Error and Predicativity

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Abstract. The article surveys ideas emerging within the predicative tradition in the foundations of mathematics, and attempts a reading of predicativity constraints as highlighting different levels of understanding in mathematics. A connection is made with two kinds of error which appear in mathematics: local and foundational errors. The suggestion is that ideas originating in the predicativity debate as a reply to foundational errors are now having profound influence to the way we try to address the issue of local errors. Here fundamental new interactions between computer science and mathematics emerge.

1 Certainty and Certification

Mathematics is often considered the most exact of all sciences, but error is not unusual even in print. Errors are also costly and unwelcome in computer science, where program verification is increasingly appealed to in order to minimize failure in hardware and software.

I would like to suggest a distinction between two types of error which can appear in mathematics. First of all there are what I should like to call *"local"* errors. These are errors which plague individual proofs, or possibly a relatively small group of proofs which share a similarity in structure. These mistakes are somehow confined to a small portion of the mathematical enterprise and, if corrigible, they can be amended without introducing any substantial revision of the underlying mathematical principles one appeals to when devising the given proof. Secondly, there are *"foundational"* mistakes, which instead relate to the very principles of proofs and the axioms; these occur when inconsistencies arise within our foundational systems.

Nowadays emphasis is on local errors. The ever growing specialization of mathematics has made proofs much harder not only to obtain, but also to verify. In addition to the complexity, the mere size of some proofs demands new strategies for their verification. A substantial debate on the role of computers for both the discovery and the verification processes in mathematics is presently ongoing within the mathematical community, with an increasing number of

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mathematicians hoping that a fruitful interaction between mathematics and computer science will produce substantial benefits for today's mathematics.[1]

As to the second kind of error, a large number of mathematicians seem by now quite confident that we have obtained inductive corroboration of our mathematical practice and also of our main mathematical systems; for example, theories like ZFC seem to have undergone sufficient scrutiny over the years to be considered reliable by most. However, a genuine concern regarding the trustworthiness of our mathematical methodology and of our foundational systems was voiced at the turn of the 20th century, as a direct reply to the deep methodological changes that mathematics was undergoing at the time, as well as the rise of the set-theoretic paradoxes.

The main topic of this article is *predicative mathematics*, a form of mathematics which originated at the beginning of the 20th century in attempts to address the threat arising from potential foundational errors, by proposing methodological constraints. Predicativity constraints are motivated by a varied family of concepts, and, as further hinted at below, inspire a number of rather different forms of mathematics. This complexity makes it very difficult to discuss predicativity within the limits of a short article and venture to draw some general conclusions. However, I would like to highlight some common themes which run through the debate on predicativity since its earliest times. I would also like to recall the difficulties encountered in understanding the demarcation between the notions of predicative and impredicative. Notwithstanding these difficulties, the hope is to be able to suggest a reading of predicativity constraints as an instrument for singling out (substantial) portions of classical mathematics which are amenable to a less abstract, and a more concrete treatment. A predicative treatment of those portions of mathematics then is often seen by its proponents as providing full conviction for the correctness of the results. That is, by restricting the methodology to more stringent canons, one ought to gain more detailed insight of the constructions carried out within a proof, and fuller grasp of the results. In the case of constructive predicativity as represented by the tradition arising within Martin-Löf type theory, the ensuing mathematics obtains a distinctive *direct* computational content. Here predicativity constraints enable an identification of mathematics with programming which has inspired fundamental research at the intersection between mathematics and computer science. In fact, recent years have seen the rise of attempts to proof-checking portions of mathematics with the support of computer systems, opening up new paths for the verification of mathematics. The suggestion I would like to make, then, is that ideas originating in the predicativity debate as a reply to foundational errors are now having profound influence to the way we try to address the issue of local errors.

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[1] Dana Scott in his opening talk at the The Vienna Summer of Logic (9th–24th July 2014) suggested that we are now witnessing a paradigm change in logic and mathematics. At least in certain areas of mathematics, there is an urgent need to solve complex and large proofs, and this requires computers and logic to work together to make progress. See Dana Scott's e-mail to the Foundations of Mathematics mailing list of 28–07–14 (http://www.cs.nyu.edu/mailman/listinfo/fom).

