

$$\langle x \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N x_n, \quad (2.9)$$

with x_n being the values of x for the different events. Due to the chaotic behavior of turbulent flow, the velocity field $\mathbf{u}(\mathbf{x}, t)$ can be taken as such a random variable, more precisely a random field. Using the ensemble average, one can split the velocity field into a mean value and a fluctuating component in the form

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t), \quad (2.10)$$

with $\bar{\mathbf{u}}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle$. From this definition, it directly follows that $\langle \bar{\mathbf{u}}(\mathbf{x}, t) \rangle = \bar{\mathbf{u}}(\mathbf{x}, t)$ and $\langle \mathbf{u}'(\mathbf{x}, t) \rangle = 0$. Inserting the above decomposition into the continuity and

Navier-Stokes Eqs. (2.3) and (2.8) and taking the ensemble average leads to

$$\begin{aligned} \nabla \cdot \bar{\mathbf{u}}(\mathbf{x}, t) &= 0, \\ \frac{\partial}{\partial t} \bar{\mathbf{u}}(\mathbf{x}, t) + (\bar{\mathbf{u}}(\mathbf{x}, t) \cdot \nabla) \bar{\mathbf{u}}(\mathbf{x}, t) + \langle (\mathbf{u}'(\mathbf{x}, t) \cdot \nabla) \mathbf{u}'(\mathbf{x}, t) \rangle &= -\nabla \langle p(\mathbf{x}, t) \rangle + \frac{1}{\text{Re}} \Delta \bar{\mathbf{u}}(\mathbf{x}, t) \\ &\quad + \langle \mathbf{f}(\mathbf{x}, t) \rangle. \end{aligned}$$

This new set of four equations comprises seven variables. Aside from the mean flow variables $\bar{\mathbf{u}}(\mathbf{x}, t)$ and $\langle p(\mathbf{x}, t) \rangle$, the last term on the left hand side depends quadratically on the velocity fluctuations $\mathbf{u}'(\mathbf{x}, t)$. It is usually written in the form

$$\langle (\mathbf{u}'(\mathbf{x}, t) \cdot \nabla) \mathbf{u}'(\mathbf{x}, t) \rangle_i = \sum_j \frac{\partial}{\partial x_j} \langle u'_j(\mathbf{x}, t) u'_i(\mathbf{x}, t) \rangle, \quad (2.11)$$

where the components of $\langle u'_j(\mathbf{x}, t) u'_i(\mathbf{x}, t) \rangle$ are known as the *Reynolds stresses*. Determining the mean velocity $\bar{\mathbf{u}}(\mathbf{x}, t)$ thus requires the knowledge of the second moment of the velocity fluctuations, meaning that the equations for $\bar{\mathbf{u}}(\mathbf{x}, t)$ are not closed. This *closure problem* prevents a statistical description of turbulence solely based on the Navier-Stokes equations (Kraichnan 1961), and some modeling is required in order to formulate results (for an overview, see e.g. Pope 2000, Chaps. 10–14).

2.2 Kolmogorov's Theory (1941)

The works published by Kolmogorov in 1941 are amongst the most important achievements in turbulence research (Frisch 1995; Kolmogorov 1941a). He built the foundation of today's understanding of turbulent flow and provided one of the few exact results derived from the Navier-Stokes equations.

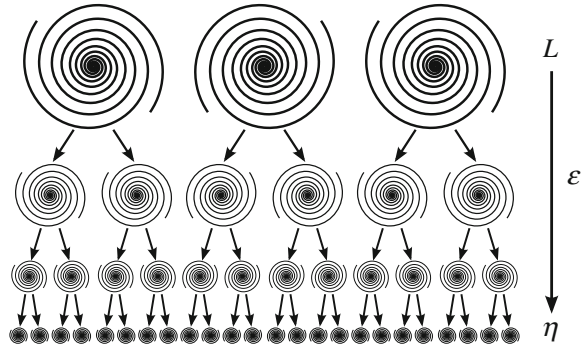
Kolmogorov refined the picture of the *turbulence energy cascade*, first promoted by Richardson (1922), and used it as a foundation for his work (Kolmogorov 1941a). According to this picture, energy injected into a three-dimensional (3D) turbulent flow will form eddies² of some large scale L (e.g. the size of a propeller stirring the fluid). These eddies break up into smaller eddies, thereby transferring their energy to smaller scales with an *energy transfer rate* ε . This process continues until a length scale η is reached at which viscous dissipation dominates (Fig. 2.1). Kolmogorov further assumed that a statistical decoupling happens during this energy transfer. As a consequence, eddies much smaller than the injection scale L should be unaffected by the nature of the injection. From this he concluded his *first hypothesis of similarity*:

Statistical quantities depending only on the smallest scales of the flow are fully determined by the energy transfer rate ε and the kinetic viscosity ν .³

² An eddy is to be understood as “a component of motion with a certain length scale, i.e. an arbitrary flow pattern characterized by size alone” (Batchelor 1950). It is not to be confused with a whirl as seen near a drain.

³ Due to the mathematical nature of the original hypotheses by Kolmogorov (1941a), a more verbose formulation is chosen here.

