Chapter 2
Zero-Dimensional Spaces

This chapter is devoted to 0-dimensional topological spaces. It follows from the definition of topological dimension given in Chap. 1 that a zero-dimensional topological space admits arbitrarily fine open partitions. As every element of an open partition is a clopen subset, i.e., a subset that is both closed and open, this suggests that any zero-dimensional space must contain many clopen subsets and hence be very disconnected since the abundance of clopen subsets reflects the discontinuous nature of a topological space. We shall study the relationship between the class of zero-dimensional topological spaces and other classes of highly-disconnected topological spaces such as the class of scattered spaces, the class of totally disconnected spaces, and the class of totally separated spaces.

2.1 The Cantor Set

In this section, we first describe the construction of the Cantor set, which is a fundamental example of a compact metrizable space with zero topological dimension.

Let a and b be real numbers such that $a < b$. The open interval

$$\left(a + \frac{b-a}{3}, b - \frac{b-a}{3}\right) = \left(\frac{2a+b}{3}, \frac{a+2b}{3}\right)$$

is called the *middle third* of the segment $[a, b]$. We denote by $T([a, b])$ the set obtained by deleting from the segment $[a, b]$ its middle third. Thus, we have

$$T([a, b]) := \left[a, a + \frac{b-a}{3}\right] \cup \left[b - \frac{b-a}{3}, b\right] = \left[a, \frac{2a+b}{3}\right] \cup \left[\frac{a+2b}{3}, b\right].$$

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More generally, for every subset $A \subset \mathbb{R}$ which is the union of a finite family $([a_i, b_i])_{1 \le i \le k}$ of pairwise disjoint segments, we set

$$T(A) := \bigcup_{i=1}^{k} T([a_i, b_i]).$$

Let us inductively define a decreasing sequence $(K_n)_{n \in \mathbb{N}}$ of closed subsets of $[0, 1]$ by setting

$$\begin{aligned} K_0 &:= [0, 1], \\ K_{n+1} &:= T(K_n) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

We therefore have

$$\begin{aligned} K_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \\ K_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \\ K_3 &= \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \\ &\quad \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right], \text{ etc.} \end{aligned}$$

Observe that the set K_n is the union of 2^n pairwise disjoint segments of length $1/3^n$. These segments are the connected components of K_n (see Fig. 2.1).

The set

$$K := \bigcap_{n \in \mathbb{N}} K_n$$

is called the *Cantor ternary set* or simply the *Cantor set*. A topological space that is homeomorphic to the Cantor ternary set K is called a *Cantor space*.

Proposition 2.1.1 *The Cantor set K is a compact subset of $\mathbb{R}$ with empty interior.*

Proof As the sets K_n are closed in $[0, 1]$, the Cantor set is closed in $[0, 1]$ and hence compact.

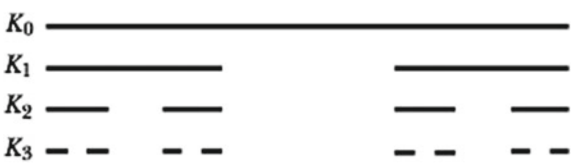

Fig. 2.1 Construction of the Cantor set

Let I be an interval of $\mathbb{R}$ such that $I \subset K$. The fact that I is connected implies that, for each $n \in \mathbb{N}$, the set I is contained in one of the 2^n connected components of K_n. We deduce that the length of I is smaller than or equal to $1/3^n$ for all $n \in \mathbb{N}$. As $1/3^n$ tends to 0 as n goes to infinity, it follows that I has zero length, i.e., is either empty or reduced to a single point. This shows that K has empty interior. $\square$

Proposition 2.1.2 *The Cantor set K has topological dimension* $\dim(K) = 0$.

Proof The set K_n is the disjoint union of 2^n segments $\Sigma_n(i)$, $1 \leq i \leq 2^n$, which are clopen in K_n. Let us set $U_n(i) := K \cap \Sigma_n(i)$. The family $\alpha_n := \big(U_n(i)\big)_{1 \leq i \leq 2^n}$ is a finite open partition of K. Therefore, we have $\operatorname{ord}(\alpha_n) = 0$. As $\operatorname{mesh}(\alpha_n) = 1/3^n$ tends to 0 as n goes to infinity, we deduce that $\dim(K) = 0$ by applying Proposition 1.4.4 (observe that the set K is not empty since we clearly have $0 \in K$). $\square$

Recall that every real number $x \in [0, 1]$ admits a *ternary expansion*, that is, a sequence $(u_k)_{k \in \mathbb{N}} \in \{0, 1, 2\}^{\mathbb{N}}$ such that

$$x = \sum_{k=0}^{\infty} \frac{u_k}{3^{k+1}}.$$

We will also write this equality under the form

$$x = \overline{0, u_0 u_1 u_2 \cdots u_k \cdots}.$$

When x is not a triadic rational number of the form $n/3^m$ with n and m integers satisfying $1 \leq n \leq 3^m - 1$, such an expansion is unique. In the case when $x = n/3^m$ with n and m integers such that $1 \leq n \leq 3^m - 1$, the number x admits two ternary expansions: a first one, called the *proper ternary expansion* of x, whose terms are eventually equal to 0 and another one, called the *unproper ternary expansion* of x, whose terms are eventually equal to 2. For example, we have

$$\frac{1}{4} = \overline{0, 02020202\ldots}$$

and

$$\frac{7}{9} = \overline{0, 210000\ldots} = \overline{0, 202222\ldots}.$$

The set K_n consists of all numbers $x \in [0, 1]$ that admit a ternary expansion $(u_k)_{k \in \mathbb{N}}$ such that $u_k \in \{0, 2\}$ for all $k \leq n - 1$. We deduce that the Cantor set K is the set consisting of the numbers $x \in [0, 1]$ that admit a ternary expansion whose terms all belong to the set $\{0, 2\}$. Thus, the ternary expansions given above show that both $1/4$ and $7/9$ belong to K.

Proposition 2.1.3 *The map* $\varphi\colon \{0,1\}^{\mathbb{N}} \to K$ *defined by*

$$\varphi(u) := \sum_{k=0}^{\infty} \frac{2u_k}{3^{k+1}}$$

for all $u = (u_k) \in \{0,1\}^{\mathbb{N}}$ *is a homeomorphism from the product space* $\{0,1\}^{\mathbb{N}}$ *onto the Cantor set* K.

Proof The fact that the map φ is well defined and bijective follows from the previous observations. Let us fix a sequence $u \in \{0,1\}^{\mathbb{N}}$. For each integer $n \geq 0$, the set $V_n(u) \subset \{0,1\}^{\mathbb{N}}$ consisting of all sequences v such that $v_k = u_k$ for all $k \leq n$ is an open neighborhood of u. For all $v \in V_n(u)$, we have that

$$|\varphi(u) - \varphi(v)| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^{k+1}} = \frac{1}{3^{n+1}}.$$

Since $1/3^{n+1}$ tends to 0 as n goes to infinity, we deduce that φ is continuous. The space $\{0,1\}^{\mathbb{N}}$ is compact as it is a product of compact spaces. Consequently, φ is a homeomorphism. □

Corollary 2.1.4 *The Cantor set is uncountable.* □

Let X be a topological space. A point $x \in X$ is called *isolated* if the singleton set $\{x\}$ is open in X. A topological space is called *perfect* if it contains no isolated points.

