Constructive Discursive Logic: Paraconsistency in Constructivism

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Abstract We propose a constructive discursive logic with strong negation *CDLSN* based on Nelson's constructive logic N^- as a constructive version of Jaśkowski's discursive logic. In *CDLSN*, discursive negation is defined similar to intuitionistic negation and discursive implication is defined as material implication using discursive negation. We give an axiomatic system and Kripke semantics with a completeness proof. We also discuss some possible applications of *CDLSN* for common-sense reasoning.

Keywords Jaśkowski · Constructive discursive logic · Common-sense reasoning

1 Introduction

Jaśkowski proposed *discursive logic* (or *discussive logic*) in 1948. It is the first formal *paraconsistent logic* which is classified as a *non-adjunctive system*; see Jaśkowski [10]. The gist of discursive logic is to consider the nature of our ordinary discourse. In a discourse, there are several *participants* who have some information, beliefs, and such. In this regard, truth is formalized by means of the sum of opinions supplied by participants. Even if each participant has consistent information, some participant could be inconsistent with other participants.

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It is reasonable to suppose that $A \wedge \sim A$ (A and $not(A)$) does not hold while both A and $\sim A$ hold to describe such situations. This means that the so-called *adjunction*, i.e., from $\vdash A, \vdash B$ to $\vdash A \wedge B$, is invalid. Here, $\vdash A$ reads "A is provable". Jaśkowski modeled the idea founded on modal logic S5 and reached the discursive logic in which adjunction and *modus ponens* cannot hold. In addition, Jaśkowski introduced discursive implication $A \rightarrow_d B$ as $\diamondsuit A \rightarrow B$ satisfying *modus ponens*, where $\diamondsuit$ denotes the possibility operator.

Akama, Abe and Nakamatsu [5] proposed a constructive discursive logic based on constructivism. It can be viewed as a constructive version of Jaśkowski's original system; also see Akama [4]. Its base is Nelson's constructive logic [17], although Jaśkowski developed his discursive logic based on classical modal logic. Our approach is seen as a new way of formalizing discursive logic.

The rest of this paper is as follows. Section 2 reviews Jaśkowski's discursive logic. In Sect. 3, we introduce constructive discursive logic with strong negation *CDLSN* with an axiomatic system. Section 4 outlines a Kripke semantics. We establish the completeness theorem. In Sect. 5, we suggest possible applications of *CDLSN* for common-sense reasoning. Section 6 concludes the paper with a discussion on future work. This paper is based on the materials in Akama, Abe and Nakamatsu [5] and Akama, Nakamatsu and Abe [6].

2 Jaśkowski's Discursive Logic

Discursive logic, due to the Polish logician S.Jaśkowski [10], is a formal system J satisfying the conditions: (a) from two contradictory propositions, it should not be possible to deduce any proposition; (b) most of the classical theses compatible with (a) should be valid; (c) J should have an intuitive interpretation.

Such a calculus has, among others, the following intuitive properties remarked by Jaśkowski himself: suppose that one desires to systematize in only one deductive system all theses defended in a discussion. In general, the participants do not confer the same meaning to some of the symbols. One would have then as theses of a deductive system that formalize such a discussion, an assertion and its negation, so both are "true" since it has a variation in the sense given to the symbols. It is thus possible to regard discursive logic as one of the so-called *paraconsistent logics*.

Jaśkowski's D_2 contains propositional formulas built from the logical symbols of classical logic. In addition, possibility operator $\diamondsuit$ in S5 is added. $\diamondsuit A$ reads "A is possible". Based on the possibility operator, three discursive logical symbols can be defined as follows:

$$\begin{array}{ll}
\text{discursive implication}: & A \rightarrow_d B =_{def} \diamondsuit A \rightarrow B \\
\text{discursive conjunction}: & A \wedge_d B =_{def} \diamondsuit A \wedge B \\
\text{discursive equivalence}: & A \leftrightarrow_d B =_{def} (A \rightarrow_d B) \wedge_d (B \rightarrow_d A)
\end{array}$$

Additionally, we can define discursive negation $\neg_d A$ as $A \to_d false$. Jaśkowski's original formulation of D_2 in [10] used the logical symbols: $\to_d, \leftrightarrow_d, \vee, \wedge, \neg$, and he later defined $\wedge_d$ in [11].

The axiomatization due to Kotas [12] has the following axioms and the rules of inference. Here, $\Box$ is the necessity operator, and is definable by $\neg\Diamond\neg$. $\Box A$ reads "A is necessary".

Axioms

(A1) $\Box(A \to (\neg A \to B))$
(A2) $\Box((A \to B) \to ((B \to C) \to (A \to C)))$
(A3) $\Box((\neg A \to A) \to A)$
(A4) $\Box(\Box A \to A)$
(A5) $\Box(\Box(A \to B) \to (\Box A \to \Box B))$
(A6) $\Box(\neg\Box A \to \Box\neg\Box A)$

Rules of Inference

(R1) substitution rule
(R2) $\Box A, \Box(A \to B)/\Box B$
(R3) $\Box A/\Box\Box A$
(R4) $\Box A/A$
(R5) $\neg\Box\neg\Box A/A$

Note that discursive implication $\to_d$ satisfies *modus ponens* in S5, but $\to$ does not. There are other axiomatizations of D_2. For example, da Costa and Dubikajtis gave an axiomatization based on the connectives $\to_d, \wedge_d, \neg$; see [8]. Semantics for discursive logic can be obtained by a Kripke semantics for modal logic S5. Jaśkowski's three conditions for J mentioned above are solved by many workers in different ways. For a comprehensive survey on discursive logic, see da Costa and Doria [9].