Fig. 2.1 Sketch of the energy cascade as used by Kolmogorov (1941a)



It follows directly that these quantities need to be homogeneous and isotropic because any preference of orientation of the largest eddies is lost at the small scales.⁴ Using only ε and ν , one is also able to construct universal time, length, and velocity scales, the *Kolmogorov microscales*:

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}}, \quad \tau_\eta = \left(\frac{\nu}{\varepsilon} \right)^{\frac{1}{2}}, \quad u_\eta = (\nu \varepsilon)^{\frac{1}{4}}. \quad (2.12)$$

By construction, these are the scales corresponding to the smallest eddies of the flow with a Reynolds number of $\text{Re}_\eta = \frac{u_\eta \eta}{\nu} = 1$. While η represents the size of the eddies, τ_η gives the *eddy turn-over time* and u_η the resulting characteristic velocity. Changes of the flow along distances much smaller than η or during time intervals much shorter than τ_η are assumed to be smooth and differentiable. The scale ratio between the large energy containing eddies and the small dissipative eddies is known to grow as $L/\eta \propto \text{Re}^{\frac{3}{4}}$ and $T_L/\tau_\eta \propto \text{Re}^{\frac{1}{2}}$ with T_L being the eddy turn-over time of the large eddies. For very strong turbulence, and thus very high Reynolds number, the scale separation L/η becomes very large. In this case, Kolmogorov suggested that there exists a range of scales, $\eta \ll r \ll L$, which is unaffected both by the large scales and by dissipation. This leads to his *second hypothesis of similarity*:

For very large Reynolds numbers, there exists an intermediate range of scales for which any statistical quantity is fully determined by the scale r and the energy transfer rate ε .

This intermediate range of scales, the *inertial range*, is therefore independent of the underlying conditions of the flow field (i.e. forcing, geometry, and viscosity) and should yield the same statistics for all flows with the same energy transfer rate. This conclusion is of major importance to the entire field of turbulence research, since it allows the comparison of results obtained from different flows.

⁴The concept of isotropic turbulence and its usefulness to compute exact relations had already been presented earlier in the important works of Taylor (1935) and Von Kármán and Howarth (1938).

Starting from these two hypotheses, one can use dimensional analysis to obtain some useful results for the limit of $\text{Re} \rightarrow \infty$. For the hypotheses to be applicable, however, the statistical quantities under consideration need to be assignable to a certain scale. A good choice for such a quantity are the *longitudinal structure functions*,

$$S_p(r) = \left\langle \left(\frac{(\mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)) \cdot \mathbf{r}}{r} \right)^p \right\rangle, \quad (2.13)$$

with p being a positive integer. Due to homogeneity, the structure functions depend only on the separation vector $\mathbf{r}$, while isotropy can be used to further simplify this dependence to the absolute value $r = |\mathbf{r}|$. Using the second hypothesis and dimensional analysis, it can be shown that

$$S_p(r) \propto (\varepsilon r)^{\frac{p}{3}}. \quad (2.14)$$

For the special case of $p = 3$, Kolmogorov even derived an exact result from the Navier-Stokes equations (Kolmogorov 1941b). Only assuming homogeneity and isotropy, he showed that in the limit of infinite Reynolds number, the third order structure function behaves as

$$S_3(r) = -\frac{4}{5}\varepsilon r, \quad (2.15)$$

where r is in the inertial range. This *4/5-law* is one of the few exact results in turbulence research and therefore plays an important role in the description of turbulence.⁵

It was later shown that an alternative formulation to the 4/5-law can be obtained by looking at the derivative of the squared relative velocity

$$\begin{aligned} \frac{1}{2} \left\langle \frac{d}{dt} [\mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)]^2 \right\rangle &= \langle [\mathbf{u}(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot [\mathbf{a}(\mathbf{x} + \mathbf{r}, t) - \mathbf{a}(\mathbf{x}, t)] \rangle \\ &= -2\varepsilon. \end{aligned} \quad (2.16)$$

This result is exact for a homogeneous flow at a high Reynolds number and with $\mathbf{r}$ in the inertial range (Mann et al. 1999; Falkovich et al. 2001; Pumir et al. 2001). Since I will use Eq. (2.16) frequently in this thesis, I will from now on use the short notation $\langle \delta \mathbf{u} \cdot \delta \mathbf{a} \rangle = -2\varepsilon$ whenever referring to this expression.

⁵It is known for many years that the scaling law in Eq. (2.14) is actually not exact for $p \neq 3$ due to intermittency effects. For a longer discussion of this topic see e.g. Frisch (1995), Chap. 8.

2.3 Turbulent Dispersion

Another area of turbulence research in which the theory of Kolmogorov can be put to great use is turbulent dispersion. The separation statistics of clusters of n particles in a turbulent flow are of crucial importance for the understanding of many fundamental processes, like the spreading of clouds or turbulent mixing (Sawford 2001; Ottino 1989). Furthermore, as opposed to single particle diffusion, the characteristic length scale of a cluster of particles dominates the statistics thereof so that Kolmogorov's theory can be applied (Batchelor 1950). The easiest case, the separation of pairs, of course plays a special role in theoretical, experimental, and numerical investigations (Sawford 2001; Salazar and Collins 2009).