Corollary 2.1.5 *The Cantor set is perfect.*

Proof Let $u \in \{0,1\}^{\mathbb{N}}$. Consider the open subsets

$$V_n(u) := \{v \in \{0,1\}^{\mathbb{N}} \mid v_k = u_k \text{ for all } k \leq n\} \subset \{0,1\}^{\mathbb{N}}.$$

By definition of the product topology, every neighborhood of u in $\{0,1\}^{\mathbb{N}}$ contains the sets $V_n(u)$ for n large enough. As the set $V_n(u)$ is infinite for every n, we deduce that u is not isolated. This shows that the space $\{0,1\}^{\mathbb{N}}$ is perfect. As K is homeomorphic to $\{0,1\}^{\mathbb{N}}$, it is also perfect. □

2.2 Scattered Spaces

In this section, we introduce the class of scattered spaces. We prove that an accessible topological space X is scattered if and only if there exists a set E such that X is homeomorphic to a subspace of the product space $\{0,1\}^E$.

Let X be a topological space. A *base* of the topological space X is a set $\mathcal{B}$ of open subsets of X such that every open subset of X can be written as a union of elements of $\mathcal{B}$.

A set $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a *neighborhood base* of x if, for every neighborhood V of x, there exists $N \in \mathcal{N}$ such that $N \subset V$. Observe that a set $\mathcal{B}$ of open subsets of X is a base of X if and only if, for every $x \in X$, the set

$$\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$$

is a neighborhood base of the point x.

If $\mathcal{B}$ is a base of a topological space X, then $\mathcal{B}$ satisfies the following two conditions:

(B1) the elements of $\mathcal{B}$ cover X;
(B2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Conversely, if X is a set and $\mathcal{B}$ is a set of subsets of X satisfying conditions (B1) and (B2) above, then there exists a unique topology on X admitting $\mathcal{B}$ as a base.

Example 2.2.1 Let X be a metric space. Then the set consisting of all open balls $B(x, 1/n)$, where $x \in X$ and $n \geq 1$ is an integer, is a base of X.

Recall that a subset of a topological space X is said to be *clopen* if it is both open and closed in X. Note that the clopen subsets of a topological space are precisely the subsets with empty boundary.

Definition 2.2.2 We say that a topological space X is *scattered* if it admits a base consisting of clopen subsets of X.

A topological space X is scattered if and only if every point of X admits a neighborhood base consisting of clopen subsets.

Example 2.2.3 Every set endowed with the discrete topology is scattered.

Remark 2.2.4 A connected space X is scattered if and only if the topology on X is the trivial one.

Note that a scattered space may fail to be accessible. For example, every set X equipped with the trivial topology is scattered. However, such a space X is not accessible as soon as X contains more than one point.

Proposition 2.2.5 *Every scattered accessible space is Hausdorff.*

Proof Let X be a scattered accessible space. Let x and y be distinct points in X. Since X is accessible, the set $X \setminus \{y\}$ is an open neighborhood of x. As X is scattered, there exists a clopen neighborhood V of x that is contained in $X \setminus \{y\}$. The sets V and $X \setminus V$ are disjoint open subsets of X containing x and y respectively. This shows that X is Hausdorff. $\square$

Proposition 2.2.6 *Every subspace of a scattered space is itself scattered.*

Proof Let X be a scattered space and $Y \subset X$. If $\mathcal{B}$ is a base of X consisting of clopen subsets, then the sets $Y \cap B$, where $B \in \mathcal{B}$, are clopen in Y and form a base of Y. Consequently, Y is scattered. □

Proposition 2.2.7 *Every product of scattered spaces is itself scattered.*

Proof Let $(X_i)_{i \in I}$ be a family of scattered spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let $\mathcal{B}_i$ be a base of X_i consisting of clopen subsets. We can assume $X_i \in \mathcal{B}_i$. Then the set $\prod_{i \in I} U_i$, where $U_i \in \mathcal{B}_i$ for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$, are clopen in X and form a base for the product topology. Therefore X is scattered. □

Every open ball of the Euclidean space $\mathbb{R}^n$ is connected. Consequently, every scattered subset of $\mathbb{R}^n$ $(n \geq 1)$ has empty interior. For the subsets of $\mathbb{R}$, the converse is also true:

Proposition 2.2.8 *Let X be a subset of the real line $\mathbb{R}$. Then X is scattered if and only if it has empty interior.*

Proof We already observed that the condition is necessary. Let us show that it is also sufficient. Suppose that X has empty interior. Let $x \in X$ and $\varepsilon > 0$. As X has empty interior, we can find real numbers a and b not in X such that $x - \varepsilon < a < x < b < x + \varepsilon$. Then the set $V := (a, b) \cap X = [a, b] \cap X$ is a clopen neighborhood of x in X satisfying $V \subset (x - \varepsilon, x + \varepsilon)$. This shows that X is scattered. □

By applying the preceding proposition, we see that the set of rational numbers $\mathbb{Q}$, the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, and the Cantor set K are all scattered.

Proposition 2.2.9 *Let X be an accessible space. Then the following conditions are equivalent:*

(a) the space X is scattered;
(b) there exists a set E such that X is homeomorphic to a subset of the product space $\{0, 1\}^E$.

Proof Given a set E, the space $\{0, 1\}^E$ is a product of discrete spaces and hence scattered by Proposition 2.2.7. As every subset of a scattered space is itself scattered by Proposition 2.2.6, this shows that (b) implies (a).

Conversely, suppose that X is a scattered space. Let E be a base of X consisting of clopen subsets. Consider the map $\varphi \colon X \to \{0, 1\}^E$ defined by $\varphi(x) = (\chi_B(x))_{B \in E}$, where $\chi_B \colon X \to \{0, 1\}$ is the characteristic map of B. As B is clopen in X, the map χ_B is continuous for each $B \in E$. It follows that φ is continuous. On the other hand, if x and y are distinct points in X, then $X \setminus \{x\}$ is an open neighborhood of y since X is accessible. Therefore, there exists a neighborhood $B_0 \in E$ of y such that $B_0 \subset X \setminus \{x\}$. This implies $\chi_{B_0}(x) \neq \chi_{B_0}(y)$ and hence $\varphi(x) \neq \varphi(y)$. We deduce that φ is injective. We have that $\varphi(B) = \varphi(X) \cap \pi_B^{-1}(1)$, where $\pi_B \colon \{0, 1\}^E \to \{0, 1\}$

is the projection map onto the B-factor of $\{0, 1\}^E$. This shows that $\varphi(B)$ is open in $\varphi(X)$ for all $B \in E$. As E is a base of X, we deduce that the image by φ of every open subset of X is open in $\varphi(X)$. Consequently, φ induces a homeomorphism from X onto $\varphi(X)$. Therefore, the space X satisfies (b). $\square$

2.3 Scatteredness of Zero-Dimensional Spaces

In this section, we give a characterization of 0-dimensional topological spaces. This characterization shows that every 0-dimensional accessible space is scattered.