3 Constructive Discursive Logic with Strong Negation

The gist of discursive logic is to use the modal logic S5 to define discursive logical connectives which can formalize a non-adjunctive system. It follows that discursive logic can be seen as a paraconsistent logic, which does not satisfy *explosion* of the form: $\{A, \neg A\} \models B$ for any A and B, where $\models$ is a consequence relation. We say that a system is *trivial* iff all the formulas are provable. Therefore, paraconsistent logic is useful to formalize inconsistent but *non-trivial* theories.

Most works on discursive logic utilize classical logic and S5 as a basis. However, we do not think that these are essential. For instance, different modal logics yield the corresponding discursive logics. We can use non-classical logics as the base. An intuitionist hopes to have a discursive system in a constructive setting. It is the starting point of Akama, Abe and Nakamatsu [5].

To make the idea formal, it is worth considering Nelson's constructive logic with strong negation N^- of Almukdad and Nelson [7]. In N^-, $\sim$ denotes *strong negation* satisfying the following axioms:

(N1) $\sim\sim A \leftrightarrow A$
(N2) $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(N3) $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
(N4) $\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$

and the axiomatization of the intuitionistic positive logic Int^+ with *modus ponens* (MP), i.e. $A, A \rightarrow B/B$ as the rule of inference.

Strong negation can express explicit negative information which cannot be described by intuitionistic negation. In this sense, strong negation is constructive, but intuitionistic negation is not. As the name shows, strong negation is stronger than intuitionistic negation in that $\sim A \rightarrow \neg A$ holds but the converse does not. Note here that N^- is paraconsistent in the sense that $\sim(A \wedge \sim A)$ and $(A \wedge \sim A) \rightarrow B$ do not hold.

If we add (N0) to N^-, we have N of Nelson [17].

$$\text{(N0)}\ (A \wedge \sim A) \rightarrow B$$

In N, *intuitionistic negation* $\neg$ can be defined as follows:

$$\neg A =_{def} A \rightarrow \ \sim A$$

If we add the law of *excluded middle*: $A \vee \sim A$ to N, the resulting system is classical logic.

Indeed, N^- is itself a paraconsistent logic; see Akama [3]. But it can also be accommodated as a version of discursive logic.

Now, we introduce the *constructive discursive logic with strong negation CDLSN*. It diverges in two ways from D_2: (1) it does not take classical logic as its starting point; and (2) it does not use the possibility operator $\Diamond$ as a modality, but use two negation operators.

CDLSN can be defined in two ways. One is to extend N^- with discursive negation $\neg_d$. The other is to weaken intuitionistic negation in N^-. We adopt the first approach.

Here, we fix the language of the logics which we use in this paper. The language of Int^+ is defined as the set of propositional variables and logical symbols: $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication). The language of *Int* is the extension of that of Int^+ with $\neg$ (intuitionistic negation). The language of N^- is the extension of that of Int^+ with $\sim$ (strong negation). The language of *CDLSN* is the extension of N^- with $\neg_d$ (discursive negation). Additionally, we use the logical constant *false* as the abbreviation of $\sim(A \rightarrow A)$.

We believe that *CDLSN* is (constructive) improvement of D_2. First, *CDLSN* uses Int^+ rather than classical logic as the base. Second, *CDLSN* simulates modality in D_2 by negations, although D_2 needs the possibility operator.

$\neg_d$ is similar to $\neg$, but these are not equivalent. The motivation of introducing $\neg_d$ is to interpret discursive negation as the negation used by an intuitionist in the discursive context. Unfortunately, intuitionistic negation is not a discursive negation. And we need to re-interpret it as $\neg_d$. Based on $\neg_d$, we can define $\rightarrow_d$ and $\wedge_d$.

Discursive implication $\rightarrow_d$ and discursive conjunction $\wedge_d$ can be respectively introduced by definition as follows.

$$A \rightarrow_d B =_{def} \neg_d A \vee B$$
$$A \wedge_d B =_{def} \sim\neg_d A \wedge B$$

Observe that $A \rightarrow (\sim A \rightarrow B)$ is not a theorem in *CDLSN* while $A \rightarrow (\neg_d A \rightarrow B)$ is a theorem in *CDLSN*. The axiomatization of *CDLSN* is that of N^- with the following three axioms.

$$\text{(CDLSN1)}\ \neg_d A \rightarrow (A \rightarrow B)$$
$$\text{(CDLSN2)}\ (A \rightarrow B) \rightarrow ((A \rightarrow \neg_d B) \rightarrow \neg_d A)$$
$$\text{(CDLSN3)}\ A \rightarrow\ \sim\neg_d A$$

Here, an explanation of these axioms may be in order. (CDLSN1) and (CDLSN2) describe basic properties of intuitionistic negation. By (CDLSN3), we show the connection of $\sim$ and $\neg_d$. The intuitive interpretation of $\sim\neg_d$ is like possibility under our semantics developed below.

