Maxwell's Equations: Continuous and Discrete

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1 Introduction

These lecture notes are meant to be a gentle introduction to the spatial discretization of electromagnetic field problems. To a large extent, emphasis is on lucidity and intuitive understanding, sometimes at the expense of rigorous developments. The reader can be assured that there is a rigorous underpinning for all results mentioned in these notes, but the details may be outside their scope and can be found in the references supplied in the beginning of each of the following sections.

A geometric perspective is favored with emphasis on structural properties of the Maxwell equations. Those become most apparent when using exterior calculus as a tool for mathematical modeling. Thus, differential forms, their discrete counterparts, and related numerical analysis techniques will play a prominent role throughout this text.

The notes are organized in three sections. The first presents Maxwell's equations from the angle of exterior calculus covering the basic equations up to variational formulations. The second section introduces finite element exterior calculus aiming for a spatial Galerkin discretization of Maxwell's equations in variational form. The final section then delves into the numerical analysis of the discretized equations in order to establish a priori convergence estimates.

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2 Maxwell's Equations

Bibliographical Notes

In this section Maxwell's equations are first put in the framework of exterior calculus of differential forms. This calculus is a core subject in differential geometry and covered in standard textbooks, see, for instance, [38, Chaps. V, X, XI, & XII] and [39, Chaps. 8 & 9]. In these books it is mainly presented from a formal algebraic and differential calculus perspective. A more geometric approach is adopted in [21, Chap. IV] and, in particular, in the work of Bossavit, see the original articles [13–16], and the review articles [18], [19, Chaps. I & II]. Since the author has been much inspired by A. Bossavit, he recommends these latter two works as supplementary reading and as a source for many more useful references. Moreover, in parts this section follows [35, Sect. 2] and some more details can be found in that survey.

2.1 Fields

Electrodynamics is a continuum field theory and, from a classical non-relativistic perspective, its key quantities, the various fields, are functions of spatial position $\boldsymbol{x}$ and time t. In this section I will try to explain in intuitive terms why viewing electromagnetic quantities as mere vectorfields $\mathbb{R}^3 \to \mathbb{R}^3$ fails to capture important structural aspects and differences.

2.1.1 The Electric Field

To grasp the nature of a physical quantity, we recommend to study ways how it is measured. There are two ways to measure the first fundamental electrodynamic quantity, the *electric field* $\mathbf{e}$ (units $1\,\frac{\mathrm{V}}{\mathrm{m}}$):

1. (Hypothetical) local measurement in point $\boldsymbol{x}$ at time t by determining the virtual work it takes to displace a test charge q by $\delta\boldsymbol{x}$:

$$\delta W = q\,\mathbf{e}(\boldsymbol{x}, t) \cdot \delta\boldsymbol{x} \ .$$

   From this perspective

$$\mathbf{e}(\boldsymbol{x}, t) \text{ is a linear mapping from displacements into } \mathbb{R}.$$

2. (Almost practical) non-local measurement from the work required to move a test charge along a directed path γ

$$W = q \int_\gamma \mathbf{e} \cdot \mathrm{d}\boldsymbol{s} \ .$$

This formula reveals that

e is a quantity that can be integrated along directed curves.

2.1.2 The Magnetic Induction

The second fundamental electrodynamic quantity, the magnetic induction **b** (units $1\frac{\mathrm{Vs}}{\mathrm{m}^2}$) can also be measured in two ways:

1. (Hypothetical) local measurement at $(\boldsymbol{x}, t)$ from the virtual work needed to turn a tiny magnetic needle (magnetic moment $\mathbf{m}$)

$$\delta w = (\mathbf{b}(\boldsymbol{x}, t) \times \mathbf{m}) \cdot \delta\boldsymbol{r} = \mathbf{b}(\boldsymbol{x}, t) \cdot (\mathbf{m} \times \delta\boldsymbol{r}) \ ,$$

   where the vector $\delta\boldsymbol{r} \in \mathbb{R}^3$ is directed along the axis of rotation and its length represents the angle of rotation, see Fig. 1a. We may conclude that

   $\mathbf{b}(\boldsymbol{x}, t)$ should be read as an anti-symmetric bilinear mapping $(\delta\boldsymbol{r}, \mathbf{m}) \to \mathbb{R}$.

2. (Almost practical) non-local measurement that relies on the work required to move a current carrying wire loop:

$$W = I \int_{\Sigma} \mathbf{b} \cdot \boldsymbol{n} \, \mathrm{d}S \ ,$$

   where I is the current and Σ is the orientable surface swept by the loop with unit normal vector field $\boldsymbol{n}$, see Fig. 1b. This leads to the interpretation that

   b is a quantity that assigns a total flux to oriented bounded surfaces.

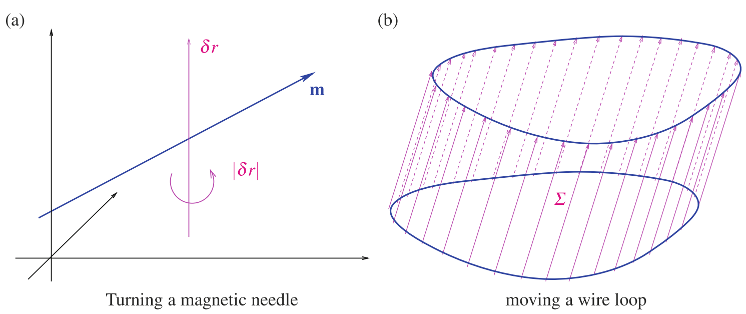

Fig. 1 Local (**a**) and non-local measurement (**b**) of the magnetic induction **b**

2.2 *Differential and Integral Forms*

Now we learn about classes of functions on a piecewise smooth n-dimensional manifold Ω that fit quantities like the electric field $\mathbf{e}$ and the magnetic induction $\mathbf{b}$ as introduced above. Of course, in classical electrodynamics Ω is a domain in $\mathbb{R}^3$, but the manifold perspective is necessary for dealing with boundary conditions properly.

2.2.1 Fundamental Concepts

The first concept is related to "non-local measurements". To state it we denote

$\mathscr{M}_\ell(\Omega)$: the set of piecewise smooth compact *oriented* ℓ-dimensional sub-manifolds of Ω, $0 \le \ell \le n$.

Notion 1 (Integral Form [35, Def. 1]) *An* (integral) ℓ-form ω, $0 \le \ell \le n$ *on* Ω *is a continuous* $(*)$ *and additive* $(**)$ *mapping* $\omega : \mathscr{M}_\ell(\Omega) \to \mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$*).*
The vector space of ℓ*-forms on* Ω *will be denoted by* $\mathscr{F}^\ell(\Omega)$*.*

$(*)$ Continuity of ω is with respect to a "deformation topology", made precise in the special field of "geometric integration theory", *cf.* [19, p. 125].
$(**)$ Additivity of ω means that its value for the union of disjoint sub-manifolds is the sum of the values for each of them.

The evaluation of an ℓ-form for a sub-manifold of suitable dimension is usually written as integration:

$$\omega \in \mathscr{F}^\ell(\Omega): \quad \int_\Sigma \omega := \omega(\Sigma) , \quad \Sigma \in \mathscr{M}_\ell(\Omega) .$$

In light of Notion 1, the considerations of Sects. 2.1.1 and 2.1.2 teach us that

- the electric field $\mathbf{e}$ should be viewed as a 1-form, and
- the magnetic induction can be regarded as a 2-form.

Already Maxwell had this insight, since in his 1891 "Treatise on Electricity and Magnetism" he wrote

> Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference with respect to a line, while in the other the quantity is defined with reference to an area.

Now we turn to a concept of ℓ-forms corresponding to the local measurement procedures described above. This is the usual approach in differential geometry as in [38, Sect. V.3] or [22, Sect. 2.1], because its rigorous mathematical handling is easier than that of integral forms.

Definition 1 A (continuous) *differential ℓ-form ω* on a C^1-manifold Ω is a (continuous) mapping $\omega : \Omega \to \Lambda^\ell(T_\Omega(\cdot))$, that is, ω assigns to every $\boldsymbol{x} \in \Omega$ a unique alternating ℓ-multilinear form on the tangent space $T_\Omega(\boldsymbol{x})$ at Ω in $\boldsymbol{x}$.

We write $C^0\Lambda^\ell(\Omega)/C^\infty\Lambda^\ell(\Omega)$ for the vector space of continuous/smooth differential ℓ-forms on Ω.

For a domain (open subset) $\Omega \subset \mathbb{R}^n$ we find $T_\Omega(\boldsymbol{x}) = \mathbb{R}^n$ for every $\boldsymbol{x} \in \Omega$ so that an ℓ-form on Ω is a function with values in $\Lambda^\ell(\mathbb{R}^n)$.

Simple formal considerations establish the connection between integral and differential ℓ-forms and connect the non-local and local point of view. Tacitly smoothness is assumed.

Differential Form → Integral Form

The integration of continuous differential forms is a standard technique, see [38, Chap. XI] and often introduced using charts (coordinates). Here, we follow [19, Rem. 6.1] and give a lucid explanation for the transition from differential forms to integral forms for $\ell = 1, 2$ and a domain $\Omega \subset \mathbb{R}^n$. It goes without saying that there is a close link between the local and integral point of view: every piecewise smooth curve can be arbitrarily well approximated by tiny line segments. Similarly, any oriented surface can be tiled with flat triangles, which inherit its orientation, *cf.* Fig. 2 for $n = 3$. The next step can be viewed as Riemann summation. For $\ell = 1$ we just sum up the values that ω assigns to the line segments, where the position arguments are taken as their midpoints. For $\ell = 2$, we feed the vectors spanning the parallelograms to the differential 2-form ω evaluated at their centers of gravity, and then add the values returned.

Integral Form → Differential Form

Again, we restrict ourselves to a domain $\Omega \subset \mathbb{R}^n$ with the simple tangent space $T_\Omega(\boldsymbol{x}) = \mathbb{R}^n$ for all $\boldsymbol{x} \in \Omega$. Then we can perform localization as follows: For

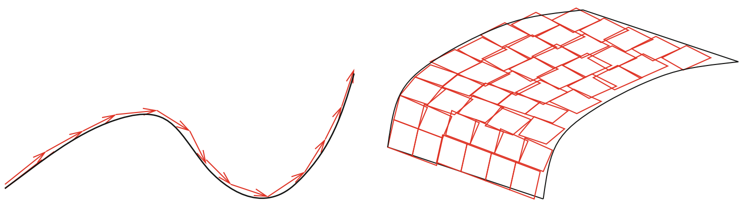

Fig. 2 Flat tilings plus Riemann integration switch from local to integral forms, *cf.* [19, Fig. 6.1]

"continuous $(*)$" $\omega \in \mathscr{F}^\ell(\Omega)$ we can define for any $\mathbf{v}_j \in \mathbb{R}^n, j = 1, \ldots, \ell$,

$$\omega(\boldsymbol{x})(\mathbf{v}_1, \ldots, \mathbf{v}_\ell) := \lim_{t \to 0} \frac{\ell!}{t^\ell} \int_{\Sigma_t} \omega \,,$$
$$\Sigma_t = \operatorname{convex}\{\boldsymbol{x}, \boldsymbol{x} + t\mathbf{v}_1, \ldots, \boldsymbol{x} + t\mathbf{v}_\ell\} \,, \tag{1}$$

where the integral is set to zero, in case Σ_t collapses to a lower-dimensional patch. If the limit exists, obviously $\omega(\boldsymbol{x}) \in \Lambda^\ell(\mathbb{R}^n)$, because swapping two spanning vectors changes the orientation of Σ_t.

2.2.2 Euclidean Vector Proxies in 3D

For a domain $\Omega \subset \mathbb{R}^n$ consider $\omega \in C^0\Lambda^\ell(\Omega) := \{\Omega \to \Lambda^\ell(\mathbb{R}^n) \text{ continuous}\}$. From linear algebra we know that $\dim \Lambda^\ell(\mathbb{R}^n) = \binom{n}{\ell}$. Hence, after picking an *arbitrary* basis of $\Lambda^\ell(\mathbb{R}^n)$, ω can be represented by its $\binom{n}{\ell}$ coefficient functions. In other words, (continuous) vector fields provide an isomorphic model of $C^0\Lambda^\ell(\Omega)$. Clearly, the concrete vector field representative for $\omega \in C^0\Lambda^\ell(\Omega)$ will depend on the choice of basis. In other words, it will depend on coordinates. Admittedly, the vector field model captures entire exterior calculus. However, the involvement of coordinates often conceals essential coordinate-independent properties and the different nature of quantities like the electric field and magnetic induction.

A special choice of basis for $n = 3$ is stipulated by orthogonality requirements and the resulting vector field representatives have been dubbed "vector proxies" by Bossavit [14, Sect. 1.4]. The concrete definition of the underlying isomorphism can be inferred from Table 1. Usually, vector proxies will be distinguished by an overset arrow ($\vec{\mathbf{e}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{h}}$, $\vec{\mathbf{d}}$, $\vec{u}$ etc.). Occasionally, we will use the notation $\mathrm{V.P.}(\omega)$ for the Euclidean vector proxy of a differential form ω.

Table 1 Relationship between differential forms and vectorfields in three-dimensional Euclidean space ($\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$), *cf.* Table 2.1 in [35]

Differential form		Related function u/vectorfield $\mathbf{u}$	
$\ell = 0$	$\mathbf{x} \mapsto \omega(\mathbf{x})$	$\vec{u}(\mathbf{x}) := \omega(\mathbf{x})$	$\vec{u} : \Omega \to \mathbb{R}$
$\ell = 1$	$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \omega(\mathbf{x})(\mathbf{v})\}$	$\vec{\mathbf{u}}(\mathbf{x}) \cdot \mathbf{v} := \omega(\mathbf{x})(\mathbf{v})$	$\vec{\mathbf{u}} : \Omega \to \mathbb{R}^3$
$\ell = 2$	$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\vec{\mathbf{u}}(\mathbf{x}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)$	$\vec{\mathbf{u}} : \Omega \to \mathbb{R}^3$
$\ell = 3$	$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$\vec{u}(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$	$\vec{u} : \Omega \to \mathbb{R}$

The operation "$\cdot$" is the canonical inner product in Euclidean space, "$\times$" the cross product. See also [5, Table 2.2]

The integration of differential forms expressed in terms of their vector proxies u/$\mathbf{u}$ according to Table 1 gives rise to familiar integrals:

$$\begin{array}{llcll} \text{0-form} & \omega \in C^0\Lambda^0(\Omega) & : & \int_{\boldsymbol{x}} \omega = u(\boldsymbol{x}) & \forall \boldsymbol{x} \in \Omega \;, \\ \text{1-form} & \omega \in C^0\Lambda^1(\Omega) & : & \int_{\gamma} \omega = \int_{\gamma} \mathbf{u} \cdot \mathrm{d}\mathbf{s} & \forall \gamma \in \mathscr{M}_1(\Omega) \;, \\ \text{2-form} & \omega \in C^0\Lambda^2(\Omega) & : & \int_{\Sigma} \omega = \int_{\Sigma} \mathbf{u} \cdot \boldsymbol{n} \, \mathrm{d}S & \forall \Sigma \in \mathscr{M}_2(\Omega) \;, \\ \text{3-form} & \omega \in C^0\Lambda^3(\Omega) & : & \int_{V} \omega = \int_{V} u(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} & \forall V \in \mathscr{M}_3(\Omega) \;. \end{array} \tag{2}$$

Here, $\boldsymbol{n}$ is a unit normal vector field to Σ, whose direction is induced by the orientation of Σ.

2.2.3 Transformation of Forms

Let $\boldsymbol{\Phi}$ stand for a diffeomorphism mapping the n-dimensional manifold $\hat{\Omega}$ onto Ω. It can be used to "pull back" any integral form on Ω to $\hat{\Omega}$ according to the following definition [39, Sect. 8.2.1]:

Definition 2 Given $\omega \in \mathscr{F}^\ell(\Omega)$ its *pullback* $\boldsymbol{\Phi}^*\omega \in \mathscr{F}^\ell(\hat{\Omega})$ is defined by

$$\int_{\hat{\Sigma}} \boldsymbol{\Phi}^*\omega := \int_{\boldsymbol{\Phi}(\hat{\Sigma})} \omega \quad \forall \hat{\Sigma} \in \mathscr{M}_\ell(\hat{\Omega}) \;.$$

This induces a linear isomorphism $\boldsymbol{\Phi}^* : \mathscr{F}^\ell(\Omega) \to \mathscr{F}^\ell(\hat{\Omega})$.

There is a local version of the pullback for differential forms and it reads

$$(\boldsymbol{\Phi}^*\omega)(\hat{\boldsymbol{x}})(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_\ell) := \omega(\boldsymbol{\Phi}(\hat{\boldsymbol{x}}))(D\boldsymbol{\Phi}(\hat{\boldsymbol{x}})\hat{\mathbf{v}}_1, \ldots, D\boldsymbol{\Phi}(\hat{\boldsymbol{x}})\hat{\mathbf{v}}_\ell) \quad \begin{array}{l} \hat{\boldsymbol{x}} \in \hat{\Omega} \;, \\ \hat{\mathbf{v}}_j \in T_{\hat{\Omega}}(\hat{\boldsymbol{x}}) \;, \end{array} \tag{3}$$

where $D\boldsymbol{\Phi}$ is the differential of $\boldsymbol{\Phi}$. The pullback for Euclidean vector proxies in 3D can be computed from (3) and the corresponding vector analytic operations are listed in Table 2.

If $\Sigma \subset \Omega$ is a sub-manifold of Ω, the pullback associated with the canonical embedding $\iota_\Sigma : \Sigma \to \Omega$ provides the *trace operators* $\mathbf{t}_\Sigma := \iota_\Sigma^* : \mathscr{F}^\ell(\Omega) \to \mathscr{F}^\ell(\Sigma)$. For Euclidean vector proxies in 3D they become point trace, tangential trace, and normal component, respectively, see Table 2. The notation $\mathbf{t}_\Sigma$ is used for forms and vector proxies, alike.

Equality of traces on interfaces supplies suitable compatibility conditions that make it possible to glue integral forms across the interface.

Table 2 Pullback and trace of Euclidean vector proxies differential forms of degree ℓ on $\Omega \subset \mathbb{R}^3$, [35, (2.16)–(2.19)]

Forms/vector proxies		Pullback	Trace onto $\partial\Omega$
$\ell = 0$	$u = \mathrm{V.P.}(\omega)$, $\vec{v} = \mathrm{V.P.}(\boldsymbol{\Phi}^*\omega)$	$\vec{v}(\hat{\boldsymbol{x}}) = \vec{u}(\boldsymbol{x})$	$\mathbf{t}_{\partial\Omega}\vec{u}(\boldsymbol{x}) = \vec{u}(\boldsymbol{x})$
$\ell = 1$	$\vec{\mathbf{u}} = \mathrm{V.P.}(\omega)$, $\vec{\mathbf{v}} = \mathrm{V.P.}(\boldsymbol{\Phi}^*\omega)$	$\vec{\mathbf{v}}(\hat{\boldsymbol{x}}) = D\boldsymbol{\Phi}(\hat{\boldsymbol{x}})^\top \vec{u}(\boldsymbol{x})$	$\mathbf{t}_{\partial\Omega}\vec{\mathbf{u}}(\boldsymbol{x}) = \vec{\mathbf{u}}_t(\boldsymbol{x})$
$\ell = 2$	$\vec{\mathbf{u}} = \mathrm{V.P.}(\omega)$, $\vec{\mathbf{v}} = \mathrm{V.P.}(\boldsymbol{\Phi}^*\omega)$	$\vec{\mathbf{v}}(\hat{\boldsymbol{x}}) = \det D\boldsymbol{\Phi}(\hat{\boldsymbol{x}}) D\boldsymbol{\Phi}(\hat{\boldsymbol{x}})^{-1}\vec{\mathbf{u}}(\boldsymbol{x})$	$\mathbf{t}_{\partial\Omega}\vec{\mathbf{u}}(\boldsymbol{x}) = \vec{\mathbf{u}}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})$
$\ell = 3$	$\vec{u} = \mathrm{V.P.}(\omega)$, $\vec{v} = \mathrm{V.P.}(\boldsymbol{\Phi}^*\omega)$	$\vec{v}(\hat{\boldsymbol{x}}) = \det D\boldsymbol{\Phi}(\hat{\boldsymbol{x}})\vec{u}(\boldsymbol{x})$	–

Lemma 1 (Compatibility Condition for Integral Forms [19, Sect. 7], [32, Thm. 8]) *Given a partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ of a manifold Ω into "nice" sub-manifolds Ω_1 and Ω_2, $\Omega_1 \cap \Omega_2 = \emptyset$, and two integral forms $\omega_1 \in \mathscr{F}^\ell(\Omega_1)$, $\omega_2 \in \mathscr{F}^\ell(\Omega_2)$, we have for*

$$\omega := \begin{cases} \omega_1 & \text{on } \Omega_1 \\ \omega_2 & \text{on } \Omega_2 \end{cases} \quad \text{that} \quad \omega \in \mathscr{F}^\ell(\Omega) \quad \Leftrightarrow \quad \mathbf{t}_\Gamma \omega_1 = \mathbf{t}_\Gamma \omega_2 \ ,$$

where $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$.

The idea behind Lemma 1 is to consider ℓ-dimensional oriented sub-manifolds of Ω that are contained in Γ. The value ω assigns to those must be unique.

2.3 Topological Electrodynamic Laws

2.3.1 Circulation and Flux Laws

Let $\{\Sigma(t)\}_{t\in\mathbb{R}}$, be a family of orientable, compact, and piecewise smooth 2-surfaces, whose elements vary smoothly with time t, thus forming a "space-time tube". Then the first "axiom" of electrodynamics, *Faraday's law* can be stated as (for any $t_1, t_2 \in \mathbb{R}$)

$$\int_{t_1}^{t_2} \int_{\partial\Sigma(\tau)} \mathbf{e}(\tau)\,\mathrm{d}\tau = \int_{\Sigma(t_1)} \mathbf{b}(t_1) - \int_{\Sigma(t_2)} \mathbf{b}(t_2) \quad \Leftrightarrow \quad \int_{\partial\Sigma(t)} \mathbf{e}(t) = -\frac{d}{dt}\int_{\Sigma(t)} \mathbf{b}(t) \ . \tag{FL}$$

Faraday's law links electric field and magnetic induction through integrals that perfectly fit the integral forms interpretation of the fields, recall Sect. 2.2.

The second law, that we treat as another "axiom" is *Ampere's law* and it links two electrodynamic quantities that have not been mentioned so far, the magnetic field **h** (units $1\frac{A}{m}$), the electric displacement **d** (units $1\frac{\mathrm{As}}{\mathrm{m}^2}$), and the electric current **j** (units $1\frac{\mathrm{A}}{\mathrm{m}^2}$):

$$\int_{t_1}^{t_2}\int_{\partial\Sigma(\tau)} \mathbf{h}(\tau)\,\mathrm{d}\tau = \int_{\Sigma(t_2)} \mathbf{d}(t_2) - \int_{\Sigma(t_1)} \mathbf{d}(t_1) + \int_{t_1}^{t_2}\int_{\Sigma(\tau)} \mathbf{j}(\tau)\,\mathrm{d}\tau$$

$$\Leftrightarrow \int_{\partial\Sigma(t)} \mathbf{h}(t) = \frac{d}{dt}\int_{\Sigma(t)} \mathbf{d}(t) + \int_{\Sigma(t)} \mathbf{j}(t)\,. \tag{AL}$$

Ampere's law expects us to consider integrals of the magnetic field **h** along curves, whereas **d** and **j** enter through their fluxes through surfaces. Matching this with our notion of (integral) forms, we find

- that the magnetic field **h** should be regarded as a *1-form*,
- that *2-forms* are the right device to describe both **d** and **j**.

Remark 1 The electric current can play the role of sources in electrodynamic models. Then **j** will be a prescribed quantity reflecting the interaction of electromagnetic fields with other physical systems. Hence, from now, think of **j** as given.

Remark 2 Another subtle distinction can be made labeling the quantities in Ampere's law *twisted forms*, see [16, Sect. 2] and [21, Sect. 28]. This is not needed for our purposes and I am not going to dwell on this.