Theorem 2.3.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) $\dim(X) = 0$*;*
(b) for every pair of disjoint closed subsets A and B of X, there exist disjoint open subsets U and V of X such that $X = U \cup V$, $A \subset U$ and $B \subset V$;
(c) for every closed subset A of X and every open subset U of X such that $A \subset U$, there exists a clopen subset V of X such that $A \subset V \subset U$.

Proof Suppose first that $\dim(X) = 0$. Let A and B be disjoint closed subsets of X. Consider the open cover $\alpha = \{X \setminus A, X \setminus B\}$. As $\dim(X) = 0$, there exists a finite open partition β of X such that $\beta \succ \alpha$. Note that no element of β can meet both A and B. Denote by U the union of all the elements of β that meet A and let $V := X \setminus U$. The sets U and V form an open partition of X. Moreover, we have that $A \subset U$ and $B \subset V$. This shows that (a) implies (b).

Let us show now that (b) implies (c). Suppose that X satisfies (b). Let A be a closed subset of X and U an open subset of X such that $A \subset U$. Then $B := X \setminus U$ is a closed subset that does not meet A. By (b), it follows that there exists a partition of X into two open subsets V and W such that $A \subset V$ and $B \subset W$. Then the set V is a clopen subset of X and we have $A \subset V \subset U$. This shows that X satisfies (c).

Finally, let us prove that (c) implies (a). Suppose that X satisfies (c). Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X. As X satisfies (c), it follows from Proposition 1.5.2 that X is normal. By applying Corollary 1.6.4, we deduce that there exists a closed cover $(F_i)_{i \in I}$ of X such that $F_i \subset U_i$ for all $i \in I$. Since X satisfies (c), we can find, for each $i \in I$, a clopen subset V_i of X such that $F_i \subset V_i \subset U_i$. Without loss of generality, we may assume that $I = \{1, \ldots, n\}$. Consider the family $\beta = (W_i)_{i \in I}$ of subsets of X defined by $W_1 := V_1$ and

$$W_i := V_i \setminus (V_1 \cup \cdots \cup V_{i-1})$$

for all $i \in \{2, \ldots, n\}$. Clearly $\beta := (W_i)_{i \in I}$ is an open partition of X. Moreover, we have that $\beta \succ \alpha$ since $W_i \subset V_i \subset U_i$ for all $i \in I$. This shows that $\dim(X) = 0$. $\square$

Corollary 2.3.2 *Every topological space X satisfying* $\dim(X) = 0$ *is normal.*

Proof A topological space X such that $\dim(X) = 0$ satisfies condition (b) in the preceding theorem and is therefore normal. □

Corollary 2.3.3 *Every accessible topological space X satisfying* $\dim(X) = 0$ *is scattered.*

Proof Let X be an accessible space such that $\dim(X) = 0$. Let V be a neighborhood of a point $x \in X$. The singleton $\{x\}$ is closed in X since X is accessible. As $\dim(X) = 0$, the space X satisfies condition (c) of the preceding theorem. Therefore, there exists a clopen subset U of X such that $x \in U \subset V$. Consequently, every point of X admits a neighborhood base consisting of clopen subsets of X. This shows that X is scattered. □

Corollary 2.3.4 *If X is an accessible topological space such that* $\dim(X) = 0$, *then X is Hausdorff.*

Proof Every scattered accessible space is Hausdorff by Proposition 2.2.5. □

Remark 2.3.5 Corollary 2.3.4 can also be deduced from Corollary 2.3.2 since, as already observed in Sect. 1.5, every normal accessible space is clearly Hausdorff.

In Sect. 5.4, we shall give an example of a locally compact Hausdorff space that is scattered but not normal. Such a space has positive topological dimension by Corollary 2.3.2.

2.4 Lindelöf Spaces

In this section, we introduce the class of Lindelöf spaces and we prove that every non-empty scattered Lindelöf space X has topological dimension $\dim(X) = 0$.

Definition 2.4.1 A topological space X is called a *Lindelöf space* if every open cover of X admits a countable subcover.

Example 2.4.2 Every countable topological space is Lindelöf. Indeed, suppose that X is a countable topological space. Let $\alpha = (U_i)_{i \in I}$ be an open cover of X. Choose, for each $x \in X$, an index $i(x) \in I$ such that $x \in U_{i(x)}$. Let $J := \{i(x) \mid x \in X\}$. Then $\beta := (U_i)_{i \in J}$ is a countable subcover of α.

Example 2.4.3 Every compact space is Lindelöf. Indeed, by definition, a topological space X is compact if and only if every open cover of X admits a finite subcover.

Example 2.4.4 Every topological space that is a union of a countable family of subsets that are Lindelöf (for the induced topology) is Lindelöf. In particular, every σ-compact space is Lindelöf (recall that a topological space is called *σ-compact* if it is the union of a countable family of compact subsets). Thus, the Euclidean space $\mathbb{R}^n$ is Lindelöf for any integer $n \geq 1$ since it is σ-compact.

Example 2.4.5 If an uncountable set X is endowed with its discrete topology, then X is not Lindelöf. Indeed, the open cover $\alpha := (\{x\})_{x \in X}$ admits no countable subcovers. Note that X is metrizable (a metric inducing the topology on X is given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise) and locally compact.

A subset of a Lindelöf space is not necessarily Lindelöf (see the example in Sect. 5.4). However, we have the following result.

Proposition 2.4.6 *Every closed subset of a Lindelöf space is itself Lindelöf.*

Proof Let X be a Lindelöf space and F a closed subset of X. Let $\alpha = (U_i)_{i \in I}$ be an open cover of F. Then we can find, for each $i \in I$, an open subset V_i of X such that $U_i = V_i \cap F$. As the family $(V_i)_{i \in I} \cup \{X \setminus F\}$ is an open cover of X and X is Lindelöf, there exists a countable subset $J \subset I$ such that the family $(V_j)_{j \in J} \cup \{X \setminus F\}$ covers X. Then the family $(U_j)_{j \in J}$ is a countable subcover of α. This shows that F is Lindelöf. □

Remark 2.4.7 The product of two Lindelöf spaces may fail to be Lindelöf (see Sect. 5.5).

Definition 2.4.8 A topological space is said to be *second-countable* if it admits a countable base.

For example, the Euclidean space $\mathbb{R}^n$ is second-countable since the open balls $B(x, 1/m)$, where $x \in \mathbb{Q}^n$ and $m \geq 1$ is an integer, form a countable base of $\mathbb{R}^n$.

A topological space X is called *first-countable* if every point of X admits a countable neighborhood base. Clearly every second-countable topological space is also first-countable. On the other hand, a first-countable space is not necessarily second-countable. For example, an uncountable set equipped with its discrete topology is first-countable but not second-countable.

