

Chapter 2

Overview of Elliptic–Hyperbolic PDE

Abstract The variety and broad applicability of elliptic–hyperbolic equations are illustrated. Included are brief discussions of: the essentials and history of equation type; a “zoo” of elliptic–hyperbolic equations; systems of elliptic–hyperbolic equations; a quasilinear example having multiple sonic lines, with an application to a recent problem in geometry; the issue of local canonical forms, with particular reference to why one of the two canonical forms has much less regularity than the other.

Keywords Elliptic–hyperbolic equation · Boussinesq systems · Tricomi equation · Keldysh equation · Operator of real principal type

2.1 Remarks on Equation Type

The 19th century’s great contribution to the theory of partial differential equations, the Cauchy–Kovalevsky Theorem, did not explicitly address the issue of equation type. But in the subsequent century, research into partial differential equations was dominated by the hypothesis of uniform type. Indeed, by the latter half of the 20th century people had not only come to specialize in either elliptic, hyperbolic, or parabolic equations, but specialists in elliptic theory had become culturally isolated from those who studied hyperbolic equations, and specialists in parabolic equations often found that they had more to say to probabilists than to specialists in either of the other two equation types. As an initial orientation it is useful to ask why it became so important to classify partial differential equations by elliptic, hyperbolic, or parabolic type to the exclusion of other properties. There are two answers to this question, a pragmatic one and a formal mathematical one.

Pragmatically, we might ask why we solve partial differential equations at all. A solution to a partial differential equation can do, in general, one of three things. It can propagate as a wave, it can diffuse as heat, or it can go nowhere at all. Because combinations of these three things give a pretty good local description of nature, we conclude that solving partial differential equations is generally a good thing to do. The *type* of the equation—hyperbolic, parabolic, or elliptic—corresponds in general to which of the three things the solution does: propagate, diffuse, or oscillate.

In order to look at the matter a little more closely, we write down a linear partial differential equation having the form

$$\alpha u_{xx} + 2\beta u_{xy} + \gamma u_{yy} + \text{lower order terms} = 0, \quad (2.1)$$

where α , β , and γ are given functions of the coordinates $(x, y) \in \Omega \subset \mathbf{R}^2$, and u is a sufficiently smooth unknown function of x and y . We assume that α , β , and γ are not simultaneously zero at any point of Ω , so that it makes sense to speak of the *order* of Eq. (2.1) being two.

Recall that in order for solutions of Eq. (2.1) to propagate as waves, the characteristics of the equation need to be real-valued; that is, the differential equation

$$\alpha dy^2 - 2\beta dx dy + \gamma dx^2 = 0 \quad (2.2)$$

must have a solution in $\mathbf{R}^2$. If Eq. (2.2) has two, independent real-valued solutions, we say that the partial differential equation is of *hyperbolic type*.

We can understand the requirements for equation type better from a mathematical point of view if we rewrite the higher-order part of Eq. (2.1) as a first-order system

$$A^1 \mathbf{u}_x + A^2 \mathbf{u}_y = 0 \quad (2.3)$$

for $\mathbf{u} = (u_x, u_y)$,

$$A^1 = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$A^2 = \begin{pmatrix} 2\beta & \gamma \\ -1 & 0 \end{pmatrix}.$$

We find that the condition that a solution be wave-like is equivalent to the condition that the quadratic form $\tilde{Q} = |A^1 - \lambda A^2|$ associated to the system (2.3) have distinct real roots. If the roots of the system (2.3) are complex-valued, then the solution is not wave-like, and we find it convenient to work in the complex plane. In that case we say that the system is of *elliptic type*. If the eigenvalues are real but the associated eigenvectors do not span $\mathbf{R}^2$, then the system is said to be *parabolic*.

The condition on the roots of $\tilde{Q}$ can be interpreted, in terms of Eq. (2.1), as the conditions

$$\alpha\gamma - \beta^2 \begin{cases} > \\ < \\ = \end{cases} 0, \quad (2.4)$$

corresponding to equations of elliptic, hyperbolic, or parabolic type, respectively. A detailed exposition of this classification is given in Sect. III.2 of [21].

A more formal answer to the question of why we tend to focus on equation type arises from the so-called *Basic Problem* (see, e.g., [78]) in the theory of partial differential equations: We are given a domain Ω , a partial differential equation defined on Ω , and some information about the solution on all or part of the boundary $\partial\Omega$. That information might include, for example, the value of the solution, the values of some of its derivatives, or the value of some quantity (or quantities) which depend on the solution and/or on some of its derivatives. The problem asks, “When does there exist a unique solution in $\overline{\Omega}$ which varies continuously with continuous variation of the boundary data?”

A solution to the Basic Problem is called a *well-posed boundary value problem*. The problem is generally ascribed to Hadamard, who posed it in the early decades of the twentieth century [32].

Elliptic, hyperbolic, and parabolic equations are associated with classes of equations for which the Basic Problem has a solution in terms of Dirichlet/Neumann, Cauchy, or initial-boundary-value problems, respectively. That association is however, valid only in a rather naïve sense. It does not seem to be true, for example, that the n -dimensional Cauchy problem is well-posed for the entire class of hyperbolic equations as defined, for example, in [33], and additional hypotheses are imposed. In solving the Basic Problem it may be more useful to impose the hypothesis of *hypoellipticity* (Sect. 6.2.1), which cuts across type to include the heat operator, or the property of being of *real principle type* (Sect. 2.5.1), which does not really distinguish between elliptic and hyperbolic classes, than to impose the hypothesis of elliptic, hyperbolic, or parabolic type. The identification of the elliptic, hyperbolic, or parabolic types with appropriate classes of boundary data could be more accurately described as a goal (of preceding generations) than as a defining property of these types.