2.3.2 Exterior Derivative

Integration of forms over boundaries features prominently both in (FL) and (AL). Recall that the boundary of an oriented piecewise smooth manifold of dimension d is an orientable $d-1$-dimensional manifold that can be equipped with an induced orientation, see Fig. 3. This induced orientation is implicitly imposed through the boundary operator ∂. For an in-depth discussion of orientation refer to [19, Sect. 4].

Definition 3 (Exterior Derivative) Let Ω be an n-dimensional manifold. Then the *exterior derivative* $\mathsf{d}_\ell : \mathscr{F}^\ell(\Omega) \to \mathscr{F}^{\ell+1}(\Omega)$, $0 \le \ell < n$, is defined by

$$\int_\Sigma \mathsf{d}_\ell\,\omega := \int_{\partial\Sigma} \omega \quad \forall \Sigma \in \mathscr{M}_{\ell+1}(\Omega)\,,$$

and $\mathsf{d}_n\,\omega := 0$ for $\omega \in \mathscr{F}^n(\Omega)$.

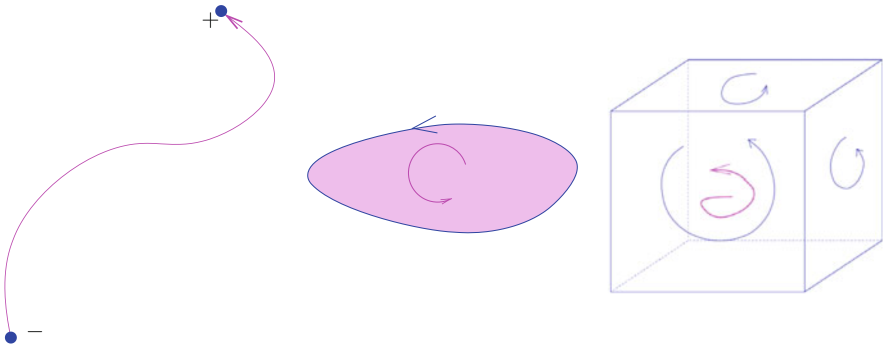

Fig. 3 One-, two-, and three-dimensional submanifolds of $\mathbb{R}^3$ and the induced orientation of their boundaries. Remember that the orientation of a path is given by its direction, the orientation of a surface by an internal sense of turning, and the orientation of a volume by a corkscrew rule

By the very definition of d_1 we can now state (FL) and (AL) concisely as

Faraday's law	$\mathsf{d}_1\,\mathbf{e} = -\partial_t\mathbf{b}\;,$	(FL)
Ampere's law	$\mathsf{d}_1\,\mathbf{h} = \partial_t\mathbf{d} + \mathbf{j}\;.$	(AL)

We highlight an immediate consequence of Definition 3:

Corollary 1 *The exterior derivative* $\mathsf{d}_\ell : \mathscr{F}^\ell(\Omega) \to \mathscr{F}^{\ell+1}(\Omega)$ *is a linear operator and commutes with the pullback:* $\boldsymbol{\Phi}^* \circ \mathsf{d}_\ell = \hat{\mathsf{d}}_\ell \circ \boldsymbol{\Phi}^*$ *for any diffeomorphism* $\boldsymbol{\Phi} : \hat{\Omega} \to \Omega$ *(*$\hat{\mathsf{d}}_\ell$ *is the exterior derivative on* $\mathscr{F}^\ell(\hat{\Omega})$*).*

Hence, if **e**, **b** solve (FL) and **h**, **d** satisfy (AL), then the transformed fields $\boldsymbol{\Phi}^*\mathbf{e}$, $\boldsymbol{\Phi}^*\mathbf{b}$, and $\boldsymbol{\Phi}^*\mathbf{h}$, $\boldsymbol{\Phi}^*\mathbf{d}$ again solve (FL) and (AL), respectively, where $\boldsymbol{\Phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is *any* diffeomorphism. In other words, we can warp space in arbitrary ways and the induced transformations take solutions of Maxwell's equations to other solutions of Maxwell's equations. Therefore (FL) and (AL) have been labelled "topological laws"; their set of solutions is invariant under arbitrary pullbacks connected with diffeomorphic deformations of space.

The evident fact that "the boundary of a boundary is empty", $\partial \circ \partial = \emptyset$, permits us to conclude a fundamental property of the exterior derivative:

Theorem 2 *For* $\ell \in \{0, \dots, n-1\}$ *holds* $\boxed{\mathsf{d}_{\ell+1} \circ \mathsf{d}_\ell = 0}$.

Thus, a simple consequence of applying d_2 to (AL) is the *continuity equation*

$$0 = \partial_t\,\mathsf{d}_2\,\mathbf{d} + \mathsf{d}_2\,\mathbf{j} = \partial_t\boldsymbol{\rho} + \mathsf{d}_2\,\mathbf{j}\;, \tag{4}$$

where $\boldsymbol{\rho} := \mathsf{d}_2\,\mathbf{d} \in \mathscr{F}^3(\Omega)$ is a 3-form modeling the density of electric charges.

Assuming "smoothness" of an (integral) form, the exterior derivative can be localized [22, Sect. 2.3], [38, Sect. V.3]:

Theorem 3 (Generalized Stokes' Theorem) *On a domain $\Omega \subset \mathbb{R}^n$ the exterior derivative of a differential ℓ-form $\omega \in C^1\Lambda^\ell(\Omega)$ is*

$$(\mathsf{d}_\ell\,\omega)(\boldsymbol{x})(\mathbf{v}_1,\ldots,\mathbf{v}_{\ell+1}) := \sum_{k=1}^{\ell+1}(-1)^k (D\omega)(\boldsymbol{x})\mathbf{v}_k(\mathbf{v}_1,\ldots,\mathbf{v}_{k-1},\mathbf{v}_{k+1},\ldots,\mathbf{v}_{\ell+1})\ ,$$

for all $\boldsymbol{x} \in \Omega$ and "tangent vectors" $\mathbf{v}_k \in \mathbb{R}^n$. Here $D\omega : \Omega \to L(\mathbb{R}^n, \Lambda^\ell(\mathbb{R}^n))$ is the (Fréchet) derivative of ω.

This paves the way for computing the vector proxy incarnations of the exterior derivatives [5, Sect. 2.3]:

$$\mathrm{V.P.}(\mathsf{d}_\ell\,\omega) = \begin{cases} \mathbf{grad}\,u & ,\ \text{for } \ell = 0, \\ \mathbf{curl}\,\mathbf{u} & ,\ \text{for } \ell = 1, \\ \operatorname{div}\mathbf{u} & ,\ \text{for } \ell = 2, \end{cases} \quad , \quad u/\mathbf{u} := \mathrm{V.P.}(\omega),\ \omega \in C^1\Lambda^\ell(\Omega)\ . \tag{5}$$

The classical Gauss' and Stokes' theorem confirm that these operators comply with Definition 3. By (5), for vector proxies $\vec{\mathbf{e}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{h}}$, $\vec{\mathbf{d}}$, and $\vec{\mathbf{j}}$ of the various electromagnetic fields, the local versions Faraday's and Ampere's law read

$$\boxed{\begin{array}{lcl} \text{(FL)} & \Rightarrow & \mathbf{curl}\,\vec{\mathbf{e}} = -\partial_t\vec{\mathbf{b}}\ , \\ \text{(AL)} & \Rightarrow & \mathbf{curl}\,\vec{\mathbf{h}} = \partial_t\vec{\mathbf{d}} + \vec{\mathbf{j}}\ . \end{array}} \begin{array}{r} (6) \\ (7) \end{array}$$

This is the classical form of Maxwell's equations written as first order partial differential equations for vector fields with three components.

Remark 3 The use of exterior calculus for the description of electromagnetic fields and the statement of electromagnetic models is well established, see [8], [39, Sect. 9.8], [21, Chap. VI], or [31, Sect. 3.5]. Surprisingly, as discovered in [37], the perspective of differential forms also sheds fresh light on boundary integral formulations for acoustics and electromagnetics.

2.3.3 Potentials

The converse of Theorem 2 holds under some assumption on the topological class of the manifold on which the forms are defined.

Theorem 4 (Existence of Potentials) *If the manifold Ω has trivial topology, that is, all Betti numbers except the first vanish, then*

$$\operatorname{Ker}(\mathsf{d}_\ell) := \{\omega \in \mathscr{F}^\ell(\Omega) : \mathsf{d}_\ell\, \omega = 0\} = \mathsf{d}_{\ell-1}\, \mathscr{F}^{\ell-1}(\Omega) .$$

The $\ell - 1$-form whose exterior derivative yields an ℓ-form ω with $\mathsf{d}_\ell\, \omega = 0$, is called a *potential* for ω. The proof of this theorem for differential forms makes use of so-called Poincaré liftings, see [38, Sect. V.4].

Let us sketch a formal justification of Theorem 4 for $\Omega = \mathbb{R}^n$ and $\ell = 1$. For every $\boldsymbol{x} \in \Omega$ let $\gamma(\boldsymbol{x})$ be the line segment connecting $\boldsymbol{x}$ and $\mathbf{0}$. Given $\eta \in \mathscr{F}^1(\Omega)$, define $\omega \in \mathscr{F}^0(\Omega)$ (a plain function) by $\omega(\boldsymbol{x}) := \int_{\gamma(\boldsymbol{x})} \eta$. For any directed path π with endpoints $\boldsymbol{x}_0, \boldsymbol{x}_1$ this 0-form satisfies

$$\int_\pi \mathsf{d}_0\, \omega = \omega(\boldsymbol{x}_1) - \omega(\boldsymbol{x}_0) = \int_{\gamma(\boldsymbol{x}_1)} \eta - \int_{\gamma(\boldsymbol{x}_0)} \eta = \int_\pi \eta ,$$
$$\text{since} \quad 0 = \int_\Sigma \mathsf{d}_1\, \eta = \int_{\partial\Sigma} \eta = \int_\pi \eta - \int_{\gamma(\boldsymbol{x}_1)} \eta + \int_{\gamma(\boldsymbol{x}_0)} \eta ,$$

where Σ is the 2-surface bounded by π, $\gamma(\boldsymbol{x}_0)$, and $\gamma(\boldsymbol{x}_1)$ (with suitable orientation). For general Ω this surface may not exist owing to topological obstructions.

A similar argument settles the case $\ell = 2$ for $\Omega = \mathbb{R}^n$. Now, write $\Sigma(\gamma)$ for the oriented surface generated by retracting the directed path γ to $\mathbf{0}$. By the retract of a set to $\mathbf{0}$ we mean the union of all line segments connecting points of the set with $\mathbf{0}$. Given $\eta \in \mathscr{F}^2(\Omega)$ with $\mathsf{d}_2\, \eta = 0$ we fix $\omega \in \mathscr{F}^1(\Omega)$ by

$$\int_\gamma \omega := \int_{\Sigma(\gamma)} \eta \quad \forall \gamma \in \mathscr{M}_1(\Omega) .$$

For an arbitrary $\Gamma \in \mathscr{M}_2(\Omega)$ let V stand for the volume defined by its retract to $\mathbf{0}$. Then

$$\int_\Gamma \mathsf{d}_1\, \omega = \int_{\partial\Gamma} \omega = \int_{\Sigma(\partial\Gamma)} \eta = \int_{\partial V} \eta + \int_\Gamma \eta = \underbrace{\int_V \mathsf{d}_2\, \eta}_{=0} + \int_\Gamma \eta .$$

Also here, topology may thwart the existence of a suitable V.

There is also a local version of Theorem 4 for differential forms, and in terms of Euclidean vector proxies it tells us that in $\Omega = \mathbb{R}^3$

$$\mathbf{curl}\, \mathbf{u} = 0 \quad \Rightarrow \quad \exists f : \Omega \to \mathbb{R} : \quad \mathbf{u} = \mathbf{grad} f \qquad (f \text{ is a scalar potential.}), \tag{8}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \Rightarrow \quad \exists \mathbf{f} : \Omega \to \mathbb{R}^3 : \quad \mathbf{u} = \mathbf{curl}\, \mathbf{f} \qquad (\mathbf{f} \text{ is a vector potential.}) \tag{9}$$

Remark 4 For general Ω an ℓ-form in $\mathrm{Ker}(\mathsf{d}_\ell)$ is still the exterior derivative of some $\eta \in \mathscr{F}^{\ell-1}(\Omega)$ after adding a correction from a finite-dimensional *cohomology space*. Since $\Omega = \mathbb{R}^3$ for Maxwell's equations, we need not worry about topological obstructions. The situation is completely different in the case of so-called magneto-quasistatic models (eddy current models), where scalar potentials for **curl**-free magnetic fields outside conductors may fail to exist.

Electromagnetic Potentials

Another axiom in electrodynamics is the non-existence of magnetic monopoles, that is, $\mathsf{d}_2\,\mathbf{b} = 0$ at "initial time" $t = 0$. Then we conclude from Theorem 2 and Faraday's law (FL) that

$$\mathsf{d}_1\,\mathbf{e} = -\partial_t \mathbf{b} \quad \overset{\mathsf{d}_2}{\Longrightarrow} \quad \partial_t\,\mathsf{d}_2\,\mathbf{b} = 0 \quad \overset{\mathsf{d}_2\,\mathbf{b}(0)=0}{\Longrightarrow} \quad \mathsf{d}_2\,\mathbf{b} = 0 \quad \forall t\,.$$

As a consequence, there exists a *magnetic vector potential* $\mathbf{a} \in \mathscr{F}^1(\Omega)$ such that $\mathbf{b} = \mathsf{d}_1\,\mathbf{a}$. Plugging the vector potential into Faraday's law, we arrive at

$$\mathsf{d}_1\,\mathbf{e} = -\partial_t\,\mathsf{d}_1\,\mathbf{a} \implies \mathsf{d}_1(\mathbf{e} + \partial_t \mathbf{a}) = 0 \overset{\text{Thm. 4}}{\Longrightarrow} \exists v \in \mathscr{F}^0(\Omega):\ \mathbf{e} = -\partial_t \mathbf{a} - \mathsf{d}_0\,v\,.$$

This 0-form (= function) v is known as *electric scalar potential*. In vector proxy notation the two potentials satisfy

$$\vec{\mathbf{b}} = \mathbf{curl}\,\vec{\mathbf{a}} \quad , \quad \vec{\mathbf{e}} + \partial_t \vec{\mathbf{a}} = -\mathbf{grad}\,v\,, \tag{10}$$

where we identified the function v and 0-form v.

Gauge Freedom

Even for given fields **e** and **b**, the potentials will not be unique, because for any $w \in \mathscr{F}^0(\Omega)$ holds

$$\begin{aligned} \mathbf{b} &= \mathsf{d}_1\,\mathbf{a}\,, \\ \mathbf{e} &= -\partial_t \mathbf{a} - \mathsf{d}_0\,v \end{aligned} \quad , \quad \begin{aligned} v' &= v + w \\ \mathbf{a}' &= \mathbf{a} + \int_0^t w\,\mathrm{d}t \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{b} &= \mathsf{d}_1\,\mathbf{a}'\,, \\ \mathbf{e} &= -\partial_t \mathbf{a}' - \mathsf{d}_0\,v'\,. \end{aligned} \tag{11}$$

This possibility to modify the potentials without affecting the fields proper is known as *gauge freedom*. It takes so-called *gauge conditions*, that is, extra constraints on the potentials, to render them unique [17].

2.4 Energies and Material Laws

2.4.1 The Exterior Product

There is a special bilinear way to combine two alternating forms into another alternating form whose degree is the sum of the degrees of the factors. This binary operation is called the *exterior product* (wedge product). By pointwise definition it can be extended to continuous differential forms on a manifold Ω [5, Sect. 2.1]

$$\wedge : \begin{cases} C^0\Lambda^\ell(\Omega) \times C^0\Lambda^m(\Omega) \to C^0\Lambda^{\ell+m}(\Omega) \\ \qquad (\omega, \eta) \qquad\qquad \mapsto \omega \wedge \eta \; . \end{cases}$$

The most important formulas connecting the exterior product and other operations on differential forms are ($\omega \in C^0\Lambda^\ell(\Omega)$,$\eta \in C^0\Lambda^m(\Omega)$, $0 \leq \ell, m \leq n$)

$$\text{(Anti-)commutativity:} \qquad \omega \wedge \eta = (-1)^{\ell m}(\eta \wedge \omega) \; , \tag{12}$$

$$\text{Commutes with pullback:} \qquad \boldsymbol{\Phi}^*\omega \wedge \boldsymbol{\Phi}^*\eta = \boldsymbol{\Phi}^*(\omega \wedge \eta) \; , \tag{13}$$

$$\text{Leibniz rule:} \qquad \mathsf{d}_{\ell+m}(\omega \wedge \eta) = \mathsf{d}_\ell\, \omega \wedge \eta + (-1)^\ell(\omega \wedge \mathsf{d}_m\, \eta) \; . \tag{14}$$

Standard bilinear pointwise operations are recovered when considering $\wedge$ on the side of Euclidean vector proxies:

$$\mathrm{V.P.}(\omega \wedge \eta) = \begin{cases} \vec{\mathbf{u}} \times \vec{\mathbf{v}} & \text{, for } \ell = m = 1, \\ \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} & \text{, for } \ell = 2, m = 1 \\ \vec{u}\, \vec{\mathbf{v}} & \text{, for } \ell = 0, m = 1, 2 \end{cases} \; , \quad \begin{array}{l} \vec{u}/\vec{\mathbf{u}} := \mathrm{V.P.}(\omega),\ \omega \in C^1\Lambda^\ell(\Omega) \; , \\ \vec{\mathbf{v}} := \mathrm{V.P.}(\eta),\ \eta \in C^1\Lambda^m(\Omega) \; . \end{array} \tag{15}$$

Following [5, Sect. 2.2], we introduce Hilbert spaces of ℓ-forms on a piecewise smooth manifold Ω

$$L^2\Lambda^\ell(\Omega) := \left\{ \begin{array}{l} \omega \in \mathscr{F}^\ell(\Omega) : \boldsymbol{x} \mapsto \omega(\boldsymbol{x})(\mathbf{v}) \in L^2(\Omega) \\ \text{for every smooth vectorfield } \mathbf{v} \text{ on } \Omega \end{array} \right\} \; .$$

For a domain $\Omega \subset \mathbb{R}^3$ a form on Ω is in $L^2\Lambda^\ell(\Omega)$, if and only if its vector proxy belongs to $(L^2(\Omega))^{\binom{n}{\ell}}$.

As a consequence of the Riesz representation theorem the exterior product allows to express duality in spaces of differential forms:

Theorem 5 *The exterior product* $\wedge : C^0\Lambda^\ell(\Omega) \times C^0\Lambda^m(\Omega) \to C^0\Lambda^{\ell+m}(\Omega)$ *can be extended to* $L^2\Lambda^\ell(\Omega) \times L^2\Lambda^m(\Omega)$ *by continuity. This extension provides a* duality pairing *between* $L^2\Lambda^\ell(\Omega)$ *and* $L^2\Lambda^{n-\ell}(\Omega)$ *through the bilinear form*

$$(\omega, \eta) \mapsto \int_\Omega \omega \wedge \eta \; .$$

2.4.2 Field Energies

Mathematically speaking, in electrodynamics an energy is a mapping from fields to non-negative numbers. Therefore, for a bounded domain $\Omega \subset \mathbb{R}^3$ we introduce

$$\begin{aligned} &\text{electric field energy:} && \mathscr{E}_{\mathrm{el}} : L^2\Lambda^2(\Omega) \to \mathbb{R}_{\geq 0} \ , \\ &\text{magnetic field energy:} && \mathscr{E}_{\mathrm{mag}} : L^2\Lambda^2(\Omega) \to \mathbb{R}_{\geq 0} \ . \end{aligned}$$

Then, the values $\mathscr{E}_{\mathrm{el}}(\mathbf{d})$ and $\mathscr{E}_{\mathrm{mag}}(\mathbf{b})$ (unit J) provide the energy content of the fields $\mathbf{d}$ and $\mathbf{b}$.

Assumption 6 (Properties of Field Energies) *Both $\mathscr{E}_{el}$ and $\mathscr{E}_{mag}$ are Fréchet-differentiable and* strictly convex.

This ensures that the Fréchet derivatives

$$D\mathscr{E}_{\mathrm{el}}, D\mathscr{E}_{\mathrm{mag}} : L^2\Lambda^2(\Omega) \to (L^2\Lambda^2(\Omega))' = L^2\Lambda^1(\Omega) \quad \text{(by Thm. 5)}$$

are strictly monotone operators and, hence, isomorphisms, see [44, Sect. 10.3.2].

In many settings the field energies are *localized* in the sense that there are two functions ("energy densities")

$$E_{\mathrm{el}}, E_{\mathrm{mag}} : \Omega \times \Lambda^2(\mathbb{R}^3) \to \mathbb{R}$$

such that

$$\mathscr{E}_{\mathrm{el}}(\mathbf{d}) = \int_\Omega E_{\mathrm{el}}(\boldsymbol{x}, \mathbf{d}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} \quad , \quad \mathscr{E}_{\mathrm{mag}}(\mathbf{b}) = \int_\Omega E_{\mathrm{mag}}(\boldsymbol{x}, \mathbf{b}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} \ .$$

If E_{el} and E_{mag} are differentiable and $\boldsymbol{x}$-uniformly strictly convex in their second argument, Assumption 6 is satisfied. Moreover, the Fréchet derivatives with respect to the second argument $D_2E_{\mathrm{el}}(\boldsymbol{x}, \mathbf{d})$ and $D_2E_{\mathrm{mag}}(\boldsymbol{x}, \mathbf{b})$ are isomorphisms $\Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$.

Finally, writing $\langle\cdot,\cdot\rangle$ for the duality pairings in $L^2\Lambda^2(\Omega)/\Lambda^2(\mathbb{R}^3)$, and appealing to Theorem 5 we find that for all $\mathbf{d}' \in C^\infty\Lambda^2(\Omega)$

$$\begin{aligned} \left\langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}), \mathbf{d}' \right\rangle &= \int_\Omega \left\langle D_2E_{\mathrm{el}}(\boldsymbol{x}, \mathbf{d}(\boldsymbol{x})), \mathbf{d}'(\boldsymbol{x}) \right\rangle \mathrm{d}\boldsymbol{x} \\ &= \int_\Omega D_2E_{\mathrm{el}}(\boldsymbol{x}, \mathbf{d}(\boldsymbol{x})) \wedge \mathbf{d}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ . \end{aligned}$$

A very special, but common case, is *local quadratic* field energies, where

$$
\left.\begin{aligned}
E_{\mathrm{el}}(\boldsymbol{x},\mathbf{d}(\boldsymbol{x})) &= \tfrac{1}{2}\beta_{\mathrm{el}}(\boldsymbol{x})(\mathbf{d}(\boldsymbol{x}),\mathbf{d}(\boldsymbol{x}))\ , \\
E_{\mathrm{mag}}(\boldsymbol{x},\mathbf{b}(\boldsymbol{x})) &= \tfrac{1}{2}\beta_{\mathrm{mag}}(\boldsymbol{x})(\mathbf{b}(\boldsymbol{x}),\mathbf{b}(\boldsymbol{x}))\ ,
\end{aligned}\right\}\quad \boldsymbol{x}\in\Omega\ ,
\tag{16a,16b}
$$

with $\boldsymbol{x}$-uniformly positive definite *bilinear* forms $\beta_{\mathrm{el}}, \beta_{\mathrm{mag}} : \Omega \to L(\Lambda^2(\mathbb{R}^3) \times \Lambda^2(\mathbb{R}^3), \mathbb{R})$. In this case, switching to Euclidean vector proxies, we may write

$$
\left.\begin{aligned}
E_{\mathrm{el}}(\boldsymbol{x},\mathbf{d}(\boldsymbol{x})) &= \tfrac{1}{2}\overrightarrow{\mathbf{d}}(\boldsymbol{x})^{\top}\boldsymbol{\varepsilon}^{-1}(\boldsymbol{x})\overrightarrow{\mathbf{d}}(\boldsymbol{x})\ , \\
E_{\mathrm{mag}}(\boldsymbol{x},\mathbf{b}(\boldsymbol{x})) &= \tfrac{1}{2}\overrightarrow{\mathbf{b}}(\boldsymbol{x})^{\top}\boldsymbol{\mu}^{-1}(\boldsymbol{x})\overrightarrow{\mathbf{b}}(\boldsymbol{x})\ ,
\end{aligned}\right\}\quad \boldsymbol{x}\in\Omega\ ,
\tag{17a,17b}
$$

where $\boldsymbol{\varepsilon} : \Omega \to \mathbb{R}^{3,3}$ and $\boldsymbol{\mu} : \Omega \to \mathbb{R}^{3,3}$ are position dependent symmetric positive definite (spd) 3×3-matrices, the *dielectric tensor* and the *magnetic permeability tensor*, respectively.