$\neg_d$ is weaker than $\neg$. Vorob'ev [20] proposed a constructive logic having both strong and intuitionistic negation. It extends N with the following two axioms:

$$\sim\neg A \leftrightarrow A$$
$$\sim A \rightarrow \neg A,$$

where A is atomic

If we replace (CDLSN3) by the axiom of the form $\sim\neg_d A \leftrightarrow A$ and add the axiom $\sim A \rightarrow \neg_d A$, then $\neg_d$ agrees with $\neg$. Thus, it is not possible to identify $\neg$ and $\neg_d$ in our axiomatization.

We use $\vdash A$ to mean that A is a theorem in *CDLSN*. Here, the notion of a proof is defined as usual. Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas and A be a formula. Then, $\Gamma \vdash A$ iff $\vdash \Gamma \rightarrow A$.

Notice that $\neg_d$ has some similarities with $\neg$, as the following lemma indicates.

Lemma 1 *The following formulas are provable in CDLSN.*

(1) $\vdash A \rightarrow \neg_d\neg_d A$
(2) $\vdash (A \rightarrow B) \rightarrow (\neg_d B \rightarrow \neg_d A)$
(3) $\vdash (A \wedge \neg_d A) \rightarrow B$
(4) $\vdash \neg_d(A \wedge \neg_d A)$
(5) $\vdash (A \rightarrow \neg_d A) \rightarrow \neg_d A$

Proof Ad(1): From (CDSLN1) and Int^+ i.e. $\vdash (A \to (B \to C)) \to (B \to (A \to C)))$, we have (i).

(i) $\vdash A \to (\neg_d A \to A)$

(ii) is an instance of (CDLSN2).
$\vdash (\neg_d A \to A) \to ((\neg_d A \to \neg_d A) \to \neg_d \neg_d A)$

(iii) is a theorem of Int^+, i.e., $\vdash (A \to B) \to ((B \to C) \to (A \to C)))$
$\vdash (A \to (\neg_d A \to A)) \to (((\neg_d A \to A) \to ((\neg_d A \to \neg_d A)$
$\to \neg_d \neg_d A)) \to (A \to ((\neg_d A \to \neg_d A) \to \neg_d \neg_d A)))$
From (i) and (iii) by (MP), we have (iv).

(iv) $\vdash (((\neg_d A \to A) \to ((\neg_d A \to \neg_d A) \to \neg_d \neg_d A))$
$\to (A \to ((\neg_d A \to \neg_d A) \to \neg_d \neg_d A)))$
From (ii) and (iv) by (MP), we have (v).

(v) $\vdash A \to ((\neg_d A \to \neg_d A) \to \neg_d \neg_d A))$
by $\vdash (A \to (B \to C)) \to (B \to (A \to C))$ we can derive (vi)

(vi) $\vdash (\neg_d A \to \neg_d A) \to (A \to \neg_d \neg_d A)$
since $\vdash A \to A$ we have (vii)

(vii) $\vdash \neg_d A \to \neg_d A$From (vi) and (vii) by (MP), we can finally obtain (viii).

(viii) $\vdash A \to \neg_d \neg_d A$

Ad(2): By (CDLSN2), we have (i).

(i) $\vdash (A \to B) \to ((A \to \neg_d B) \to \neg_d A)$

(ii) is a theorem of Int^+.
$\vdash (\neg_d B \to (A \to \neg_d B)) \to (((A \to \neg_d B) \to \neg_d A)$
$\to (\neg_d B \to \neg_d A))$

(iii) is an instance of $A \to (B \to A)$ which is the axiom of Int^+
$\vdash \neg_d B \to (A \to \neg_d B)$
From (ii) and (iii) by (MP), (iv) is obtained.

(iv) $\vdash ((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A)$

(v) is a theorem of Int^+.
$\vdash ((A \to B) \to ((A \to \neg_d B) \to \neg_d A)$
$\to ((((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A))$
$\to ((A \to B) \to (\neg_d B \to \neg_d A)))$
From (i) and (v) by (MP), (vi) can be proved.

(vi) $\vdash ((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A))$
$\to ((A \to B) \to (\neg_d B \to \neg_d A))$
From (iv) and (vi) by (MP), we can reach (vii).

(vii) $\vdash (A \to B) \to (\neg_d B \to \neg_d A)$

Ad(3): By (CDLSN1), we have (i).

(i) $\vdash \neg_d A \to (A \to B)$
From $\vdash (A \to (B \to C)) \to (B \to (A \to C))$, we can derive (ii).

(ii) $\vdash A \to (\neg_d A \to B)$
since $\vdash (A \to (B \to C)) \to ((A \wedge B) \to C)$, we have (iii).

(iii) $\vdash(A \to (\neg_d A \to B)) \to ((A \wedge \neg_d A) \to B)$
From (ii) and (iii) by (MP), we can obtain (iv).
(iv) $\vdash(A \wedge \neg_d A) \to B$
Ad(4): By (3), we have (i) and (ii).
(i) $\vdash(A \wedge \neg_d A) \to B$
(ii) $\vdash(A \wedge \neg_d A) \to \neg_d B$
From (CDLSN2), (iii) holds.
(iii) $((A \wedge \neg_d A) \to B) \to (((A \wedge \neg_d A) \to \neg_d B) \to \neg_d (A \wedge \neg_d A))$
From (i) and (iii) by (MP), we have (iv).
(iv) $((A \wedge \neg_d A) \to \neg_d B) \to \neg_d (A \wedge \neg_A)$
From (ii) and (iv) by (MP), we can derive (v).
(v) $\vdash\neg_d (A \wedge \neg_d A)$
Ad(5): By (CDLSN2), we have (i).
(i) $\vdash(A \to A) \to ((A \to \neg_d A) \to \neg_d A)$
(ii) is a theorem of Int^+.
$\vdash A \to A$
From (i) and (ii) by (MP), we can obtain (iii).
(iii) $(A \to \neg_d A) \to \neg_d A$ □

It should be, however, pointed out that the following formulas are not provable in *CDLSN*.