2 Predicativity

Predicativity has its origins in the writings of Poincaré and Russell, and is only one of a number of influential programmes which arose at the beginning of the past century in an attempt to bring clarity to a fast changing mathematics. Mathematics, in fact, had undergone deep methodological alterations during the 19th century which soon prompted a lively foundational debate. The paradoxes that were discovered in Cantor's and Frege's set theories in the early 20th century were one of the principal motivations for the very rich discussions between Poincaré and Russell, within which the concept of predicativity was forged (see for example [19,22,23]). These saw impredicativity as the main source of the paradoxes, and attempted to clarify a notion of predicativity, adherence to which would hinder inconsistencies. According to one rendering of this notion, a definition is impredicative if it defines an object by quantifying on a totality which includes the object to be defined. Through Russell and Poincaré's confrontation a number of ways of capturing impredicativity and explaining its perceived problematic character emerged. One influential thought (originating in Richard and, via Poincaré, adopted and particularly pressed further by Russell) saw impredicativity as engendering from a vicious circularity, or self-reference.[2] According to this view, a vicious circle arises if we suppose that a collection of objects may contain members which can *only* be defined by means of the collection as a whole, thus bearing reference to the definiendum. As a response to these difficulties Russell introduced his well-known vicious circle principle, which in one formulation states that: "whatever in any way concerns all or any or some of a class must not be itself one of the members of a class." [24, p.198]

Perhaps an example could help clarify the issue of impredicativity. The most paradigmatic instance of antinomy is Russell's paradox, which was discovered by Russell in Frege's Grundgesetze in 1901. A modern rendering of the paradox amounts to forming Russell's set, $R = \{x \mid x \notin x\}$, by unrestricted comprehension. One then obtains: $R \in R$ if and only if $R \notin R$. A circularity arises here from the fact that R is defined by reference to (i.e. quantification on) the whole universe of sets, to which R itself would belong. Russell's vicious circle principle, then, endeavours to prevent R from selecting a collection. The perceived difficulty arising from this kind of circularity can be elucidated from a number of perspectives. For example, according to one view, we ought to have access to a well-determined meaning for the condition appearing in the above instance of the comprehension principle (i.e. $x \notin x$). The difficulty is then realated to the fact that we seem to be unable to grant this independently of whether or not there exists a set R as specified above [4].

The analysis of the paradoxes turned out to be extremely fruitful for the development of mathematical logic[3], starting from Russell's own implementation of the vicious circle principle through his type theory. In the mature version

[2] Another analysis proposed by Poincaré [20] stressed a form of "invariance" as characteristic of predicativity: a predicative set cannot be "disturbed" by the introduction of new elements, contrary to an impredicative set [3,11].

[3] See [3] for a rich discussion of the impact of the paradoxes on mathematical logic.

of [23] two crucial ideas are interwoven: that of a type restriction and of ramification. By combining these two aspects ramified type theory seems to block all vicious circularity, and thus paradoxes of both set-theoretic and semantic nature.

Russell's type theory is a first fundamental contribution to the clarification of the complex question of what is predicativity in precise, logico-mathematical terms. However, as a way of developing a predicative form of mathematics Russell's type theory encountered substantial difficulties; it eventually surrendered to the assumption, in Principia Mathematica [29], of the axiom of reducibility, whose effect (in that context) was to restore full impredicativity. However, another attempt to develop analysis from a predicative point of view was proposed by Weyl [28], who showed how to carry out (a portion of) analysis on the basis of the bare assumption of the natural number structure. Weyl's crucial idea was to take the natural number structure with mathematical induction as given, as an ultimate foundation of mathematical thought, which can not be further reduced. Restrictions motivated by predicativity concerns were then imposed at the next level of idealization: the continuum. Weyl, in fact, introduced restrictions on how we form *subsets* of the natural numbers; in today's terminology, he saw as justified only those subsets of the natural numbers of the form $\{x : \varphi(x)\}$ if the formula φ is arithmetical, that is, it does not quantify over sets (but may quantify over natural numbers). The idea was that the natural numbers with full mathematical induction constitute an intuitively given category of mathematical objects; we can then use this and some immediately exhibited properties of and relations between the objects of this category (as obtained by arithmetical comprehension) to ascend to sets of natural numbers. In this way one also avoids vicious circularity in defining subsets of the natural numbers, as the restriction to number quantifiers in the comprehension principle does not allow for the definition of a new set by quantifying over a totality of sets to which the definiendum belongs.