Remark 5 The concept of energy content of a field in the presence of matter is inherently *macroscopic* (phenomenological), because it ignores very complex interactions at the atomic level.

2.4.3 Material Laws

Material laws state a one-to-one correspondence between the electric field $\mathbf{e}$ and the displacement current $\mathbf{d}$, and between the magnetic induction $\mathbf{b}$ and the magnetic field $\mathbf{h}$. In concrete terms we stipulate

$$
\boxed{\begin{aligned}
\mathbf{e} = \mathbf{e}(\mathbf{d}) &= D\mathscr{E}_{\mathrm{el}}(\mathbf{d}) \in L^2\Lambda^1(\Omega)\ , \\
\mathbf{h} = \mathbf{h}(\mathbf{b}) &= D\mathscr{E}_{\mathrm{mag}}(\mathbf{b}) \in L^2\Lambda^1(\Omega)\ .
\end{aligned}}
\tag{18a,18b}
$$

The inverses of these material laws can be stated as

$$
\mathbf{d}(\mathbf{e}) = D\mathscr{E}^{*}_{\mathrm{el}}(\mathbf{e}) \quad , \quad \mathbf{b}(\mathbf{e}) = D\mathscr{E}^{*}_{\mathrm{mag}}(\mathbf{h})\ , \tag{19}
$$

where $\mathscr{E}^{*}_{\mathrm{el}} : L^2\Lambda^1(\Omega) \to \mathbb{R}$ and $\mathscr{E}^{*}_{\mathrm{mag}} : L^2\Lambda^1(\Omega) \to \mathbb{R}$ are the strictly convex Fenchel conjugates of $\mathscr{E}_{\mathrm{el}}$ and $\mathscr{E}_{\mathrm{mag}}$, called co-energies in physics [19, Def. 12.2].

Relying on the duality from Theorem 5, we can cast the material laws (18a), (18b), and (19) into variational form.

$$
\int_{\Omega} \mathbf{e}(\mathbf{d}) \wedge \mathbf{d}' = \left\langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}), \mathbf{d}' \right\rangle \qquad \forall \mathbf{d}' \in L^2\Lambda^2(\Omega)\ , \tag{20a}
$$

$$
\int_{\Omega} \mathbf{h}(\mathbf{b}) \wedge \mathbf{b}' = \left\langle D\mathscr{E}_{\mathrm{mag}}(\mathbf{b}), \mathbf{b}' \right\rangle \qquad \forall \mathbf{b}' \in L^2\Lambda^2(\Omega)\ , \tag{20b}
$$

$$\int_\Omega \mathbf{d}(\mathbf{e}) \wedge \mathbf{e}' = \left\langle D\mathscr{E}^*_{\mathrm{el}}(\mathbf{e}), \mathbf{e}' \right\rangle \qquad \forall \mathbf{e}' \in L^2\Lambda^1(\Omega)\ , \tag{20c}$$

$$\int_\Omega \mathbf{b}(\mathbf{h}) \wedge \mathbf{h}' = \left\langle D\mathscr{E}^*_{\mathrm{mag}}(\mathbf{h}), \mathbf{h}' \right\rangle \qquad \forall \mathbf{h}' \in L^2\Lambda^1(\Omega)\ , \tag{20d}$$

Special Case: Local Quadratic Energies

For energies given by (16a) and (16b), which are still continuous on $L^2\Lambda^2(\Omega)$, the general formulas (18a), (18b), and (19) become

$$\left.\begin{aligned} \mathbf{e}(\boldsymbol{x}) = M_{\mathrm{el}}(\mathbf{d}(\boldsymbol{x})) \quad &\Leftrightarrow \quad \mathbf{d}(\boldsymbol{x}) = M_{\mathrm{el}}^{-1}(\mathbf{d}(\boldsymbol{x}))\ , & \text{(21a)} \\ \mathbf{h}(\boldsymbol{x}) = M_{\mathrm{mag}}(\mathbf{b}(\boldsymbol{x})) \quad &\Leftrightarrow \quad \mathbf{b}(\boldsymbol{x}) = M_{\mathrm{mag}}^{-1}(\mathbf{h}(\boldsymbol{x}))\ , & \text{(21b)} \end{aligned}\right\} \text{for almost all } \boldsymbol{x} \in \Omega\ ,$$

where both M_{el} and M_{mag} are bounded *linear* operators $\Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$. They are specimens of *Hodge operators*, which, in the general case, induce isomorphisms $\Lambda^\ell(\mathbb{R}^n) \cong \Lambda^{n-\ell}(\mathbb{R}^n)$. By pointwise application Hodge operators can be defined for continuous differential forms, and then can be extended to $L^2\Lambda^\ell(\Omega)$. For these Hodge operators we adopt the customary notation $\star$ and write for (21a), (21b)

$$\mathbf{e} = \star_{\varepsilon^{-1}}\mathbf{d} \ \Leftrightarrow\ \mathbf{d} = \star_\varepsilon \mathbf{e} \quad , \quad \mathbf{h} = \star_{\mu^{-1}}\mathbf{b} \ \Leftrightarrow\ \mathbf{b} = \star_\mu \mathbf{h}\ . \tag{22}$$

Then the field energies can be expressed by

$$\mathscr{E}_{\mathrm{el}} = \tfrac{1}{2}\int_\Omega \star_{\varepsilon^{-1}}\mathbf{d} \wedge \mathbf{d} = \tfrac{1}{2}\int_\Omega \star_\varepsilon \mathbf{e} \wedge \mathbf{e}\ ,$$

$$\mathscr{E}_{\mathrm{mag}} = \tfrac{1}{2}\int_\Omega \star_{\mu^{-1}}\mathbf{b} \wedge \mathbf{b} = \tfrac{1}{2}\int_\Omega \star_\mu \mathbf{h} \wedge \mathbf{h}\ .$$

Remark 6 The notation in (22) hints that the Hodge operators emerge from the material tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ introduced in (17a) and (17b). These tensors can be viewed as coordinate representations of a Riemannian metric on Ω. Indeed, the usual definition of Hodge operators on $\Lambda^\ell(\mathbb{R}^n)$ relies on inner products in $\mathbb{R}^n$ [18, Sect. 4].

The vector proxy form of (22) is immediate from (17):

$$\left.\begin{aligned} \vec{\mathbf{d}}(\boldsymbol{x}) = \boldsymbol{\varepsilon}(\boldsymbol{x})\vec{\mathbf{e}}(\boldsymbol{x}) \quad &\Leftrightarrow \quad \vec{\mathbf{e}}(\boldsymbol{x}) = \boldsymbol{\varepsilon}(\boldsymbol{x})^{-1}\vec{\mathbf{d}}(\boldsymbol{x})\ , & \text{(23a)} \\ \vec{\mathbf{b}}(\boldsymbol{x}) = \boldsymbol{\mu}(\boldsymbol{x})\vec{\mathbf{h}}(\boldsymbol{x}) \quad &\Leftrightarrow \quad \vec{\mathbf{h}}(\boldsymbol{x}) = \boldsymbol{\mu}(\boldsymbol{x})^{-1}\vec{\mathbf{b}}(\boldsymbol{x})\ , & \text{(23b)} \end{aligned}\right\} \text{a.e. in } \Omega\ .$$

Thus, the variational material laws (20) can be expressed as

$$\int_{\Omega} \vec{\mathbf{d}}(\boldsymbol{x}) \cdot \vec{\mathbf{e}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{x})\vec{\mathbf{e}}(\boldsymbol{x}) \cdot \vec{\mathbf{e}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall \vec{\mathbf{e}}' \in (L^2(\Omega))^3 ,$$
$$\Leftrightarrow \qquad \int_{\Omega} \vec{\mathbf{e}}(\boldsymbol{x}) \cdot \vec{\mathbf{d}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{\varepsilon}^{-1}(\boldsymbol{x})\vec{\mathbf{d}}(\boldsymbol{x}) \cdot \vec{\mathbf{d}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall \vec{\mathbf{d}}' \in (L^2(\Omega))^3 , \tag{24a}$$

$$\int_{\Omega} \vec{\mathbf{b}}(\boldsymbol{x}) \cdot \vec{\mathbf{h}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \mu(\boldsymbol{x})\vec{\mathbf{h}}(\boldsymbol{x}) \cdot \vec{\mathbf{h}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall \vec{\mathbf{h}}' \in (L^2(\Omega))^3 ,$$
$$\Leftrightarrow \qquad \int_{\Omega} \vec{\mathbf{h}}(\boldsymbol{x}) \cdot \vec{\mathbf{b}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \mu^{-1}(\boldsymbol{x})\vec{\mathbf{b}}(\boldsymbol{x}) \cdot \vec{\mathbf{b}}'(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall \vec{\mathbf{b}}' \in (L^2(\Omega))^3 . \tag{24b}$$

Remark 7 There is another local material law that is often encountered in electromagnetic field models, known as *Ohm's law*. It links the electric field $\mathbf{e}$ and the current $\mathbf{j}$ according to

$$\mathbf{j} = \star_{\sigma}\mathbf{e} . \tag{25}$$

Here, $\sigma = \sigma(\boldsymbol{x})$ is another metric tensor called the *conductivity*. Expressed in terms of vector proxies (25) reads

$$\vec{\mathbf{j}}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})\vec{\mathbf{e}}(\boldsymbol{x}) \quad \text{a.e. in } \Omega , \tag{26}$$

with $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3,3}$ uniformly spd.

2.4.4 Energy Balance (Poynting's Theorem)

If the fields $\mathbf{e}$, $\mathbf{b}$, $\mathbf{d}$, and $\mathbf{h}$ satisfy Maxwell's equations, the total field energy $\mathscr{E}_{\mathrm{tot}} := \mathscr{E}_{\mathrm{el}} + \mathscr{E}_{\mathrm{mag}}$ fulfills [19, Prop 12.1]

$$\begin{aligned}\tfrac{d}{dt}\mathscr{E}_{\mathrm{tot}}(t) &= \tfrac{d}{dt}\big(\mathscr{E}_{\mathrm{el}}(\mathbf{d}(t)) + \mathscr{E}_{\mathrm{mag}}(\mathbf{b}(t))\big) \\ &= \langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}(t)), \partial_t\mathbf{d}(t)\rangle + \big\langle D\mathscr{E}_{\mathrm{mag}}(\mathbf{b}(t)), \partial_t\mathbf{b}(t)\big\rangle \\ &= \int_{\Omega} \mathbf{e}(t) \wedge \partial_t\mathbf{d}(t) + \mathbf{h}(t) \wedge \partial_t\mathbf{b}(t) \;,\end{aligned}$$

using Ampere's law (AL) to eliminate $\partial_t\mathbf{d}$ and Faraday's law (FL) on $\partial_t\mathbf{b}$,

$$= \int_{\Omega} \mathbf{e}(t) \wedge (\mathsf{d}_1\,\mathbf{h}(t) - \mathbf{j}) + \mathbf{h}(t) \wedge (-\,\mathsf{d}_1\,\mathbf{e})$$

$$
\begin{aligned}
&= \int_{\Omega} (\mathbf{e} \wedge \mathsf{d}_1 \mathbf{h} - \mathbf{h} \wedge \mathsf{d}_1 \mathbf{e})(t) - (\mathbf{e} \wedge \mathbf{j})(t) \\
&= \int_{\partial\Omega} (\mathbf{e} \wedge \mathbf{h})(t) - \int_{\Omega} (\mathbf{e} \wedge \mathbf{j})(t) \ .
\end{aligned}
$$

In the last step we used integration by parts, that is, we combined Definition 3 and the Leibniz rule (14). The first term is the *Poynting vector* 2-form, whose integral supplies the flow of electromagnetic energy through a surface. The second term is the power consumed by dissipation.

2.5 Maxwell's Equations: Variational Approach

Next, we derive the weak form of well-posed boundary value problems for Maxwell's equations (FL) and (AL) equipped with general material laws in weak form given in (20). Throughout we focus on a bounded domain $\Omega \subset \mathbb{R}^3$. Analogous considerations for time-harmonic fields can be found in [35, Sect. 2.3]. As a key tool we recall the integration by parts formula for $\omega \in C^1\Lambda^\ell(\Omega)$, $\eta \in C^1\Lambda^k(\Omega)$:

$$
\int_{\Omega} \mathsf{d}_\ell\, \omega \wedge \eta + (-1)^\ell(\omega \wedge \mathsf{d}_k\, \eta) = \int_{\partial\Omega} \omega \wedge \eta \ . \tag{27}
$$

2.5.1 a-Based Variational Formulation

As in Sect. 2.3.3 we employ the electromagnetic potentials, but do so in a particular way, using the gauge freedom (11) to drop the scalar electric potential v ("temporal gauge", $v = 0$), which leaves us with a vector potential $\mathbf{a} \in \mathscr{F}^1(\Omega)$ that is just a temporal primitive of the electric field and satisfies

$$
\mathbf{e}(t) = -\partial_t \mathbf{a}(t) \quad \text{and} \quad \mathbf{b}(t) = \mathsf{d}_1\, \mathbf{a}(t) \quad \text{in } \Omega \ . \tag{28}
$$

First, test (AL) with $\mathbf{a}' \in C^\infty\Lambda^1(\Omega)$ (independent of time) and integrate by parts according to (27), which yields

$$
\int_{\Omega} \mathbf{h}(t) \wedge \mathsf{d}_1\, \mathbf{a}' + \int_{\partial\Omega} \mathbf{h}(t) \wedge \mathbf{a}' = \partial_t \int_{\Omega} \mathbf{d}(t) \wedge \mathbf{a}' + \int_{\Omega} \mathbf{j}(t) \wedge \mathbf{a}' \ .
$$

Next, use (20b) and (20c) to rewrite the first terms on both sides,

$$
\left\langle D\mathscr{E}_{\mathrm{mag}}(\mathbf{b}(t)), \mathsf{d}_1\, \mathbf{a}' \right\rangle + \int_{\partial\Omega} \mathbf{h}(t) \wedge \mathbf{a}' = \partial_t \left\langle D\mathscr{E}_{\mathrm{el}}^*(\mathbf{e}(t)), \mathbf{a}' \right\rangle + \int_{\Omega} \mathbf{j}(t) \wedge \mathbf{a}' \ ,
$$

and then plug in (28):

$$\begin{aligned}&\left\langle D\mathscr{E}_{\text{mag}}(\mathsf{d}_1\,\mathbf{a}(t)),\mathsf{d}_1\,\mathbf{a}'\right\rangle+\int_{\partial\Omega}\mathbf{h}(t)\wedge\mathbf{a}'\\&\qquad=\partial_t\left\langle D\mathscr{E}^*_{\text{el}}(-\partial_t\mathbf{a}(t)),\mathbf{a}'\right\rangle+\int_{\Omega}\mathbf{j}(t)\wedge\mathbf{a}'\;,\end{aligned}\tag{29}$$

which is supposed to hold for all $\mathbf{a}'\in C^\infty\Lambda^1(\Omega)$. Formally, this is a non-linear second-order evolution problem for the unknown 1-form valued function $\mathbf{a}=\mathbf{a}(t)$. Initial conditions $\mathbf{a}(0)$ and $\partial_t\mathbf{a}(0)$ have to be supplied.

For local linear material laws (22) we seek $\mathbf{a}=\mathbf{a}(t)$ such that

$$\begin{aligned}&\int_{\Omega}(\star_{\mu^{-1}}\mathsf{d}_1\,\mathbf{a}(t))\wedge\mathsf{d}_1\,\mathbf{a}'\;+\partial_t^2\int_{\Omega}(\star_{\varepsilon}\mathbf{a}(t))\wedge\mathbf{a}'\\&\qquad=-\int_{\partial\Omega}\mathbf{h}(t)\wedge\mathbf{a}'\;+\int_{\Omega}\mathbf{j}(t)\wedge\mathbf{a}'\;.\end{aligned}\tag{30}$$

This is a linear 2nd-order evolution problem, posed on the "energy space"

$$H\Lambda^\ell(\Omega):=\{\omega\in L^2\Lambda^\ell(\Omega):\;\mathsf{d}_\ell\,\omega\in L^2\Lambda^{\ell+1}(\Omega)\}\quad\text{for }\ell=1\;.\tag{31}$$

The spaces $H\Lambda^\ell(\Omega)$ are Sobolev spaces of differential forms on Ω [5]. They are Hilbert spaces with inner product ($\star$ is the Euclidean Hodge operator)

$$(\omega,\eta)_{H\Lambda^\ell(\Omega)}:=\int_{\Omega}(\star\omega)\wedge\eta+(\star\,\mathsf{d}_\ell\,\omega)\wedge\mathsf{d}_\ell\,\eta\;,\quad\omega,\eta\in H\Lambda^\ell(\Omega)\;.\tag{32}$$

The spaces $C^\infty\Lambda^\ell(\Omega)$ are dense in $H\Lambda^\ell(\Omega)$. For a domain $\Omega\subset\mathbb{R}^3$ and $\ell=1$ the Hilbert space of vector proxies isomorphic to $H\Lambda^1(\Omega)$ is the well-known Sobolev space $\boldsymbol{H}(\mathbf{curl},\Omega)$, for $\ell=2$ we get $\boldsymbol{H}(\mathrm{div},\Omega)$, and for $\ell=0$ the function space $H^1(\Omega)$, see [35, Sect. 2.4].

Thus, in terms of vector proxies the electrodynamic evolution problem in the $\mathbf{a}$-based formulation reads: find $\overrightarrow{\mathbf{a}}(t)\in\boldsymbol{H}(\mathbf{curl},\Omega)$ with

$$\begin{aligned}&\int\limits_{\Omega}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\overrightarrow{\mathbf{a}}(t)\cdot\mathbf{curl}\,\overrightarrow{\mathbf{a}}'\,\mathrm{d}\boldsymbol{x}+\partial_t^2\int\limits_{\Omega}\boldsymbol{\varepsilon}\,\overrightarrow{\mathbf{a}}(t)\cdot\overrightarrow{\mathbf{a}}'\,\mathrm{d}\boldsymbol{x}\\&\qquad=-\int\limits_{\partial\Omega}(\overrightarrow{\mathbf{h}}(t)\times\overrightarrow{\mathbf{a}}')\cdot\boldsymbol{n}\,\mathrm{d}S+\int\limits_{\Omega}\overrightarrow{\mathbf{j}}(t)\cdot\overrightarrow{\mathbf{a}}'\,\mathrm{d}\boldsymbol{x}\end{aligned}\tag{33}$$

for all $\overrightarrow{\mathbf{a}}'\in\boldsymbol{H}(\mathbf{curl},\Omega)$.

2.5.2 h-Based Variational Formulation

Alternatively, we may test Faraday's law (FL) with $\mathbf{h}' \in C^\infty \Lambda^1(\Omega)$ (independent of time), which, after integration by parts (27), yields

$$\int_\Omega \mathbf{e}(t) \wedge \mathsf{d}_1 \mathbf{h}' + \int_{\partial\Omega} \mathbf{e}(t) \wedge \mathbf{h}' = -\partial_t \int_\Omega \mathbf{b}(t) \wedge \mathbf{h}' \ . \tag{34}$$

We use the material laws (20a) and (20d) to replace the two integrals over Ω:

$$\left\langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}(t)), \mathsf{d}_1 \mathbf{h}' \right\rangle + \int_{\partial\Omega} \mathbf{e}(t) \wedge \mathbf{h}' = -\partial_t \left\langle D\mathscr{E}^*_{\mathrm{mag}}(\mathbf{h}(t)), \mathbf{h}' \right\rangle \ .$$

Then replace $\mathbf{d}$ by means of Ampere's law (integrated in time) and obtain the variational problem: seek $\tilde{\mathbf{h}} = \tilde{\mathbf{h}}(t)$ such that

$$\begin{aligned} &\left\langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}(0) + \mathsf{d}_1 \tilde{\mathbf{h}}(t) - \int_0^t \mathbf{j}(\tau)\,\mathrm{d}\tau), \mathsf{d}_1 \mathbf{h}' \right\rangle + \int_{\partial\Omega} \mathbf{e}(t) \wedge \mathbf{h}' \\ &\quad = -\partial_t \left\langle D\mathscr{E}^*_{\mathrm{mag}}(\partial_t \tilde{\mathbf{h}}(t)), \mathbf{h}' \right\rangle \ , \end{aligned} \tag{35}$$

for all $\mathbf{h}' \in C^\infty \Lambda^1(\Omega)$. The unknown field $\tilde{\mathbf{h}}(t)$ is a temporal primitive of $\mathbf{h}$: $\tilde{\mathbf{h}}(t) = \int_0^t \mathbf{h}(\tau)\,\mathrm{d}\tau$; in particular $\tilde{\mathbf{h}}(0) = 0$.

Using the local linear material laws (22) we recover a special variant of (35): Seek $\tilde{\mathbf{h}}(t) \in H\Lambda^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega (\star_{\varepsilon^{-1}} \mathsf{d}_1 \tilde{\mathbf{h}}(t)) \wedge \mathsf{d}_1 \mathbf{h}' + \partial_t^2 \int_\Omega (\star_\mu \tilde{\mathbf{h}}(t)) \wedge \mathbf{h}'\,\mathrm{d}\boldsymbol{x} \\ &\quad = -\int_{\partial\Omega} \mathbf{e}(t) \wedge \mathbf{h}' - \int_\Omega \star_{\varepsilon^{-1}} \big(\mathbf{d}(0) + \int_0^t \mathbf{j}(\tau)\,\mathrm{d}\tau\big) \wedge \mathsf{d}_1 \mathbf{h}' \end{aligned} \tag{36}$$

for all $\mathbf{h}' \in H\Lambda^1(\Omega)$. Rewriting this for vector proxies gives us: Find $\vec{\tilde{\mathbf{h}}}(t) \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ with

$$\begin{aligned} &\int_\Omega \boldsymbol{\varepsilon}^{-1} \mathbf{curl}\,\vec{\tilde{\mathbf{h}}}(t) \cdot \mathbf{curl}\,\vec{\mathbf{h}}'\,\mathrm{d}\boldsymbol{x} + \partial_t^2 \int_\Omega \boldsymbol{\mu}\,\vec{\tilde{\mathbf{h}}}(t) \cdot \vec{\mathbf{h}}'\,\mathrm{d}\boldsymbol{x} \\ &\quad = -\int_{\partial\Omega} (\vec{\mathbf{e}}(t) \times \vec{\mathbf{h}}') \cdot \boldsymbol{n}\,\mathrm{d}S + \int_\Omega \boldsymbol{\varepsilon}^{-1} \big(\vec{\mathbf{d}}(0) + \int_0^t \vec{\mathbf{j}}(\tau)\,\mathrm{d}\tau\big) \cdot \mathbf{curl}\,\vec{\mathbf{h}}'\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{37}$$

for all $\vec{\mathbf{h}}' \in \boldsymbol{H}(\mathbf{curl}, \Omega)$.

Remark 8 We frequently have the possibility to cast a single boundary value problem or evolution problem into several variational forms. The standard example is the standard (primal) and mixed (dual) variational formulation of scalar second-order elliptic boundary value problems. For a more general discussion refer to [11, Sect. 1.3].

2.5.3 Boundary Conditions

Both variational formulations (29) and (35) feature undetermined boundary terms and have to be supplemented with boundary conditions. To that end, we partition $\Gamma := \partial\Omega$ into three parts $\Gamma = \Gamma_e \dot{\cup} \Gamma_m \dot{\cup} \Gamma_i$ with disjoint interiors. Each part may not be present and collapse to $\emptyset$.