2.3.1 Eulerian and Lagrangian Descriptions of Turbulence

Studying particle separation requires to follow particle trajectories through a flow. This is best done by the so-called *Lagrangian description* of turbulence (Yeung 2002; Toschi and Bodenschatz 2009). As opposed to the Eulerian framework in which the flow field is given at fixed points in space and time, the Lagrangian framework describes flow properties along the trajectories of fluid particles. Here, I use the term *fluid particle*, or *fluid element*, to describe a single point in the fluid continuum which follows the velocity field. It is an abstract mathematical concept, not to be confused with e.g. a molecule. If such a fluid particle starts at a position $\mathbf{y}$ at time $t = 0$, then its position at a later time t will be denoted by $\mathbf{X}(t|\mathbf{y}, 0)$ with $\mathbf{X}(0|\mathbf{y}, 0) = \mathbf{y}$. By construction, the velocity of a fluid particle is always identical to the velocity field at its position, that is, $\frac{d}{dt}\mathbf{X}(t|\mathbf{y}, 0) = \mathbf{U}(t|\mathbf{y}, 0) = \mathbf{u}(\mathbf{X}(t|\mathbf{y}, 0), t)$. Experimentally, tracer particles with the same density as the fluid and a diameter $d \ll \eta$ much smaller than the Kolmogorov scale can be treated as fluid particles.

2.3.2 Pair Dispersion

The displacement statistics of a pair of fluid particles can be described by the probability density function (p.d.f.) $P(\mathbf{x}_1, \mathbf{x}_2, t|\mathbf{y}_1, \mathbf{y}_2, 0)$ of finding the particles at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ at time t under the condition that they have been at positions $\mathbf{y}_1$ and $\mathbf{y}_2$ at time $t = 0$ (Batchelor 1952). In the Lagrangian notation introduced above, this means that $\mathbf{X}(t|\mathbf{y}_1, 0) = \mathbf{x}_1$ and $\mathbf{X}(t|\mathbf{y}_2, 0) = \mathbf{x}_2$. Separating the relative positioning $\mathbf{R} = \mathbf{x}_2 - \mathbf{x}_1$ and $\mathbf{R}_0 = \mathbf{y}_2 - \mathbf{y}_1$ from the general movement $\mathbf{M} = \mathbf{x}_1 - \mathbf{y}_1$, we obtain the more useful p.d.f. $P(\mathbf{M}, \mathbf{R}, t|\mathbf{R}_0, 0)$, where the dependence on the initial positions $\mathbf{y}_1$ and $\mathbf{y}_2$ is omitted under the assumption of a homogeneous flow. In many cases, as for example the growth of a particle cloud, the collective motion of the two particles is of no interest, and the separation p.d.f.

$$P_r(\mathbf{R}, t|\mathbf{R}_0, 0) = \int P(\mathbf{M}, \mathbf{R}, t|\mathbf{R}_0, 0)d\mathbf{M} \quad (2.17)$$

provides an adequate description. For an isotropic homogeneous flow, the first moment of this p.d.f. vanishes and the second moment is given by

$$\left\langle [\mathbf{X}(t|\mathbf{y}_1, 0) - \mathbf{X}(t|\mathbf{y}_2, 0)]^2 \right\rangle \equiv \langle \mathbf{R}(t)^2 \rangle = \int \mathbf{R}^2 P_r(\mathbf{R}, t|\mathbf{R}_0, 0)d\mathbf{R}. \quad (2.18)$$

This second moment, the *mean squared separation*, gives the relative dispersion of the two fluid particles and its evolution can be written in the very general form

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle = F(R_0, t), \quad (2.19)$$

where, due to the assumption of homogeneity and isotropy, the function F can only depend on the absolute value $R_0 = |\mathbf{R}_0|$ of the initial separation. With this, Kolmogorov's hypotheses can be applied to learn more about the function F and I will loosely follow Batchelor (1950) for this procedure.

As a prerequisite for the first similarity hypothesis, $|\mathbf{R}(t)|$ needs to be much smaller than the energy injection scale L , which results in the requirement that t and R_0 be “not too large.” Under these conditions, Eq. (2.19) can be expressed more precisely as

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle = \nu F \left(\frac{R_0 \varepsilon^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}, \frac{t \varepsilon^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \right), \quad (2.20)$$

where the arguments of F are now dimensionless. In the case where R_0 , and thus $|\mathbf{R}(t)|$, is in the inertial range, the second similarity hypothesis states that F can only depend on ε and the involved time and spatial scales. This leads to

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle = \varepsilon t^2 F \left(\frac{R_0}{\varepsilon^{\frac{1}{2}} t^{\frac{3}{2}}} \right), \quad (2.21)$$

where the only constructible dimensionless variable depends on both R_0 and t . Aside from restricting R_0 , we can of course also constrain the time variable:

very small t: The mean squared separation can always be rewritten in terms of the particle velocities as

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle = 2 \left\langle \left(\mathbf{R}(0) + \int_0^t \mathbf{V}(t') dt' \right) \cdot \mathbf{V}(t) \right\rangle, \quad (2.22)$$

with $\mathbf{V}(t) = \mathbf{U}(t|\mathbf{y}_1, 0) - \mathbf{U}(t|\mathbf{y}_2, 0)$ being the relative velocity between the two particles. In his original work, Batchelor (1950) assumed the initial separation and the velocity to be uncorrelated, so that $\langle \mathbf{R}(0) \cdot \mathbf{V}(t) \rangle$ can be set to zero. Furthermore, in the limit where t is very small, one finds that $\mathbf{V}(t') \approx \mathbf{V}(t) \approx \mathbf{V}(0)$ within

the limits of the integration. This should be valid as long as the two velocities are still strongly correlated, meaning that $t \ll t_0$, where $t_0 = (R_0^2/\varepsilon)^{\frac{1}{3}}$ is the characteristic Kolmogorov time for an eddy of size R_0 .⁶ For R_0 in the inertial range, Eq. (2.21) then reduces to