Proposition 2.4.9 *Every subset of a second-countable topological space is itself second-countable.*

Proof If X is a topological space admitting a countable base $\mathcal{B}$ and $Y \subset X$, then the set consisting of all the subsets of the form $Y \cap B$, where B runs over $\mathcal{B}$, is clearly a countable base for Y. □

Proposition 2.4.10 *Every countable product of second-countable spaces is itself second-countable.*

Proof Let $(X_i)_{i \in I}$ be a countable family of second-countable spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let $\mathcal{B}_i$ be a countable base of X_i. We can assume $X_i \in \mathcal{B}_i$. Then the sets $\prod_{i \in I} U_i$, where $U_i \in \mathcal{B}_i$ for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$, form a countable base for the product topology. Therefore X is second-countable. □

Proposition 2.4.11 (Lindelöf) *Every second-countable topological space is Lindelöf.*

Proof Let X be a topological space admitting a countable base $\mathcal{B}$. Let $\alpha = (U_i)_{i \in I}$ be an open cover of X. Denote by $\mathcal{B}'$ the set consisting of all $B \in \mathcal{B}$ such that there exists $i \in I$ satisfying $B \subset U_i$. Define a map $\varphi \colon \mathcal{B}' \to I$ by choosing, for each $B \in \mathcal{B}'$, an index $\varphi(B) \in I$ such that $B \subset U_{\varphi(B)}$. Then the image set $J = \varphi(\mathcal{B}') \subset I$ is countable. Let $x \in X$. As α covers X, we can find an index $i(x) \in I$ such that $x \in U_{i(x)}$. Since $\mathcal{B}$ is a base of X, there exists an open subset $B(x) \in \mathcal{B}$ such that $x \in B(x) \subset U_{i(x)}$. We have that $B(x) \in \mathcal{B}'$, by definition of $\mathcal{B}'$, and $x \in B(x) \subset U_{\varphi(B(x))}$. It follows that $(U_i)_{i \in J}$ is a countable cover of X. This shows that X is Lindelöf. □

Definition 2.4.12 A topological space is said to be *separable* if it admits a countable dense subset.

Proposition 2.4.13 *Every second-countable topological space is separable.*

Proof Let X be a topological space and $\mathcal{B}$ a base of X. Let us choose, for each $B \in \mathcal{B}$ with $B \neq \varnothing$, a point $x_B \in B$ and denote by Y the set consisting of all such points x_B. Since $\mathcal{B}$ is a base for X, every non-empty open subset of X contains a point of Y. Consequently, Y is dense in X. If $\mathcal{B}$ is countable, then Y is also countable and hence X is separable. □

From Propositions 2.4.9, 2.4.11 and 2.4.13, we immediately deduce the following result.

Corollary 2.4.14 *Every subset of a second-countable space is separable and Lindelöf. In particular, every subset of the Euclidean space $\mathbb{R}^n$ is separable and Lindelöf.* □

The following example shows that a separable compact Hausdorff space may fail to be first-countable.

Example 2.4.15 Let X denote the set consisting of all maps from $\mathbb{R}$ into the unit segment $[0, 1]$. We equip X with the topology of pointwise convergence. Thus, the space X may be identified with the product space $[0, 1]^{\mathbb{R}}$ and is a compact Hausdorff space by Tychonoff's theorem. Let $f \in X$. By definition of the topology of pointwise convergence, for every $\varepsilon > 0$ and every finite subset $A \subset \mathbb{R}$, the set

$$V(f, \varepsilon, A) := \{g \in X \mid |f(x) - g(x)| < \varepsilon \text{ for all } x \in A\}$$

is an open neighborhood of f. Moreover, the sets $V(f, \varepsilon, A)$, where $\varepsilon > 0$ and $A \subset \mathbb{R}$ is a finite subset, form a neighborhood base of f. Let D denote the subset of X consisting of all finite linear combinations with rational coefficients of characteristic maps of segments of $\mathbb{R}$ with rational endpoints. Clearly D is dense in X. As D is countable, this shows that X is separable. However, X is not first-countable. Otherwise, every $f \in X$ would admit a countable neighborhood base W_n, $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, there would exist $\varepsilon_n > 0$ and a finite subset $A_n \subset \mathbb{R}$ such that $V(f, \varepsilon_n, A_n) \subset W_n$. The set $E := \bigcup_{n \in \mathbb{N}} A_n$ would be countable and hence we would have $\mathbb{R}\backslash E \neq \varnothing$. Taking a point $x_0 \in \mathbb{R}\backslash E$, any map $g\colon \mathbb{R} \to [0, 1]$ such that $g(x_0) \neq f(x_0)$ and $g(x) = f(x)$ for all $x \in E$ would satisfy $g \in W_n$ for all $n \in \mathbb{N}$. As X is Hausdorff, this would imply $g = f$, which contradicts $g(x_0) \neq f(x_0)$. Consequently, X is not first-countable and hence not second-countable either.

Remark 2.4.16 The topological space in the preceding example is not metrizable. Indeed, every metrizable space is first-countable since, in a metric space X, every point $x \in X$ admits a countable neighborhood base, e.g., the one formed by the open balls $B(x, 1/n)$, $n \geq 1$.

Remark 2.4.17 The space X in Example 2.4.15 is Lindelöf since it is compact. In Sect. 5.5, we will describe a first-countable separable Lindelöf Hausdorff space S which is not second-countable (see Proposition 5.5.1 and Corollary 5.5.7).

For metrizable spaces, we have the following equivalent conditions.

Proposition 2.4.18 *Let X be a metrizable space. Then the following conditions are equivalent:*

(a) X is second-countable;
(b) X is Lindelöf;
(c) X is separable;
(d) X is homeomorphic to a subset of the Hilbert cube $[0, 1]^{\mathbb{N}}$.

Proof The fact that (a) implies (b) follows from Proposition 2.4.11.

Let us fix a metric d on X compatible with its topology.

Suppose (b). Given an integer $n \geq 1$, consider the cover of X formed by the open balls $B(x, 1/n)$, $x \in X$. As X is Lindelöf, there exists a countable subset $Y_n \subset X$ such that the balls $B(y, 1/n)$, $y \in Y_n$, cover X. The set $Y := \bigcup_{n \geq 1} Y_n$ is countable and dense in X. Consequently, X is separable. This shows that (b) implies (c).

The unit segment $[0, 1] \subset \mathbb{R}$ is second-countable. Thus, condition (d) implies (a) since any countable product of second-countable topological spaces is second-countable by Proposition 2.4.10 and any subset of a second-countable space is second-countable by Proposition 2.4.9.