On these parts of Γ different boundary conditions on the fields are imposed by means of the trace operators $\mathbf{t}_\Gamma$ from Sect. 2.2.3:

- *Electric boundary conditions* on Γ_e: $\mathbf{t}_{\Gamma_e}\mathbf{e}(t) = \mathbf{g}_e(t) \in \mathscr{F}^1(\Gamma_e)$.
- *Magnetic boundary conditions* on Γ_m: $\mathbf{t}_{\Gamma_m}\mathbf{h}(t) = \mathbf{g}_m(t) \in \mathscr{F}^1(\Gamma_m)$.
- *Impedance boundary conditions* on Γ_i: $\mathbf{t}_{\Gamma_i}\mathbf{h}(t) = Z(\mathbf{t}_{\Gamma_i}\mathbf{e}(t))$,

where $Z : \mathscr{F}^1(\Gamma_i) \to \mathscr{F}^1(\Gamma_i)$ is a local or non-local impedance map, which boils down to a surface Hodge operator in the simplest case.
❶ For the **a**-based variational formulation (29)

☞ electric boundary conditions are *essential* and have to be enforced on the trial 1-forms and (in their homogeneous variant) on the test 1-forms,
☞ magnetic boundary conditions are *natural* and taken into account on the right hand side of the variational formulation,
☞ impedance boundary conditions give rise to another term on the left hand side of (29).

Assuming benign nonlinearity of $D\mathscr{E}_{\mathrm{el}}$ and $D\mathscr{E}_{\mathrm{mag}}$, we arrive at the following variational evolution problem: seek $\mathbf{a}(t) \in H\Lambda^1(\Omega)$ with $(\mathbf{t}_{\Gamma_e}\mathbf{a})(t) = -\int_0^t \mathbf{g}_e(\tau)\,\mathrm{d}\tau$ such that

$$\begin{aligned}\left\langle D\mathscr{E}_{\mathrm{mag}}(\mathsf{d}_1\,\mathbf{a}(t)), \mathsf{d}_1\,\mathbf{a}'\right\rangle - \partial_t\left\langle D\mathscr{E}^*_{\mathrm{el}}(-\partial_t\mathbf{a}(t)), \mathbf{a}'\right\rangle + \int_{\Gamma_i} Z(-\partial_t\mathbf{t}_{\Gamma_i}\mathbf{a}(t)) \wedge \mathbf{a}' \\ = -\int_{\Gamma_m}\mathbf{g}_m(t)\wedge\mathbf{a}' + \int_\Omega \mathbf{j}(t)\wedge\mathbf{a}' \ , \end{aligned} \tag{38}$$

for all $\mathbf{a}' \in H\Lambda^1(\Omega)$ satisfying $\mathbf{t}_{\Gamma_e}\mathbf{a}' = 0$.
❷ In the case of the **h**-based variational formulation (35)

☞ electric boundary conditions become *natural* boundary conditions and show up on the right hand side of the variational formulation,
☞ magnetic boundary conditions have to be imposed on trial and test 1-forms, that is, they are *essential*,

☞ impedance boundary conditions engender another contribution to the left hand side of the variational formulation.

Hence, taking into account the various boundary conditions, the variational formulation becomes: Seek a temporal primitive $\tilde{\mathbf{h}}(t) \in H\Lambda^1(\Omega)$ of the magnetic field with $\mathbf{t}_{\Gamma_m}\tilde{\mathbf{h}}(t) = \int_0^t \mathbf{g}_m(\tau)\,\mathrm{d}\tau$, such that

$$\left\langle D\mathscr{E}_{\mathrm{el}}(\mathbf{d}(0) + \mathsf{d}_1\,\tilde{\mathbf{h}}(t) - \int_0^t \mathbf{j}(\tau)\,\mathrm{d}\tau), \mathsf{d}_1\,\mathbf{h}'\right\rangle + \partial_t\left\langle D\mathscr{E}^*_{\mathrm{mag}}(\partial_t\tilde{\mathbf{h}}(t)), \mathbf{h}'\right\rangle$$
$$+ \int_{\Gamma_i} Z^{-1}(\partial_t\tilde{\mathbf{h}}(t)) \wedge \mathbf{h}' = -\int_{\Gamma_e} \mathbf{g}_e(t) \wedge \mathbf{h}' \ , \tag{39}$$

for all $\mathbf{h}' \in H\Lambda^1(\Omega)$ with $\mathbf{t}_{\Gamma_m}\mathbf{h}' = 0$.

Remark 9 When viewing electric boundary conditions as Dirichlet boundary conditions, magnetic boundary conditions as Neumann boundary conditions, and relating impedance boundary conditions to Robin boundary conditions, striking similarities between Maxwell's equations and scalar second-order elliptic evolution problems become apparent. This is not a coincidence, because both Maxwell's equations and the scalar wave equation belong to a single family of evolution problems. Using exterior calculus, they can even be stated in a unified way. Some details are give in Sect. 4.1 and a comprehensive discussion can be found in [33, Sect. 2].

3 Co-chains and Whitney Forms

Now we are concerned with the discretization of electromagnetic fields. The key insight from Sect. 2.1 was that, from a non-local point of view, fields are integral ℓ-forms, *cf.* Definition 1, assigning (real/complex) values to oriented ℓ-dimensional submanifolds of $\mathbb{R}^3$. Discretization means that we switch to a description of the fields involving only finitely many degrees of freedom. To begin with, the choice of these degrees of freedom will be guided by our understanding of integral forms. Then, in the spirit of finite element exterior calculus (FEEC), we pursue the construction of *discrete differential forms* that are valid integral forms, uniquely determined by the degrees of freedom, and satisfy fundamental algebraic properties with respect to the exterior derivative. We also study a key tool in FEEC: commuting projectors.

Bibliographical Notes

The topics of these section are covered in [5, Sects. 2–5], and [35, Sect. 3], and some aspects are addressed in [19, Chap. IV]. A complete survey of discrete differential forms is given in the Periodic Table of Finite Elements by D. Arnold [4]. Using vector proxies, discrete differential forms can be treated as classical (mixed, vector valued) finite element functions. This is the perspective adopted in [11, Sects. 2.3–2.6] and [40, Chaps. 5–6]. Some of the ideas and results presented below are fairly recent and covered only in research articles, which are cited locally.

3.1 Meshes

We aim for discrete fields that are mappings from a finite number of oriented ℓ-dimensional submanifolds of $\mathbb{R}^3$ to $\mathbb{R}$ (or $\mathbb{C}$). However, arbitrary sets of submanifolds will usually not be eligible, because Maxwell's equations in integral form as stated in (FL) and (AL) rely on the concept of a boundary of a surface. Thus, the set of submanifolds in the representation of discrete fields must be closed with respect to the boundary operator ∂. Such special sets are given in the next definition, *cf.* [12, Sect. 5.2.1], [19, Sect. 14]. In fact, it describes special instances of so-called cell complexes [7, Sect. 3.1].

Definition 4 (Mesh/Triangulation [35, Def. 3]) A *mesh/triangulation* $\mathscr{T}_h$ of a bounded domain $\Omega \subset \mathbb{R}^3$ is a finite collection of *oriented* cells ($\rightarrow$ set $\mathscr{F}_3(\mathscr{T}_h)$ of 3-facets), faces ($\rightarrow$ set $\mathscr{F}_2(\mathscr{T}_h)$ of 2-facets), edges ($\rightarrow$ set $\mathscr{F}_1(\mathscr{T}_h)$ of 1-facets), and vertices ($\rightarrow$ set $\mathscr{F}_0(\mathscr{T}_h)$ of 0-facets) such that

(i) every ℓ-facet $f \in \mathscr{F}_\ell(\mathscr{T}_h)$ is the diffeomorphic image of an open non-degenerate polytope in $\mathbb{R}^\ell$,
(ii) $\mathscr{F}_0(\mathscr{T}_h) \cup \mathscr{F}_1(\mathscr{T}_h) \cup \mathscr{F}_2(\mathscr{T}_h) \cup \mathscr{F}_3(\mathscr{T}_h)$ is a partition of $\overline{\Omega}$,
(iii) for every $F \in \mathscr{F}_\ell(\mathscr{T}_h)$, $0 < \ell \leq 3$, there are $f_1, \dots, f_m \in \mathscr{F}_{\ell-1}(\mathscr{T}_h)$ such that $\partial F = \overline{f}_1 \cup \dots \cup \overline{f}_m$,
(iv) for each $f \in \mathscr{F}_\ell(\mathscr{T}_h)$, $0 \leq \ell < 3$, there is a $F \in \mathscr{F}_{\ell+1}(\mathscr{T}_h)$ such that $f \subset \partial F$.

The generic term for the elements of $\mathscr{F}_\ell(\mathscr{T}_h)$ is ℓ-facets.

A special type of meshes are tetrahedral meshes, whose faces are (flat) triangles, whereas all cells are tetrahedra (Fig. 4). Another special case are tensor product meshes, for which the cells are axis aligned bricks and the faces are squares. Of course, all the meshes can be subject to global homeomorphisms of $\mathbb{R}^3$ and will remain valid meshes under this transformation.

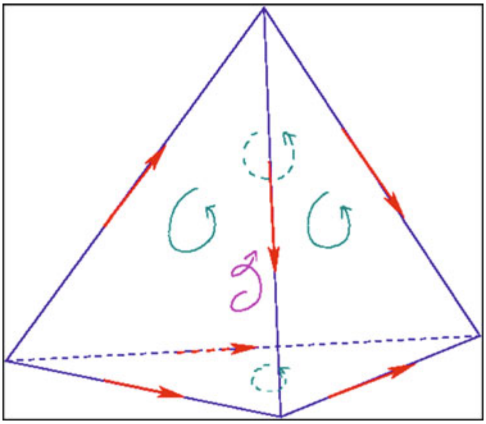

Fig. 4 Oriented tetrahedron: cell of a tetrahedral mesh. The orientation of the edges is given by their directions, the orientation of the face by "sense to turn on the tangential plane", the orientation of the tetrahedron by a corkscrew rule [19, Fig. 14.2]

3.2 Co-chains

3.2.1 Definition

Sloppily speaking, co-chains are discrete versions of integral forms [35, Sect. 3.1].

Definition 5 (Co-chain [35, Def. 4]) An ℓ-co-chain $\check{\omega}$, $\ell \in \{0, 1, 2, 3\}$, on a mesh $\mathscr{T}_h$ of Ω is a mapping $\check{\omega} : \mathscr{F}_\ell(\mathscr{T}_h) \to \mathbb{R}$.

The values an ℓ-co-chain assigns to ℓ-facets are sometimes called coefficients or degrees of freedom (d.o.f.). Figure 5 illustrates the phrase "the d.o.f. of an ℓ-co-chain are located on the ℓ-facets". Obviously, the ℓ-co-chains on a fixed mesh $\mathscr{T}_h$ form a vector space $\mathscr{C}^\ell(\mathscr{T}_h)$ with dimension

$$\dim \mathscr{C}^\ell(\mathscr{T}_h) = \sharp\mathscr{F}_\ell(\mathscr{T}_h) \ . \tag{40}$$

Thus, after ordering the ℓ-facets of $\mathscr{T}_h$, we can identify $\mathscr{C}^\ell(\mathscr{T}_h) \cong \mathbb{R}^{\sharp\mathscr{F}_\ell(\mathscr{T}_h)}$.

3.2.2 Co-chain Calculus

In Sects. 2.2.3 and 2.3.2 we learned about fundamental concepts in the calculus of (differential) forms, the trace and the exterior derivative. Those remain meaningful for co-chains. For instance, the *trace* of a co-chain $\check{\omega} \in \mathscr{C}^\ell(\mathscr{T}_h)$, $\ell \in \{0, 1, 2\}$, onto $\partial\Omega$ is just the restriction of $\check{\omega}$ to $\{f \in \mathscr{F}_\ell(\mathscr{T}_h) : f \subset \partial\Omega\}$.

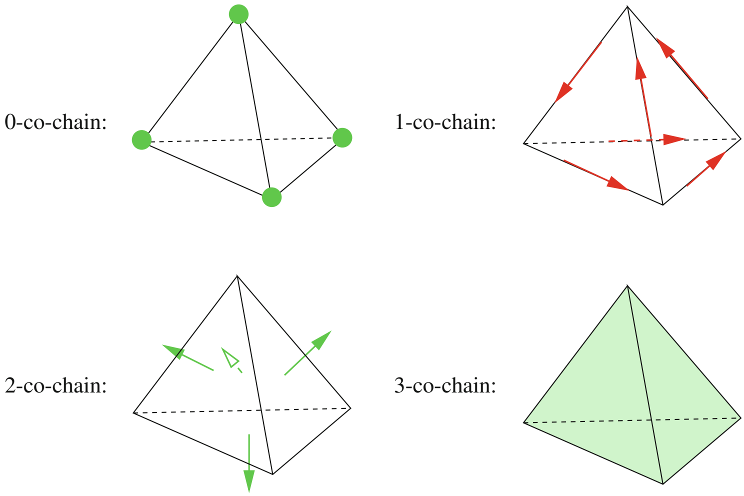

Fig. 5 "Locations" of degrees of freedoms for different co-chains

To define the *(discrete) exterior derivative* of co-chains, we need the notion of relative orientation of two facets $F \in \mathscr{F}_\ell(\mathscr{T}_h)$ and $f \in \mathscr{F}_{\ell-1}(\mathscr{T}_h)$, $0 < \ell \leq 3$:

$$\sigma_r(f,F) := \begin{cases} 1 & \text{, if } f \subset \partial F \text{ and orientations of } f \text{ and } \partial F \text{ match,} \\ -1 & \text{, if } f \subset \partial F \text{ and orientations of } f \text{ and } \partial F \text{ do not match,} \\ 0 & \text{, if } f \not\subset \partial F. \end{cases} \tag{41}$$

Definition 6 (Discrete Exterior Derivative) The *discrete exterior derivative* of co-chains is a mapping $\mathbf{d}_\ell : \mathscr{C}^\ell(\mathscr{T}_h) \to \mathscr{C}^{\ell+1}(\mathscr{T}_h)$, $0 \leq \ell < 3$, defined by

$$(\mathbf{d}_\ell\, \check{\omega})(F) := \sum_{f \in \mathscr{F}_\ell(\mathscr{T}_h)} \sigma_r(f,F)\, \check{\omega}(f) \;, \quad F \in \mathscr{F}_{\ell+1}(\mathscr{T}_h) \;.$$

Figure 6 illustrates the action of the discrete exterior derivatives on the d.o.f.s of co-chains. Obviously, $\mathbf{d}_\ell : \mathscr{C}^\ell(\mathscr{T}_h) \to \mathscr{C}^{\ell+1}(\mathscr{T}_h)$ is a linear operator. Thus, assuming an ordering of the facets, $\mathbf{d}_\ell$ can be represented by a matrix $\mathbf{D}_\ell \in \{-1,0,1\}^{N_{\ell+1},N_\ell}$, $N_j := \dim \mathscr{C}^j(\mathscr{T}_h) = \sharp\mathscr{F}_j(\mathscr{T}_h)$. These matrices are the so-called *incidence matrices* of the mesh [19, Sect. 14], see also Fig. 6.

A simple computation shows that the analogue of Theorem 2 holds for the discrete exterior derivative:

Theorem 7 $\mathbf{d}_{\ell+1} \circ \mathbf{d}_\ell = 0 \quad \Leftrightarrow \quad \mathbf{D}_{\ell+1}\mathbf{D}_\ell = 0, \quad \ell \in \{0,1,2\}.$

There is also a counterpart of Theorem 4 for co-chains [35, Thm. 3.1]:

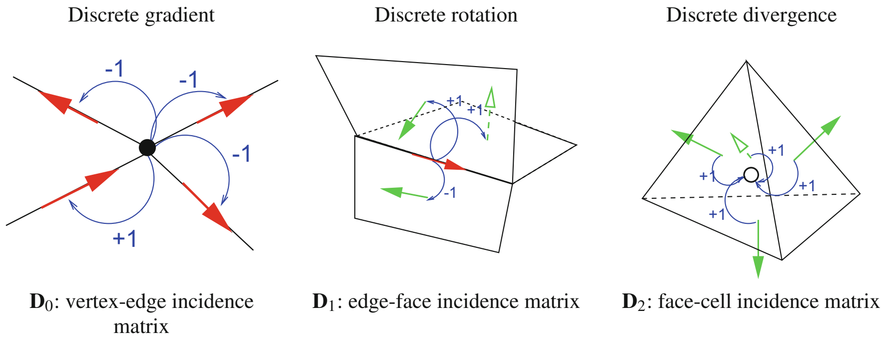

Fig. 6 Visualization of the action of the discrete exterior derivatives of co-chains on a 3D mesh through local stencils [35, Fig. 3.1]

Theorem 8 (Co-chain Potentials) *If Ω has trivial topology, then*

$$\mathrm{Ker}(\mathbf{d}_\ell) := \{\check{\omega} \in \mathscr{C}^\ell(\mathscr{T}_h) : \mathbf{d}_\ell\, \check{\omega} = 0\} = \mathbf{d}_{\ell-1}\, \mathscr{C}^{\ell-1}(\mathscr{T}_h) \;\Leftrightarrow\; \mathrm{Ker}\mathbf{D}_\ell = \mathrm{Im}\mathbf{D}_{\ell-1} .$$

3.3 Discrete Electrodynamic Laws

Since the co-chain calculus furnishes a counterpart of the exterior derivative, the topological electrodynamic laws (FL) and (AL) can be lifted to the discrete setting. To do so, we consider a mesh $\mathscr{T}_h$ of a bounded domain $\Omega \subset \mathbb{R}^3$ and introduce the *co-chain sampling operators* (also called de Rham maps)

$$\mathsf{S}_\ell : \begin{cases} \mathscr{F}^\ell(\Omega) \to \mathscr{C}^\ell(\mathscr{T}_h) \\ \quad \omega \quad \mapsto \left(\int_f \omega\right)_{f \in \mathscr{F}_\ell(\mathscr{T}_h)} \end{cases} , \quad \ell \in \{0, 1, 2, 3\} . \tag{42}$$

These operators evaluate integral forms for the special submanifolds provided by the mesh $\mathscr{T}_h$. Owing to the compatibility of the exterior derivative d_ℓ (Definition 3) and of its co-chain version $\mathbf{d}_\ell$ (Definition 6), there holds

$$\mathbf{d}_\ell \circ \mathsf{S}_\ell = \mathsf{S}_{\ell+1} \circ \mathsf{d}_\ell \quad \text{on} \quad \mathscr{F}^\ell(\Omega) . \tag{43}$$

This renders S_ℓ the perfect tool for "projecting" Maxwell's equations (FL), (AL) onto co-chains. To do this, let the (integral) forms $\mathbf{e}, \mathbf{b}, \mathbf{h}, \mathbf{d}, \mathbf{j}$ solve (FL) and (AL) on Ω. Then we may define their *co-chain interpolants*:

$$\check{\mathbf{e}} := \mathsf{S}_1\mathbf{e} , \quad \check{\mathbf{b}} := \mathsf{S}_2\mathbf{b} , \quad \check{\mathbf{h}} := \mathsf{S}_1\mathbf{h} , \quad \check{\mathbf{d}} := \mathsf{S}_2\mathbf{d} , \quad \check{\mathbf{j}} := \mathsf{S}_2\mathbf{j} .$$

From the integral forms (FL) and (AL) and (42) it is immediate that these co-chains fulfill:

$$\mathbf{D}_1\check{\mathbf{e}} = -\partial_t\check{\mathbf{b}}\ , \tag{44a}$$

$$\mathbf{D}_1\check{\mathbf{h}} = \partial_t\check{\mathbf{d}} + \check{\mathbf{j}}\ . \tag{44b}$$

Here, the co-chains are viewed as vectors of d.o.f.s and the discrete exterior derivatives have been replaced with the corresponding incidence matrices. Equation (44) may be regarded as circuit equations; for instance, (44a) is an electric network formed by the edges of the mesh with faces defining the loops. Sometimes the fact that the interpolants exactly satisfy the circuit equations is advertised as "perfect consistency of the co-chain model".

It goes without saying that Theorems 7 and 8 lead to co-chain versions of the continuity equation (4) and discrete electromagnetic potentials in $\mathscr{C}^1(\mathscr{T}_h)$ and $\mathscr{C}^0(\mathscr{T}_h)$, respectively, satisfying (10) on the co-chain side. Also for co-chain potentials we have gauge freedom similar to (11).

3.4 *Whitney Forms*

3.4.1 Whitney Map

Co-chain calculus could capture the topological electrodynamic laws, but cannot accommodate the material laws (18), which require square integrable differential form arguments defined almost everywhere in Ω. Thus we need an "interpolation" device for reconstructing (integral) forms $\in L^2\Lambda^\ell(\Omega)$ from co-chains $\in \mathscr{C}^\ell(\mathscr{T}_h)$. This will be accomplished by linear "extension operators"

$$\mathsf{W}_\ell : \mathscr{C}^\ell(\mathscr{T}_h) \to L^2\Lambda^\ell(\Omega)\ , \tag{45}$$

called *Whitney maps* in [47]. We dub its range the space of *Whitney ℓ-forms* and write

$$\mathscr{W}^\ell(\mathscr{T}_h) := \mathsf{W}_\ell(\mathscr{C}^\ell(\mathscr{T}_h))\ . \tag{46}$$

Before we delve into concrete constructions, we state a few fundamental algebraic properties of W_ℓ as guidelines [35, Sect. 3.2]

(W1) Extension property:

$$\mathsf{S}_\ell \circ \mathsf{W}_\ell = \mathsf{Id} \quad \text{on} \quad \mathscr{C}^\ell(\mathscr{T}_h)\ . \tag{47}$$

(If one interpolates an extended co-chain, the same co-chain is recovered.)

(W2) Compatibility with exterior derivatives:

$$\mathsf{d}_\ell \circ \mathsf{W}_\ell = \mathsf{W}_{\ell+1} \circ \mathbf{d}_\ell \quad \text{on} \quad \mathscr{C}^\ell(\mathscr{T}_h) \; . \tag{48}$$

(Extending the discrete exterior derivative of a co-chain yields the same as the exterior derivative of the extended co-chain.)

(W3) Locality: for all $T \in \mathscr{T}_h$ and $\check{\omega} \in \mathscr{C}^\ell(\mathscr{T}_h)$

$$\check{\omega}(f) = 0 \; \forall f \in \mathscr{F}_\ell(\mathscr{T}_h), f \subset \overline{T} \quad \Rightarrow \quad \mathsf{W}_\ell \check{\omega}|_T = 0 \; . \tag{49}$$

(If a co-chain is zero on all ℓ-facets contained in the boundary of a mesh cell, its extension must vanish on the whole cell.)

(W4) Polynomial:

$$\forall \check{\omega} \in \mathscr{C}^\ell(\mathscr{T}_h),\; T \in \mathscr{T}_h : \quad \begin{array}{l} \mathsf{W}_\ell \check{\omega}|_T \in C^\infty \Lambda^\ell(T) \quad \text{and} \\ \boldsymbol{x} \in T \mapsto \big((\mathsf{W}_\ell \check{\omega})(\boldsymbol{x})\big)(\boldsymbol{v}) \text{ affine linear } \forall \boldsymbol{v} \in \mathbb{R}^3 \; . \end{array} \tag{50}$$

(On each cell of the mesh the extended form is a valid smooth differential form according to Definition 1 with affine linear vector proxies.)

A projection, called the *nodal interpolation operator* is obtain by combining extension with sampling

$$\mathsf{I}_\ell : \mathscr{F}^\ell(\Omega) \to \mathscr{W}^\ell \quad , \quad \mathsf{I}_\ell := \mathsf{W}_\ell \circ \mathsf{S}_\ell \; . \tag{51}$$

The projection property is straightforward from (47). From (48) and (43) we infer that nodal interpolation meshes well with the exterior derivative.

Lemma 2 (Commuting Diagram Property for Nodal Interpolation)

$$\boxed{\mathsf{d}_\ell \circ \mathsf{I}_\ell = \mathsf{I}_{\ell+1} \circ \mathsf{d}_\ell} \quad \textit{on} \quad \mathscr{F}^\ell(\Omega) \; .$$

The locality property **(W3)** of the Whitney map implies strict locality of nodal interpolation.