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle \approx 2t \langle \mathbf{V}(0)^2 \rangle = 2 \frac{11}{3} C_2 (\varepsilon R_0)^{\frac{2}{3}} t. \quad (2.23)$$

This scaling law for the particle separation is known as *Batchelor scaling*. Note that the prescribed linearity in time as a result of t being small, together with the assumption of R_0 being in the inertial range, are sufficient to completely determine F up to an unknown constant C_2 .

intermediate t: For a sufficiently high Reynolds number and R_0 in the inertial range, there is a range of intermediate times, $t > t_0$, for which $|\mathbf{R}(t)|$ is still in the inertial range but much larger than the initial separation. In this case, one can argue that $\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle$ does not depend on the exact value of R_0 , so that Eq. (2.21) simplifies to

$$\frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle \approx 3g\varepsilon t^2. \quad (2.24)$$

Again, the evolution of the pair separation is fully determined except for a constant g , termed Richardson constant. Equation (2.24) is respectively known as the *Richardson-Obukhov law* (Richardson 1926; Obukhov 1941).

Apart from these early findings, another more recent consideration shall be presented here. As mentioned above, Batchelor (1950) assumed the initial separation and relative velocity to be uncorrelated. Without this assumption, Eq. (2.23) changes to

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{R}(t)^2 \rangle &\approx 2 \langle \mathbf{R}(0) \cdot \mathbf{V}(0) \rangle + 2t \langle \mathbf{V}(0)^2 \rangle \\ &\approx 2 \langle \mathbf{R}(0) \cdot \mathbf{V}(0) \rangle + C_2 (\varepsilon R_0)^{\frac{2}{3}} t, \end{aligned} \quad (2.25)$$

which can be re-expressed in the form of the *mean squared change of separation*

$$\frac{d}{dt} \langle \delta \mathbf{R}^2(t) \rangle \equiv \frac{d}{dt} \langle [\mathbf{R}(t) - \mathbf{R}(0)]^2 \rangle \approx C_2 (\varepsilon R_0)^{\frac{2}{3}} t. \quad (2.26)$$

For a perfectly homogeneous and isotropic flow, Eqs. (2.24) and (2.26) are identical. There is some evidence, however, that for real flows, both experimentally and numerically, this condition is only approximately fulfilled. In this case, the mean squared change of separation shows much clearer scaling than the original expression by Batchelor and should be used instead (Ouellette et al. 2006).

⁶One can also think of t_0 as the *eddy-turn-over time* of an eddy of size R_0 . After one turn-over, the eddy loses its coherence and the velocities of two fluid particles belonging to the eddy become uncorrelated.

The Richardson-Obukhov law, on the other hand, is identical for both the mean squared separation and the mean squared change of separation since an independence from the initial separation was assumed in the derivation.

Please note that in the Lagrangian notation, the relation for $\langle \delta \mathbf{u} \cdot \delta \mathbf{a} \rangle$ in Eq. (2.16) can be conveniently written as

$$\frac{1}{2} \left\langle \frac{d}{dt} \mathbf{V}(t)^2 \right|_{t=0} \rangle = \langle \mathbf{V}(0) \cdot \mathbf{A}(0) \rangle = -2\varepsilon, \quad (2.27)$$

where $\mathbf{A}(t) = \frac{d}{dt} \mathbf{V}(t)$ is the relative acceleration between the two particles. This expression can be understood as the Lagrangian equivalent of the 4/5-law.

2.3.3 Backward Dispersion

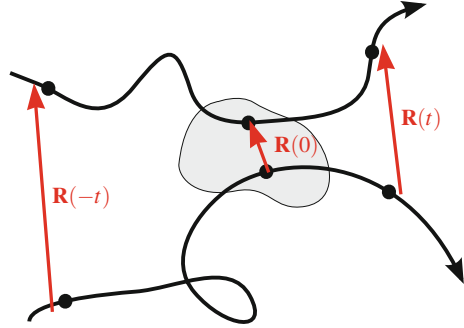
I have thus far focused on the dispersion of particle pairs over time that start with some fixed initial separation. Naming this process *forward dispersion*, one can distinguish another concept, where a fixed separation is prescribed at a later time and the dispersion of particle pairs prior to that time is analyzed. This is called *backward dispersion*. As shown above, the forward dispersion of a pair of fluid particles is given by $\langle \mathbf{R}(t)^2 \rangle = \int \mathbf{R}^2 P_r(\mathbf{R}, t | \mathbf{R}_0, 0) d\mathbf{R}$, with $P_r(\mathbf{R}, t | \mathbf{R}_0, 0)$ being the probability that $\mathbf{X}(t | \mathbf{y}_2, 0) - \mathbf{X}(t | \mathbf{y}_1, 0) = \mathbf{R}$ for $\mathbf{y}_2 - \mathbf{y}_1 = \mathbf{R}_0$. Accordingly, the backward dispersion can be written as

$$\langle \mathbf{R}(-t)^2 \rangle = \int \mathbf{R}^2 P_r(\mathbf{R}, -t | \mathbf{R}_0, 0) d\mathbf{R} = \int \mathbf{R}^2 P_r(\mathbf{R}_0, t | \mathbf{R}, 0) d\mathbf{R}, \quad (2.28)$$