I wish to highlight two aspects of Weyl's contribution. First, his approach to the question of the limit of predicativity went directly at the core of the mathematical practice, to show that large parts of 19th century analysis could be recovered on the basis of this restricted methodology. He thus succeeded in reducing to predicative methodology a conspicuous segment of mathematics, including portions which prima facie required impredicativity. Second, Weyl saw only this part of classical mathematics as fully justified; as he quickly became aware that not all of classical mathematics could be so recovered, he was ready to give up the rest, as (so far) not fully justified.

After Poincaré, Russell's and Weyl's fundamental contributions, predicativity lost momentum until the 1950's, when fresh attempts were made to obtain a clearer demarcation of the boundary between predicative and impredicative mathematics. The literature from the period shows the complexity of the task, but also witnesses the fruitfulness of the mathematical methodology for the philosophy of mathematics. The celebrated upshot of that research is the logical analysis of predicativity[4] by Feferman and Schütte (independently) following

[4] According to a notion of predicativity given the natural numbers which is discussed in the next section.

lines indicated by Kreisel [4,10,25,26]. Here Russell's original idea of ramification had a crucial role, as a transfinite progression of systems of ramified second order arithmetic indexed by ordinals was used to determine a precise limit for predicativity. This turned out to be expressed in terms of an ordinal, called Γ_0, which was the least non-predicatively provable ordinal. A formal system was then considered predicatively justifiable if it is proof-theoretically reducible to a system of ramified second order arithmetic indexed by an ordinal less than Γ_0.[5]

Another crucial contribution to the clarification of the extent of predicativity was the mathematical analysis of predicativity, aiming at elucidating which parts of mathematics can be expressed in predicative terms [5,27]. Work by Feferman, as well as results obtained within Friedman and Simpson's programme of Reverse Mathematics have shown that large parts of contemporary mathematics can be framed within (weak) predicative systems. Ensuing these results, Feferman has put forth the working hypothesis that all of scientifically applicable analysis can be developed in the system W of [5], which codifies in mordern terms Weyl's system in Das Kontinuum. These recent developments help better understand the reach of predicative mathematics, and reveal that predicativity goes much further than previously thought.

3 Plurality of Predicativity

As clarified by Feferman [4,6], the logical analysis of predicativity aimed at determining the limits of a notion of predicativity *given the natural numbers*. That is, one here takes an approach to predicativity similar to Weyl's, in assuming for given the structure of the natural numbers with full induction, and then imposing appropriate predicativity constraints on the formation of subsets of the natural numbers.[6] With Kreisel and Feferman the study of predicativity becomes thus an attempt to clarify what is implicit in the acceptance of the natural number structure (with full induction).[7]

Different incarnations of predicativity have however appeared in the literature, giving rise to very different forms of mathematics. For example, predicativity constraints have motivated Nelson's predicative arithmetic [16] and Parsons' criticism of the impredicativity of standard explanations of the notion of natural number [18]. According to Nelson already the whole system of the natural numbers equipped with full mathematical induction is predicatively problematic on grounds of circularity [16]: "The induction principle assumes that the natural number system is given. A number is conceived to be an object satisfying

[5] See [6] for an informal account of this notion of predicativity and for further references.

[6] The resulting notion of predicativity is, in fact, more generous than in Weyl's original proposal. The proof theoretic strength of a modern version of Weyl's system, like, for example, Feferman's system W from [5], equates that of Peano Arithmetic, and thus lays well below Γ_0.