Why we solve the Basic Problem itself is not difficult to explain. Any solution to a differential equation, whether ordinary or partial, reduces to an integration method. Iterated multiple integration produces arbitrary functions; specifying the value of the solution a priori on some part of the domain is an attempt to determine those functions uniquely. A solution that is not unique would not describe the results of experiments in classical physics, because classical physics is a deterministic theory: experiments repeated under identical conditions are expected to give identical results. A solution that did not vary continuously with the prescribed data would also fail to describe the results of actual experiments, because even classical experimental conditions cannot be imposed without error—if only due to thermal oscillations. The Basic Problem has led mathematicians to search for the “right” class of boundary value problems for a given differential equation, which engineers sometimes criticize as an imposition of mathematical aesthetics on nature. It is that to some extent, but it is also an attempt to find a class of problems that will not violate the constraints of classical physics.

An interesting side question is whether the data in the Basic Problem need to be given on the boundary, rather than on some other part of the domain. Neuberger [63] has suggested alternative ways to prescribe data, which have advantages from the perspective of numerical analysis.

2.2 A Zoo of Elliptic–Hyperbolic Equations

The Law of Trichotomy applied to (2.4) implies that equations of the form (2.1) will always fall into one of the three types at every point of their domain. The behavior in higher dimensions is more complex. For example, the Gauss–Codazzi–Ricci system for the Riemannian curvature R_{ijkl} , which arises in the embedding of higher-dimensional Riemannian manifolds into Euclidean space [13, 14], has no type. Relatively little is known about elliptic–hyperbolic equations in dimensions exceeding 2, and the primitive typology generated by condition (2.4) will include all the equations studied in this course. For the general case represented by equations of arbitrary order in arbitrary dimension, see [38].

Although the Law of Trichotomy implies that every equation of the form (2.1) will fall into one of the three types at a given point, it certainly does not imply that an equation will be of uniform type on its entire domain. For example:

- Separating variables in Laplace’s equation in toroidal coordinates produces a partial differential equation which is elliptic inside the unit disc and hyperbolic on the $\mathbf{R}^2$ -complement of that disc. This problem goes back to Riemann [80], and was considered by Neumann, Hicks, Basset, and Heine in the 19th century (see Sect. 10.1 of [8] and the references cited therein);
- the equations for a steady, irrotational, isentropic, compressible flow are elliptic for velocities below the speed of sound in the medium, and hyperbolic at velocities above the speed of sound [7];
- the continuity equations for shallow water are elliptic when the flow speed is below a certain value (the *Froude number*) and hyperbolic when the speed exceeds that value (see [79] for one space dimension and Sect. 10.12 of [83] for two space dimensions);
- the governing equations in a continuum model for two-lane traffic flow in opposing directions change from hyperbolic to elliptic type over a region of phase space, provided a parameter is introduced which represents a degree of dependence between the flow density in each lane and the flow density in the opposing lane. The area of the elliptic region depends on the strength of this interaction parameter [10];
- the governing equations for ideal magnetohydrodynamic flow in a nozzle [18] change between elliptic and hyperbolic type *three* times: at the sonic speed c , at the Alfvén speed a , and at the speed $ca/\sqrt{c^2 + a^2}$;
- the complex eikonal equation is of hyperbolic type in the interior of a circular caustic and of elliptic type in the region exterior to the caustic. See the discussion of Eqs. (5.26, 5.27) in [69] and the references [47, 51–56];
- the equations for small-amplitude electromagnetic wave propagation in zero-temperature plasma are elliptic at certain wave frequencies and hyperbolic at other frequencies [28, 49, 52, 61, 64, 65, 67, 68, 76, 88, 90]; see also Chap. 4 of [69];
- the Laplace–Beltrami equation on the extended projective disc $\mathbf{P}^2$ is elliptic inside the Beltrami disc but hyperbolic outside of it [40, 41, 57, 66]; see also Sects. 6.1–6.3 of [69];

- by a series of approximations, the partition function for quantum electrodynamics can be reduced to a relativistic model in which the quarks are coupled to a pair of classical gauge fields. Additional choices reduce the analysis to a problem in nonlinear electrostatics in which the governing equation changes from elliptic to hyperbolic type depending on the value of a (highly nonlinear) function of the flux, its gradient, and the radial coordinate [2, 3];
- according to the Hartle–Hawking model [37], the Laplace–Beltrami equation on a matter-free region of space-time would have been of elliptic type in the early universe and would have changed to an equation of hyperbolic type across a hypersurface which is space-like when viewed from the present; see also Sect. 6.4.5 of [69];
- some models for multiphase flow change from hyperbolic to elliptic type [43];
- in certain models for plasma flow in a tokamak [50], the governing equation changes its type twice: from elliptic to hyperbolic and back again, with increasing flow velocity; see also the very recent elliptic–hyperbolic model for transonic MHD flow introduced in [30];
- the wave equation in Minkowski space-time, in a reference frame rotating with constant angular velocity with respect to another reference frame, is elliptic for certain values of the radial coordinate and hyperbolic for other values [82];
- the helically reduced wave equation on a torus (a qualitative linear model for the reduction of the Einstein equations by a helical Killing vector field) is elliptic for certain values of the radial coordinate and hyperbolic for other values [45, 84, 85];
- an equation arising in the design of automobile windshields changes from elliptic to hyperbolic type as a certain shape parameter of the windshield is varied [11];
- the governing equations of a model [23] for longitudinal (or for pure shearing) motions are modeled in [75] by a simple elliptic–hyperbolic system;
- the equation that determines the existence of an isometric embedding of a Riemannian 2-manifold into 3-dimensional Euclidean space changes from hyperbolic to elliptic when the sign of the Gauss curvature of the manifold changes from negative to positive [34]. This problem goes back to a conjecture by Schläfli [81] in 1873. (An analogous dependence characterizes the equation governing the existence of a surface having prescribed Gauss curvature [36]);