$\nvdash \sim (A \wedge \sim A)$

$\nvdash A \vee \sim A$

$\nvdash (A \to B) \to (\sim B \to \sim A)$

$\nvdash \neg_d \neg_d A \to A$

$\nvdash A \vee \neg_d A$

$\nvdash (\neg_d A \to A) \to A$

$\nvdash \sim \neg_d A \to A$

$\nvdash A \to_d A$

4 Kripke Semantics

It is possible to give a Kripke semantics for *CDLSN* which is a discursive modification of that for *N*. A Kripke semantics for *N* can be formalized as an extension of that for intuitionistic logic. It first provided by Thomason [19]; also see Akama [1, 2]. Akama [3]studied a Kripke semantics for N^-.

Now, we define a Kripke model for *CDLSN*. Let *PV* be a set of propositional variables and *p* be a propositional variable, and *For* be a set of formulas. A *CDLSN-model* is a tuple $\langle W, w_0, R, V \rangle$, where $W \neq \emptyset$ is a set of *worlds*, $w_0 \in W$ satisfying $\forall w(w_0 R w)$, $R \subseteq W \times W$ is a reflexive and transitive relation, and $V : PV \times W \to \{0, 1\}$ is a partial *valuation* satisfying:

$$V(p,w) = 1 \text{ and } wRv \ \Rightarrow \ V(p,v) = 1$$
$$V(p,w) = 0 \text{ and } wRv \ \Rightarrow \ V(p,v) = 0$$

for any formula $p \in PV$ and $w, v \in W$. Here, $V(p,w) = 1$ is read “p is true at w” and $V(p,w) = 0$ is read “p is false at w”, respectively. Both truth and falsity are independently given by a constructive setting.

We can now extend V for any formula A, B in a tandem way as follows.

$$\begin{array}{ll}
V(\sim A, w) = 1 & \text{iff } V(A,w) = 0 \\
V(A \wedge B, w) = 1 & \text{iff } V(A,w) = 1 \text{ and } V(B,w) = 1 \\
V(A \vee B, w) = 1 & \text{iff } V(A,w) = 1 \text{ or } V(B,w) = 1 \\
V(A \rightarrow B, w) = 1 & \text{iff } \forall v(wRv \text{ and } V(A,v) = 1 \Rightarrow V(B,v) = 1) \\
V(\neg_d A, w) = 1 & \text{iff } \forall v(wRv \Rightarrow V(A,v) = 0) \\
V(\sim A, w) = 0 & \text{iff } V(A,w) = 1 \\
V(A \wedge B, w) = 0 & \text{iff } V(A,w) = 0 \text{ or } V(B,w) = 0 \\
V(A \vee B, w) = 0 & \text{iff } V(A,w) = 0 \text{ and } V(B,w) = 0 \\
V(A \rightarrow B, w) = 0 & \text{iff } V(A,w) = 1 \text{ and } V(B,w) = 0 \\
V(\neg_d A, w) = 0 & \text{iff } \exists v(wRv \text{ and } V(A,v) = 1)
\end{array}$$

Additionally, we need the following condition:

$V(A \wedge \sim A, w) = 1$ for some A and some w.

This condition is used to invalidate $(A \wedge \sim A) \rightarrow B$, and guarantees the paraconsistency of $\sim$ in *CDSLN*.

Here, observe that truth and falsity conditions for $\sim\neg_d A$ are implicit in the above clauses from the equivalences such that $V(\sim\neg_d A, w) = 1$ iff $V(\neg_d A, w) = 0$, and $V(\sim\neg_d A, w) = 0$ iff $V(\neg_d A, w) = 1$. One can claim that $\sim\neg_d$ behaves as a modality. In this regard, we do not need to introduce a possibility operator into *CDLSN* as a primitive.

We say that A is *valid*, written $\vDash A$, iff $V(A, w_0) = 1$ in all *CDLSN*-models. Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas. Then, we say that Γ *entails* A, written $\Gamma \vDash A$, iff $\Gamma \rightarrow A$ is valid.

Lemma 2 states the monotonicity of valuation in a Kripke model.

Lemma 2 *The following hold for any formula A which is not of the form* $\sim\neg_d B$, *and any worlds* $w, v \in W$.

$$V(A,w) = 1 \text{ and } wRv \ \Rightarrow \ V(A,v) = 1,$$
$$V(A,w) = 0 \text{ and } wRv \ \Rightarrow \ V(A,v) = 0.$$

Proof By induction on A.

Ad($\sim$): Suppose $V(\sim A, w) = 1$and wRv. Then, we have that $V(A,w) = 0$ and wRv. By induction hypothesis (IH), we have that $V(A,v) = 0$, i.e $V(\sim A, v) = 1$.