where stationarity and incompressibility⁷ of the flow are used for the second equality. The alternative formulations in Eq. (2.28) correspond to the two notions that the particle pair is either tracked from its initial condition backwards in time towards an earlier state, or from an earlier state towards a fixed end state. Both interpretations are equally useful and show an important difference to forward dispersion as illustrated in Fig. 2.2. Assuming that the fluid particles are part of a material cloud at $t = 0$, forward dispersion describes the spreading of the cloud whereas backward dispersion describes its formation. The latter provides valuable information about how advected material coming from different locations of a flow is brought together, which is apparently closely connected to scalar concentration fields and turbulent mixing (Corrsin 1952; Durbin 1980; Frisch et al. 1999; Thomson 2003; Celani et al. 2004; Sawford et al. 2005).

⁷For an incompressible flow, it can be shown that the displacement probability of two particles obeys $P(\mathbf{x}_1, \mathbf{x}_2, t | \mathbf{y}_1, \mathbf{y}_2, 0) = P(\mathbf{y}_1, \mathbf{y}_2, 0 | \mathbf{x}_1, \mathbf{x}_2, t)$ (Lundgren 1981), which directly leads to $P_r(\mathbf{R}, t | \mathbf{R}_0, 0) = P_r(\mathbf{R}_0, 0 | \mathbf{R}, t)$.

Fig. 2.2 Sketch of the relative motion of two fluid particles with a fixed separation $|\mathbf{R}(0)| = R_0$ at $t = 0$. If the particles are located inside a cloud of material at $t = 0$, forward dispersion captures the dispersion of the cloud with time while backward dispersion is connected to the concentration field at $t = 0$



While forward dispersion has been studied extensively since the early works of Taylor (1922) and Richardson (1926), backward dispersion has only recently become focus of turbulence research. For the relative dispersion of fluid particles with an initial separation R_0 in the inertial range, recent studies found that dispersion happens faster backwards in time than forwards (Sawford et al. 2005; Berg et al. 2006; Bragg et al. 2014). Sawford et al. (2005) proposed that this time asymmetry can be captured by extending Richardson scaling towards the backward case with

$$\mathbf{R}^2(t) = \begin{cases} R_0^2 + g_f \varepsilon t^3 & \text{for } t > 0 \\ R_0^2 + g_b \varepsilon |t|^3 & \text{for } t < 0, \end{cases} \quad (2.29)$$

where the two scaling constants have been experimentally measured to be $g_f = 0.55$ and $g_b = 1.15$ (Berg et al. 2006). The reliability of these result will be discussed later in this thesis.

2.3.4 Multi-particle Dispersion

In recent years, new experimental techniques and ever-growing computer clusters have enabled us to go beyond pair statistics and take first steps in the direction of multi-particle dispersion (Chertkov et al. 1999; Falkovich et al. 2001; Biferale et al. 2005; Lüthi et al. 2007; Hackl et al. 2011; Xu et al. 2011; Pumir et al. 2013). This overview will be restricted to the theoretical essentials for describing n -particle dispersion with a focus on $n = 4$. Further developments and ideas will be covered in Chap. 5 as well as the discussion section of this thesis.

The major benefit of looking at clusters of $n > 2$ fluid particles is that beyond the size of the cluster, there is a multitude of additional shape information. In order to capture all of this information, a clever description of the cluster is needed. In a homogeneous, d -dimensional flow, a cloud of n particles can be described by a set of $n - 1$ position vectors. In the field of turbulence research, the usual notation for this set of vectors is

$$\rho^{(m)}(t) = \sqrt{\frac{m}{m+1}} \left[\mathbf{X}(t|\mathbf{y}^{(m+1)}, 0) - \frac{1}{m} \sum_{l=1}^m \mathbf{X}(t, \mathbf{y}^{(l)}, 0) \right], \quad m \in \{1, 2, \dots, n-1\}, \quad (2.30)$$

with $\rho^{(m)}$ being the m th ρ -vector. One can now construct a $d \times (n-1)$ matrix

$$\mathbf{P}(t) = \left(\rho^{(1)}(t) \rho^{(2)}(t) \dots \rho^{(n-1)}(t) \right), \quad (2.31)$$

which contains the full geometrical information of the cluster. The singular value decomposition of $\mathbf{P}(t)$ is then given by

$$\mathbf{P} = \mathbf{U}(t) \text{diag}(\sigma_1(t), \sigma_2(t), \dots, \sigma_{\min(d, n-1)}(T)) \mathbf{W}^T(t). \quad (2.32)$$

Here, $\mathbf{W}(t)$ is an $(n-1) \times (n-1)$ orthogonal matrix and gives the orientation in pseudo-space, a vector space spanned by the numbering of the ρ -vectors. The singular values, σ_i , contain the elongation of the cluster along different axes, and the $d \times d$ orthogonal matrix $\mathbf{U}(t)$ gives the orientation in real space. In an isotropic flow, the orientation of the cluster in real space does not matter, so that the full cluster can be described by $\min(d, n)$ singular values and $(n-1)(n-2)/2$ Euler angles in pseudo-space. Since $\mathbf{P}$ does not transform as a tensor, however, it is helpful to define the *moment of inertia* or *shape tensor*,

$$\mathbf{G}(t)_{ij} = \sum_{a=1}^n \rho_i^{(a)}(t) \rho_j^{(a)}(t) = (\mathbf{P}(t) \mathbf{P}^T(t))_{ij}, \quad (2.33)$$

with $i, j \in \{1, 2, \dots, d\}$ being spatial indices.⁸ The eigenvalues $g_i(t)$ of the shape tensor are given by the squares of the singular values of $\mathbf{P}(t)$ and thus must be real and non-negative. With this definition, the shape of any cluster of n particles can be fully described by the eigenvalues of the shape tensor together with $(n-1)(n-2)/2$ Euler angles. It is helpful to sort the eigenvalues by size, with $g_1(t) \geq g_2(t) \geq \dots \geq g_{n-1}(t)$.