etc., etc., etc.... It is clearly quite common for solutions to change their nature (e.g., to oscillatory from propagating) across a smooth curve in their domain. That is the motivation for these notes.

Perhaps the simplest explicit example is the *Lavrent’ev–Bitsadze* equation [48]

$$\operatorname{sgn}[y]u_{xx} + u_{yy} = 0, \quad (2.5)$$

which is Laplace’s equation on the upper half-plane and the wave equation on the lower half-plane. We have already seen this equation in example (1.1) with the type-change function (1.2). Equation (2.5) has been extensively studied; for very recent examples see, e.g., [58–60].

If the coefficients α , β , and γ in Eq. (2.1) are allowed to depend on the solution u and its derivatives u_x and u_y , but not on any second-order derivatives, then the equation will be linear only in its highest derivatives, in the sense that the coefficients of the highest-order terms will not depend on those terms, but may depend on lower-order terms. Such equations are said to be *quasilinear*. An example of a quasilinear elliptic–hyperbolic equation is the *nonlinear Lavrent’ev–Bitsadze* equation [16]

$$u_{xx} + \operatorname{sgn}[u]u_{yy} = 0. \quad (2.6)$$

Although it was clear from inspection that Eq. (2.5) changed type on the x -axis, it is not possible to identify the curve Ω_0 in the xy -plane on which Eq. (2.6) changes type without solving the equation. This will lead us to approach this equation using methods originally introduced for free boundary problems, in which the unknown curve Ω_0 functions as a free boundary, to be obtained as part of the solution. This reminds us of an opinion by Friedrichs [24], that conventional boundary value problems even for linear elliptic–hyperbolic equations tend to be over-determined because conditions on the transition from elliptic to hyperbolic type—e.g., that the solution be continuous on the curve Ω_0 —act as an extra boundary condition; see Sect. 2.2 of [69] or the appendix to [1] for illustrations of that view.

The curve Ω_0 on which a planar elliptic–hyperbolic equation changes type is also the curve on which the equation is of parabolic type. For that reason it has been called the *parabolic curve*, and the change of type along that curve has been called the *parabolic transition*. But that terminology is confusing, partly because in some cases, the curve Ω_0 actually *is* a parabola (Sects. 4.2 and 4.3). Perhaps for this reason the terms *sonic line* and *sonic transition*, derived from fluid dynamics, tend to be applied regardless of physical context. As is typically the case when one uses standard terminology, it is quite possible for the sonic “line” to have nonzero curvature.

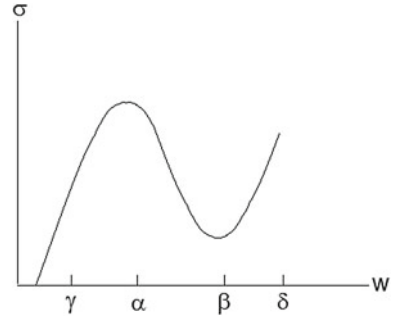
2.3 A Simple Elliptic–Hyperbolic System

Consider scalar equations having the form

$$u_{tt} = \sigma(u_x)_x, \quad (2.7)$$

where in the simplest case the unknown function $u(x, t)$ represents displacement along a line and thus depends on a spatial coordinate $x \in \mathbf{R}$ and a time coordinate $t \in \mathbf{R}^+$. It is clear that the analytic properties of Eq. (2.7) will depend on the hypotheses placed on the function $\sigma(w)$. We will assume as in [9, 23, 75] that there are numbers α , β , γ , and δ are such that $w \in \Omega \equiv [\gamma, \delta]$; $\sigma'(w) > 0$ on the complement in Ω of $[\alpha, \beta]$; $\sigma'(w) \leq 0 \forall w \in [\alpha, \beta]$; $\sigma''(w) < 0$ for $w < \alpha$; $\sigma''(w) > 0$ for $w > \beta$; see Fig. 2.1. In that case the change in equation type will correspond geometrically to a change in concavity of the function σ .

Fig. 2.1 A “stress function” $\sigma(w)$ for Eq. (2.7)



Just as the term “sonic line” tends to be applied to the transition curve Ω_0 regardless of the application, it is not uncommon to call $\sigma(w)$ the *stress function*, regardless of application. This term is based on an analogy to nonlinear elasticity in which u would represent the displacement of a string under longitudinal or pure shearing motions.