Suppose $V(\sim A, w) = 0$and wRv. Then, we have that $V(A, w) = 1$ and wRv. By (IH), we have that $V(A, v) = 1$, i.e. $V(\sim A, v) = 0$.

Ad(∧): Suppose $V(A \wedge B, w) = 1$ and wRv. Then, we have $V(A, w) = 1$ and $V(B, w) = 1$. By (IH), $V(A, v) = 1$ and $V(B, v) = 1$, i.e. $V(A \wedge B, v) = 1$.
Suppose $V(A \wedge B, w) = 0$ and wRv. Then, we have $V(A, w) = 0$ or $V(B, w) = 0$. By (IH), $V(A, v) = 0$ or $V(B, v) = 0$, i.e. $V(A \wedge B, v) = 0$.

Ad(∨): Suppose $V(A \vee B, w) = 1$ and $wRv) = 1$. Then, we have $V(A, w) = 1$ or $V(B, w) = 1$. By (IH), $V(A, v) = 1$ or $V(B, v) = 1$, i.e. $V(A \vee B, v) = 1$.
Suppose $V(A \vee B, w) = 0$ and wRv. Then, we have $V(A, w) = 0$ and $V(B, w) = 0$. By (IH), $V(A, v) = 0$ and $V(B, v) = 0$, i.e. $V(A \vee B, v) = 0$.

Ad(→): Suppose $V(A \rightarrow B) = 1$ and wRv. Then, we have $\forall v(wRv \text{ and } V(A, v) = 1 \Rightarrow V(B, v) = 1)$. By (IH) and the transitivity of R $\forall z(vRz \text{ and } V(A, z) = 1 \Rightarrow V(B, z) = 1)$, i.e. $V(A \rightarrow B, v) = 1$.
Suppose $V(A \rightarrow B, w) = 0$ and wRv. Then, we have $V(A, w) = 1$ and $V(B, w) = 0$. By (IH), $V(A, v) = 1$ and $V(B, v) = 0$, i.e. $V(A \rightarrow B, v) = 0$. □

Lemma 2 does not hold for the formula of the form $\sim \neg_d A$. We can easily construct a counter model. We only treat the case of $V(\sim \neg_d A, w) = 1$. The case of $V(\sim \neg_d A, w) = 0$ is similar. Assume that $V(\sim \neg_d A, w) = 1$ and wRv. Then, $V(\neg_d A, w) = 0$ iff $\exists u(wRu \text{ and } V(A, u) = 1)$. Now, suppose that there exists a world t distinct from u such that vRt and a valuation such that $V(A, t) = 0$. This means that $V(\sim \neg_d A, v) = 0$. Thus, $V(\sim \neg_d A, w) = 1$ and wRv, but $V(\sim \neg_d A, v) = 0$.

We think that the fact is intuitive because $\sim \neg_d A$ behaves as possibility. There are no reasons for possibility in discourse to satisfy the monotonicity.

Next, we present a soundness theorem.

Theorem 1 (soundness)$\vdash A \Rightarrow \vDash A$

Proof It suffices to check that (CDLSN1), (CDLSN2) and (CDLSN3) are valid and (MP) preserves validity. The proof of preservation of validity under (MP) is well-known in constructive and intuitionistic logic. Thus, we here prove the validity of three axioms.

Ad(CDLSN1): Suppose it is not valid. Then, $V(\neg_d A, w_0) = 1$ and $V(A \rightarrow B, w_0) \neq 1$. From the first conjunct, $\forall v(w_0 Rv \Rightarrow V(A, v) \neq 1)$holds. From the second conjunct, $\exists v(w_0 Rv \text{ and } V(A, v) = 1 \text{ and } V(B, v) \neq 1)$. However, $V(A, v) = 1$ and $V(A, v) \neq 1$are contradictory.

Ad(CDLSN2): Suppose it is not valid. Then, $V(A \rightarrow B, w_0) = 1$and $V(A \rightarrow \neg_d B, w_0) = 1$ and $V(\neg_d A, w_0) \neq 1$. From the first conjunct, $\forall v(w_0 Rv \text{ and } V(A, v) = 1 \Rightarrow V(B, v) = 1)$ holds. From the second conjunct, $\forall v(w_0 Rv \text{ and } V(A,v) = 1 \Rightarrow V(\neg_d B, v) = 1)$iff $\forall v(wRv \text{ and } V(A,v) = 1 \Rightarrow \forall z(vRz \Rightarrow V(A,z) \neq 1)$. From the third conjunct, $\exists v(w_0 Rv \text{ and } V(A,v) = 1$holds. However, $V(A,v) = 1$ and $V(A,z) \neq 1$ for any z such that vRz are contradictory.

Ad(CDLSN3): Suppose it is not valid. Then, $V(A,w_0) = 1$ and $V(\sim\neg_d A, w_0) \neq 1$. From the second conjunct, we have $V(\neg_d A, w_0) \neq 0$ iff $\forall v(w_0 Rv \Rightarrow V(A,v) \neq 1)$. However, $V(A,w_0) = 1$ and $V(A,v) \neq 1$or any v such that $w_0 Rv$ are contradictory. □

Theorem 3 Theorem 3 can be generalized as a strong form, i.e. $\Gamma \vdash A \Rightarrow \Gamma \vDash A$.