Lemma 3 (Locality of Nodal Interpolation)

$$\forall \omega \in \mathscr{F}^\ell(\Omega),\; T \in \mathscr{T}_h : \quad \omega|_T = 0 \quad \Rightarrow \quad (\mathsf{I}_\ell \omega)|_T = 0 \; .$$

3.4.2 Local Construction of Simplicial Whitney Forms

We consider a simplicial mesh, whose cells are tetrahedra and single out a tetrahedron $T := \mathrm{convex}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$. On T we build concrete instances of Whitney maps W_ℓ, $0 \leq \ell < 3$, following an idea of Bossavit [19, Sect. 23].

❶ $\ell = 0$: We are given values $\check{\omega}(\boldsymbol{a}_j)$, $j = 1,2,3,4$, for a 0-co-chain in the vertices of T and seek to extend them linearly. Of course, this will boil down to standard linear interpolation, but we are going to view it from a different angle.

A point $\boldsymbol{x} \in T$ can be written as a "weighted combination of the vertices":

$$\boldsymbol{x} = \sum_{j=1}^{4} \lambda_j(\boldsymbol{x}) \ , \tag{52}$$

where the functions $\lambda_j : T \to [0,1]$ are the barycentric coordinates of T: $\lambda_j(\boldsymbol{a}_k) = \delta_{jk}$, δ the Kronecker symbol [5, Sect. 4.1]. Inspired by this formula, we express $\mathsf{W}_0\check{\omega}$ as a corresponding linear combination of vertex values:

$$(\mathsf{W}_0\check{\omega})(\boldsymbol{x}) := \sum_{j=1}^{4} \lambda_j(\boldsymbol{x})\check{\omega}(\boldsymbol{a}_j) \ , \tag{53}$$

which results in plain old linear interpolation.

❷ $\ell = 1$: Now we are given edge values $\check{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j])$, where $[\boldsymbol{a}_i, \boldsymbol{a}_j]$ stands for the edge connecting $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. Adapting the idea from $\ell = 0$, in analogy to (52), we write an arbitrary line segment $[\boldsymbol{x}, \boldsymbol{y}] \subset T$ as a "weighted sum of edges of T", see Fig. 7:

$$\begin{aligned}
[\boldsymbol{x},\boldsymbol{y}] &= \{t\boldsymbol{x} + (1-t)\boldsymbol{y} \ ; \ 0 \leq t \leq 1\} \\
&= \left\{ \sum_i \left(t\lambda_i(\boldsymbol{x}) + (1-t)\lambda_i(\boldsymbol{y})\right) \boldsymbol{a}_i \ ; \ 0 \leq t \leq 1 \right\} \\
&= \left\{ \sum_i \left(t \sum_j \lambda_j(\boldsymbol{y})\lambda_i(\boldsymbol{x}) + (1-t) \sum_j \lambda_j(\boldsymbol{x})\lambda_i(\boldsymbol{y}) \right) \boldsymbol{a}_i \ ; \ 0 \leq t \leq 1 \right\} \\
&= \left\{ \sum_i \sum_j \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y}) \left(t\boldsymbol{a}_i + (1-t)\boldsymbol{a}_j\right) \ ; \ 0 \leq t \leq 1 \right\} \ .
\end{aligned} \tag{54}$$

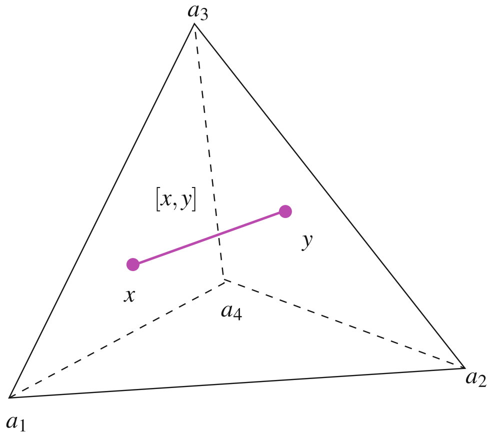

Fig. 7 A line segment $[\boldsymbol{x}, \boldsymbol{y}]$ inside the tetrahedron T. The set equation (54) writes it as a weighted sum of edges of T

Taking the cue from (53) this suggests the definition

$$\begin{aligned}\int_{[\mathbf{x},\mathbf{y}]} \mathsf{W}_1\breve{\omega} &:= \sum_i \sum_j \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\breve{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j]) \\ &:= \sum_{i<j} \big(\lambda_i(\mathbf{x})\lambda_j(\boldsymbol{y}) - \lambda_i(\mathbf{y})\lambda_j(\mathbf{x})\big)\, \breve{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j]) \;.\end{aligned} \tag{55}$$

❸ $\ell = 2$: The 2-co-chain $\breve{\omega}$ is determined on T by the values $\breve{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k])$ it assigns to the faces of T, here designated by a triple of vertices. Similar to (54) we may write an oriented triangle $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ as a combination of faces of T weighted with products barycentric coordinate functions. The formula, which we skip here, suggests the definition

$$\begin{aligned}\int_{[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \mathsf{W}_2\breve{\omega} &:= \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\lambda_k(\boldsymbol{z})\breve{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k]) \\ &:= \sum_{i<j<k} \big(\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\lambda_k(\boldsymbol{z}) + \lambda_k(\boldsymbol{x})\lambda_i(\boldsymbol{y})\lambda_j(\boldsymbol{z}) \\ &\qquad + \lambda_j(\boldsymbol{x})\lambda_k(\boldsymbol{y})\lambda_i(\boldsymbol{z})\big)\, \breve{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k])\end{aligned} \tag{56}$$

The forms obtained through (55) and (56) are clearly smooth and by Formula (1) we can recover the associated differential forms according to Definition 1. For instance, from (55) we obtain for $\boldsymbol{x} \in T$ and all $\boldsymbol{v} \in \mathbb{R}^3$

$$(\mathsf{W}_1\breve{\omega})(\boldsymbol{x})(\boldsymbol{v}) = \lim_{t\to 0} \frac{1}{t} \int_{[\boldsymbol{x},\boldsymbol{x}+t\boldsymbol{v}]} \mathsf{W}_1\breve{\omega}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \sum_{i<j} (\lambda_i(\boldsymbol{x}) \frac{\lambda_j(\boldsymbol{x}+t\boldsymbol{v}) - \lambda_j(\boldsymbol{x})}{t} \\
&\quad - \lambda_j(\boldsymbol{x}) \frac{\lambda_i(\boldsymbol{x}+t\boldsymbol{v}) - \lambda_i(\boldsymbol{x})}{t}) \check{\omega}([\boldsymbol{a}_i,\boldsymbol{a}_j]) \\
&= \sum_{i<j} (\lambda_i(\boldsymbol{x})\, \mathsf{d}_0\, \lambda_j(\boldsymbol{x})(\boldsymbol{v}) - \lambda_j(\boldsymbol{x})\, \mathsf{d}_0\, \lambda_i(\boldsymbol{x})(\boldsymbol{v})) \check{\omega}([\boldsymbol{a}_i,\boldsymbol{a}_j]) \ .
\end{aligned}
$$

The 1-forms $\beta^1_{i,j} := \lambda_i\, \mathsf{d}_0\, \lambda_j - \lambda_j\, \mathsf{d}_0\, \lambda_i$ play the role of "local shape functions" or "local basis forms". They arise from extending a "unit 1-co-chain" and their Euclidean vector proxies read, see (5) and (15),

$$
\mathrm{V.\,P.}(\beta^1_{i,j}) = \lambda_i\, \mathbf{grad}\, \lambda_j - \lambda_j\, \mathbf{grad}\, \lambda_i \ . \tag{57}
$$

The same manipulation succeeds for (56) and yields for $\boldsymbol{x} \in T$ and any two vectors $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^3$

$$
\begin{aligned}
(\mathsf{W}_2\check{\omega})(\boldsymbol{x})(\boldsymbol{v},\boldsymbol{w}) &= \lim_{t\to 0} \frac{2}{t^2} \int_{[\boldsymbol{x},\boldsymbol{x}+t\boldsymbol{v},\boldsymbol{x}+t\boldsymbol{w}]} \mathsf{W}_2\check{\omega} \\
&= 2 \sum_{i<j<k} \Big(\lambda_i(\boldsymbol{x}) \big(\mathsf{d}_0\, \lambda_j(\boldsymbol{x}) \wedge \mathsf{d}_0\, \lambda_k(\boldsymbol{x})\big)(\boldsymbol{v},\boldsymbol{w}) \\
&\quad - \lambda_j(\boldsymbol{x}) \big(\mathsf{d}_0\, \lambda_i(\boldsymbol{x}) \wedge \mathsf{d}_0\, \lambda_k(\boldsymbol{x})\big)(\boldsymbol{v},\boldsymbol{w}) \\
&\quad + \lambda_k(\boldsymbol{x}) \big(\mathsf{d}_0\, \lambda_i(\boldsymbol{x}) \wedge \mathsf{d}_0\, \lambda_j(\boldsymbol{x})\big)(\boldsymbol{v},\boldsymbol{w}) \Big) \check{\omega}([\boldsymbol{a}_i,\boldsymbol{a}_j,\boldsymbol{a}_k]) \ .
\end{aligned}
$$

We can read off the local basis 2-forms

$$
\beta^2_{i,j,k} := \lambda_i \big(\mathsf{d}_0\, \lambda_j \wedge \mathsf{d}_0\, \lambda_k\big) - \lambda_j \big(\mathsf{d}_0\, \lambda_i \wedge \mathsf{d}_0\, \lambda_k\big) + \lambda_k \big(\mathsf{d}_0\, \lambda_i \wedge \mathsf{d}_0\, \lambda_j\big) \ ,
$$

whose vector proxies are

$$
\begin{aligned}
\mathrm{V.\,P.}(\beta^2_{i,j,k}) &= \lambda_i\, \mathbf{grad}\, \lambda_j \times \mathbf{grad}\, \lambda_k + \lambda_j\, \mathbf{grad}\, \lambda_k \\
&\quad \times \mathbf{grad}\, \lambda_i + \lambda_k\, \mathbf{grad}\, \lambda_i \times \mathbf{grad}\, \lambda_j \ .
\end{aligned} \tag{58}
$$

From the vector proxies of the local basis forms we get alternative vector analytic representations of the spaces spanned by them, see Table 3. We point out that all the local spaces contain the constant functions and all the vector proxies are linear functions, as we demanded in property **(W4)**.

Table 3 also gives the linear functionals underlying the sampling operators S_ℓ, the so-called "local degrees of freedom". Their form in vector proxy notation can be deduced from (2).

Table 3 Vector analytic (vector proxy) formulas for the local spaces on a tetrahedron spanned by the local basis forms

Degree	Local spaces	Local d.o.f.
$\ell = 0$	$\mathscr{W}^0(T) = \{\boldsymbol{x} \mapsto \boldsymbol{a} \cdot \boldsymbol{x} + \beta,\ \boldsymbol{a} \in \mathbb{R}^3, \beta \in \mathbb{R}\}$	$\vec{u} \mapsto \vec{u}(\boldsymbol{a}_i)$
$\ell = 1$	$\mathscr{W}^1(T) = \{\boldsymbol{x} \mapsto \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b},\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$	$\vec{\mathbf{u}} \mapsto \int_{[\boldsymbol{a}_i, \boldsymbol{a}_j]} \vec{\mathbf{u}} \cdot \mathrm{d}\mathbf{s}$
$\ell = 2$	$\mathscr{W}^2(T) = \{\boldsymbol{x} \mapsto \alpha \boldsymbol{x} + \boldsymbol{b},\ \alpha \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\}$	$\vec{\mathbf{u}} \mapsto \int_{[\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k]} \vec{\mathbf{u}} \cdot \boldsymbol{n}\, \mathrm{d}S$
$\ell = 3$	$\mathscr{W}^3(T) = \{\boldsymbol{x} \mapsto \alpha,\ \alpha \in \mathbb{R}\}$	$\vec{u} \mapsto \int_T \vec{u}\, \mathrm{d}\mathbf{x}$

Remark 10 This procedure can even be generalized to ℓ-co-chains in arbitrary dimension n and an n-simplex $T \subset \mathbb{R}^n$, $n \in \mathbb{N}$:

$$(\mathsf{W}_\ell \breve{\omega})(\mathbf{x}) = \sum_I \sum_{j=0}^{\ell} (-1)^j \Big(\underbrace{\lambda_{i_j}\, \mathsf{d}_0 \lambda_{i_0} \wedge \ldots \wedge \cancel{\mathsf{d}_0 \lambda_{i_j}} \wedge \ldots \wedge \mathsf{d}_0 \lambda_{i_\ell}}_{\beta_I^\ell} \Big) \cdot \breve{\omega}([\boldsymbol{a}_I])$$

where $I = (i_0, \ldots, i_\ell)$, $0 \le l \le n$, runs through all $\ell+1$-subsets of $\{0, \ldots, n\}$ and the ordering is induced by the orientation of the corresponding ℓ-facet $[\mathbf{a}_{i_0}, \ldots, \mathbf{a}_{i_\ell}]$. The symbol $\breve{\omega}([\boldsymbol{a}_I])$ stands for value assigned by the ℓ-co-chain coefficient associated with that facet. Of course, the β_I^ℓ can be regarded as local basis forms.

3.4.3 Local Commuting Diagram Property

Let us examine the commuting property (48) for the extensions defined by (53), (55), and (56). We do this locally on a tetrahedron T. First, for $\ell = 0$, given a 0-co-chain $\breve{\omega} \in \mathscr{C}^0(\mathscr{T}_h)$ and $\boldsymbol{x}, \boldsymbol{y} \in T$, we find, thanks to $\sum_{j=1}^4 \lambda_j \equiv 1$,

$$\begin{aligned}
\int_{[\boldsymbol{x},\boldsymbol{y}]} \mathsf{W}_1(\mathbf{d}_0\, \breve{\omega}) &= \sum_{i=1}^4 \sum_{j=1}^4 \lambda_i(\boldsymbol{x}) \lambda_j(\boldsymbol{y}) (\mathbf{d}_0\, \breve{\omega})([\boldsymbol{a}_i, \boldsymbol{a}_j]) \\
&= \sum_{i=1}^4 \sum_{j=1}^4 \lambda_i(\boldsymbol{x}) \lambda_j(\boldsymbol{y}) \big(\breve{\omega}(\boldsymbol{a}_j) - \breve{\omega}(\boldsymbol{a}_i) \big) \\
&= \sum_{j=1}^4 \lambda_j(\boldsymbol{y}) \breve{\omega}(\boldsymbol{a}_j) - \sum_{i=1}^4 \lambda_i(\boldsymbol{x}) \breve{\omega}(\boldsymbol{a}_i) \\
&= (\mathsf{W}_0(\breve{\omega}))(\boldsymbol{y}) - (\mathsf{W}_0(\breve{\omega}))(\boldsymbol{x}) = \int_{\partial[\boldsymbol{x},\boldsymbol{y}]} \mathsf{W}_0 \breve{\omega} = \int_{[\boldsymbol{x},\boldsymbol{y}]} \mathsf{d}_0\, \mathsf{W}_0 \breve{\omega}\ ,
\end{aligned}$$

by definition of the exterior derivative. This amounts to (48) for $\ell = 0$, because $\boldsymbol{x}$ and $\boldsymbol{y}$ have been arbitrary.

In the case $\ell = 1$ we proceed along similar lines and pick $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in T$, write $\triangle := [\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ and get on T for $\check{\omega} \in \mathscr{C}^2(\mathscr{T}_h)$

$$\begin{aligned}
\int_{\triangle} \mathsf{W}_2(\mathbf{d}_1\, \check{\omega}) &= \sum_{i,j,k} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\lambda_k(\boldsymbol{z})\big(\mathbf{d}_1(\check{\omega})\big)([\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k]) \\
&= \sum_{i,j,k} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\lambda_k(\boldsymbol{z})\, \big(\check{\omega}([\boldsymbol{a}_i.\boldsymbol{a}_j]) + \check{\omega}([\boldsymbol{a}_j.\boldsymbol{a}_k]) + \check{\omega}([\boldsymbol{a}_k.\boldsymbol{a}_i])\big) \\
&= \sum_{i,j} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\check{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j]) + \sum_{j,k} \lambda_j(\boldsymbol{y})\lambda_k(\boldsymbol{z})\check{\omega}([\boldsymbol{a}_j, \boldsymbol{a}_k]) \\
&\quad + \sum_{k,i} \lambda_k(\boldsymbol{z})\lambda_i(\boldsymbol{x})\check{\omega}([\boldsymbol{a}_k, \boldsymbol{a}_i]) \\
&= \int_{[\boldsymbol{x},\boldsymbol{y}]} \mathsf{W}_1\check{\omega} + \int_{[\boldsymbol{y},\boldsymbol{z}]} \mathsf{W}_1\check{\omega} + \int_{[\boldsymbol{z},\boldsymbol{x}]} \mathsf{W}_1\check{\omega} = \int_{\partial[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \mathsf{W}_1\check{\omega} = \int_{[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \mathsf{d}_1\, \mathsf{W}_1\check{\omega}\ .
\end{aligned}$$

Since this holds for any triangle inside T we conclude (48) for $\ell = 1$. Of course, when adopting the above construction of W_ℓ in any dimension, (48) will always hold.

3.4.4 Global Whitney Forms

Thus far, the construction of W_ℓ has been utterly local. We aim for a integral form on the entire domain Ω, however. According to Lemma 1 we have to verify that the traces of the local co-chain extensions agree on both sides of all faces of the mesh. This is ensured, once we can demonstrate that the trace of W_ℓ, $0 \le \ell < 3$, onto a face $f \in \mathscr{F}_2(\mathscr{T}_h)$ depends only on the (unique) co-chain coefficients associated with that face.

To discuss this for $\ell = 1$ we pick a tetrahedron $T = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$ and, without loss of generality, the face $f = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$. For any $\boldsymbol{x}, \boldsymbol{y} \in f$ the construction gives

$$\begin{aligned}
\sum_{[\boldsymbol{x},\boldsymbol{y}]} \mathsf{W}_1(\check{\omega}) &= \sum_{i=1}^{4}\sum_{j=1}^{4} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\, \check{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j]) \\
&= \sum_{i:\boldsymbol{a}_i\in f}\ \sum_{j:\boldsymbol{a}_j\in f} \lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{y})\, \check{\omega}([\boldsymbol{a}_i, \boldsymbol{a}_j])\ ,
\end{aligned}$$

because barycentric coordinate functions not belonging to vertices of f vanish on that face. Hence $\mathbf{t}_f\mathsf{W}_1\check{\omega}$ depends only on $\check{\omega}|_f$ and will be independent of the adjacent tetrahedron on which we have built the Whitney map (Fig. 8).

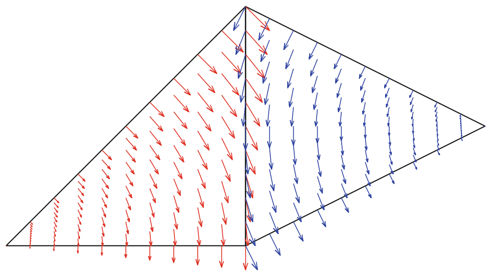

Fig. 8 The compatibility of local extensions of 1-co-chains in 2D by means of the Whitney map. The underlying co-chain had coefficient 1 for the vertical edge, 0 on all other edges. The tangential continuity of the vector proxies hints at the agreement of traces of the local extensions

Summing up, the local constructions introduced in the previous section define integral forms on Ω and, thus, spaces $\mathscr{W}^\ell(\mathscr{T}_h) \subset L^2\Lambda^\ell(\Omega)$ of Whitney ℓ-forms on the tetrahedral mesh $\mathscr{T}_h$. By construction, the elements of $\mathscr{W}^\ell(\mathscr{T}_h)$ are valid integral forms, because their integrals make sense for any ℓ-dimensional oriented sub-manifold of Ω.

3.4.5 Affine Equivalence

There is a unique *affine* map $\boldsymbol{\Phi}$ between any two non-degenerate tetrahedra T and $\hat{T}$ such that $T = \boldsymbol{\Phi}(\hat{T})$. To begin with, since the pullback operator $\boldsymbol{\Phi}^*$ from Definition 2 commutes with both the exterior derivative and the wedge product, we readily infer from the formulas for the local basis forms given in Sect. 3.4.2 that those are mapped onto each other under pullback

$$\hat{\beta}^\ell_{\hat{f}} = \boldsymbol{\Phi}^*\beta^\ell_f \ , \quad f = \boldsymbol{\Phi}(\hat{f}) \ .$$

Here $\hat{\beta}^\ell_{\hat{f}}$ is the basis form on $\hat{T}$ associated with the ℓ-dimensional facet $\hat{f}$ of $\hat{T}$, and β^ℓ_f a basis form on T belonging to the ℓ-facet f of T. This is a key property, known as "affine equivalence" in the theory of finite elements [24, Sect. 2.3]. As a consequence the pullback also transforms local spaces of Whitney forms into each other.

Remark 11 Invariance under pullback paves the way for defining parametric Whitney forms, *cf.* [24, Sect. 4.3]. If the cells of a mesh are obtained as diffeomorphic images of a single simplex, inverse pullbacks of Whitney forms on that simplex supply the building blocks for piecewise smooth integral forms with facet integrals as degrees of freedom.

3.4.6 General Discrete Differential Forms

The Whitney forms introduced above are just the simplest (lowest order) representatives of families of discrete differential forms on meshes that comprise members of any local polynomial degree, the *higher order* discrete differential forms.

There is a unified way to obtain generalizations of Whitney forms on simplicial meshes in any dimension and of arbitrarily high polynomial degree. This was pioneered in [32] and fully elaborated in [5, Sect. 4], using a device from cohomology theory, the Koszul lifting. Possible degrees of freedom for these discrete differential forms are weighted traces on suitable facets of the mesh ("moments"), but there might be more "geometric" choices, see [19, Sect. 25] and [43]. Yet, the principal concern when choosing local basis functions for higher order discrete differential are computational aspects like ease of evaluation [2], conditioning of the resulting linear systems [1], or separation of functions in the kernel of d_ℓ [34, 48].

Counterparts of Whitney forms have been found for tensor product meshes, see [6] for their construction, and even on hybrid meshes comprising tetrahedra, hexahedra, and prisms [9, 10, 41]. Suitable higher order extensions are also described in these articles.

3.5 *Commuting Projections*

3.5.1 Nodal Interpolation

The nodal interpolation operators I_ℓ from (51) for Whitney forms satisfy the very special properties of perfect locality (Lemma 3) and that they commute with the exterior derivative (Lemma 2). These extraordinary features are somewhat marred by the fact that, in 3D,

> except for $\ell = 3$, the nodal interpolation operators I_ℓ are not bounded on the energy space $H\Lambda^\ell(\Omega)$, as defined in (31).

This is well known for $\ell = 0$: the point sampling operator S_0 is not bounded on the standard Sobolev space $H^1(\Omega)$, because there is no continuous embedding of $H^1(\Omega)$ into $C^0(\overline{\Omega})$, as $H^1(\Omega)$ contains unbounded functions. In 3D counterexamples can easily be constructed for $\ell = 1, 2$, too. Even worse, for $\ell = 1$ the operator I_1 fails to be bounded even on the space of vectorfields with components in $H^1(\Omega)$, though the norm of this space is clearly stronger than that of $\boldsymbol{H}(\mathbf{curl}, \Omega)$.

This flaw thwarts interpolation error estimates of the form $\left\| \vec{\mathbf{u}} - \mathsf{I}_1 \vec{\mathbf{u}} \right\|_{L^2(\Omega)} \leq Ch \left\| \vec{\mathbf{u}} \right\|_{H^1(\Omega)}$ with a constant independent of $\vec{\mathbf{u}}$ (h is the mesh width of $\mathscr{T}_h$). We have to settle for estimates like [35, Thm. 3.14]

$$\left\| \vec{\mathbf{u}} - \mathsf{I}_1 \vec{\mathbf{u}} \right\|_{L^2(\Omega)} \leq Ch \left(\left\| \vec{\mathbf{u}} \right\|_{H^1(\Omega)} + \left\| \mathbf{curl}\, \vec{\mathbf{u}} \right\|_{H^1(\Omega)} \right) \quad \forall \vec{\mathbf{u}} \text{ sufficiently smooth}\,, \tag{59}$$

where $C > 0$ depends only on the shape regularity of the mesh. (The notion of shape regularity of a mesh is presented, e.g., in [35, Sect. 3.6] following [24, Sect. 3.1].) Yet, for many purposes in numerical analysis (59) is not sufficient.