We associate with (2.7) a first-order system in w and v , satisfying the initial value problem

$$\begin{aligned} w_t &= v_x, \\ v_t &= \sigma(w)_x, \\ w(x, 0) &= w_0(x), \\ v(x, 0) &= v_0(x). \end{aligned} \tag{2.8}$$

Equating mixed second partial derivatives of v , we obtain as a consistency condition for (2.8) a second-order scalar equation in w having the form

$$w_{tt} = \sigma(w)_{xx}. \tag{2.9}$$

Alternatively, we may choose in Eq. (2.8) the substitutions $w = u_x$ and $v = u_t$, in which case the second of Eqs. (2.8), expressed in terms of u , is exactly Eq. (2.7). The associated physical models may be expressed either via (2.7), as in [75], or via (2.9), essentially as in [9].

We obtain from the first two equations of (2.8) the matrix equation

$$\begin{pmatrix} w \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ \sigma'(w) & 0 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}_x.$$

This is of course a matrix equation of the form $AX_x - IX_t = 0$. In order to compute the eigenvalues of the matrix A , we solve the scalar equation ([21], Sect. III.2.2)

$$|A + \lambda I| = \begin{vmatrix} \lambda & 1 \\ \sigma'(w) & \lambda \end{vmatrix} = \lambda^2 - \sigma'(w) = 0.$$

Because the eigenvalues

$$\lambda = \pm \sqrt{\sigma'(w)}$$

are real-valued and nondegenerate when σ' is positive, we take this to be the region of hyperbolicity for the system (2.8). Explicitly, the system is hyperbolic on the union of the sets $\{w|\gamma < w < \alpha\}$ and $\{w|\beta < w < \delta\}$, as expected. The system is elliptic on set $\{w|\alpha < w < \beta\}$.

2.3.1 Boussinesq Systems

Making the choice $\sigma(t) = t^2$ in the preceding discussion leads to equations of *Boussinesq type*. Following Sect. 3.2 of [12], we view these as first-order systems

$$u_t + B(u)u_x = 0, \quad (2.10)$$

taking $u = (h, w)^T$, where

$$B(u) = \begin{pmatrix} -hw & \frac{1-h^2}{2} \\ \frac{1-w^2}{2} & -hw \end{pmatrix}. \quad (2.11)$$

The domain of hyperbolicity coincides with that region of the domain of (2.10) on which the eigenvalues of B —that is, the numbers

$$\lambda_B^\pm = -hw \pm \frac{\sqrt{(1-h^2)(1-w^2)}}{2},$$

are real and distinct, having eigenvectors

$$v_B^\pm = \begin{pmatrix} \sqrt{1-h^2} \\ \pm \sqrt{1-w^2} \end{pmatrix}$$

which span $\mathbf{R}^2$. The conditions for wave-like solutions are satisfied, for example, in the rectangular region in which $|h|$ and $|w|$ are both less than 1. That rectangle does not lie in the xt -plane but in the “phase space” of the hw -plane, and so cannot be explicitly determined without solving the system.

Elliptic–hyperbolic Boussinesq systems of this kind also arise in the context of Hamiltonian dynamics. In the simplest useful case, that of a potential $u(x, t)$ satisfying

$$\ddot{x}(t) = -u(x, t)_x,$$

one may seek a first integral having the form

$$a_0(x, t) + a_1(x, t) \dot{x} + \cdots + a_n(x, t) \dot{x}^n.$$

One obtains by substitution a recursive system of $n + 2$ equations in $n + 2$ unknowns [46] which in the case $n = 3$ yields the system

$$\begin{aligned}(a_0)_x &= -3u_t, \\ (a_0)_t &= 3uu_x,\end{aligned}$$

or

$$u_{tt} + (uu_x)_x = 0.$$

This is Eq. (2.9) with the choice

$$\sigma(w) = -\frac{w^2}{2},$$

essentially the dispersionless Boussinesq equation mentioned earlier; see [9] and references cited therein for extensions. A microlocal analysis of a planar Hamiltonian system arising in the study of Eq. (2.9) with the choice $\sigma(w) = -tw$ is given in [70], a result generalized in various directions in the papers [71–74]; see also [26, 27] and Sect. 2.5.1, below. Equation (2.9) with this choice of σ is called the “Tricomi equation;” c.f. Eq. (2.17).

2.4 A Quasilinear Elliptic–Hyperbolic Equation Having Multiple Sonic Lines

In his 1929 paper, Bateman [7] considered the solution of boundary value problems for the following quasilinear elliptic–hyperbolic equation. Using the famous notation of Lagrange,

$$p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy},$$

Bateman introduces further short-hand notation

$$\begin{aligned}a &= y^2 - q^2 (x^2 - y^2), \quad b = x^2 - p^2 (x^2 - y^2), \\ cy &= q [x^2 - y^2 - (px + qy)^2],\end{aligned}$$

and

$$h = xy + pq(x^2 - y^2),$$

to express the differential equation to be studied in a simple algebraic form:

$$ar + 2hs + bt = c. \quad (2.12)$$

The discriminant

$$ab - h^2 = (y^2 - x^2)(py + qx)^2,$$

indicates that the equation changes from elliptic to hyperbolic type when the coordinates (x, y) change from the region $x^2 < y^2$ to the region $x^2 > y^2$ (Fig. 2.2).