To complete the proof, it suffices to show that (c) implies (d). Suppose (c). Let $A = \{a_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X. After possibly replacing $d(x, y)$ by the metric $\min(d(x, y), 1)$, which is also compatible with the topology on X, we can assume that $\operatorname{diam}(X) \leq 1$. Consider the map $F\colon X \to [0, 1]^{\mathbb{N}}$ defined by

$$F(x) = (d(x, a_n))_{n \in \mathbb{N}}\,.$$

The map F is continuous since all maps $x \mapsto d(x, a_n)$ are continuous. As every point of X is the limit of some sequence of points in A, it follows that F is injective (uniqueness of the limit in Hausdorff spaces). Let now $x_0 \in X$ and $\varepsilon > 0$. As A is dense in X, there exists an integer $n_0 \geq 0$ such that $d(x_0, a_{n_0}) < \varepsilon/2$. Then the subset $U \subset [0, 1]^{\mathbb{N}}$ consisting of all sequences $(u_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ such that $u_{n_0} < \varepsilon/2$ is an open neighborhood of $F(x_0)$. If $x \in X$ is such that $F(x) \in U$, then x satisfies $d(x, a_{n_0}) < \varepsilon/2$ and hence

$$d(x, x_0) \leq d(x, a_{n_0}) + d(x_0, a_{n_0}) < \varepsilon,$$

by applying the triangle inequality. Consequently, we have that

$$F^{-1}(U) \subset B(x_0, \varepsilon).$$

We deduce that F induces a homeomorphism from X onto $F(X)$. This shows that X satisfies (d). □

As every compact space is Lindelöf, we immediately get the following:

Corollary 2.4.19 *Every compact metrizable space is second-countable and hence separable.*

It follows from Corollary 2.3.3 that every accessible topological space X with $\dim(X) = 0$ is scattered. The following theorem states that the converse holds in the class of Lindelöf spaces. This is very useful for showing that certain spaces are zero-dimensional.

Theorem 2.4.20 *Let X be a non-empty scattered Lindelöf space. Then one has* $\dim(X) = 0$.

Proof As X is scattered, it admits a base $\mathcal{B}$ consisting of clopen subsets. Consider a finite open cover $\alpha = (U_i)_{i \in I}$ of X. For every $x \in X$, we can find an index $i(x) \in I$ such that $x \in U_{i(x)}$. As $\mathcal{B}$ is a base of X, there exists $B(x) \in \mathcal{B}$ such that $x \in B(x) \subset U_{i(x)}$. The subsets $B(x)$, $x \in X$, form an open cover of X. Since X is Lindelöf, this open cover admits a countable subcover. Therefore there exists a cover $\beta = (B_n)_{n \in \mathbb{N}}$ of X such that $\beta \succ \alpha$ and $B_n \in \mathcal{B}$ for all n.

Consider the sequence $\gamma = (C_n)_{n \in \mathbb{N}}$ of subsets of X defined by $C_0 := B_0$ and

$$C_n := B_n \setminus (B_0 \cup B_1 \cup \cdots \cup B_{n-1}),$$

for every integer $n \geq 1$. As the subsets B_n are clopen and cover X, it is clear that γ is an open partition of X. On the other hand, we have that $\gamma \succ \beta \succ \alpha$. By applying Proposition 1.1.6, we deduce that $D(\alpha) = 0$. Thus, we have $\dim(X) = \sup_\alpha D(\alpha) = 0$. □

Remark 2.4.21 As mentioned earlier, we shall give in Sect. 5.4 an example of a scattered locally compact Hausdorff space with positive topological dimension.

By Corollary 2.3.3, every accessible space X with $\dim(X) = 0$ is scattered. Combining this result with the previous theorem, we get the following.

Corollary 2.4.22 *Let X be an accessible Lindelöf space (e.g., a separable metrizable space or a compact Hausdorff space) with $X \neq \varnothing$. Then the following conditions are equivalent:*

(a) $\dim(X) = 0$*;*
(b) X *is scattered.* □

Example 2.4.23 We deduce from Corollary 2.4.22 and Proposition 2.2.8 that a non-empty subset $X \subset \mathbb{R}$ satisfies $\dim(X) = 0$ if and only if X has empty interior in $\mathbb{R}$. This shows in particular that the set $\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers satisfies $\dim(\mathbb{R}\setminus\mathbb{Q})=0$.

As an immediate consequence of Corollary 2.4.22, we obtain the following results.

Corollary 2.4.24 *Let* $(X_i)_{i\in I}$ *be a family of compact Hausdorff spaces with* $\dim(X_i) = 0$ *for all* $i \in I$*. Then the product space* $X := \prod_{i\in I} X_i$ *satisfies* $\dim(X) = 0$.

Proof By Proposition 2.2.7, the space X is scattered since it is a product of scattered spaces. On the other hand, X is a product of compact Hausdorff spaces and hence also compact and Hausdorff. □

Corollary 2.4.25 *Let* $(X_i)_{i\in I}$ *be a family of non-empty finite discrete spaces. Then the product space* $X := \prod_{i\in I} X_i$ *satisfies* $\dim(X) = 0$.

Proof This immediately follows from Corollary 2.4.24 since each X_i is a compact Hausdorff space with $\dim(X_i) = 0$. □

By taking $X_i = \{0, 1\}$ for all $i \in I$ in Corollary 2.4.25, we get the following.

Corollary 2.4.26 *One has* $\dim(\{0, 1\}^E) = 0$ *for any set* E. □

Example 2.4.27 We have $\dim(\{0, 1\}^{\mathbb{N}}) = 0$. As $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set K by Proposition 2.1.3, we recover the fact that $\dim(K) = 0$ (cf. Proposition 2.1.2).

Corollary 2.4.28 *Let* $(X_i)_{i\in I}$ *be a countable family of separable metrizable spaces such that* $\dim(X_i) = 0$ *for all* $i \in I$*. Then the product space* $X := \prod_{i\in I} X_i$ *satisfies* $\dim(X) = 0$.

Proof By Proposition 2.2.7, the space X is scattered since it is a product of scattered spaces. On the other hand, X is a product of countably many separable metrizable spaces and hence also separable and metrizable. □

The following example shows that the product of two zero-dimensional topological spaces may fail to be zero-dimensional.

Example 2.4.29 Let $X = \{x_0, x_1\}$ be a set with cardinality 2. Equip X with the topology for which the open sets are $\varnothing$, $\{x_0\}$ and X. We have that $\dim(X) = 0$ since the open cover of X reduced to X is finer than any open cover of X. In fact, X is the space described in Example 1.1.11 for $n = 0$. Consider now the set $X \times X$ equipped with the product topology. The open subsets of $X \times X$ are $\varnothing$ and all the subsets of $X \times X$ that contain (x_0, x_0). Thus, we have that $\dim(X \times X) = 2$ by applying the result in Example 1.1.11 for $n = 2$.

Remark 2.4.30 The topological space X in the previous example is not Hausdorff, not even accessible since $\{x_0\}$ is not closed in X. In Sect. 5.5, we shall give an example of a normal Hausdorff space X such that $\dim(X) = 0$ and $\dim(X \times X) \neq 0$.

2.5 Totally Disconnected Spaces

Let X be a topological space. Recall that the *connected component* of a point $x \in X$ is the union of all the connected subsets of X containing x. The connected components of the points of X form a partition of X. Moreover, every connected component is connected and closed in X.

Definition 2.5.1 We say that a topological space X is *totally disconnected* if the connected component of every point $x \in X$ is the singleton set reduced to the point x.

In other words, a topological space X is totally disconnected if and only if the only non-empty connected subsets of X are the subsets that are reduced to a single point.

Example 2.5.2 Every discrete space is totally disconnected.