Fortunately, there is a very special interpolation error estimate for $\ell = 1$ that often comes handy [35, Lemma 4.6]:

Lemma 4 *The interpolation operator* $\mathsf{I}_1 : C^\infty(\Lambda^1(\Omega)) \to \mathscr{W}^1$ *can be extended to a bounded operator on* $\{\vec{\mathbf{u}} \in (H^1(\Omega))^3 : \mathbf{curl}\, \vec{\mathbf{u}} \in \mathscr{W}^2(\mathscr{T}_h)\}$ *(a space of vectorfields with components in* $H^1(\Omega)$*, whose* **curl***s are piecewise constant) and satisfies*

$$\left\| \vec{\mathbf{u}} - \mathsf{I}_1 \vec{\mathbf{u}} \right\|_{L^2(\Omega)} \leq Ch \left| \vec{\mathbf{u}} \right|_{H^1(\Omega)} \quad \forall \vec{\mathbf{u}} \in (H^1(\Omega))^3,\ \mathbf{curl}\, \vec{\mathbf{u}} \in \mathscr{W}^2(\mathscr{T}_h)\,,$$

with a constant $C > 0$ *depending only on the shape regularity of the mesh* $\mathscr{T}_h$.

Proof Pick one tetrahedron $T \in \mathscr{T}_h$ and, without loss of generality, assume $0 \in T$. Then define the lifting operator, *cf.* the "Koszul lifting" [5, Sect. 3.2],

$$\vec{\mathbf{w}} \mapsto \mathsf{K}\vec{\mathbf{w}}\,, \quad \mathsf{K}\vec{\mathbf{w}}(\boldsymbol{x}) := \tfrac{1}{3}\vec{\mathbf{w}}(\boldsymbol{x}) \times \boldsymbol{x}\,, \quad \boldsymbol{x} \in T\,. \tag{60}$$

Elementary calculations reveal that for any constant vectorfield $\vec{\mathbf{w}} \equiv$ const.

$$\mathbf{curl}\mathsf{K}\vec{\mathbf{w}} = \vec{\mathbf{w}}\,, \tag{61}$$

$$\left\| \mathsf{K}\vec{\mathbf{w}} \right\|_{L^2(T)} \leq h_T \left\| \vec{\mathbf{w}} \right\|_{L^2(T)}\,, \tag{62}$$

$$\mathsf{K}\vec{\mathbf{w}} \in \mathscr{W}^1(T)\,. \tag{63}$$

Here h_T is the size of T. The continuity (62) permits us to extend K to $(L^2(T))^3$.

Given $\vec{\mathbf{u}} \in (H^1(T))^3$ with $\mathbf{curl}\, \vec{\mathbf{u}} \equiv \text{const}^3$, by (63) we know that $\mathsf{K}\mathbf{curl}\, \vec{\mathbf{u}}$ is a linear function. Thus, an inverse inequality yields

$$\left| \mathsf{K}\mathbf{curl}\, \vec{\mathbf{u}} \right|_{H^1(T)} \leq Ch_T^{-1} \left\| \mathsf{K}\mathbf{curl}\, \vec{\mathbf{u}} \right\|_{L^2(T)} \overset{(62)}{\leq} C \left\| \mathbf{curl}\, \vec{\mathbf{u}} \right\|_{L^2(T)}\,, \tag{64}$$

with $C > 0$ depending only on shape regularity of T. Next, by (61) and the existence of a scalar potential, see (8),

$$\mathbf{curl}(\vec{\mathbf{u}} - \mathsf{K}\mathbf{curl}\, \vec{\mathbf{u}}) = 0 \quad \Rightarrow \quad \exists \vec{p} \in H^1(T) : \quad \vec{\mathbf{u}} - \mathsf{K}\mathbf{curl}\, \vec{\mathbf{u}} = \mathbf{grad}\, \vec{p}\,. \tag{65}$$

From (64) we conclude that $\vec{p} \in H^2(T)$ and $|\vec{p}|_{H^2(T)} \leq C\left|\vec{\mathbf{u}}\right|_{H^1(T)}$. Moreover, thanks to the commuting diagram property we have

$$\vec{\mathbf{u}} - \mathsf{I}_1\vec{\mathbf{u}} = \underbrace{\mathsf{K}\mathbf{curl}\,\vec{\mathbf{u}} - \mathsf{I}_1\mathsf{K}\mathbf{curl}\,\vec{\mathbf{u}}}_{=0 \text{ by } (63)} + \mathbf{grad}(\vec{p} - \mathsf{I}_0\vec{p}) \ . \tag{66}$$

Next, recall that I_0 agrees with standard linear interpolation on a tetrahedron. That is bounded on $H^2(T)$ and its interpolation error satisfies $|\vec{p} - \mathsf{I}_0\vec{p}|_{H^1(T)} \leq Ch_T|\vec{p}|_{H^2(T)}$. Thus, we arrive at

$$\left\|\vec{\mathbf{u}} - \mathsf{I}_1\vec{\mathbf{u}}\right\|_{L^2(T)} = |\vec{p} - \mathsf{I}_0\vec{p}|_{H^1(T)} \leq Ch_T|\vec{p}|_{H^2(T)} \leq Ch_T\left|\vec{\mathbf{u}}\right|_{H^1(T)} \ .$$

Summation over all tetrahedra of the mesh finishes the proof.

3.5.2 Decomposition Based Projections

If boundedness on $H\Lambda^\ell(\Omega)$ (defined in (31)) and the commuting diagram property matter most and one can dispense with locality (Lemma 3), there is a simple replacement for nodal interpolation. We review its construction for a bounded domain $\Omega \subset \mathbb{R}^3$ with trivial topology, *cf.* Theorem 4, equipped with a tetrahedral mesh $\mathscr{T}_h$. As tools we use

1. the $L^2\Lambda^\ell(\Omega)$-orthogonal *Helmholtz decomposition* [40, Sect. 3.7]:

$$L^2\Lambda^\ell(\Omega) = \underbrace{\mathsf{d}_{\ell-1}\, H\Lambda^{\ell-1}(\Omega)}_{=\mathrm{Ker}(\mathsf{d}_\ell)\cap\mathscr{W}^\ell(\mathscr{T}_h)} \oplus \mathscr{X}^\ell(\Omega) \ . \tag{67}$$

   For $\ell = 0$, the first space should be replaced by the set of constant functions. On the complement of $\mathrm{Ker}(\mathsf{d}_\ell)$ there holds, *cf.* [35, Cor. 4.4],

$$\|\omega\|_{L^2\Lambda^\ell(\Omega)} \leq C\|\mathsf{d}_\ell\,\omega\|_{L^2\Lambda^{\ell+1}(\Omega)} \quad \forall \omega \in \mathscr{X}^\ell(\Omega) \ , \tag{68}$$

   with constants $C > 0$ depending only on Ω.
2. the $L^2\Lambda^\ell(\Omega)$-orthogonal *discrete Helmholtz decomposition*

$$\mathscr{W}^\ell(\mathscr{T}_h) = \mathsf{d}_{\ell-1}\,\mathscr{W}^{\ell-1}(\mathscr{T}_h) \oplus \mathscr{X}^\ell(\mathscr{T}_h) \ , \tag{69}$$

   where the first space coincides with the kernel of d_ℓ in $\mathscr{W}^\ell$ for $\ell > 0$, and, again, is the constant functions for $\ell = 0$.

The estimate (68) remains true for Whitney forms. As a tool the proof uses Lemma 4 and so-called regular decompositions that will be introduced later in Sect. 4.4, p. 45.

Lemma 5 (Discrete Friedrichs Inequality [35, Thm. 4.7]) *With a constant depending only on Ω and the shape regularity of the mesh $\mathscr{T}_h$*

$$\|\omega_h\|_{L^2\Lambda^\ell(\Omega)} \le C\|\mathsf{d}_\ell\, \omega_h\|_{L^2\Lambda^{\ell+1}(\Omega)} \quad \forall \omega_h \in \mathscr{X}^\ell(\mathscr{T}_h)\ .$$

Next, we introduce a lifting operator $\mathsf{L}_\ell : H\Lambda^\ell(\Omega) \to \mathscr{X}^{\ell-1}(\mathscr{T}_h)$ by

$$(\mathsf{d}_{\ell-1}\,\mathsf{L}_\ell\omega - \omega, \mathsf{d}_{\ell-1}\,\eta_h)_{L^2\Lambda^\ell(\Omega)} = 0 \quad \forall \eta_h \in \mathscr{X}^{\ell-1}(\mathscr{T}_h)\ . \tag{70}$$

Since the kernel of $\mathsf{d}_{\ell-1}$ has been removed from $\mathscr{X}^{\ell-1}(\mathscr{T}_h)$, this is a valid definition. It is the key ingredient in

$$\mathsf{P}_\ell := \mathsf{d}_{\ell-1} \circ \mathsf{L}_\ell + \mathsf{L}_{\ell+1} \circ \mathsf{d}_\ell\ . \tag{71}$$

Lemma 6 ("Helmholtz Projection") *The linear operator P_ℓ according to (71) is a bounded projector $H\Lambda^\ell(\Omega) \to \mathscr{W}^\ell(\mathscr{T}_h)$ and commutes with the exterior derivative.*

Proof To see that ${\mathsf{P}_\ell}^2 = \mathsf{P}_\ell$ note that $\mathsf{d}_{\ell-1}(\mathsf{L}_\ell\omega_h) = \omega_h$ for all $\omega_h \in \mathrm{Ker}(\mathsf{d}_\ell) \cap \mathscr{W}^\ell(\mathscr{T}_h)$ and $\mathsf{L}_{\ell+1}\,\mathsf{d}_\ell\,\omega_h = \omega_h$ for all $\omega_h \in \mathscr{X}^\ell(\mathscr{T}_h)$.

Clearly, $\|\mathsf{d}_{\ell-1}\,\mathsf{L}_\ell\omega\|_{L^2\Lambda^\ell(\Omega)} \le \|\omega\|_{L^2\Lambda^\ell(\Omega)}$ for every $\omega \in L^2\Lambda^\ell(\Omega)$. Then, by virtue of Lemma 5, $\mathsf{L}_\ell : H\Lambda^\ell(\Omega) \to H\Lambda^{\ell-1}(\Omega)$ is bounded. The boundedness of $\mathsf{P}_\ell : H\Lambda^\ell(\Omega) \to H\Lambda^\ell(\Omega)$ is an immediate consequence.

The commuting diagram property follows from $\mathsf{d}_\ell \circ \mathsf{d}_{\ell-1} = 0$:

$$\mathsf{d}_\ell \circ \mathsf{P}_\ell = \mathsf{d}_\ell \circ \mathsf{L}_{\ell+1} \circ \mathsf{d}_\ell = (\mathsf{d}_\ell \circ \mathsf{L}_{\ell+1} + \mathsf{L}_{\ell+2} \circ \mathsf{d}_{\ell+1}) \circ \mathsf{d}_\ell = \mathsf{P}_{\ell+1} \circ \mathsf{d}_\ell\ .$$

Evidently, both Helmholtz decompositions have a distinctly non-local character, because they both rely on the $L^2\Lambda^\ell(\Omega)$ inner product on Ω. Thus, P_ℓ cannot be local in the sense that $\mathsf{P}_\ell\omega|_T$ for $T \in \mathscr{T}_h$ depends only on ω restricted to a neighborhood of T.

3.5.3 Local Quasi-Interpolation

The first to achieve a breakthrough was Schöberl in [46], a manuscript that was published only as a technical report. He was inspired by the well-known so-called quasi-interpolation operator, see [42, Sect. 2.1.1]

$$\mathsf{Q}_0 : \begin{cases} L^2\Lambda^0(\Omega) \to \mathscr{W}^0(\mathscr{T}_h)\ , \\ \quad \omega \quad \mapsto \sum\limits_{\boldsymbol{p}\in\mathscr{F}_0(\mathscr{T}_h)} \int_{T_{\boldsymbol{p}}} w_{\boldsymbol{p}}(\boldsymbol{x})\omega(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \cdot \beta^0_{\boldsymbol{p}}\ , \end{cases} \tag{72}$$

where $T_{\boldsymbol{p}} \in \mathscr{F}_3(\mathscr{T}_h)$ is a cell abutting the vertex $\boldsymbol{p}$, and $w_{\boldsymbol{p}} \in L^\infty(\Omega)$ is a function supported on $T_{\boldsymbol{p}}$ that satisfies

$$\int_\Omega w_{\boldsymbol{p}} \beta^0_{\boldsymbol{q}} \,\mathrm{d}\boldsymbol{x} = \begin{cases} 1 & \text{, if } \boldsymbol{p} = \boldsymbol{q} \,, \\ 0 & \text{, otherwise,} \end{cases} \qquad \boldsymbol{p}, \boldsymbol{q} \in \mathscr{F}_0(\mathscr{T}_h) \,. \tag{73}$$

These properties ensure that Q_0 is a bounded projector: $\mathsf{Q}_0^2 = \mathsf{Q}_0$. Moreover, functions that are constant in a local neighborhood of T are preserved on T.

Schöberl's feat was to generalize Q_0 to a family of bounded operators $\mathsf{Q}_\ell : L^2\Lambda^\ell(\Omega) \to \mathscr{W}^\ell(\mathscr{T}_h)$ defined as

$$\mathsf{Q}_1\omega := \sum_{[\boldsymbol{x},\boldsymbol{y}] \in \mathscr{F}_1(\mathscr{T}_h)} \left(\int_{T_x} \int_{T_y} w_{\boldsymbol{x}}(\boldsymbol{x}') w_{\boldsymbol{y}}(\boldsymbol{y}') \{ \int_{[\boldsymbol{x}',\boldsymbol{y}']} \omega \} \,\mathrm{d}\boldsymbol{y}' \,\mathrm{d}\boldsymbol{x}' \right) \cdot \beta^1_{[\boldsymbol{x},\boldsymbol{y}]} \,, \tag{74}$$

$$\mathsf{Q}_2\omega := \sum_{[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}] \in \mathscr{F}_2(\mathscr{T}_h)} \left(\int_{T_x} \int_{T_y} \int_{T_z} w_{\boldsymbol{x}}(\boldsymbol{x}') w_{\boldsymbol{y}}(\boldsymbol{y}') w_{\boldsymbol{z}}(\boldsymbol{z}') \{ \int_{[\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}']} \omega \} \,\mathrm{d}\boldsymbol{z}' \,\mathrm{d}\boldsymbol{y}' \,\mathrm{d}\boldsymbol{x}' \right) \cdot \beta^2_{[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]} \,. \tag{75}$$

All these operators satisfy $\mathsf{d}_\ell \circ \mathsf{Q}_\ell = \mathsf{Q}_{\ell+1} \circ \mathsf{d}\,\ell$ and enjoy the approximation property

$$\|\omega - \mathsf{Q}_\ell\omega\|_{L^2\Lambda^\ell(\Omega)} \le Ch \,|\omega|_{H^1\Lambda^\ell(\Omega)} \;, \quad \forall \omega \in H^1\Lambda^\ell(\Omega) \,, \tag{76}$$

with $C > 0$ depending only on shape regularity. Here $H^1\Lambda^\ell(\Omega)$ designates the space of ℓ-forms with vector proxy components in $H^1(\Omega)$. However, except for Q_0 the other quasi-interpolation operators are **no** projections. The same flaw also marred a mollifier based construction presented in [23].

The ultimate solution, an explicit formula providing *bounded, local, commuting, projectors* that map $H\Lambda^\ell(\Omega) \to \mathscr{W}^\ell(\mathscr{T}_h)$ for any simplicial mesh in *any* dimension n and $0 \le \ell \le n$, was only recently discovered by Falk and Winther in [29, Eq. (4.2)]. As Schöberl's invention, it is a quasi-interpolation based on weighted local integrals. Unfortunately, the scheme is too complicated to be covered here in detail and we merely summarize the result, which is an immensely powerful tool in the numerical analysis of discrete differential forms.

Theorem 9 (Falk-Winther Local Commuting Projections [29, Thms. 4.5, 4.7, 5.2]) *For any simplicial mesh $\mathscr{T}_h$ of $\Omega \subset \mathbb{R}^n$ there is a family of linear* bounded projection operators $\mathsf{Q}_\ell : H\Lambda^\ell(\Omega) \to \mathscr{W}^\ell(\mathscr{T}_h)$, $\ell \in \{0, \ldots, n\}$*, such that*

(i) they commute with the exterior derivative

$$\mathsf{Q}_{\ell+1} \circ \mathsf{d}_\ell = \mathsf{d}_\ell \circ \mathsf{Q}_\ell \quad \textit{on} \quad H\Lambda^\ell(\Omega) \,, \tag{77}$$

(ii) *they are quasi-local: for all $T \in \mathscr{T}_h$ the restriction $\mathsf{Q}_\omega|_T$ depends only on ω restricted to a mesh-neighborhood Ω_T of T,*

(iii) *they satisfy the approximation property*

$$\|\omega - \mathsf{Q}_\ell \omega\|_{L^2 \Lambda^\ell(T)} \leq C h_T \, |\omega|_{H^1 \Lambda^\ell(\Omega_T)} \ , \quad \forall \omega \in H^1 \Lambda^\ell(\Omega_T) \ , \tag{78}$$

where $C > 0$ depends only on the shape regularity of $\mathscr{T}_h$ and ℓ.

Indeed, the construction in [29] even covers higher-order generalizations of Whitney forms. A simplified presentation for Whitney forms in 2D is given in [30].

4 Whitney Form Galerkin Discretization of the Maxwell Cavity Problem

In this section we perform an a priori convergence analysis for the Galerkin discretization of a particular Maxwell boundary value problem in frequency domain. Trial and test spaces are supplied by Whitney 1-forms, aka lowest order edge elements. This will allow us to discuss a few fundamental considerations and techniques. Of course, only a tiny fraction of the numerical analysis developed for computational electromagnetism can be covered.

Throughout this section $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron, equipped with a simplicial mesh $\mathscr{T}_h$.

Bibliographical Notes

The main references for this section are [35, Sect. 5] and [40, Chap. 7]. Refined duality estimates are given in [50], whereas for the analysis of edge element discretizations of the time-dependent linear Maxwell equations like (33) we refer to [25, 49].

4.1 Maxwell Cavity Problem

We consider Maxwell's equations on Ω with local linear material laws (22) that can be expressed by means of Hodge operators. Their vector proxy representation is given in (23). Moreover, we rely on a *frequency domain model*, that is, the evolution equations are subject to a continuous Fourier transform in time, which amounts to replacing every temporal derivative ∂_t with a multiplication with $\imath\omega$, ω the angular frequency and $\imath$ the imaginary unit. The unknowns will be complex valued forms on Ω ("phasors"), for which we retain the same symbols as in Sect. 2.3:

$$\left.\begin{aligned} \mathsf{d}_\ell\,\mathbf{e} &= -\imath\omega \star_\mu \mathbf{h} && \text{(79a)}\\ \mathsf{d}_{2-\ell}\,\mathbf{h} &= \imath\omega \star_\varepsilon \mathbf{e} + \mathbf{j} && \text{(79b)} \end{aligned}\right\} \quad \text{in } \Omega \quad , \quad \ell = 1 \; .$$

We impose impedance boundary conditions as a simple version of so-called absorbing (transparent) boundary conditions:

$$\mathbf{t}_{\partial\Omega}\mathbf{h} = \star_\lambda \mathbf{t}_{\partial\Omega}\mathbf{e} \quad \text{on } \partial\Omega \; , \tag{80}$$

with an impedance λ, which is a Riemannian metric on $\partial\Omega$.

For the remainder of this section we switch to the vector proxy perspective introduced in Sect. 2.2.2. Then the **a**-based variational formulation (33) (in frequency domain) reads: seek $\vec{\mathbf{a}} \in V$ such that

$$\begin{aligned} \mathsf{a}_M(\vec{\mathbf{a}}, \vec{\mathbf{a}}') := \int_\Omega \boldsymbol{\mu}^{-1}(\boldsymbol{x})\mathbf{curl}\,\vec{\mathbf{a}} \cdot \mathbf{curl}\,\vec{\mathbf{a}}' - \omega^2 \boldsymbol{\varepsilon}(\boldsymbol{x})\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}' \,\mathrm{d}\boldsymbol{x} \\ - \imath\omega \int_{\partial\Omega} \boldsymbol{\lambda}(\boldsymbol{x})\vec{\mathbf{a}}_t \cdot \vec{\mathbf{a}}'_t \,\mathrm{d}S = \imath\omega \int_\Omega \vec{\mathbf{j}} \cdot \vec{\mathbf{a}}' \,\mathrm{d}\boldsymbol{x} \quad \forall \vec{\mathbf{a}}' \in V \; , \end{aligned} \tag{81}$$

posed on the Hilbert space (subscript t tags a tangential component trace, *cf.* Table 2)

$$V = \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) := \{\vec{\mathbf{w}} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \; \vec{\mathbf{w}}_t \in (L^2(\partial\Omega))^3\} \; , \tag{82}$$

with norm

$$\left\|\vec{\mathbf{w}}\right\|^2_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} := \left\|\vec{\mathbf{w}}\right\|^2_{L^2(\Omega)} + \left\|\mathbf{curl}\,\vec{\mathbf{w}}\right\|^2_{L^2(\Omega)} + \left\|\vec{\mathbf{w}}_t\right\|^2_{L^2(\partial\Omega)} \; . \tag{83}$$

This is the maximal Hilbert space on which the bilinear form a_M of (81) is still continuous. We remind that $\boldsymbol{\mu}$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\lambda}$ are bounded and uniformly positive definite tensor coefficients.

Deliberately, the degree of the form **e** (and, indirectly, **h**) was retained as parameter ℓ in (79a). When we set $\ell = 0$, that is, **e** is read as a 0-form, **h** as a 2-form, and **j** as a 3-form, then we arrive at the equations of the *acoustic cavity problem* in frequency domain. Its "**a**-based" variational formulations reads in vector proxies: seek $\vec{u} \in H^1(\Omega)$

$$\begin{aligned} \mathsf{a}_H(\vec{u}, \vec{u}') := \int_\Omega \mu^{-1}(\boldsymbol{x})\mathbf{grad}\,\vec{u} \cdot \mathbf{grad}\,\vec{u}' - \omega^2 \boldsymbol{\varepsilon}(\boldsymbol{x})\vec{u}\,\vec{u}' \,\mathrm{d}\boldsymbol{x} \\ + \imath\omega \int_{\partial\Omega} \boldsymbol{\lambda}(\boldsymbol{x})\vec{u}\,\vec{u}' \,\mathrm{d}S = \int_\Omega f\,\vec{u}' \,\mathrm{d}\boldsymbol{x} \quad \forall \vec{u}' \in H^1(\Omega) \; . \end{aligned} \tag{84}$$

Evidently, there are sweeping structural similarities between (81) and (84), of course. Yet, in one respect the acoustic boundary value problem will be substantially simpler than its electromagnetic counterpart. Hence, it makes didactic sense, to discuss (84) before addressing the more difficult (81).

For the sake of simplicity, in the sequel we restrict ourselves to the case $\boldsymbol{\mu} = \boldsymbol{\varepsilon} = \boldsymbol{\lambda} \equiv 1$ of constant coefficients scaled to unity. This does not affect the gist of any argument.

4.2 Splittings of $\boldsymbol{H}(\mathbf{curl}, \Omega)$

In this section we provide decompositions of vectorfields in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ into **curl**-free components and some complement spaces. They have turned out to be pivotal tools in the mathematical and numerical analysis of Maxwell's equations. For the sake of simplicity, we assume trivial topology of Ω throughout, *cf.* Theorem 4.