The size of the cluster can be described by the *radius of gyration*, which is defined as the squared distances of the individual particles from the center of mass of the cluster, $\mathbf{X}_{\text{com}}(t)$. In terms of the shape tensor, the radius of gyration is simply given by

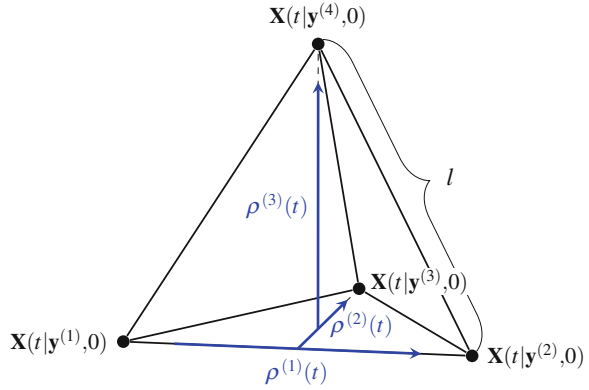
$$R^2(t) = \sum_i \left(\mathbf{X}(t|\mathbf{y}^{(m)}, 0) - \mathbf{X}_{\text{com}}(t) \right)^2 = \text{tr}(\mathbf{G}(t)) = \sum_i g_i(t). \quad (2.34)$$

Four-Particle Clusters

A cluster of four particles in a 3D flow is of particular interest. Four points, forming a tetrahedron, is the minimum constellation needed to describe a 3D volume. The

⁸Instead of the shape tensor, one can also define the *dispersion tensor* $\mathbf{C}(t) = \mathbf{P}^T(t) \mathbf{P}(t) = \mathbf{W}(t) \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_{\min(d, n-1)}^2) \mathbf{W}^T(t)$. It has the same eigenvalues as $\mathbf{G}(t)$ and serves the same purpose (Hackl et al. 2011).

Fig. 2.3 A regular tetrahedron formed by a cluster of four fluid particles. All edges have the same length, l . Three ρ -vectors are needed to fully describe the tetrahedron in a homogeneous flow



number of needed ρ -vectors to span the tetrahedron is given by $n - 1 = 3$ and equals the dimension of the flow. The ρ -vectors are given by

$$\begin{aligned}\rho^{(1)}(t) &= \sqrt{\frac{1}{2}} \left[\mathbf{X}(t|\mathbf{y}^{(2)}, 0) - \mathbf{X}(t|\mathbf{y}^{(1)}, 0) \right], \\ \rho^{(2)}(t) &= \sqrt{\frac{2}{3}} \left[\mathbf{X}(t|\mathbf{y}^{(3)}, 0) - \frac{1}{2}\mathbf{X}(t|\mathbf{y}^{(1)}, 0) - \frac{1}{2}\mathbf{X}(t|\mathbf{y}^{(2)}, 0) \right], \\ \rho^{(3)}(t) &= \sqrt{\frac{3}{4}} \left[\mathbf{X}(t|\mathbf{y}^{(4)}, 0) - \frac{1}{3}\mathbf{X}(t|\mathbf{y}^{(1)}, 0) - \frac{1}{3}\mathbf{X}(t|\mathbf{y}^{(2)}, 0) - \frac{1}{3}\mathbf{X}(t|\mathbf{y}^{(3)}, 0) \right].\end{aligned}\quad (2.35)$$

Figure 2.3 shows the special configuration of a regular tetrahedron, together with the corresponding ρ -vectors. For a regular tetrahedron, all edges have the same length l and the ρ -vectors are perpendicular to each other with $\rho^{(n)} \cdot \rho^{(m)} = \frac{l^2}{2} \delta_{nm}$, where δ_{nm} is the Kronecker delta.

Due to $(n - 1) = d = 3$, one finds that there are exactly three eigenvalues $g_1 \geq g_2 \geq g_3$ and three Euler angles needed to fully describe the tetrahedron. The eigenvalues alone suffice for some classification:

- For $g_1 = g_2 = g_3$ the tetrahedron is regular.
- For $g_1 \approx g_2 \gg g_3$ the tetrahedron is close to two-dimensional (2D) (“pancake-like”).
- For $g_1 \gg g_2 \approx g_3$ the tetrahedron is close to one-dimensional (1D) (“needle-like”).