Example 2.5.3 The only connected subsets of $\mathbb{R}$ are the intervals. It follows that a subset $X \subset \mathbb{R}$ is totally disconnected if and only if X has empty interior.

Proposition 2.5.4 *Every subset of a totally disconnected space is itself totally disconnected.*

Proof This immediately follows from the observation that if Y is a subset of a topological space X and $y \in Y$ then the connected component of y in Y is contained in the connected component of y in X. □

Proposition 2.5.5 *Every product of totally disconnected spaces is itself totally disconnected.*

Proof Let $(X_i)_{i \in I}$ be a family of totally disconnected spaces and consider their direct product $X := \prod_{i \in I} X_i$. Let C be a non-empty connected subset of X. As the continuous image of a connected space is itself connected, the projection of C on each X_i is connected and hence reduced to a single point since X_i is totally disconnected. This implies that C itself is reduced to a single point. □

Proposition 2.5.6 *Every totally disconnected space is accessible.*

Proof In a topological space, every connected component is closed. Consequently, if the topological space X is totally disconnected then $\{x\}$ is closed in X for all $x \in X$. □

The following example shows that a totally disconnected space may fail to be Hausdorff.

Example 2.5.7 Let X be an infinite set. Let us fix two distinct points $a, b \in X$ and let $Y := X \setminus \{a, b\}$. Let $\mathcal{T}$ denote the set consisting of all $U \subset X$ satisfying one of the following two conditions:

(1) $U \subset Y$;
(2) $U = U_1 \cup U_2$, where U_1 is a non-empty subset of $\{a, b\}$ and $U_2 \subset Y$ is such that $Y \setminus U_2$ is a finite set.

It is straightforward to verify that $\mathcal{T}$ is the set of open sets for a topology on X. Let us equip X with this topology. Suppose that $A \subset X$ has more than one point. If we can find a point $y_0 \in A \cap Y$, then the singleton set $\{y_0\}$ is clopen in A. Otherwise, we have that $A = \{a, b\}$ and then $\{a\}$ is clopen in A. It follows that A is not connected. Thus, the space X is totally disconnected. However, X is not Hausdorff since every open neighborhood of a meets every open neighborhood of b.

2.6 Totally Separated Spaces

In this section, we introduce the class of totally separated spaces. We prove that every totally separated space is totally disconnected and that every scattered accessible space is totally separated.

Let X be a topological space. The *quasi-component* of a point $x \in X$ is the intersection of all clopen neighborhoods of x. Note that the quasi-component of every point $x \in X$ is a closed subset of X containing x.

Definition 2.6.1 We say that a topological space X is *totally separated* if the quasi-component of every point $x \in X$ is the singleton set reduced to the point x.

Remark 2.6.2 A topological space X is totally separated if and only if it satisfies the following condition: for every pair of distinct points x and y in X, there exists a partition of X into two open subsets U and V such that $x \in U$ and $y \in V$.

Proposition 2.6.3 *Every totally separated space is Hausdorff.*

Proof This immediately follows from the preceding remark. □

Proposition 2.6.4 *Let X be a topological space and x a point in X. Then the connected component of x is contained in the quasi-component of x.*

Proof Denote by C_x the connected component of x and by Q_x its quasi-component. Consider a clopen neighborhood V of x in X. Then $C_x \cap V$ is a clopen subset of C_x that is not empty since it contains x. By connectedness of C_x, we deduce that $C_x \cap V = C_x$, that is, $C_x \subset V$. It follows that $C_x \subset Q_x$. □

Corollary 2.6.5 *Every totally separated space is totally disconnected.* □

A totally disconnected space is not necessarily totally separated. Indeed, we have described in Example 2.5.7 a totally disconnected space that is not Hausdorff. Such a space is not totally separated since, by Proposition 2.6.3, every totally separated space is Hausdorff. In Sect. 5.2, we shall give an example of a totally disconnected separable metrizable space that is not totally separated.

Proposition 2.6.6 *Every scattered accessible space is totally separated and hence totally disconnected.*

Proof Let X be a scattered accessible space. Let $\mathcal{B}$ be a base of X consisting of clopen subsets of X. Consider a point x in X. As $\mathcal{B}$ is a base of X, the set $\mathcal{B}_x$ consisting of all elements of $\mathcal{B}$ containing x is a neighborhood base of x. The intersection of all the neighborhoods of x is reduced to the point x since X is accessible. This implies that the intersection of the elements of $\mathcal{B}_x$ is also reduced to x. Consequently, the quasi-component of x is the singleton set $\{x\}$. This shows that X is totally separated. □

The accessibility hypothesis in Proposition 2.6.6 cannot be removed. Indeed, a set having more than one point equipped with its trivial topology is scattered but not totally separated (not even totally disconnected).

Let us note also that the converse of Proposition 2.6.6 is false. Indeed, we will give in Sect. 5.1 an example of a separable metrizable space that is totally separated but not scattered. However, as we shall see, the converse of Proposition 2.6.6 becomes true if we restrict ourselves to locally compact Hausdorff spaces. Let us first establish the following result.

Lemma 2.6.7 *Let X be a compact Hausdorff space. Let x be a point in X. Then the connected component of x coincides with its quasi-component.*

Proof Denote by C_x the connected component of x and by Q_x its quasi-component. We have that $C_x \subset Q_x$ by Proposition 2.6.4. Thus, it suffices to prove that Q_x is connected. Let A and B be disjoint closed subsets of Q_x such that $A \cup B = Q_x$. We can assume that $x \in A$. As Q_x is closed in X, the sets A and B are closed in X. On the other hand, since X is a compact Hausdorff space, it is normal by Proposition 1.5.4. Consequently, there exist disjoint open subsets V and W of X such that $A \subset V$ and $B \subset W$. Denote by $\mathcal{E}$ the set consisting of all clopen neighborhoods of x in X. We have that

$$\bigcap_{U \in \mathcal{E}} U = Q_x \subset V \cup W.$$

Therefore, the open subsets $X \backslash U$, $U \in \mathcal{E}$, cover $X \backslash (V \cup W)$. As $X \backslash (V \cup W)$ is compact, there exists a finite sequence $U_1, \ldots, U_n$ of elements of $\mathcal{E}$ such that

$$X \backslash (V \cup W) \subset (X \backslash U_1) \cup \cdots \cup (X \backslash U_n).$$

By setting $\Omega := U_1 \cap \cdots \cap U_n$, this amounts to saying that $\Omega \subset V \cup W$. As V and W are disjoint, we deduce that $\Omega \cap V = \Omega \backslash W$. Consequently, the set $\Omega \cap V$ is a clopen neighborhood of x in X. It follows that $Q_x \subset \Omega \cap V$. Therefore we have that $Q_x = A$. This shows that Q_x is connected. □

Proposition 2.6.8 *Let X be a locally compact Hausdorff space. Then the following conditions are equivalent:*

(a) X is scattered;
(b) X is totally separated;
(c) X is totally disconnected.

Proof The fact that (a) implies (b) follows from Proposition 2.6.6. On the other hand, Corollary 2.6.5 shows that (b) implies (c).