Bateman proposed an explicit solution

$$z = \sqrt{x^2 + \csc^2 x - y^2} - \cot x, \quad (2.13)$$

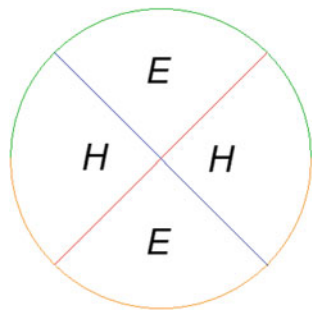
which exists provided the y^2 term is sufficiently small relative to the other terms under the radical. He noticed that on the unit disc the greatest value of y^2 , for any given value of x^2 , is $1 - x^2$ and that, substituting that value into Eq. (2.13), the quantity under the radical becomes

$$2x^2 + \csc^2 x - 1 = 2x^2 + \cot^2 x,$$

which is positive on the unit circle. This leads him to conjecture that the Dirichlet problem is well-posed on the unit disc despite the mixed elliptic–hyperbolic nature of the equation, a conjecture which, in its full generality, remains a topic of current interest [69].

Moreover it has become apparent, in the more than eight decades since Bateman's paper appeared, that equations having properties analogous to those of Fig. 2.2 yield information on an important geometric question.

Fig. 2.2 E denotes the elliptic regions of Eq. (2.12); H denotes the hyperbolic regions of that equation. The diagonal lines denote sonic transitions in the case for which the x -axis bisects the regions H and the y -axis bisects the regions E



2.4.1 Surfaces of Prescribed Gauss Curvature

Given a function $K(x, y)$ defined in a neighborhood of the origin of $\mathbf{R}^2$, does there exist a graph $z(x, y)$ having Gauss curvature K ? Because every surface may be expressed locally as a graph, this is the problem of *locally prescribing the curvature* of a surface Σ . This problem can be reduced, by methods similar to those to be discussed in Sect. 6.1, to the question of whether a smooth solution to a fully nonlinear “Monge–Ampère” equation having the form [44]

$$\det \partial_{ij} z = K \left(1 + |\nabla z|^2 \right)^2, \quad (2.14)$$

exists locally, where the subscripted index denotes partial differentiation in the direction of the indexed variable x^i , $i = 1, 2$, for $(x^1, x^2) = (x, y)$. In studying Eq. (2.14), it is convenient to assume that if the Gauss curvature changes sign, it does so “cleanly”. That is to say, it is assumed that the curvature changes sign to first order across the (unique) curve $\Omega_0 \in \Sigma$ on which K vanishes: if $K = 0$ on Ω_0 , it is assumed that $\nabla K \neq 0$ on Ω_0 . Recently, Han and Khuri [35, 36] considered in various contexts the more complex case in which K changes sign to finite order on two intersecting curves on the surface. In that case, the problem of showing the existence of a smooth solution to Eq. (2.14) is replaced by the problem of showing the existence of a smooth solution $u(x, y)$, defined in a neighborhood of the origin, to an equation having the general form ([36], Eq. 1.1)

$$u_{xx}u_{yy} - u_{xy}^2 = (x^2 - y^2) \Psi(x, y, u, u_x, u_y), \quad (2.15)$$

where Ψ is a smooth positive function in $B_1 \times \mathbf{R} \times \mathbf{R}^2$ and B_1 is the unit disc in $\mathbf{R}^2$. It turns out to be sufficient, using methods to be described in Sect. 6.2.1, to demonstrate the existence of smooth solutions to the linear equation ([36], Eq. 1.2)

$$u_{yy} + (x^2 - y^2) u_{xx} = f. \quad (2.16)$$

This linear equation changes type in a manner analogous to Fig. 2.2, but with the positions of E and H exchanged. See also [89].

Multiple sonic lines have also arisen recently in a linear model for transonic plasma flow; see Eq. (22) of [30]. However, in that case only one of the sonic lines is known in advance; the other must be determined with the solution. For a partial review of the literature on multiple sonic lines in elliptic–hyperbolic equations, see Appendix B, item 3, of [69].

2.5 Local Canonical Forms

The local canonical forms for linear elliptic–hyperbolic equations in the plane are discussed extensively in Sects. 3.1 and 3.2 of [69]. After briefly reviewing that material, we will consider a question that was not considered in [69]: the reason for the weaker regularity possessed by equations of Keldysh type.

In 1923 Tricomi proposed the equation

$$yu(x, y)_{xx} + u(x, y)_{yy} = 0 \quad (2.17)$$

as a local canonical form for all equations of the form (2.1) having sufficiently smooth coefficients, and which change from elliptic to hyperbolic type on a smooth curve. He defined the equation on a domain $\Omega \subset \subset \mathbf{R}^2$ bounded by a smooth Jordan curve Γ_0 in the upper half plane and by intersecting characteristics in the lower half-plane, as shown in Fig. 2.3. Tricomi proved [87] that a unique classical solution to Eq. (2.17) exists on Ω , provided the values of $u(x, y)$ are smoothly prescribed on the elliptic arc Γ_0 and on the characteristic Γ_1 , and no data are prescribed on the characteristic Γ_2 . Such a problem is now called the *Tricomi problem*.