Recently, it was found that tetrahedra starting with a regular shape with edge length $l \ll L$ at $t = 0$ deform into elongated pancake-like shapes with $g_1 > g_2 \gg g_3$ at later times (Pumir et al. 2000; Biferale et al. 2005; Xu et al. 2008). This deformation can be explained by the concept of the *perceived velocity gradient tensor* (Chertkov et al. 1999). In analogy to the usual velocity gradient tensor, $A_{ij}(\mathbf{x}, t) = \frac{\partial u_i(\mathbf{x}, t)}{\partial x_j}$,

the perceived velocity gradient tensor, $M(t)$, is defined over the four points of a tetrahedron by

$$\mathbf{v}^{(a)}(t) = \boldsymbol{\rho}^{(a)}(t)^T M(t), \quad (2.36)$$

with $\mathbf{v}^{(a)}(t) = \frac{d}{dt}\boldsymbol{\rho}^{(a)}(t)$ being the change in the ρ -vectors. From this, a *perceived rate of strain* and *perceived rate of rotation tensor* can be derived as

$$S(t) = \frac{1}{2}[M(t) + M(t)^T] \quad \text{and} \quad \Omega(t) = \frac{1}{2}[M(t) - M(t)^T]. \quad (2.37)$$

For l in the dissipative range, these quantities are identical to the usual rate of strain and rate of rotation tensors, $S_A(\mathbf{x}, t)$ and $\Omega_A(\mathbf{x}, t)$, respectively. For an incompressible flow, one can show that the velocity gradient tensor, and thus also the rate of strain tensor, are traceless, that is $\text{tr}(A(t)) = \text{tr}(S_A(t)) = 0$. This leads to the fact that the largest eigenvalue of $S_A(t)$, s_1 , is positive and the smallest one, s_3 , is negative. Most importantly, however, Betchov (1956) showed that, on average, the intermediate eigenvalue $\langle s_2 \rangle$ is positive as well. As a consequence, there exist two straining directions and one compressing direction for most of the time, leading to a coplanar transformation of a fluid element. Even though this result was derived for the dissipative range, it was also confirmed both experimentally and numerically for the perceived rate of strain tensor, $S(t)$, with a somewhat smaller but still positive value of $\langle s_2 \rangle$ (Lüthi et al. 2007; Xu et al. 2008; Pumir et al. 2013). Consequently, also for tetrahedra in the inertial range, a deformation into co-planar structures is expected.

Aside from the qualitative results for the shape deformation, a universal behavior of the shape eigenvalues was observed. If time is rescaled with τ_η , for R_0 in the dissipative range, or t_0 , for R_0 in the inertial range, the obtained results for various values of initial separation and Reynolds number collapse onto one curve (Pumir et al. 2000; Xu et al. 2008). Here, $t_0 = (l^2/\varepsilon)^{\frac{1}{3}}$ is again the characteristic time for an eddy of size l . Furthermore, in analogy to pair dispersion, a Richardson t^3 scaling is expected for the eigenvalues g_i and for the radius of gyration $\langle R^2(t) \rangle$ at times $t \gg \tau_\eta$. So far, however, this scaling behavior was not observed unambiguously (Pumir et al. 2000; Biferale et al. 2005; Xu et al. 2008).

References

- Argyris, J., Faust, G., Haase, M., Friedrich, R.: Die Erforschung des Chaos. Springer, New York (2010)
- Avila, K., Moxey, D., de Lozar, A., Avila, M., Barkley, D., Hof, B.: The onset of turbulence in pipe flow. *Science* **333**, 192–196 (2011)
- Batchelor, G.: The application of the similarity theory of turbulence to atmospheric diffusion. *Q. J. R. Meteorol. Soc.* **76**, 133–146 (1950)
- Batchelor, G.: Diffusion in a field of homogeneous turbulence. *Proc. Camb. Philos. Soc.* **48**, 345–362 (1952)
- Berg, J., Lüthi, B., Mann, J., Ott, S.: Backwards and forwards relative dispersion in turbulent flow: an experimental investigation. *Phys. Rev. E* **74**, 016304 (2006)