Now, we give a completeness proof. We say that a set of formulas Γ^* is a *maximal non-trivial discursive theory* (mntdt) iff (1) Γ^* is a theory, (2) Γ^* is *non-trivial*, i.e. $\Gamma^* \nvdash B$ for some B, (3) Γ^* is *maximal*, i.e. $A \in \Gamma^*$ or $A \notin \Gamma^*$, (4) Γ^* is *discursive*, i.e. $\neg_d A \notin \Gamma^*$ iff $\sim\neg_d A \in \Gamma^*$. Here, discursiveness is needed to capture the property of discursive negation.

Lemma 3 For any mntdt Γ and any formula A, B the following hold:

(1) $A \wedge B \in \Gamma$ iff $A \in \Gamma$and $B \in \Gamma$
(2) $A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$
(3) $A \rightarrow B \in \Gamma$ iff $\forall \Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta \Rightarrow B \in \Delta)$
(4) $\neg_d A \in \Gamma$ iff $\forall \Delta(\Gamma \subseteq \Delta \Rightarrow A \notin \Delta)$
(5) $\sim(A \wedge B) \in \Gamma$ iff $\sim A \in \Gamma$ or $\sim B \in \Gamma$
(6) $\sim(A \vee B) \in \Gamma$ iff $\sim A \in \Gamma$ and $\sim B \in \Gamma$
(7) $\sim(A \rightarrow B) \in \Gamma$ iff $A \in \Gamma$ and $\sim B \in \Gamma$
(8) $\sim\sim A \in \Gamma$ iff $A \in \Gamma$
(9) $\sim\neg_d A \in \Gamma$ iff $\exists \Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta)$.

Proof We only prove (4) and (9). Other cases are similarly justified from the literature on constructive logic (cf. Thomason [19]).

Ad(4): $\neg_d A \in \Gamma$ iff (by axiom (CDLSN1)) iff (by axiom (CDLSN1)) $A \rightarrow B \in \Gamma$ iff (by Lemma 3 (3)) $\forall \Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta \Rightarrow B \in \Delta)$. Since Γ is non-trivial, $B \notin \Gamma$ for some B. Thus, $B \in \Delta$ does not always hold, i.e.$\forall \Delta(\Gamma \subseteq \Delta$ and $A \in \Delta \Rightarrow \mathit{false})$iff $\forall \Delta(\Gamma \subseteq \Delta \Rightarrow A \notin \Delta)$.

Ad(9): We prove it by contraposition from (4). Contraposition can derive $\exists \Delta(\Gamma \subseteq \Delta$ and $A \in \Delta)$ by negating the left and right sides of (4). Then, it is shown to be equivalent to $\neg_d A \notin \Gamma$. By (discursiveness), $\neg_d A \notin \Gamma$ iff $\sim\neg_d A \in \Gamma$.

Based on the maximal non-trivial discursive theory, we can define a canonical model $(\Gamma, \subseteq, V)$ such that Γ is a mntdt, $\subseteq$ is the subset relation, and V is a valuation satisfying the conditions that $V(p, \Gamma) = 1$ iff $p \in \Gamma$ and that $V(p, \Gamma) = 0$ iff $\sim p \in \Gamma$.□

The next lemma is a truth lemma.

Lemma 4 (truth lemma) *For any mntdt* Γ *and any* A, *we have the following*:

$V(A,\Gamma)=1$ *iff* $A\in\Gamma$

$V(A,\Gamma)=0$ *iff* $\sim A\in\Gamma$

Proof It suffices to check the case $A=\neg_d B$.

$$\begin{array}{ll} V(\neg_d B,\Gamma)=1 & \textit{iff}\ \forall\Delta\in\Gamma^*(\Gamma\subseteq\Delta\Rightarrow V(B,\Delta)\neq 1) \\ (\mathit{IH}) & \textit{iff}\ \forall\Delta\in\Gamma^*(\Gamma\subseteq\Delta\Rightarrow B\notin\Delta) \\ (\mathit{Lemma}\,4\,(4)) & \textit{iff}\ \neg_d B\in\Gamma \end{array}$$

$$\begin{array}{ll} V(\neg_d B,\Gamma)=0 & \textit{iff}\ \exists\Delta\in\Gamma^*(\Gamma\subseteq\Delta\ \text{and}\ V(B,\Delta)=1) \\ (\mathit{IH}) & \textit{iff}\ \exists\Delta\in\Gamma^*(\Gamma\subseteq\Delta\ \text{and}\ B\in\Delta) \\ (\mathit{Lemma}\,4\,(9)) & \textit{iff}\ \sim\neg_d B\in\Gamma \end{array}$$

□

Then, we can state the (strong) completeness of *CDLSN* as follows:

Theorem 2 (completeness). $\Gamma\vDash A\ \Rightarrow\ \Gamma\vdash A$

Proof Assume $\Gamma\nvdash A$. Then, by the Lindenbaum lemma, there is a mntdt Γ such that $A\notin\Gamma$. By using a canonical model defined above, we have $V(A,\Gamma)\neq 1$ by Lemma 4. Consequently, completeness follows. □

Finally, we justify the formal properties of *CDLSN* as a discursive logic. It is extremely important because we can understand the differences of *CDLSN* and standard discursive logics like D_2. As mentioned in Sect. 1, Jaśkowski suggested three conditions of discursive logics. We check them here.