4.3 Helmholtz Decomposition

The Helmholtz decomposition of (67) can be restricted to $H\Lambda^\ell(\Omega)$ and then provides an $H\Lambda^\ell(\Omega)$-orthogonal splitting of $H\Lambda^\ell(\Omega)$. The important observation is that the $\mathscr{X}^\ell$-component will enjoy some smoothness. Let us look at Helmholtz decompositions from the angle of vector proxies in 3D: For $\ell = 1$ we get

$$\boldsymbol{H}(\mathbf{curl}, \Omega) = \mathbf{grad} H^1(\Omega) \oplus \underbrace{(\boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}\, 0, \Omega))}_{=:\boldsymbol{X}_T(\Omega)} , \tag{85}$$

where

$$\boldsymbol{H}_0(\mathrm{div}\, 0, \Omega) := \{\vec{\mathbf{w}} \in (L^2(\Omega))^3 : \mathrm{div}\, \vec{\mathbf{w}} = 0,\ \vec{\mathbf{w}} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\} , \tag{86}$$

and for $\ell = 2$ the Helmholtz decomposition becomes

$$\boldsymbol{H}(\mathrm{div}, \Omega) = \mathbf{curl} \boldsymbol{H}(\mathbf{curl}, \Omega) \oplus (\boldsymbol{H}(\mathrm{div}, \Omega) \cap \boldsymbol{H}_0(\mathbf{curl}\, 0, \Omega)) , \tag{87}$$

with

$$\boldsymbol{H}_0(\mathbf{curl}\, 0, \Omega) := \{\vec{\mathbf{w}} \in (L^2(\Omega))^3 : \mathbf{curl} \vec{\mathbf{w}} = 0,\ \vec{\mathbf{w}}_t = 0 \text{ on } \partial\Omega\} . \tag{88}$$

The enhanced smoothness of the complement spaces like $\boldsymbol{X}_T(\Omega)$ is asserted in the following result, see [3, Sect. 2]:

Theorem 10 (Regularity of Complements in Helmholtz Decomposition) *If Ω has* $C^{1,1}$-smooth boundary *or Ω is* convex*, then* $\boldsymbol{X}_T(\Omega) := \boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}\,0, \Omega)$ *(equipped with the norm of $\boldsymbol{H}(\mathbf{curl}, \Omega)$) is continuously embedded into* $(H^1(\Omega))^3$.

Proof (outline) For Ω with smooth boundary, integration by parts and manipulations of surface differential operators lead to the identity

$$\left|\overrightarrow{\mathbf{w}}\right|^2_{H^1(\Omega)} + \int_{\partial\Omega} \mathscr{B}(\overrightarrow{\mathbf{w}} \times \boldsymbol{n}, \overrightarrow{\mathbf{w}} \times \boldsymbol{n})\,\mathrm{d}S = \left\|\mathbf{curl}\,\overrightarrow{\mathbf{w}}\right\|^2_{L^2(\Omega)} + \left\|\mathrm{div}\,\overrightarrow{\mathbf{w}}\right\|^2_{L^2(\Omega)} ,$$

where $\mathscr{B}$ is the curvature tensor on $\partial\Omega$. The second ingredient for the proof is the density of $(H^1(\Omega))^3$ in $\boldsymbol{X}_T(\Omega)$. This holds true, provided that the Neumann problem for $-\Delta$ is 2-regular on Ω, which is guaranteed under the assumptions of the theorem, see [3, Lemma 2.10].

A simple counterexample demonstrates that the assumptions of the theorem are necessary:

In the geometric setting depicted in Fig. 9, consider the function, given in cylindrical coordinates,

$$\psi(r, \phi, z) = r^{\pi/\omega} \cos(\frac{\pi}{\phi^*}\phi) , \quad 0 \le \phi \le \phi^*,\ r > 0 .$$

We find that in a neighborhood of the edge $\overrightarrow{\mathbf{w}} := \mathbf{grad}\,\psi$ satisfies

- $\mathbf{curl}\,\overrightarrow{\mathbf{w}} = 0$ and $\mathrm{div}\,\overrightarrow{\mathbf{w}} = 0$,
- $\overrightarrow{\mathbf{w}} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$,
- but $\overrightarrow{\mathbf{w}} \notin (H^1(\Omega))^3$, because "$\int_\Omega |\mathbf{grad}\,\overrightarrow{\mathbf{w}}|^2\,\mathrm{d}\mathbf{x} = \infty$".

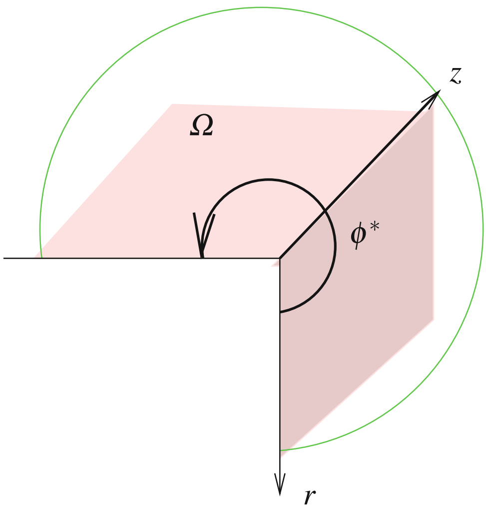

Fig. 9 A domain Ω generated by forming the tensor product of a 2D polygon with a re-entrant corner (angle $\phi^* > \pi$) with an interval (in z-direction). The *shaded planes* correspond to $\partial\Omega$

As a consequence, functions in $\boldsymbol{X}_T(\Omega)$ may fail to belong to $(H^1(\Omega))^3$ in case of non-smooth Ω with reentrant ("non-convex") edges.

4.4 Regular Decomposition

To remedy the potential loss of $H^1(\Omega)$-regularity of functions in $\boldsymbol{X}_T(\Omega)$, we can sacrifice the strict $L^2(\Omega)$-orthogonality of the Helmholtz decomposition (85) and settle for decompositions that are "merely" stable. Those will be dubbed "regular decompositions" in the sequel [35, Lemma 2.4].

Theorem 11 (Regular Decomposition) *There are* continuous *linear mappings* $\mathsf{R} : \boldsymbol{H}(\mathbf{curl}, \Omega) \to (H^1(\Omega))^3$ *and* $\mathsf{N} : \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}(\mathbf{curl}\, 0, \Omega) := \boldsymbol{H}(\mathbf{curl}, \Omega) \cap \mathrm{Ker}(\mathbf{curl})$ *such that*

$$\mathsf{Id} = \mathsf{R} + \mathsf{N} \quad \textit{and} \quad \left\| \mathsf{R}\vec{\mathbf{w}} \right\|_{H^1(\Omega)} \leq C \left\| \mathbf{curl}\, \vec{\mathbf{w}} \right\|_{L^2(\Omega)} .$$

This theorem can be remembered as

$$\text{“} \quad \boldsymbol{H}(\mathbf{curl}, \Omega) = (H^1(\Omega))^3 + \mathbf{curl}\text{-free} \quad \text{”} .$$

Existence of regular decompositions can be established easily using a powerful lifting operator that has been discovered recently [27]:

Theorem 12 (Regularized Poincaré Lifting) *There is a* continuous *operator* $\mathsf{Y} : (L^2(\Omega))^3 \to (H^1(\Omega))^3$ *that satisfies*

$$\mathbf{curl}\mathsf{Y}(\vec{\mathbf{w}}) = \vec{\mathbf{w}} \quad \forall \vec{\mathbf{w}} \in \boldsymbol{H}(\mathrm{div}\, 0, \Omega) .$$

Proof (of Theorem 11) We simply define $\mathsf{R} := \mathsf{Y} \circ \mathbf{curl}$ and $\mathsf{N} := \mathsf{Id} - \mathsf{R}$. The mapping properties of R are immediate from those of Y.

To demonstrate an application of regular decompositions, we use them to prove the discrete Friedrichs inequality from Lemma 5 for $\ell = 1$.

Proof (of Lemma 5 for $\ell = 1$) Pick $\vec{\mathbf{x}}_h \in \mathscr{X}^1(\mathscr{T}_h)$ and rewrite

$$\begin{aligned} \left\| \vec{\mathbf{x}}_h \right\|^2_{L^2(\Omega)} = \left(\vec{\mathbf{x}}_h - \mathsf{I}_1 \mathsf{R} \vec{\mathbf{x}}_h, \vec{\mathbf{x}}_h \right)_{L^2(\Omega)} + \left((\mathsf{I}_1 - \mathsf{Id}) \mathsf{R} \vec{\mathbf{x}}_h, \vec{\mathbf{x}}_h \right)_{L^2(\Omega)} \\ + \left(\mathsf{R} \vec{\mathbf{x}}_h, \vec{\mathbf{x}}_h \right)_{L^2(\Omega)} . \end{aligned} \tag{89}$$

Since, by the commuting diagram property for nodal interpolation from Lemma 2,

$$\mathbf{curl}(\vec{\mathbf{x}}_h - \mathsf{I}_1 \mathsf{R} \vec{\mathbf{x}}_h) = \mathbf{curl}\, \vec{\mathbf{x}}_h - \mathsf{I}_2(\mathbf{curl} \mathsf{R} \vec{\mathbf{x}}_h) = \mathbf{curl}\, \vec{\mathbf{x}}_h - \mathsf{I}_2(\mathbf{curl}\, \vec{\mathbf{x}}_h) = 0 ,$$

the first term in (89) vanishes, because $\vec{\mathbf{x}}_h$ is orthogonal to $\mathrm{Ker}(\mathbf{curl}) \cap \mathscr{W}^1(\mathscr{T}_h)$. To estimate the second term Lemma 4 and the continuity of R come handy and yield

$$\left\| (\mathsf{I}_1 - \mathsf{Id})\mathsf{R}\vec{\mathbf{x}}_h \right\|_{L^2(\Omega)} \le Ch \left| \mathsf{R}\vec{\mathbf{x}}_h \right|_{H^1(\Omega)} \le Ch \left\| \mathbf{curl}\,\vec{\mathbf{x}}_h \right\|_{L^2(\Omega)} . \tag{90}$$

Again citing the continuity of R to bound the third term, we finally arrive at

$$\left\| \vec{\mathbf{x}}_h \right\|^2_{L^2(\Omega)} \le C(h+1) \left\| \mathbf{curl}\,\vec{\mathbf{x}}_h \right\|_{L^2(\Omega)} \left\| \vec{\mathbf{x}}_h \right\|_{L^2(\Omega)} .$$

All constants may depend only on the shape regularity of the mesh and the domain Ω.

Remark 12 Since tangential traces of vectorfields in $(H^1(\Omega))^3$ are contained in $(L^2(\partial\Omega))^3$, we find that $\mathsf{R}(\boldsymbol{H}(\mathbf{curl}, \Omega)) \subset \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$ for the operator R from Theorem 11. This enables us to restrict regular decompositions to $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$ using the restrictions of the operators of Theorem 11 to $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$. Below we will tacitly use these "$\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$-restricted" regular decomposition operators when needed.

4.5 Helmholtz Cavity Problem: Well-Posedness

We first tackle the well-posedness of the simpler Helmholtz cavity variational problem (84). The key tool will be a Fredholm alternative argument [28, Theorem D.5], [40, Sect. 2.2.4].

Theorem 13 (Fredholm Alternative) *Let X, Y be Banach spaces,* $\mathsf{T} : X \to Y$ *a* bijective *bounded linear operator, and* $\mathsf{K} : X \to Y$ *a* compact *linear operator. Then for* $\mathsf{T} + \mathsf{K}$ *it is equivalent*

$$\mathsf{T} + \mathsf{K} \text{ injective} \quad \Leftrightarrow \quad \mathsf{T} + \mathsf{K} \text{ bijective} \quad \Leftrightarrow \quad \mathsf{T} + \mathsf{K} \text{ surjective} .$$

In order to apply this theorem we split the bilinear form a_H into

$$\left.\begin{aligned} \mathsf{t}_H(\vec{u}, \vec{u}') &:= \int_\Omega \mathbf{grad}\,\vec{u} \cdot \mathbf{grad}\,\vec{u}' + \vec{u}\,\vec{u}'\,\mathrm{d}\boldsymbol{x} , \\ \mathsf{k}_H(\vec{u}, \vec{u}') &:= -\int_\Omega (\omega^2+1)\vec{u}\,\vec{u}'\,\mathrm{d}\boldsymbol{x} + \imath\omega \int_{\partial\Omega} \vec{u}\,\vec{u}'\,\mathrm{d}S , \end{aligned}\right\} \quad \vec{u}, \vec{u}' \in H^1(\Omega) .$$

Obviously, $\mathsf{a}_H = \mathsf{t}_H + \mathsf{k}_H$ and the operator $\mathsf{T}_H : H^1(\Omega) \to (H^1(\Omega))'$ associated with t_H clearly is an isomorphism. To see the compactness of the operator K_H :

$H^1(\Omega) \to (H^1(\Omega))'$ induced by k_H we appeal to (generalized) Rellich compactness theorems [28, Sect. 5.7]:

Theorem 14 ((Generalized) Rellich Compactness Theorem) *The following embeddings are* compact*:* $H^1(\Omega) \subset L^2(\Omega)$ *and* $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$.

Since the two parts of k_H are continuous on $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively, and the point trace $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is continuous, from Theorem 14 we conclude the compactness of K_H. Thus, the assumptions of the Fredholm alternative from Theorem 13 are satisfied for $\mathsf{T}_H + \mathsf{K}_H : H^1(\Omega) \to (H^1(\Omega))'$.

Lemma 7 (Injectivity of a_H) *The operator $\mathsf{A}_H : H^1(\Omega) \to (H^1(\Omega))'$ induced by the bilinear form a_H of the Helmholtz cavity problem is injective.*

Proof We have to show that $\mathsf{A}_H u = 0$ implies $u = 0$, or, equivalently

$$\mathsf{a}_H(\vec{u}, \vec{u}') = 0 \quad \forall \vec{u}' \in H^1(\Omega) \quad \Rightarrow \quad \vec{u} = 0 \,.$$

If, for all $\vec{u}' \in H^1(\Omega)$,

$$\int\limits_{\Omega} \mathbf{grad}\,\vec{u} \cdot \mathbf{grad}\,\vec{u}' - \omega^2 \vec{u}\,\vec{u}' \,\mathrm{d}\boldsymbol{x} + \imath\omega \int\limits_{\partial\Omega} \vec{u}\,\vec{u}' \,\mathrm{d}S = 0 \,,$$

then we can infer

(i) $\vec{u} \in H^1_0(\Omega)$, when choosing $\vec{u}' = \vec{u}$ and considering the imaginary part,
(ii) $-\Delta\vec{u} - \omega^2\vec{u} = 0$ in the sense of distributions, by testing with $\vec{u}' \in C_0^\infty(\Omega)$,
(iii) $\mathbf{grad}\,\vec{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ from (ii) and testing with $\vec{u}' \in C^\infty(\overline{\Omega})$.

Thus, since both $\vec{u} = 0$ and $\mathbf{grad}\,\vec{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, extension by zero to $\mathbb{R}^3$ gives a solution $\tilde{\vec{u}} \in H^1(\mathbb{R}^3)$ of $-\Delta\tilde{\vec{u}} - \omega^2\tilde{\vec{u}} = 0$. It is known that such a function must vanish everywhere, see [26, Sect. 3.2] for a discussion of the uniqueness of solutions of acoustic scattering problems.

Summing up, $\mathsf{A}_H : H^1(\Omega) \to (H^1(\Omega))'$ is an isomorphism, which implies the continuous dependence of the solution $u \in H^1(\Omega)$ of (84) on the data $f \in L^2(\Omega)$ (even $f \in (H^1(\Omega))'$).

Remark 13 What we have established for a_H is often stated as a so-called Gårding inequality:

Corollary 2 (Gårding Inequality) *There is a compact operator $\mathsf{K}_H : H^1(\Omega) \to (H^1(\Omega))'$ such that*

$$\exists C > 0: \quad \mathrm{Re}\{\mathsf{a}_H(\vec{u}, \overline{\vec{u}}) + \left\langle \mathsf{K}_H \vec{u}, \overline{\vec{u}} \right\rangle\} \geq C \|\vec{u}\|^2_{H^1(\Omega)} \quad \forall \vec{u} \in H^1(\Omega) \,.$$

4.6 Maxwell Cavity Problem: Well-Posedness

Contrasting the bilinear form a_M from (81) for the Maxwell cavity problem with its Helmholtz counterpart a_H from (84), we find a striking difference. When separating the three parts of a_M

$$\mathsf{a}_C(\vec{\mathbf{w}}, \vec{\mathbf{w}}') := \int_\Omega \mathbf{curl}\,\vec{\mathbf{w}} \cdot \mathbf{curl}\,\vec{\mathbf{w}}' + \vec{\mathbf{w}} \cdot \vec{\mathbf{w}}' \,\mathrm{d}\boldsymbol{x} \;,$$

$$\mathsf{a}_Z(\vec{\mathbf{w}}, \vec{\mathbf{w}}') := -\int_\Omega (\omega^2 + 1)\vec{\mathbf{w}} \cdot \vec{\mathbf{w}}' \,\mathrm{d}\boldsymbol{x} \;,$$

$$\mathsf{a}_\partial(\vec{\mathbf{w}}, \vec{\mathbf{w}}') := \imath\omega \int_{\partial\Omega} \vec{\mathbf{w}}_t \cdot \vec{\mathbf{w}}_t' \,\mathrm{d}S$$

with $\mathsf{a}_M = \mathsf{a}_C + \mathsf{a}_Z + \mathsf{a}_\partial$, and writing $\mathsf{C}, \mathsf{Z}, \mathsf{B} : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)'$ ($\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$ defined in (82)) for the associated operators, we discover that neither Z nor B are compact! The reason is that Ker($\mathbf{curl}$) has infinite dimension so that $\boldsymbol{H}(\mathbf{curl}, \Omega)$ cannot be compactly embedded in $(L^2(\Omega))^3$.

4.6.1 Generalized Gårding Inequality

The first insight is that Ker($\mathbf{curl}$) requires a special treatment in the analysis of a_M, because on Ker($\mathbf{curl}$) the "zero-order" a_Z-part of the bilinear form will have to be taken into account in the bijective operator T when applying the Fredholm alternative of Theorem 13. The most elementary criterion for invertibility in a variational framework (in Banach spaces) is uniform positivity of (the real part) of a bilinear form. Awkwardly, the reversed sign of a_Z compared to the "second-order" a_C-part initially foils this simple argument. Sloppily speaking, on Ker($\mathbf{curl}$) the sign of a_Z has to be "corrected" first. This can be accomplished by the regular decomposition from Theorem 11, because it can serve as a tool to separate Ker($\mathbf{curl}$) $\subset \boldsymbol{H}(\mathbf{curl}, \Omega)$ of a complement space, on which a bilinear form that is continuous on $(L^2(\Omega))^3$ gives rise to a compact operator.

The details are as follows: recall that Theorem 11 together with Remark 12 provides bounded operators $\mathsf{R} : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to (H^1(\Omega))^3$ and $\mathsf{N} : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \cap \mathrm{Ker}(\mathbf{curl})$ such that $\mathsf{R} + \mathsf{N} = \mathsf{Id}$. Based on these operators, we define the *sign-flip isomorphism*

$$\mathsf{F} := \mathsf{R} - \mathsf{N} = 2\mathsf{R} - \mathsf{Id} : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \;. \tag{91}$$

For the sake of brevity, we choose the following tags for the components of a regular decomposition of $\vec{\mathbf{w}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$:

$$\vec{\mathbf{w}}^* := \mathsf{R}\vec{\mathbf{w}}\,, \quad \vec{\mathbf{w}}^0 := \mathsf{N}\vec{\mathbf{w}} \quad \Rightarrow \quad \vec{\mathbf{w}} = \vec{\mathbf{w}}^* + \vec{\mathbf{w}}^0\,. \tag{92}$$

In this notation we have—mind the −-sign!—

$$\mathsf{F}\vec{\mathbf{w}} = \mathsf{F}(\vec{\mathbf{w}}^* + \vec{\mathbf{w}}^0) = \vec{\mathbf{w}}^* - \vec{\mathbf{w}}^0\,. \tag{93}$$

Now, let us scrutinize the parts of a_M under the lens of a regular decomposition after the sign-flip isomorphism has been applied to the test function:

(C) From $\mathbf{curl}\mathsf{R}(\vec{\mathbf{w}}) = \mathbf{curl}\,\vec{\mathbf{w}}$ and $\mathbf{curl}\,\vec{\mathbf{w}}^0 = 0$ it is immediate that

$$\mathsf{a}_C(\vec{\mathbf{w}}, \mathsf{F}\vec{\mathbf{q}}) = \left(\mathbf{curl}\,\vec{\mathbf{w}}^*, \mathbf{curl}\,\vec{\mathbf{q}}^*\right)_{L^2(\Omega)}\,. \tag{94}$$

(Z) By (93) and bilinearity

$$\begin{aligned}\mathsf{a}_Z(\vec{\mathbf{w}}, \mathsf{F}(\vec{\mathbf{q}})) = \omega^2\Big(&-\left(\vec{\mathbf{w}}^*, \vec{\mathbf{q}}^*\right)_{L^2(\Omega)} + \left(\vec{\mathbf{w}}^*, \vec{\mathbf{q}}^0\right)_{L^2(\Omega)} \\ &- \left(\vec{\mathbf{w}}^0, \vec{\mathbf{q}}^*\right)_{L^2(\Omega)} + \left(\vec{\mathbf{w}}^0, \vec{\mathbf{q}}^0\right)_{L^2(\Omega)}\Big)\,.\end{aligned} \tag{95}$$

The key conclusion from the continuity of $\mathsf{R} : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to (H^1(\Omega))^3$ and the Rellich compactness theorem (Theorem 14) is that the bilinear form on $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$, given by

$$(\vec{\mathbf{w}}, \vec{\mathbf{q}}) \mapsto -\left(\vec{\mathbf{w}}^*, \vec{\mathbf{q}}^*\right)_{L^2(\Omega)} + \left(\vec{\mathbf{w}}^*, \vec{\mathbf{q}}^0\right)_{L^2(\Omega)} - \left(\vec{\mathbf{w}}^0, \vec{\mathbf{q}}^*\right)_{L^2(\Omega)}\,, \tag{96}$$

spawns a compact operator, because at least one of the arguments in the $L^2(\Omega)$ inner products belongs to $(H^1(\Omega))^3$.

(B) Similarly, for the boundary part we obtain

$$\begin{aligned}\mathsf{a}_\partial(\vec{\mathbf{w}}, \mathsf{F}(\vec{\mathbf{q}})) = \imath\omega\Big(&\left(\vec{\mathbf{w}}_t^*, \vec{\mathbf{q}}_t^*\right)_{L^2(\partial\Omega)} - \left(\vec{\mathbf{w}}_t^*, \vec{\mathbf{q}}_t^0\right)_{L^2(\partial\Omega)} \\ &+ \left(\vec{\mathbf{w}}_t^0, \vec{\mathbf{q}}_t^*\right)_{L^2(\partial\Omega)} - \left(\vec{\mathbf{w}}_t^0, \vec{\mathbf{q}}_t^0\right)_{L^2(\partial\Omega)}\Big)\,.\end{aligned} \tag{97}$$

Observe that $\vec{\mathbf{w}}_t^*$ and $\vec{\mathbf{q}}_t^*$ are tangential traces of vector fields in $(H^1(\Omega))^3$. They belong to a space of tangential vector fields on $\partial\Omega$ that is compactly embedded in $(L^2(\partial\Omega))^3$. Consequently, invoking Theorem 14 again, the operator $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega) \rightarrow (\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega))'$ induced by the bilinear form on $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$

$$(\vec{\mathbf{w}},\vec{\mathbf{q}}) \mapsto \left(\vec{\mathbf{w}}_t^*,\vec{\mathbf{q}}_t^*\right)_{L^2(\partial\Omega)} - \left(\vec{\mathbf{w}}_t^*,\vec{\mathbf{q}}_t^0\right)_{L^2(\partial\Omega)} + \left(\vec{\mathbf{w}}_t^0,\vec{\mathbf{q}}_t^*\right)_{L^2(\partial\Omega)} \tag{98}$$

will be compact.