- Betchov, R.: An inequality concerning the production of vorticity in isotropic turbulence. *J. Fluid Mech.* **1**, 497–504 (1956)
- Biferale, L., Boffetta, G., Celani, A., Devenish, J., Lanotte, A.: Multiparticle dispersion in fully developed turbulence. *Phys. Fluids* **17**, 111701 (2005)
- Bragg, A.D., Ireland, P.J., Collins, L.R.: Forward and backward in time dispersion of fluid and inertial particles in isotropic turbulence. [arXiv:1403.5502v1](https://arxiv.org/abs/1403.5502v1) (2014)
- Celani, A., Cencini, M., Mazzino, A., Vergassola, M.: Active and passive fields face to face. *New J. Phys.* **6**, 72 (2004)
- Chertkov, M., Pumir, A., Shraiman, B.: Lagrangian tetrad dynamics and the phenomenology of turbulence. *Phys. Fluids* **11**, 2394–2410 (1999)
- Corrsin, S.: Heat transfer in isotropic turbulence. *J. Appl. Phys.* **23**, 113–118 (1952)
- Durbin, P.: A stochastic model of two-particle dispersion and concentration fluctuations in homogeneous turbulence. *J. Fluid Mech.* **100**, 279–302 (1980)
- Falkovich, G., Gawedzki, K., Vergassola, M.: Particles and fields in fluid turbulence. *Rev. Mod. Phys.* **73**, 913–975 (2001)
- Frisch, U.: *Turbulence*. Cambridge University Press, Cambridge (1995)
- Frisch, U., Mazzino, A., Noullez, A., Vergassola, M.: Lagrangian method for multiple correlations in passive scalar advection. *Phys. Fluids* **11**, 2178–2186 (1999)
- Hackl, J.F., Yeung, P.K., Sawford, B.L.: Multi-particle and tetrad statistics in numerical simulations of turbulent relative dispersion. *Phys. Fluids* **23**, 065103 (2011)
- Kolmogorov, A.N.: The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Dokl. Akad. Nauk SSSR* **30**, 299–303 (1941a)
- Kolmogorov, A.N.: Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 16–18 (1941b)
- Kraichnan, R.: The closure problem of turbulence theory. Technical report, DTIC Document (1961)
- Lorenz, E.N.: Deterministic nonperiodic flow. *J. Atmos. Sci.* **20**, 130–141 (1963)
- Lundgren, T.: Turbulent pair dispersion and scalar diffusion. *J. Fluid Mech.* **111**, 27–57 (1981)
- Lüthi, B., Ott, S., Berg, J., Mann, J.: Lagrangian multi-particle statistics. *J. Turbul.* **8**, 45 (2007)
- Mann, J., Ott, S., Andersen, J.S.: Experimental study of relative, turbulent diffusion. Technical report. (Risø-R-1036(EN)) (1999)
- Monin, A.S., Yaglom, A.M.: *Statistical Fluid Mechanics—Mechanics of Turbulence*, vol. 1, 2. Dover Publications, New York (2007)
- Navier, C.L.M.: Sur les lois du mouvement des fluides. *Comptes Rendus des Seances de l'Academie des Sciences* **6**, 389–440 (1827)
- Obukhov, A.: Spectral energy distribution in a turbulent flow. *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz.* **5**, 453–466 (1941)
- Ottino, J.: *The Kinematics of Mixing: Stretching, Chaos, and Transport*. Cambridge University Press, Cambridge (1989)
- Ouellette, N.T., Xu, H., Bourgoin, M., Bodenschatz, E.: An experimental study of turbulent relative dispersion models. *New J. Phys.* **8**, 109 (2006)
- Pope, S.B.: *Turbulent Flows*. Cambridge University Press, Cambridge (2000)
- Pumir, A., Shraiman, B., Chertkov, M.: Geometry of Lagrangian dispersion in turbulence. *Phys. Rev. Lett.* **85**, 5324–5327 (2000)
- Pumir, A., Shraiman, B., Chertkov, M.: The Lagrangian view of energy transfer in turbulent flow. *Europhys. Lett.* **56**, 379–385 (2001)
- Pumir, A., Bodenschatz, E., Xu, H.: Tetrahedron deformation and alignment of perceived vorticity and strain in a turbulent flow. *Phys. Fluids* **25**, 035101 (2013)
- Reynolds, O.: An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Philos. Trans. R. Soc. Lond. A* **174**, 935–982 (1883)
- Reynolds, O.: On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Philos. Trans. R. Soc. Lond. A* **186**, 123–164 (1895)

- Richardson, L.F.: *Weather Prediction by Numerical Process*. Cambridge University Press, Cambridge (1922)
- Richardson, L.F.: Atmospheric diffusion shown on a distance-neighbour graph. *Proc. R. Soc. Lond. A* **110**, 709–737 (1926)
- Salazar, J.P., Collins, L.R.: Two-particle dispersion in isotropic turbulent flows. *Annu. Rev. Fluid Mech.* **41**, 405–432 (2009)
- Sawford, B.L.: Turbulent relative dispersion. *Annu. Rev. Fluid Mech.* **33**, 289–317 (2001)
- Sawford, B.L., Yeung, P.K., Borgas, M.S.: Comparison of backwards and forwards relative dispersion in turbulence. *Phys. Fluids* **17**, 095109 (2005)
- Siebert, H., Franke, H., Lehmann, K., Maser, R., Saw, E.W., Schell, D., Shaw, R.A., Wendisch, M.: Probing finescale dynamics and microphysics of clouds with helicopter-borne measurements. *Bull. Am. Meteorol. Soc.* **87**, 1727–1738 (2006)
- Stokes, G.G.: On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Trans. Camb. Philos. Soc.* **8**, 287–305 (1845)
- Taylor, G.I.: Diffusion by continuous movements. *Proc. Lond. Math. Soc.* **2**, 196–212 (1922)
- Taylor, G.I.: Statistical theory of turbulence. *Proc. R. Soc. Lond. A* **151**, 421–444 (1935)
- Thomson, D.J.: Dispersion of particle pairs and decay of scalar fields in isotropic turbulence. *Phys. Fluids* **15**, 801–813 (2003)
- Toschi, F., Bodenschatz, E.: Lagrangian properties of particles in turbulence. *Annu. Rev. Fluid Mech.* **41**, 375–404 (2009)
- Von Kármán, T., Howarth, L.: On the statistical theory of isotropic turbulence. *Proc. R. Soc. Lond. A* **164**, 192–215 (1938)
- Xu, H., Ouellette, N.T., Bodenschatz, E.: Evolution of geometric structures in intense turbulence. *New J. Phys.* **10**, 013012 (2008)
- Xu, H., Pumir, A., Bodenschatz, E.: The pirouette effect in turbulent flows. *Nat. Phys.* **7**, 709–712 (2011)
- Yeung, P.K.: Lagrangian investigation of turbulence. *Annu. Rev. Fluid Mech.* **34**, 115–142 (2002)

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Jucha, J.

2015, XII, 113 p. 42 illus., 22 illus. in color., Hardcover

ISBN: 978-3-319-19191-1