Suppose that X is totally disconnected. Let x be a point in X and let V be a neighborhood of x. As X is locally compact, there exists a compact neighborhood W of x such that $W \subset V$. Denote by U the interior of W in X and by $\mathcal{E}$ the set consisting of all clopen neighborhoods of x in W. As W is totally disconnected by Proposition 2.5.4, it follows from Lemma 2.6.7 that $\{x\} = \bigcap_{F \in \mathcal{E}} F$. This implies that the family

$$\alpha := \{U\} \cup \{W \setminus F \mid F \in \mathcal{E}\}$$

is an open cover of W. Since W is compact, α admits a finite subcover. This means that there exists a finite sequence $F_1, \dots, F_n \in \mathcal{E}$ such that the set $A := F_1 \cap \dots \cap F_n$ satisfies $A \subset U$. Each F_i, $1 \leq i \leq n$, is closed in W and hence in X since W is closed in X. On the other hand, A is open in U and hence open in X. It follows that A is clopen in X. As $x \in A \subset V$, we deduce that the neighborhoods of x that are clopen in X form a neighborhood base of x. This shows that X is scattered. Thus, (c) implies (a). □

2.7 Zero-Dimensional Compact Hausdorff Spaces

By combining results obtained in the previous sections, we get the following characterizations of zero-dimensional compact Hausdorff spaces.

Theorem 2.7.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) X is a compact Hausdorff space with $\dim(X) = 0$*;*
(b) X is a scattered compact Hausdorff space;
(c) X is a totally separated compact Hausdorff space;
(d) X is a totally disconnected compact Hausdorff space;
(e) there exists a set E such that X is homeomorphic to a closed subset of the product space $\{0, 1\}^E$.

Proof Conditions (a) and (b) are equivalent by virtue of Corollary 2.4.22. On the other hand, conditions (b), (c) and (d) are equivalent by Proposition 2.6.8. Finally, the equivalence of (b) and (e) is an immediate consequence of Proposition 2.2.9 since the product space $\{0, 1\}^E$ is a compact Hausdorff space for any set E by Tychonoff's theorem. □

2.8 Zero-Dimensional Separable Metrizable Spaces

We also get the following characterizations of zero-dimensional separable metrizable spaces.

Theorem 2.8.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) X is a separable metrizable space with $\dim(X) = 0$*;*
(b) X is a scattered separable metrizable space;
(c) X is a separable metrizable space that admits a countable base consisting of clopen subsets;
(d) X is homeomorphic to a subset of $\{0, 1\}^{\mathbb{N}}$*;*
(e) X is homeomorphic to a subset of the Cantor set.

Proof Conditions (a) and (b) are equivalent by Corollary 2.4.22.

Suppose that X is a scattered separable metric space. Let $\mathcal{B}$ be a base of X consisting of clopen subsets. As X is separable, we can find a countable dense subset $Y \subset X$. Let us choose, for each $y \in Y$ and each integer $n \geq 1$, a neighborhood $B_{y,n} \in \mathcal{B}$ of y contained in the open ball of radius $1/n$ centered at y. Then the subsets $B_{y,n}$ form a countable base of X. This shows that (b) implies (c).

Let us now show that (c) implies (d) (cf. the proof of Proposition 2.2.9). Suppose that X is a separable metric space and that $(B_n)_{n \in \mathbb{N}}$ is a base of X consisting of clopen subsets. Let $\chi_n \colon X \to \{0, 1\}$ denote the characteristic map of B_n. Consider the map $\varphi \colon X \to \{0, 1\}^{\mathbb{N}}$ defined by $\varphi(x) = (\chi_n(x))_{n \in \mathbb{N}}$ for all $x \in X$. As B_n is clopen, the map χ_n is continuous for every $n \in \mathbb{N}$. This implies that φ is continuous. As X is Hausdorff, the injectivity of φ follows from the fact that the subsets B_n, $n \in \mathbb{N}$, form a base of X. We have that $\varphi(B_n) = \varphi(X) \cap \pi_n^{-1}(1)$, where $\pi_n \colon \{0, 1\}^{\mathbb{N}} \to \{0, 1\}$ is the projection onto the n-factor of $\{0, 1\}^{\mathbb{N}}$. This shows that $\varphi(B_n)$ is open in $\varphi(X)$. As the subsets B_n form a base of X, we deduce that the image by φ of any open subset of X is open in $\varphi(X)$. Consequently, φ induces a homeomorphism from X onto $\varphi(X)$. This shows that X satisfies (d).

To complete the proof, it suffices to observe that (d) implies (b) by Proposition 2.2.9 and that (d) and (e) are equivalent since the space $\{0, 1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set by Proposition 2.1.3. □

Remark 2.8.2 As already mentioned above, we will give in Sect. 5.1 an example of a separable metrizable space that is totally separated (and hence totally disconnected) but not scattered.

2.9 Zero-Dimensional Compact Metrizable Spaces

Every compact metrizable space is both Hausdorff and separable. By combining Theorems 2.7.1 and 2.8.1, we obtain the following statement (Table 2.1).

Table 2.1 Summary Table (X non-empty)

NOTES

$$\boxed{\dim(X) = 0} \Rightarrow \boxed{\text{scattered}} \Rightarrow \boxed{\text{totally separated}} \Rightarrow \boxed{\text{totally disconnected}}$$

Hausdorff spaces

$$\boxed{\dim(X) = 0} \Leftrightarrow \boxed{\text{scattered}} \Rightarrow \boxed{\text{totally separated}} \Rightarrow \boxed{\text{totally disconnected}}$$

separable metrizable spaces

$$\boxed{\dim(X) = 0} \Rightarrow \boxed{\text{scattered}} \Leftrightarrow \boxed{\text{totally separated}} \Leftrightarrow \boxed{\text{totally disconnected}}$$

locally compact Hausdorff spaces

$$\boxed{\dim(X) = 0} \Leftrightarrow \boxed{\text{scattered}} \Leftrightarrow \boxed{\text{totally separated}} \Leftrightarrow \boxed{\text{totally disconnected}}$$

compact Hausdorff spaces

Theorem 2.9.1 *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(a) *X is a compact metrizable space with* $\dim(X) = 0$*;*
(b) *X is a scattered compact metrizable space;*
(c) *X is a totally separated compact metrizable space;*
(d) *X is a totally disconnected compact metrizable space;*
(e) *X is a compact metrizable space that admits a countable base consisting of clopen subsets;*
(f) *X is homeomorphic to a closed subset of* $\{0, 1\}^{\mathbb{N}}$*;*
(g) *X is homeomorphic to a closed subset of the Cantor set.* □

Notes

The terminology used in this chapter follows that of Bourbaki [18]. However, the terms "scattered", "totally disconnected", and "totally separated" have sometimes different meanings in the literature. For example, spaces that are called "scattered" in the present book are called "zero-dimensional" in [102], while a "scattered" space in [102] is a topological space in which every non-empty subset admits an isolated point.

The Cantor ternary set was described by Cantor in [21, note 11 p. 46]. It can be shown that every totally disconnected compact metrizable space that is perfect is homeomorphic to the Cantor set (see for example [48, Corollary 2–98]).