In fact the Tricomi equation (2.17) is a simple example of one of the *two* local canonical forms for linear elliptic–hyperbolic equations in the plane. A simple example of the other local canonical form is the equation

$$xu_{xx} + u_{yy} = 0. \quad (2.18)$$

Equation (2.18) was introduced by Maria Cinquini-Cibrario in 1932, in a paper which derived the two local canonical forms for elliptic–hyperbolic equations [19]. Nevertheless, Eq. (2.18) is now called the *Keldysh equation*, in honor of a mathematician who studied (2.18) nearly two decades after Cinquini-Cibrario’s work, in

Fig. 2.3 A typical Tricomi domain. The elliptic region Ω^+ is bounded above by the arc Γ_0 and lies in the upper half-plane; the hyperbolic region Ω^- is bounded below by the characteristics Γ_1 , Γ_2 and lies in the lower half-plane. The hyperbolic part of the boundary $\partial\Omega^-$ is the graph of the equation $y = -\{(3/2)(1 - |x|)\}^{2/3}$

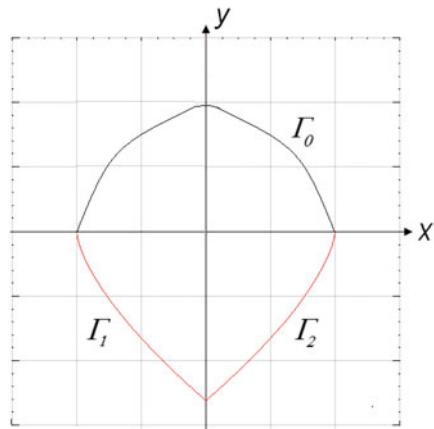
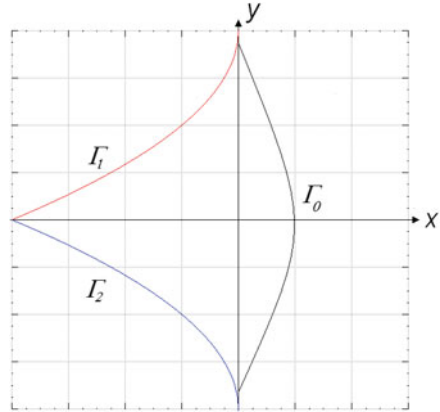


Fig. 2.4 The domain Ω of Eq. (2.18). The hyperbolic region is to the left of the y -axis [69]



the context of the degeneration of ellipticity in a singular boundary value problem [42]. (At Courant Institute a couple of generations ago, (2.18) was informally called “anti-Tricomi,” presumably to distinguish it from the equation actually studied by Keldysh, which does not change type.)

Boundary value problems for (2.18) have solutions on a domain which looks somewhat like a Tricomi domain rotated clockwise by 90° . The curves Γ_1 and Γ_2 in Fig. 2.4 are the characteristics $y = \mp\sqrt{-x} \pm 4$. Cinquini-Cibrario showed [20] that there exist classical solutions to (2.18) with data prescribed (only) on the elliptic boundary Γ_0 .

Equations (2.17) and (2.18) are local canonical forms in the following sense: Given any sufficiently regular point (x_0, y_0) of an equation having the form (2.1) on the class of domains $\Omega = \Omega^+ \cup \Omega^- \cup \Omega_0$ defined in Sect. 1.1, there is a coordinate transformation, which is nonsingular in the neighborhood of (x_0, y_0) , under which (2.1) assumes one of two forms, either

$$L_T \equiv y^{2m+1}u_{xx} + u_{yy} + \text{lower-order terms}$$

or

$$L_K \equiv u_{xx} + y^{2m+1}u_{yy} + \text{lower-order terms},$$

where m is a non-negative integer.

A differential equation for which the differential operator L can be transformed locally into an operator of the form L_T is said to be of *Tricomi type*. A differential equation for which the differential operator L can be transformed locally into an operator of the form L_K is said to be of *Keldysh type*.

We can always write a second-order linear equation with smooth coefficients, defined in an open set of $\mathbf{R}^2$, in the local form

$$Lu = \mathcal{H}(x, y)u_{xx} + u_{yy} + \text{lower-order terms}.$$

One can, by a further nonsingular transformation, arrive at a coordinate system in which the type-change function $\mathcal{K}(x, y)$ has either of two forms:

1. $\mathcal{K}(y)$, with $\mathcal{K}(0) = 0$ and $y\mathcal{K}(y) > 0$ for $y \neq 0$,

or

2. $\mathcal{K}(x)$, with $\mathcal{K}(0) = 0$ and $x\mathcal{K}(x) > 0$ for $x \neq 0$.

The former condition characterizes equations of Tricomi type and the latter, equations of Keldysh type. That equations in the form (2.1) are locally equivalent to equations with type-change function $\mathcal{K}$ having one of these two forms is demonstrated in Sect. 3.2 of [69].

Still more generally, one notices that the distinction between Tricomi and Keldysh type is only meaningful in a neighborhood of the sonic line. Equations of Keldysh type can be characterized by the degeneration of their characteristic lines, which intersect the sonic line tangentially, leading to weaker regularity. For example, both canonical forms have been shown to possess fundamental solutions, which are the main ingredient for solving the Dirichlet problem by the Green's function method. But while a fundamental solution exists in a classical sense for the Tricomi equation [4–6], a fundamental solution exists for the Keldysh equation only as the finite part of a divergent improper integral [15].

Note that the distinction between these two canonical forms is entirely local. For example, S-X. Chen recently studied boundary conditions for which the characteristics in the hyperbolic region are transversal to the transition curve Ω_0 on a subset of that curve, but are tangential to it on another subset [17].