[7] This line of research has been brought forward with Feferman's notion of unfolding, as analysed further by Feferman and Strahm e.g. in [7].

every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question." [16, p.1] From this point of view, then, already the theory of Peano Arithmetic, with its unrestricted induction, lies well beyond predicativity. Therefore Nelson's rejection of circularity leads him to justify only systems which are interpretable in a weak fragment of primitive recursive arithmetic, Robinson's system Q.[8]

Themes stemming from the original predicativity debates also play a prominent role within constructive mathematics, for example in the work of Lorenzen and Myhill [12], and in Martin-Löf type theory [14]. For constructive foundational theories, a more 'liberal' approach to predicativity, compared with that by Kreisel-Feferman-Schütte, has been suggested. Here the driving idea is that so-called *generalised inductive definitions* ought to be allowed in the realm of constructive mathematics. The intuitive justification of inductive definitions is related to the fact that they can be expressed by means of *finite rules*, and allow for a specification of sets which proceeds from the 'bottom up'. The underlying idea is to start from a well understood structure, say the natural numbers, and then use finite rules to extend this, by a process of successive iterations. We thus build a first subset of the set of natural numbers according to the rule, then use this to build a new one, and so on. The predicativity of this process is granted provided that we can ensure that at no stage in the built up of the new set, we need to presuppose a totality "outside" the set under construction. If this were the case, then, we would rely exclusively on increasingly more complex fragments of the very set under definition, and no vicious circularity would occur.[9] An important point to make is that the proof-theoretic strength of so-called theories of inductive definitions goes well beyond Feferman and Schütte's bound (and thus also very much beyond Peano Arithmetic), as shown in [1]. Following this line of reasoning, relatively strong theories are considered predicative in today's foundations of constructive mathematics [17,21].

A remarkable fact which emerges starting from the detailed logical analysis initiated in the 1950's, is that we now witness a number of different versions of predicativity, that appear to relate to very different forms of mathematics. Thus predicativity constraints motivate Nelson's strictly finitary subsystems of Peano Arithmetic, but also the much more generous predicativity given the natural numbers, which, under the analysis by Kreisel, Feferman and Schütte, extends well beyond Peano Arithmetic. Further up in the proof theoretic scale, we have constructive predicativity, which (on the basis of intuitionistic logic) reaches the strength of rather substantial subsystems of second order arithmetic [21]. In fact, the use of intuitionistic logic and its interaction with predicativity

[8] As such, Nelson's ideas have proved extremely fruitful, as they have paved the way for substantial contributions to the area of computational complexity [2].

[9] Theories of inductive definitions are discussed in [4], where they are considered unacceptable from a predicative point of view on grounds of circularity. See also [18] for an alternative view which sees inductive definitions as justified from a constructive perspective.

makes it more difficult to assess the relation between this kind of predicativity and the others. But it would seem that in all cases predicatively motivated constraints can be "applied" to different initial "bases", different mathematical structures which are taken as accepted, or granted. A possible understanding of predicativity would then see it as a (series of) methodological constraints, often motivated by the desire to avoid vicious circularity, which can be implemented on top of a previously given base, considered secure and granted. Predicativity constraints then impose methodological restrictions on the mathematical constructions which populate the next higher level of abstraction. For example, predicativity given the natural numbers takes the natural number structure with full induction as unquestionable and builds predicatively motivated restrictions on top of it, thus constraining the notion of arbitrary set.

A very significant aspect which emerges here is the crucial role of the principle of induction for debates on predicativity. In fixing the conceptual framework which we take as basis, we have to explicitly clarify how much induction we are prepared to accept. That is, it would seem that induction (possibly appropriately restricted) is a crucial component of the structure one takes as base, and, as highlighted by Nelson and Parsons, plays a crucial role in discussions of impredicativity. In less neutral terms, it would seem that when looking at the conceptual framework of reference, we need to include not only the relevant objects, for example the natural numbers, but also the way we are to reason about them. The example of constructive predicativity also seems to support similar conclusions, suggesting to include even the logic within the base one takes for granted.