Reassembling the parts, we find

$$\begin{aligned}\mathsf{a}_M(\vec{\mathbf{w}},\mathsf{F}(\vec{\mathbf{q}})) = \left(\mathbf{curl}\,\vec{\mathbf{w}}^*,\mathbf{curl}\,\vec{\mathbf{q}}^*\right)_{L^2(\Omega)} + \omega^2\left(\vec{\mathbf{w}}^0,\vec{\mathbf{q}}^0\right)_{L^2(\Omega)}&\\ - \imath\omega\left(\vec{\mathbf{w}}_t^0,\vec{\mathbf{q}}_t^0\right)_{L^2(\partial\Omega)} + \mathsf{k}_M(\vec{\mathbf{w}},\vec{\mathbf{q}})\ ,&\end{aligned} \tag{99}$$

where k_M collects the "compact remainders" from (96) and (98). This means

$$\mathsf{a}_M(\vec{\mathbf{w}},\mathsf{F}(\overline{\vec{\mathbf{w}}})) = \left\|\mathbf{curl}\,\vec{\mathbf{w}}^*\right\|_{L^2(\Omega)} + \omega^2\left\|\vec{\mathbf{w}}^0\right\|_{L^2(\Omega)} - \imath\omega\left\|\vec{\mathbf{w}}^0\right\|_{L^2(\partial\Omega)} + \mathsf{k}_M(\vec{\mathbf{w}},\overline{\vec{\mathbf{w}}})\ ,$$

and, since from $\left\|\mathsf{R}\vec{\mathbf{w}}\right\|_{L^2(\Omega)} \leq C\left\|\mathbf{curl}\,\vec{\mathbf{w}}\right\|_{L^2(\Omega)}$ we can conclude that

$$\left\|\mathbf{curl}\,\vec{\mathbf{w}}^*\right\|_{L^2(\Omega)} + \left\|\vec{\mathbf{w}}^0\right\|_{L^2(\Omega)} \geq C\left\|\vec{\mathbf{w}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\ ,$$

we have a more general version of Gårding's inequality:

Theorem 15 (Generalized Gårding Inequality for Maxwell Cavity Problem) *There is a* compact *operator* $\mathsf{K}_M : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega) \rightarrow (\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega))'$ *such that*

$$\exists C > 0 : \quad \left|\mathsf{a}_M(\vec{\mathbf{w}},\mathsf{F}(\overline{\vec{\mathbf{w}}})) + \left\langle \mathsf{K}_M\vec{\mathbf{w}},\overline{\vec{\mathbf{w}}}\right\rangle\right| \geq C\left\|\vec{\mathbf{w}}\right\|^2_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}$$

for all $\vec{\mathbf{w}} \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$*, where* F *is the sign-flip isomorphism defined in* (91) *and* $C > 0$ *depends on* Ω *only.*

4.6.2 Existence and Uniqueness of Solutions

Together with the Lax-Milgram theorem about the invertibility of operators rising from elliptic bilinear forms [40, Lemma 2.21], Theorem 15 tells us that $\mathsf{F}' \circ \mathsf{A}_M + \mathsf{K}_M : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega) \rightarrow \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)'$ is bijective. Here, A_M is the operator associated with a_M and $\mathsf{F}' : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)' \rightarrow \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)'$ is the adjoint of F. Hence, $\mathsf{A}_M + (\mathsf{F}')^{-1} \circ \mathsf{K}_M$ is bijective as well and, thus, A_M has been identified as a compact perturbation of an invertible operator; the Fredholm alternative of Theorem 13 applies!

Parallel to Sect. 4.5 we have to establish injectivity of a_M.

Lemma 8 (Injectivity of a_M [40, Thm. 4.12]) *The operator $\mathsf{A}_M : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)'$ associated with the bilinear form a_M of the Maxwell cavity problem is injective.*

Proof Along the lines of proof of Lemma 7 we conclude from

$$\left(\mathbf{curl}\,\vec{\mathbf{w}}, \mathbf{curl}\,\vec{\mathbf{w}}'\right)_{L^2(\Omega)} - \omega^2\left(\vec{\mathbf{w}}, \vec{\mathbf{w}}'\right)_{L^2(\Omega)} + \imath\omega\left(\vec{\mathbf{w}}_t, \vec{\mathbf{w}}'_t\right)_{L^2(\partial\Omega)} = 0$$

for all $\vec{\mathbf{w}}' \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$ that

(i) $\vec{\mathbf{w}}_t = 0$ on $\partial\Omega$ (through testing with $\vec{\mathbf{w}}' = \vec{\mathbf{w}}$),
(ii) $\mathbf{curl}\,\mathbf{curl}\,\vec{\mathbf{w}} - \omega^2\vec{\mathbf{w}} = 0$ (through testing with $\vec{\mathbf{w}} \in (C_0^\infty(\Omega))^3$),
(iii) $\mathbf{curl}\,\vec{\mathbf{w}} \times \boldsymbol{n} = 0$ (through testing with $\vec{\mathbf{w}} \in (C^\infty(\overline{\Omega}))^3$).

Then extending $\vec{\mathbf{w}}$ by zero outside Ω gives an entire solution of Maxwell's equations on $\mathbb{R}^3$ vanishing at ∞. Then necessarily $\vec{\mathbf{w}} = 0$ thanks to uniqueness results for electromagnetic scattering problems [26, Thm. 6.10].

Eventually, Lemma 8 together with a Fredholm alternative argument shows that (81) has a unique solution, which depends continuously on the data $\vec{\mathbf{j}} \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)'$.

4.7 *Quasi-Optimality of Whitney Form Galerkin Discretization*

The Whitney form Galerkin discretization of the variational Maxwell cavity problem (81) seeks an edge element vectorfield $\vec{\mathbf{a}}_h \in \mathscr{W}^1(\mathscr{T}_h)$, $\mathscr{T}_h$ a simplicial mesh of $\Omega \subset \mathbb{R}^3$, such that

$$\mathsf{a}_M(\vec{\mathbf{a}}_h, \vec{\mathbf{a}}'_h) = \imath\omega \int_\Omega \vec{\mathbf{j}} \cdot \vec{\mathbf{a}}'_h \,\mathrm{d}\boldsymbol{x} \quad \vec{\mathbf{a}}'_h \in \mathscr{W}^1(\mathscr{T}_h)\,. \tag{100}$$

As we saw in Sect. 3.4.4 this is a conforming Galerkin method in the sense that $\mathscr{W}^1(\mathscr{T}_h) \subset \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$.

Our a priori convergence results will be asymptotic and be valid only on "sufficiently fine" meshes. This forces us to examine the behavior of the discretization error for some infinite family of meshes $\{\mathscr{T}_h\}_{h\in\mathbb{H}}$, where $\mathbb{H}$ is a sequence of mesh widths tending to 0. A key assumption is the *h-uniform shape regularity* of $\{\mathscr{T}_h\}_{h\in\mathbb{H}}$, which makes it possible for us to demand that below none of the constants may depend on $h \in \mathbb{H}$.

4.7.1 Discrete inf-sup Conditions

In Sect. 4.6.2 we learned that the continuous variational problem (81) is well posed. As explained in [45, Sect. 2.1.6], this is equivalent to a continuous inf-sup condition, namely the existence of a constant $\gamma > 0$ such that, for all $\vec{\mathbf{w}} \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$

$$\sup_{\vec{\mathbf{w}}' \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \frac{|\mathsf{a}_M(\vec{\mathbf{w}}, \vec{\mathbf{w}}')|}{\left\|\vec{\mathbf{w}}'\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}} \geq \gamma \left\|\vec{\mathbf{w}}\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} . \tag{101}$$

On the other hand, existence, uniqueness, and quasi-optimality of Galerkin solutions of (100) can be concluded from a *discrete inf-sup condition* that asserts the existence of $\gamma_h > 0$ depending only on the shape regularity of the mesh $\mathscr{T}_h$ such that for all $\vec{\mathbf{w}}_h \in \mathscr{W}^1(\mathscr{T}_h)$

$$\boxed{\sup_{\vec{\mathbf{w}}'_h \in \mathscr{W}^1(\mathscr{T}_h)} \frac{|\mathsf{a}_M(\vec{\mathbf{w}}_h, \vec{\mathbf{w}}'_h)|}{\left\|\vec{\mathbf{w}}'_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}} \geq \gamma_h \left\|\vec{\mathbf{w}}_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}} . \tag{102}$$

More precisely we have the result [45, Thm. 4.2.1]

Theorem 16 (Generalized Cea Lemma) *If* (102) *holds, a unique solution* $\vec{\mathbf{a}}_h \in \mathscr{W}^1(\mathscr{T}_h)$ *of* (100) *exists and is* quasi-optimal*:*

$$\left\|\vec{\mathbf{a}} - \vec{\mathbf{a}}_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \leq C \inf_{\vec{\mathbf{w}}_h \in \mathscr{W}^1(\mathscr{T}_h)} \left\|\vec{\mathbf{a}} - \vec{\mathbf{w}}\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} ,$$

where $\vec{\mathbf{a}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ *is the solution of* (81) *and* $C > 0$ *depends only on the norm of* a_M *and* γ_h*.*

The customary attack on (102) picks an arbitrary $\vec{\mathbf{w}}_h \in \mathscr{W}^1(\mathscr{T}_h)$ and looks for a "candidate function" $\vec{\mathbf{w}}'_h = \vec{\mathbf{w}}'_h(\vec{\mathbf{w}}_h) \in \mathscr{W}^1(\mathscr{T}_h)$ such that, with constants enjoying the usual (in)dependencies,

$$\text{(i)} \qquad |\mathsf{a}_M(\vec{\mathbf{w}}_h, \vec{\mathbf{w}}'_h)| \geq C \left\|\vec{\mathbf{w}}\right\|^2_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} , \tag{103}$$

$$\text{(ii)} \qquad \left\|\vec{\mathbf{w}}'_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \leq C \left\|\vec{\mathbf{w}}_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} . \tag{104}$$

The search for this discrete candidate function can be guided by finding an analogous continuous candidate function for (101). It will be a gift of the Generalized Gårding inequality from Theorem 15, because, given $\vec{\mathbf{w}} \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$ we may choose the complex conjugate of

$$\vec{\mathbf{w}}' := \mathsf{F}\vec{\mathbf{w}} + (\mathsf{A}_M^*)^{-1}(\mathsf{K}_M \vec{\mathbf{w}}) , \tag{105}$$

where $\mathsf{A}_M^* : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)'$ is the (invertible) formal adjoint of A_M. Since $\left\|\vec{\mathbf{w}}'\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \leq C\left\|\vec{\mathbf{w}}\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}$ we conclude (101) from Theorem 15 by simply evaluating $\mathsf{a}_M(\vec{\mathbf{w}}, \overline{\vec{\mathbf{w}}'})$:

$$\begin{aligned}\mathrm{Re}\{\mathsf{a}_M(\vec{\mathbf{w}}, \overline{\vec{\mathbf{w}}'})\} &= \mathrm{Re}\{\mathsf{a}_M(\vec{\mathbf{w}}, \mathsf{F}\overline{\vec{\mathbf{w}}} + (\mathsf{A}_M^*)^{-1}(\mathsf{K}_M\overline{\vec{\mathbf{w}}}))\} \\ &= \mathrm{Re}\{\mathsf{a}_M(\vec{\mathbf{w}}, \mathsf{F}\overline{\vec{\mathbf{w}}}) + \left\langle \mathsf{K}\vec{\mathbf{w}}, \overline{\vec{\mathbf{w}}}\right\rangle\} \geq C\left\|\vec{\mathbf{w}}\right\|^2_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} .\end{aligned}$$

Now, the challenge is that

$$\vec{\mathbf{w}}'(\vec{\mathbf{w}}_h) \text{ will usually not belong to } \mathscr{W}^1(\mathscr{T}_h), \text{ even for } \vec{\mathbf{w}}_h \in \mathscr{W}^1(\mathscr{T}_h).$$

4.7.2 Discrete Helmholtz Cavity Problem

Let us first elucidate the strategy for the Helmholtz cavity problem (84), for which we can rely on the Gårding inequality from Corollary 2. Here, for a fixed $\vec{u} \in H^1(\Omega)$, the continuous candidate function is

$$\vec{u}'(\vec{u}) := \vec{u} + (\mathsf{A}_H^*)^{-1}(\mathsf{K}_H\vec{u}) . \tag{106}$$

Please note the difference between (105) and (106); a counterpart of the sign-flipping isomorphism F is conspicuously absent in (106). This makes it possible to obtain a suitable discrete candidate function by plain projection.

For the Helmholtz cavity problem we have to employ discrete 0-forms for Galerkin discretization, that is, the space $\mathscr{W}^0(\mathscr{T}_h)$ of piecewise linear Lagrangian finite element functions. Let G_h^H stand for the $H^1(\Omega)$-orthogonal projection $\mathsf{G}_h^H : H^1(\Omega) \to \mathscr{W}^0(\mathscr{T}_h)$. Asymptotic density of $\mathscr{W}^0(\mathscr{T}_h)$ in $H^1(\Omega)$ for $h \to 0$ implies that $\mathsf{G}_h^H \to \mathsf{Id}$ *pointwise* in $H^1(\Omega)$ for $h \to 0$. This is an important observation thanks to the following result [36, Lemma 7.1].

Lemma 9 *Let* $\{\mathsf{T}_j\}_{j\in\mathbb{N}}$ *be a sequence of bounded operators* $X \to Y$*,* X, Y *Banach spaces, with* $\lim_{j\to\infty}\mathsf{T}_j x = 0$ *for all* $x \in X$ *(pointwise convergence). If, for another Banach space* Z*,* $\mathsf{K} : Z \to X$ *is* compact*, then*

$$\lim_{j\to\infty}\left\|\mathsf{T}_j \circ \mathsf{K}\right\|_{Z\to Y} = 0 .$$

This tells us that "compactness promotes pointwise convergence to uniform convergence". We apply this lemma with $X = Y = Z = H^1(\Omega)$, $\mathsf{K} = (\mathsf{A}_H^*)^{-1} \circ \mathsf{K}_H$, which inherits compactness from K_H, and $\mathsf{T}_j \leftrightarrow \mathsf{Id} - \mathsf{G}_h^H$. Thus, we infer that

$$\lim_{h\to 0} \sup_{\vec{v}\in H^1(\Omega)} \frac{\left\| \mathsf{G}_h^H((\mathsf{A}_H^*)^{-1}(\mathsf{K}_H \vec{v})) \right\|_{H^1(\Omega)}}{\left\| \vec{v} \right\|_{H^1(\Omega)}} = 0 \,. \tag{107}$$

This makes it possible to pick as a discrete candidate function

$$\vec{u}'_h(\vec{u}_h) := \vec{u}_h + \mathsf{G}_h^H((\mathsf{A}_H^*)^{-1}(\mathsf{K}_H \vec{u}_h)) \in \mathscr{W}^0(\mathscr{T}_h) \,, \tag{108}$$

because

$$\mathsf{a}_H(\vec{u}_h, \vec{u}'_h) = \mathsf{a}_H(\vec{u}_h, (\mathsf{A}_H^*)^{-1}(\mathsf{K}_H \vec{u}_h) + (\mathsf{G}_h^H - \mathsf{Id})(\mathsf{A}_H^*)^{-1}(\mathsf{K}_H \vec{u}_h)) \,, \tag{109}$$

and limit (107) ensures that for any $c > 0$ there is a sufficiently small $h_c > 0$ with

$$\left\| (\mathsf{Id} - \mathsf{G}_h^H)((\mathsf{A}_H^*)^{-1}(\mathsf{K}_H \vec{u}_h)) \right\|_{H^1(\Omega)} \leq c \left\| \vec{u}_h \right\|_{H^1(\Omega)} \quad \forall \vec{u}_h \in \mathscr{W}^0(\mathscr{T}_h),\ \forall h < h_c \,. \tag{110}$$

Thus by picking h smaller than some threshold, we can make the constant c in

$$\mathrm{Re}\{\mathsf{a}_H(\vec{u}_h, \vec{u}'_h)\} \geq C \left\| \vec{u}_h \right\|_{H^1(\Omega)}^2 - c \left\| \mathsf{A}_H^{-1} \right\| \left\| \mathsf{K}_H \right\| \left\| \vec{u}_h \right\|_{H^1(\Omega)}{}^2 \tag{111}$$

smaller than $\frac{1}{2}C/(\left\| \mathsf{A}_H^{-1} \right\| \left\| \mathsf{K}_H \right\|)$, $C > 0$ from Corollary 2. This yields an *asymptotic discrete inf-sup condition* in the sense that it will hold on sufficiently fine meshes only.

4.7.3 Discrete Maxwell Cavity Problem

As pointed out above, in contrast to (106), neither summand in (105) lies in the finite element space $\mathscr{W}^1(\mathscr{T}_h)$. Hence, both terms have to be projected onto $\mathscr{W}^1(\mathscr{T}_h)$ in order to obtain an admissible discrete candidate function. Reusing notations from Sect. 4.7.1, let us opt for

$$\vec{\mathbf{w}}'_h(\vec{\mathbf{w}}_h) := \mathsf{I}_1(\mathsf{F}(\vec{\mathbf{w}}_h)) + \mathsf{G}_h^H\big((\mathsf{A}_M^*)^{-1}(\mathsf{K}_M \vec{\mathbf{w}}_h)\big) \,, \tag{112}$$

where $\mathsf{G}_h^H : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega) \to \mathscr{W}^1(\mathscr{T}_h)$ is the $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl}, \Omega)$-orthogonal projection onto $\mathscr{W}^1(\mathscr{T}_h)$. As all operators involved in (112) are continuous, $\left\| \vec{\mathbf{w}}'_h \right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}$ can be bounded by $\left\| \vec{\mathbf{w}}_h \right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)}$, thus satisfying (104).

In order to establish (103), we try to recover the continuous candidate function $\vec{\mathbf{w}}'(\vec{\mathbf{w}}_h)$ from (105) by "adding zero" in a clever way.

$$\begin{aligned}\mathsf{a}_M(\vec{\mathbf{w}}_h, \vec{\mathbf{w}}_h'(\vec{\mathbf{w}}_h)) &= \mathsf{a}_M\big(\vec{\mathbf{w}}_h, \mathsf{I}_1(\mathsf{F}(\vec{\mathbf{w}}_h)) + \mathsf{G}_h^H\big((\mathsf{A}_M^*)^{-1}(\mathsf{K}_M\vec{\mathbf{w}}_h)\big)\big) \\ &= \mathsf{a}_M(\vec{\mathbf{w}}_h, \vec{\mathbf{w}}'(\vec{\mathbf{w}}_h)) + \mathsf{a}_M\big(\vec{\mathbf{w}}_h, (\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h) \\ &\quad + (\mathsf{G}_h^H - \mathsf{Id})\big((\mathsf{A}_M^*)^{-1}(\mathsf{K}_M\vec{\mathbf{w}}_h)\big)\big)\ .\end{aligned}$$

Since $(\mathsf{A}_M^*)^{-1}\circ\mathsf{K}_M : \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega) \to \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$ is compact and $\mathsf{G}_h^H - \mathsf{Id} \to 0$ pointwise in $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$ for $h \to 0$, we can copy the approach of Sect. 4.7.2 for the last term and appeal to Lemma 9. This confirms the existence of $c : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{h\to 0} c(h) = 0$ such that

$$\left\|(\mathsf{G}_h^H - \mathsf{Id})\big((\mathsf{A}_M^*)^{-1}(\mathsf{K}_M\vec{\mathbf{w}}_h)\big)\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \le c(h)\left\|\vec{\mathbf{w}}_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} . \tag{113}$$

To deal with the other terms we use $\mathsf{F} = 2\mathsf{R} - \mathsf{Id}$ and the projector property of I_1:

$$(\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h) = (\mathsf{I}_1 - \mathsf{Id})(2\mathsf{R}(\vec{\mathbf{w}}_h) - \vec{\mathbf{w}}_h) = 2(\mathsf{I}_1 - \mathsf{Id})\mathsf{R}\vec{\mathbf{w}}_h\ .$$

Since $\mathbf{curl}\mathsf{R}\vec{\mathbf{w}}_h = \mathbf{curl}\vec{\mathbf{w}}_h \in \mathscr{W}^2(\mathscr{T}_h)$ and R maps into $(H^1(\Omega))^3$, the assumptions of Lemma 4 are fulfilled and we can use its interpolation error estimate:

$$\left\|(\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h)\right\|_{L^2(\Omega)} \le Ch\left\|\mathsf{R}\vec{\mathbf{w}}_h\right\|_{H^1(\Omega)} \le Ch\left\|\mathbf{curl}\vec{\mathbf{w}}_h\right\|_{L^2(\Omega)} . \tag{114}$$

Moreover, by the commuting diagram property of I_1 and the projector property of I_2

$$\mathbf{curl}(\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h) = 2(\mathsf{I}_2 - \mathsf{Id})(\mathbf{curl}\mathsf{R}(\vec{\mathbf{w}}_h)) = 0\ . \tag{115}$$

The estimate for the boundary contribution to the $\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$-norm is more subtle. Here we merely cite a consequence of [20, Lemma 16], which can be proved by interpolation in Sobolev scales.

Lemma 10 *If* $\vec{\mathbf{w}} \in \{\vec{\mathbf{u}} \in (H^1(\Omega))^3 : \mathbf{curl}\vec{\mathbf{u}} \in \mathscr{W}^2(\mathscr{T}_h)\}$ *then*

$$\left\|\vec{\mathbf{w}} - \mathsf{I}_1\vec{\mathbf{w}}\right\|_{L^2(\partial\Omega)} \le Ch^{\frac{1}{2}}\left|\vec{\mathbf{w}}\right|_{H^1(\Omega)} ,$$

with $C > 0$ *depending only on* Ω *and the shape regularity of the mesh.*

Applying this estimate and the same reasoning that led to (114), we end up with

$$\left\|(\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h)\right\|_{L^2(\partial\Omega)} \le Ch^{\frac{1}{2}}\left\|\mathbf{curl}\vec{\mathbf{w}}_h\right\|_{L^2(\Omega)} . \tag{116}$$

Eventually, combining (114), (115), and (116), we conclude

$$\left\|(\mathsf{I}_1 - \mathsf{Id})\mathsf{F}(\vec{\mathbf{w}}_h)\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \leq Ch^{\frac{1}{2}}\left\|\mathbf{curl}\,\vec{\mathbf{w}}_h\right\|_{L^2(\Omega)} . \tag{117}$$

This, together with (113) shows that existence of a function $c' : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|\mathsf{a}_M(\vec{\mathbf{w}}_h, \vec{\mathbf{w}}'_h(\vec{\mathbf{w}}_h) - \vec{\mathbf{w}}'(\vec{\mathbf{w}}_h))| \leq c'(h)\left\|\vec{\mathbf{w}}_h\right\|^2_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} ,$$

and

$$\lim_{h\to 0} c'(h) = 0 .$$

This, via (103), permits us to infer the discrete inf-sup condition (102) from the continuous inf-sup condition (101) for sufficiently small h. Thus we have verified the assumptions of Theorem 16, whose concrete assertion for the Galerkin discretization of the Maxwell cavity boundary value problem is given as a final result.

Theorem 17 (Asymptotic Quasi-Optimality of Whitney Form Galerkin Discretization of the Maxwell Cavity Problem) *For any shape-regular family of meshes $\{\mathscr{T}_h\}_{h\in\mathbb{H}}$ there is a threshold $h^* > 0$ and a constant $C > 0$ depending only on the (material) coefficients in a_M, ω, Ω, and shape-regularity, such that*

$$\left\|\vec{\mathbf{a}} - \vec{\mathbf{a}}_h\right\|_{\boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)} \leq C \inf_{\vec{\mathbf{w}}\in\mathscr{W}^1(\mathscr{T}_h)} \left\|\vec{\mathbf{a}} - \vec{\mathbf{w}}\right\| \quad \forall h \leq h^* ,$$

where $\vec{\mathbf{a}} \in \boldsymbol{H}_{\partial\Omega}(\mathbf{curl},\Omega)$ and $\vec{\mathbf{a}}_h \in \mathscr{W}^1(\mathscr{T}_h)$ solve (81) and (100), respectively.

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