A non-empty topological space X is scattered if and only if $\operatorname{ind}(X) = 0$ (see the Notes on Chap. 1, p.19, for the definition of the small inductive dimension $\operatorname{ind}(X)$). The question of the existence of scattered metrizable spaces with positive topological dimension remained open for many years (cf. [18, note 1 p. IX.119]). An affirmative answer to this question was finally given by Roy [96, 98] who constructed a scattered metrizable space X with $\dim(X) = 1$.

The notion of a totally disconnected space and that of a totally separated space were respectively introduced by Hausdorff [47] and by Sierpinski [99]. In [99], Sierpinski described a totally disconnected subset of $\mathbb{R}^2$ that is not totally separated and a totally separated subset of $\mathbb{R}^2$ with positive topological dimension.

Exercises

2.1 Does the real number $1/\pi$ belong to the Cantor set?

2.2 Show that the Cantor set has Lebesgue measure 0.

2.3 Show that every countable product of Cantor spaces is a Cantor space.

2.4 Let H denote the Hilbert space of square-summable real sequences $(u_n)_{n \geq 1}$. Show that the subset $X \subset H$ consisting of all sequences $(u_n)_{n \geq 1}$ such that $|u_n| \leq 1/n$ for all $n \geq 1$ is homeomorphic to the Hilbert cube $[0, 1]^{\mathbb{N}}$.

2.5 Let G be a group. Let $\mathcal{B}$ denote the set of all left cosets of subgroups of finite index of G, i.e., the subsets of the form gH, where $g \in G$ and $H \subset G$ is a subgroup with $[G : H] < \infty$.

(a) Show that there is a unique topology on G admitting $\mathcal{B}$ as a base. This topology is called the *profinite topology* on G.
(b) Show that the profinite topology on G is scattered.
(c) Show that the profinite topology on G is discrete if and only if G is finite.
(d) Show that the profinite topology on the additive group $\mathbb{Q}$ of rational numbers is the trivial topology.
(e) Show that the profinite topology on G is Hausdorff if and only if G is residually finite. (Recall that the group G is called *residually finite* if the intersection of all its subgroups of finite index is reduced to the identity element.)

2.6 (*Furstenberg's topological proof of the infinitude of primes* [38]). Let $\mathbb{Z}$ denote the group of integers equipped with its profinite topology (see Exercise 2.5).

(a) Show that $n\mathbb{Z}$ is a closed subset of $\mathbb{Z}$ for every $n \in \mathbb{Z}$.
(b) Show that every non-empty open subset of $\mathbb{Z}$ is infinite.

(c) Let $\mathcal{P} := \{2, 3, 5, 7, 11, \dots\}$ denote the set of prime numbers. Use the results obtained in (a) and (b) to recover Euclid's theorem that $\mathcal{P}$ is infinite. Hint: observe that $\bigcup_{p \in \mathcal{P}} p\mathbb{Z} = \mathbb{Z} \setminus \{-1, 1\}$ is not closed in $\mathbb{Z}$.

2.7 Let $f : X \to Y$ be a continuous map from a Lindelöf space X into a topological space Y. Show that $f(X)$ is a Lindelöf space.

2.8 Show that every locally compact Lindelöf space is σ-compact.

2.9 Let X be an uncountable set equipped with its cofinite topology. Show that X is not first-countable.

2.10 Show that every open subset of a separable space is separable.

2.11 Show that every subspace of a separable metrizable space is separable.

2.12 Show that the set consisting of all isolated points of a separable space is countable.

2.13 Show that every countable product of separable spaces is separable.

2.14 Let (X, d) be a separable metric space. Consider the Banach space $\ell^\infty(\mathbb{R})$ consisting of all bounded sequences of real numbers $u = (u_n)_{n \in \mathbb{N}}$ with the supremum norm $\|u\| = \sup_{n \in \mathbb{N}} |u_n|$. Fix a point $x_0 \in X$ and a sequence $(a_n)_{n \in \mathbb{N}}$ of points of X such that the set $\{a_n \mid n \in \mathbb{N}\}$ is dense in X. Show that the sequence $(d(x, a_n) - d(x_0, a_n))_{n \in \mathbb{N}}$ is in $\ell^\infty(\mathbb{R})$ for every $x \in X$ and that the map $\varphi : X \to \ell^\infty(\mathbb{R})$ defined by $\varphi(x) = (d(x, a_n) - d(x_0, a_n))_{n \in \mathbb{N}}$ is an isometric embedding.

2.15 Show that the Banach space $\ell^\infty(\mathbb{R})$ is not separable.

2.16 Show that every second-countable scattered accessible space is homeomorphic to a subset of the Cantor set.

2.17 A metric space (X, d) is called an *ultrametric space* if one has

$$d(x, y) \leq \max(d(x, z), d(y, z))$$

for all $x, y, z \in X$. Let (X, d) be a non-empty ultrametric space.

(a) Let A be a closed subset of X and $\rho > 0$. Show that the set consisting of all $x \in X$ such that $\operatorname{dist}(x, A) = \rho$ is a clopen subset of X.

(b) Let A and B be disjoint closed subsets of X. Show that the set consisting of all $x \in X$ such that $\operatorname{dist}(x, A) \leq \operatorname{dist}(x, B)$ is a clopen subset of X.

(c) Show that $\dim(X) = 0$.

(d) Show that the metric completion (X', d') of (X, d) is also an ultrametric space.

2.18 Let p be a prime integer. Every non-zero rational number $q \in \mathbb{Q} \setminus \{0\}$ can be written in the form $q = p^n \dfrac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z} \setminus p\mathbb{Z}$ are integers not divisible by p. The integer $v_p(q) := n \in \mathbb{Z}$ is well defined and called the *p-valuation* of q. Define the map $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by

$$d(x, y) := \begin{cases} p^{-v_p(x-y)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in \mathbb{Q}$.

(a) Show that $(\mathbb{Q}, d)$ is an ultrametric space.
(b) Show that the metric completion $\mathbb{Q}_p$ of $(\mathbb{Q}, d)$ satisfies $\dim(\mathbb{Q}_p) = 0$. (The set $\mathbb{Q}_p$ is the set of *p-adic numbers*.)

2.19 Show that every totally disconnected topological space that is locally connected is discrete. (Recall that a topological space X is called *locally connected* if every point $x \in X$ admits a neighborhood base consisting of connected subsets.)

2.20 Let X be a non-empty subset of $\mathbb{R}$. Show that one has $\dim(X) = 0$ if and only if X is totally disconnected.

2.21 Let X be the topological space described in Example 2.5.7.

(a) Show that X is compact.
(b) Show that X is not normal.
(c) Show that $\dim(X) = 1$.

2.22 A topological space X is called *extremally disconnected* if the closure of any open subset of X is open in X.

(a) Show that if a set X is equipped with its trivial (resp. discrete) topology then X is extremally disconnected.
(b) Show that every extremally disconnected Hausdorff space is totally separated.
(c) Show that every extremally disconnected metrizable space is discrete.

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