2.5.1 Why Do Equations of Keldysh Type Have Weaker Regularity?

Microlocal methods, in one sense, go back to Fourier, and in another sense to the work of Cauchy, Riemann, and Hadamard on the relation between singularities in the solutions of partial differential equations and the geometry of their characteristics [31]. By 1970 the modern terminology had largely emerged. Here we show how a single definition from (the soft-analytic context of) microlocal analysis, the classification of operators according to whether or not they are of real principal type, explains why one of the two local canonical forms for linear elliptic–hyperbolic equations has considerably greater regularity than the other.

Define a linear partial differential operator on Ω in the usual way via

$$Lu = A^i \frac{\partial u}{\partial x^i} + Bu, \quad (2.19)$$

where for simplicity $i = 1, 2$ and u is real-valued. The extension to higher-order operators can be effected via multi-indices, and to the complex field via multiplicative factors of the form $-\sqrt{-1}$; see, e.g., [22, 86].

We define the *principal symbol* $\tilde{\sigma}$ of L to be the function $\tilde{\sigma} : \Omega \times (\mathbf{R}^2 \setminus \{0\}) \rightarrow \mathbf{R}$ given by $(x, \xi) \rightarrow \det [A^i(x)\xi_i]$. Because $\tilde{\sigma}$ is a function of the cotangent bundle, we can introduce the Hamiltonian operator

$$H_{\tilde{\sigma}} = \frac{\partial \tilde{\sigma}}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial \tilde{\sigma}}{\partial x^i} \frac{\partial}{\partial \xi_i}.$$

The function $\tilde{\sigma}$ is a conserved quantity for the Hamiltonian flow, and hence vanishes along its integral curves. If $v : (-s, s) \rightarrow \Omega \times (\mathbf{R}^2 \setminus \{0\})$ is a curve such that $\tilde{\sigma}(v(0)) = 0$, then v is called a *null bicharacteristic*. (Note that there is some variation in terminology; see, e.g., p. 569 of [39].)

In order to compute the null bicharacteristics explicitly, we solve the system

$$\begin{aligned} \dot{x}(t) &= [\nabla_{\xi} \tilde{\sigma}](x(t), \xi(t)) \\ \dot{\xi}(t) &= -[\nabla_x \tilde{\sigma}](x(t), \xi(t)). \end{aligned} \quad (2.20)$$

Additionally, we have the initial condition $(x(0), \xi(0)) = (x_0, \xi_0)$ for which $\tilde{\sigma}(x_0, \xi_0) = 0$ but the “initial velocity vector” ξ_0 is nonzero.

In the case of the Tricomi equation (2.17), we write the vectors x and ξ in terms of their components x_1, x_2 and ξ_1, ξ_2 and obtain the symbol

$$\tilde{\sigma}(x_1, x_2; \xi_1, \xi_2) = x_2 \xi_1^2 + \xi_2^2.$$

We further obtain from Eq. (2.20) the four equations

$$\dot{x}_1(t) = \frac{\partial \tilde{\sigma}}{\partial \xi_1} = 2\xi_1 x_2; \quad (2.21)$$

$$\dot{x}_2(t) = \frac{\partial \tilde{\sigma}}{\partial \xi_2} = 2\xi_2; \quad (2.22)$$

$$\dot{\xi}_1(t) = -\frac{\partial \tilde{\sigma}}{\partial x_1} = 0; \quad (2.23)$$

$$\dot{\xi}_2(t) = -\frac{\partial \tilde{\sigma}}{\partial x_2} = -\xi_1^2. \quad (2.24)$$

Integration of Eqs. (2.23, 2.24) yields immediately

$$\xi_1(t) = \xi_{1,0} \quad (2.25)$$

and

$$\xi_2(t) = -\xi_{1,0}^2 t + \xi_{2,0}. \quad (2.26)$$

c.f. Sect.3 of [70]. The subscripted zeros in Eqs. (2.25, 2.26) denote a fixed initial value for the component, that is, an integration “constant” for the differential equation. Substituting Eq. (2.26) into Eq. (2.22) and integrating, we find that

$$x_2(t) = -\xi_{1,0}^2 t^2 + 2\xi_{2,0}t + x_{2,0}.$$

Then

$$\dot{x}_1(t) = -2\xi_{1,0}^2 t^2 + 4\xi_{1,0}\xi_{2,0}t + 2\xi_{1,0}x_{2,0},$$

yielding

$$x_1(t) = -\frac{2}{3}(\xi_{1,0}t)^3 + 2\xi_{1,0}\xi_{2,0}t^2 + 2x_{2,0}\xi_{1,0}t + x_{1,0},$$

as in Sect.3 of [70].

Notice that any bicharacteristic curve starting at the point $(x_{1,0}, x_{2,0}, \xi_{1,0}, \xi_{2,0})$ and proceeding along $x_2(t)$ in the direction $\dot{x}_2(t)$ will “escape to infinity” as t runs through values in either the positive or negative direction, unless (until) it strikes a boundary point of the domain.