CDLSN is *discursive*. First, $\sim(A\wedge\sim A)$ does not hold. The explosion also fails, i.e. $A,\sim A\nvdash B$. But, these hold for $\neg_d$ (cf. Lemma 1), and are not a problem because explosion should be valid for plausible discourses.

Note that the adjunction of the form $\vdash A,\vdash B\ \Rightarrow\ \vdash A\wedge_d B$ does not hold in *CDLSN*. But, it holds for $\wedge$.

Second, in *CDLSN*, most of the theses of constructive logic are valid. Since *CDLSN* has a constructive base, it is different from D_2 whose base is classical logic.

Third, we can give an intuitive interpretation for *CDLSN* by means of Kripke models as discussed below.

CDLSN is *constructive* because the law of excluded middle, which is a non-constructive principle, does not hold. As discussed above, N^- is a constructive logic, and the fact is not surprising.

From our Kripke semantics given above, we can give an intuitive interpretation of *CDLSN*. The interpretations of the logical symbols of N^- are obvious, and we concentrate on discursive logical symbols.

Here, it may be helpful to explain the interpretation by a brief example. Consider a *discourse* which consists of several persons who are interested in some subjects. Each person has knowledge about subjects, and a discourse is plausibly expanded by adding other persons.

In this setting, a world in our semantics could be identified with the discourse just given. So, the logical symbols can be interpreted with reference to a discourse.

Since the interpretations of $\neg_d$ are crucial, we begin with this, namely

$\neg_d A$ is true iff A is false in all plausible growing discourses,
$\neg_d A$ is false iff A is true in some plausible growing discourse.

Here, the second clause corresponds to the possibility used in discursive logic. Note here that the plausible growth of discourse implies the increase of information (or knowledge) in view of constructive setting.

Other discursive logical symbols can be read as follows:

$A \wedge_d B$ is true iff A is true in one discourse and B is true in another plausible discourse.
$A \rightarrow_d B$ is true iff if A is true in certain plausible discourse then B is true in a discourse.

The interpretations of $\vee_d$ and $\leftrightarrow_d$ can be obtained by definition. The important point here is that the primitive discursive connective is $\neg_d$.

In our approach, two kinds of negations are used and it is necessary to compare them. $\sim$ is a constructive negation which can express constructive falsity of the proposition, whereas $\neg_d$ is a discursive negation of the proposition with modal flavor, which is similar to intuitionistic negation.

They can express the possibility operator needed in discursive logic as $\sim\neg_d$. Here, $\sim$ behaves as classical-like negation and $\neg_d$ as modal-like negation. We know that in classical modal logic the following holds.

$$\Diamond A \equiv -\Box - A$$

Here, $-$ is classical negation and $\equiv$ is classical equivalence. It is therefore natural to consider two negations in classical-like and modal-like ways.

From the above discussion, *CDLSN* is shown to be a constructive discursive logic which is compatible with Jaśkowski's original ideas. It means that a constructivist can formally perform discursive reasoning.

5 Applications

Constructive discursive logic seems to have many applications for several fields. Although discursive logic was originally motivated in a philosophical tradition, it has the potential to be used for other areas. Here, we take up the so-called *common-sense reasoning* like paraconsistent and non-monotonic reasoning, which are of special importance to knowledge representation in *Artificial Intelligence* (AI).

First, we discuss paraconsistent reasoning which can appear in many real situations. That is, it can extract some conclusions in the presence of contradiction. As is well known, it is obliged to have any arbitrary conclusion from contradiction, if we use the underlying logical basis as classical (or intuitionistic) logic.

However, it is not compatible with our common-sense intuition. Human beings can usually do reasoning in a natural manner, even if they are faced with contradiction. Because human beings have obviously limited memory and reasoning capacity, their knowledge is not always consistent. And contradiction naturally arises in common-sense reasoning.

Paraconsistent logics are useful in such contexts. It is also to be noticed that paraconsistent logic can serve as a foundation for inconsistent (or paraconsistent) mathematics. *CDLSN* can describe paraconsistent reasoning in a logical setting, since it is a paraconsistent logic. Consider the following knowledge base KB_1.

$$KB_1 = \{A, \neg B, A \rightarrow B, B \rightarrow C\}$$

Here, we assume that the base logic is classical logic. From KB_1, we should conclude C using *modus ponens*. But, it is impossible in classical logic, since KB_1 produces inconsistency. In fact, both $\vdash_{KB_1} B$ and $\vdash_{KB_1} \neg B$.

However, in classical logic C, $B \wedge \neg B \vdash_C D$, where D denotes an arbitrary formula. In other words, the knowledge base KB_1 is trivial and it is not of use as a knowledge base in that no useful information is derivable. This fact reveals that classical logic is not suited for reasoning under contradiction.

But, we can derive C in *CDLSN*, as required. The reason is that $B \wedge \sim B \nvdash_{CDLSN} D$. This is a desired feature of common-sense reasoning. Normally, a knowledge base is built from incomplete knowledge due to several reasons. Thus, such a knowledge base may contain some contradictions which need to be tolerated.

Second, we show that *CDLSN* can model non-monotonic reasoning, in which old conclusions can be invalidated by new knowledge. Non-monotonic reasoning is regarded as fundamental in common-sense reasoning. But, standard logics like classical logic are monotonic. Minsky addressed the inadequacy of classical logic as the formalism for describing common-sense reasoning by pointing out that classical logic cannot express non-monotonic reasoning; see Minsky [15].