There is here some complex philosophical work which is required to justify the choice of the privileged base as well as the methodological restrictions to be put on place. It is not unusual within the literature on predicativity to find reference to the time-honoured distinction between potential and actual infinity in mathematics. Often then predicativity constraints are seen as ways of avoiding full commitment to actual infinity; this, in turn, is frequently linked to the philosophical debate on realism versus anti-realism in mathematics. From a perspective of this kind, for example, one might be prompted to accept predicativity given the natural numbers, from the desire to subscribe to some form of realism with respect to the natural number structure, while maintaining an anti-realist (e.g. a definitionist) position on arbitrary sets [6]. Here I would like to suggest another possible reading of predicativity, which cashes it out in terms of our understanding of mathematical concepts.[10] Predicativity now becomes a crucial instrument in arguing for differences in levels of understanding, and conceptual clarity. Predicativity given the natural numbers, for example, would now represent a way of vindicating a commonly preceived difference in understanding between the concept of natural number and that of real number or, more generally, of arbitrary set [6]. That is, one here attempts to capture a distinction between forms of understanding, rather than ontological status, claiming that some concepts are more fundamental, or clearer, or more evident than others.

[10] A view along similar lines is also hinted at by Feferman in [6].

Predicativity constraints then could be seen as ways of extending beyond those more fundamental concepts (the conceptual basis) in ways which are somehow already implicit in the basis itself, that is, withouth extending the very conceptual apparatus in substantial ways. Here again a difficult philosophical task lays ahead in attempting to further explicate the distinction between different forms of understanding, especially in light of the logical analysis briefly discussed above, which brings to the fore a plurality of versions of predicativity.[11] A crucial aspect of this view is that predicativity becomes a tool for clarifying different forms of mathematics and various ways of understanding, but it does not entail a claim that only predicative mathematics of some kind is justified. In fact, predicativity, like other restrictions to standard methodology, in the hands of the logician become a tool for exploring in precise terms which parts of standard mathematics are amenable to be reframed in terms of more elementary assumptions or ways of reasoning.

Predicativity is an essential component of constructive type theory [14,15]. In fact, predicativity made a very dramatic appearance within Martin-Löf type theory, which bears surprising similarities to how it entered the mathematical landscape at the beginning of the 20th century. The appeal to an impredicative type of all types in the first formulation of intuitionistic type theory, in fact, gave rise to Girard's paradox [8]. Martin-Löf promptly corrected his type theory by eliminating the all-encompassing type of all types, and introduced in its place a hierarchy of type universes, each "reflecting" on previously constructed sets and universes [14]. Type theoretic universes are indeed at the centre of the generous notion of predicativity which arises in intuitionistic type theory [21].

Martin-Löf type theory embodies the Curry–Howard isomorphism, and thus identifies propositions with types (and their proofs with the elements of the corresponding types). As a consequence, type theory is simultaneously a very general programming language and a mathematical formalism. Girard's paradox is usually read as implying that in this context impredicativity (in the form of arbitrary quantification on types) is inconsistent with the Curry–Howard isomorphism. In a sense, predicativity signs the limit of the strong identification of mathematics with programming which is at the heart of constructive type theory.[12]

An observation naturally comes to mind: recent years have seen the flourishing of research on formalization of mathematics, with the purpose of verification. Here a new interplay between computer science and mathematics emerges. For

[11] Further challenges are also posed by technical developments in proof theory which have brought Gerhard Jäger to introduce a notion of *metapredicative* [9]. A thorough analysis of predicativity also ought to clarify its relation with metapredicativity.

[12] Predicativity is also at the centre of Martin-Löf's meaning explanations for type theory, which explain the type theoretic constructions of this theory "from the bottom up". A key concept here is that of *evidence*: constructive type theory represents a form of mathematics which is, according to its proponents, intuitively evident, amenable to contentual and computational understanding. This contentual understanding is then seen as supporting the belief in the consistency of this form of mathematics [13].

example, as observed by Georges Gonthier, one strategy which proved useful in proof checking is to turn mathematical concepts into data structures or programs, thus converting proof checking into program verification. Here constructive type theory has played a pivotal role, and inspired the development of other systems, like the (impredicative) calculus of constructions which underlines the Coq system.[13] One would then be tempted to conclude that ideas which originated through the fear of foundational errors are now having profound impact on new ways of addressing the ever pressing issue of local errors.

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[13] The calculus of constructions takes an opposite route compared with Martin-Löf type theory to the impasse given by Girard's paradox: it relinquishes the Curry–Howard hisomorphism in favour of impredicative type constructions. Although the Coq sytem was originally developed on the impredicative calculus of constructions, recent versions are based on a predicative core, although they also allow for impredicative extensions.

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