Now we try to repeat the calculation for the principal symbol of Eq. (2.18). We have

$$\tilde{\sigma}(x_1, x_2; \xi_1, \xi_2) = x_1\xi_1^2 + \xi_2^2.$$

We obtain from Eq. (2.20) the four equations

$$\dot{x}_1(t) = \frac{\partial \tilde{\sigma}}{\partial \xi_1} = 2\xi_1 x_1; \quad (2.27)$$

$$\dot{x}_2(t) = \frac{\partial \tilde{\sigma}}{\partial \xi_2} = 2\xi_2; \quad (2.28)$$

$$\dot{\xi}_1(t) = -\frac{\partial \tilde{\sigma}}{\partial x_1} = -\xi_1^2; \quad (2.29)$$

$$\dot{\xi}_2(t) = -\frac{\partial \tilde{\sigma}}{\partial x_2} = 0. \quad (2.30)$$

For reasons that will become obvious, we will proceed formally, initially taking $t_0 = \varepsilon$ for small, positive ε , and subsequently attempting to take a limit as ε tends to zero. We have $\xi_2 = \xi_{2,0}$ by (2.30). Then (2.28) yields

$$x_2(t) = 2\xi_{2,0}t + x_{2,0}.$$

Equation (2.29) can be solved by inspection to yield

$$\xi_1 = \frac{1}{t}; \quad (2.31)$$

Notice that

$$\xi_{1,0} = \xi_1(\varepsilon) = \frac{1}{\varepsilon}.$$

Substitution of (2.31) into (2.27) yields the differential equation

$$\dot{x}_1(t) - 2\frac{x_1(t)}{t} = 0,$$

which is satisfied for $t \neq 0$ by

$$x_1(t) = t^2,$$

where $x_{1,0} = x_1(\varepsilon) = \varepsilon^2$.

Now letting ε tend to zero, we note that the right-hand side of (2.31) becomes singular. Recalling that, at $t = \varepsilon$, we require that $\tilde{\sigma}(x_\varepsilon, \xi_\varepsilon) = 0$ but $\xi_0 \neq 0$, we write the corresponding equation for the components:

$$\xi_{2,\varepsilon} = \pm \sqrt{x_1 \xi_{1,\varepsilon}^2}.$$

These objects become “vertical” on the sonic line $x_1 = 0$ as ε tends to zero, which can be interpreted as the sonic degeneration of characteristic lines, lifted to the cotangent bundle. The escape to infinity of an incoming or outgoing null bicharacteristic curve associated to the Tricomi equation (2.17), along a curve having a negative y -coordinate, has no analogy for the bicharacteristics of Eq. (2.18).

This motivates the following fundamental definition of linear analysis:

Denote by $\pi : \Omega \times (\mathbf{R}^2 \setminus \{0\}) \rightarrow \Omega$ the projection $\pi(x, \xi) = x$, where in this context $\{0\}$ denotes the origin of coordinates in $\mathbf{R}^2$. The operator defined in (2.19) is of *real principal type* at the point $\{0\}$ if there exists a compact subset $\tilde{\Omega} \subset \Omega$ such that $\{0\} \in \tilde{\Omega}$ and if v is a null bicharacteristic satisfying $\pi(v(0)) \in \tilde{\Omega}$, then there exists a number $T > 0$ such that $\pi(v(\pm T)) \notin \tilde{\Omega}$.

That is, L is of real principal type if every null bicharacteristic leaves $\tilde{\Omega}$ in both the forward and backwards directions, and no complete bicharacteristic is trapped over a compact set containing the point $\{0\}$. (A bicharacteristic is *complete* if its integral curves have been maximally extended with respect to their parametrization.) Note that if $\tilde{\sigma}(v(0)) = 0$ for v defined on $(-s, s)$, then $\tilde{\sigma}(v(t)) = 0 \forall t \in (-s, s)$. Obviously there is nothing special about the point $\{0\} \in \mathbf{R}^2$; the origin can be replaced by any fixed point in $\mathbf{R}^2$ without affecting the meaning of the definitions. See, e.g., [22, 25, 62, 77, 86] for alternative definitions and interpretations, which by the way are legion. Note, for example, that we do not observe here the technical

distinction between operators which are *locally* or only *microlocally* of real principal type; c.f. [71].

In the concrete case of a scalar operator having the form

$$Lu = \mathcal{K}(x, y) u_{xx} + u_{yy} + \text{lower-order},$$

we find that L is of real principal type if whenever $\mathcal{K}(x, y) = 0$, we also have $\mathcal{K}_y(x, y) \neq 0$. That is, $\mathcal{K}$ is required to change sign *cleanly*, in the sense of Sect. 2.4.1. A tedious algebraic calculation—[69], Sect. 3.2—will show directly that the operator $\mathcal{L}$ is of Tricomi type if and only if $\mathcal{K}$ and $\mathcal{K}_y$ do not vanish simultaneously.

That is to say, operators of Tricomi type (e.g., $\mathcal{K}(x, y) = y$) are of real principal type but operators of Keldysh type (e.g., $\mathcal{K}(x, y) = x$) are not. The non-trapping property of null bi-characteristics over compact sets leads to strong local solvability theorems for operators of Tricomi type via microlocalization [26, 27, 71–74]. Moreover, operators of real principal type have the property that the important analytic properties (e.g., the local solvability properties just mentioned) are independent of the lower-order terms. Because operators of Keldysh type lack this non-trapping property, a lower-order perturbation of a well-behaved operator of Keldysh type may be badly behaved. Thus existence, uniqueness, and regularity theorems for equations of Keldysh type tend to depend on hypotheses placed on the lower-order terms, whereas the corresponding theorems for equations of Tricomi type do not. This property of Keldysh-type operators was noticed in special cases long before the microlocal approach was applied; see, e.g., [29].

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