Based on the observation, Minsky considered that a logic-based approach to AI is not adequate and impossible. If we rely on classical logic as the logic, his consideration may be true. But, we can overcome the difficulty by developing a logic which is not monotonic.

In AI, there is a rich literature on *non-monotonic logics*, formalizing non-monotonic reasoning in a logical setting. For instance, McDermott and Doyle proposed a version of non-monotonic logic by extending classical logic with the consistent operator M; see McDermott and Doyle [14].

Their non-monotonic logic is very similar to modal logic. A formula of the form MA in their non-monotonic logic denotes that A is consistent. Unfortunately, their logic lacks formal semantics as discussed below. Later, McDermott [13] worked out non-monotonic logics based on modal logics, but his attempt was not successful. For example, non-monotonic S5 is shown to be monotonic S5.

We also know other interesting non-monotonic logics like the *default logic* of Reiter [18] and the *autoepistemic logic* of Moore [16]. Default logic is an extension of classical logic with default rules, describing default reasoning, i.e., reasoning by default.

Autoepistemic logic extends classical logic with the belief operator, which models beliefs of a rational agent. Since a rational agent can believe his beliefs and lack of beliefs, reasoning based on his beliefs are non-monotonic according to the increase of new beliefs.

Unfortunately, many non-monotonic logics in AI have been criticized due to the lack of theoretical foundations. This is because most non-monotonic logics rely on meta-rules whose interpretation is outside the scope of object-language. For example, the consistency expressed by McDermott and Doyle's non-monotonic logic needs meta-level reasoning. Namely, M cannot be regarded as a modal operator in modal logic in that it has a reasonable semantics in the standard sense.

Now, we see real examples. Consider the knowledge base KB_2.

$$KB_2 = \{A, A \rightarrow B, C\}$$

We can deduce B from KB_2, written $KB_2 \vdash_C B$, in the framework of classical logic. However, a knowledge base grows with new knowledge. Suppose that new knowledge base $\neg B$ is obtained from KB_2 by adding the new knowledge denoted $\neg B$.

$$KB_3 = \{A, A \rightarrow B, C, \neg B\}$$

Here, the desired reasoning is that $\neg B$ is provable, i.e., $KB_3 \vdash \neg B$. This implies that $KB_3 \nvdash B$, where the old conclusion B is withdrawn in KB_3. Classical logic concludes that $B \wedge \neg B$, i.e., contradiction is provable, however.

A typical example is as follows. Normally birds fly, which can be seen as common-sense. Tweety is a bird. Since all birds can fly, we can conclude that Tweety can fly at this stage. Later, we learn that Tweety is a penguin and all penguins cannot fly. At the stage in which new information is supplied, we naturally infer that Tweety cannot fly. The old conclusion that Tweety can fly is invalidated by the new information concerning Tweety, namely, that he is a penguin.

Non-monotonic reasoning can be formally expressed in *CDLSN*. "Normally if A then B" is described as $A \wedge \sim \neg_d B \rightarrow B$. Note here that $\sim \neg_d$ behaves like M in non-monotonic logic. Although non-monotonic logic requires the interpretation of M in the meta-level in that $\Gamma \vdash \mathrm{M}A$ iff $\Gamma \nvdash \sim A$, where Γ denotes a set of formulas, *CDLSN* dispenses with meta-level features in that $\sim \neg_d$ has the formal interpretation in Kripke semantics.

Paraconsistent and non-monotonic reasoning are closely related. Usually, in common-sense reasoning new knowledge seems to be of importance, yielding non-monotonic reasoning. However, it is not always the case. There appear to be situations in which we cannot give a priority of the old conclusion A to the new conclusion $\neg A$.

For example, assume that both A and $\neg A$ are added to a knowledge base at the same time. The case has no reason to give a priority of A and $\neg A$. We may resolve inconsistency for our purposes. But, we cannot decide whether A or $\neg A$ is appropriate in the situation. There are several ways for decision.

One possible solution is to assume that both A and $\neg A$ hold, which is not a problem in paraconsistent logic. This is because we cannot deduce arbitrary B from $A \wedge \neg A$. This aspect is, however, neglected in non-monotonic reasoning based on classical logic.

6 Concluding Remarks

We proposed a constructive discursive logic *CDLSN* with Hilbert-style axiomatization and Kripke semantics. It can be viewed as a constructive version of Jaśkowski's original system. We established some formal results of *CDLSN* including completeness. We also discussed applications to common-sense reasoning. We believe that *CDLSN* can serve as a logical foundation for paraconsistent intelligent systems.

Finally, we mention topics which remain to be worked out. First, we should extend *CDLSN* with quantifiers for dealing with many interesting problems. There seem to be no difficulties with axiomatization and Kripke semantics.

Second, for practical applications, we need efficient proof methods since a Hilbert system is not suitable. Tableau and sequent calculi are desirable as a proof method. Tableau calculi for N^- and N have been worked out in Akama [3], and they can be modified for *CDLSN*.

Third, we should elaborate on the formalization of common-sense reasoning in *CDLSN*. It is interesting to study the connections of *CDLSN* and several non-monotonic logics. Non-monotonic formalisms are also related to logic programming, and we should explore relationships in this context.

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