Chapter 2
Asymptotic Analysis of the Contact Problem for Two Bonded Elastic Layers

Abstract The first part of the chapter deals with the distributional asymptotic analysis of the contact problem of frictionless unilateral interaction of two bonded elastic layers. The case of incompressible layer materials is thoroughly treated in the second part of the chapter, beginning in Sect. 2.4.

2.1 Contact Problem Formulation

In this section we formulate the problem of frictionless unilateral contact between two uniform transversely isotropic elastic layers bonded to rigid substrates. In the case of substrates shaped like elliptic paraboloids, the general expression for the gap function is derived. The boundary conditions of unilateral contact are considered in detail, including the refined contact condition with allowance for tangential displacements on the contact interface. The contact problem is formulated as an integral equation over the contact area, which in turn should be determined by the positiveness condition imposed on the contact pressure.

2.1.1 Geometry of Surfaces in Contact

Let us consider two thin elastic layers ($n = 1, 2$), each of uniform thickness h_n, ideally bonded to rigid substrates with slightly curved surfaces (see Fig. 2.1). Introducing the Cartesian coordinate system (y_1, y_2, z), we write out the equations of the surfaces of the coating layers ($n = 1, 2$) in the form

$$z = (-1)^n \varphi_n(\mathbf{y}), \tag{2.1}$$

assuming that in the undeformed state the thin layer/substrate systems occupy domains $z \leq -\varphi_1(\mathbf{y})$ and $z \geq \varphi_2(\mathbf{y})$.

In particular, it is of great practical interest to consider the case of substrates shaped like paraboloids

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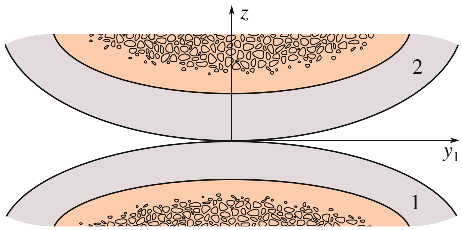


Fig. 2.1 Contact of two thin elastic layers in the initial undeformed configuration

$$\varphi_n(\mathbf{y}) = k_{11}^{(n)} y_1^2 + 2k_{12}^{(n)} y_1 y_2 + k_{22}^{(n)} y_2^2, \tag{2.2}$$

where the coefficients $k_{11}^{(n)}$, $k_{12}^{(n)}$, and $k_{22}^{(n)}$ have the dimension of reciprocal length.

The initial gap between the two surfaces is given by the gap function

$$\varphi(\mathbf{y}) = \varphi_1(\mathbf{y}) + \varphi_2(\mathbf{y}). \tag{2.3}$$

Correspondingly, in the case (2.2), we have

$$\varphi(\mathbf{y}) = k_{11} y_1^2 + 2k_{12} y_1 y_2 + k_{22} y_2^2, \tag{2.4}$$

where

$$k_{11} = k_{11}^{(1)} + k_{11}^{(2)}, \quad k_{12} = k_{12}^{(1)} + k_{12}^{(2)}, \quad k_{22} = k_{22}^{(1)} + k_{22}^{(2)}.$$

Now, let $R_1^{(n)}$ and $R_2^{(n)}$ be the principal radii of curvature of the surface of the n-th layer at its apex, such that the n-th layer surface can then be expressed as

$$\varphi_n(\mathbf{y}) = \frac{(y_1^n)^2}{2R_1^{(n)}} + \frac{(y_2^n)^2}{2R_2^{(n)}}, \tag{2.5}$$

where the directions of the local coordinate axes y_1^n and y_2^n coincide with the principal curvature directions.

The transformation of the coordinates (y_1^n, y_2^n) to the common set of axes (y_1, y_2) inclined at the angle β_n to the axis y_1^n is given by

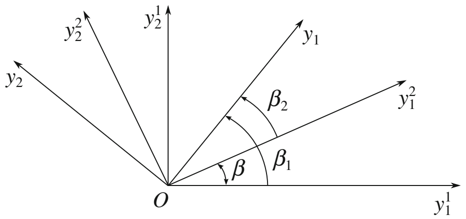
$$y_1^n = y_1 \cos\beta_n - y_2 \sin\beta_n, \quad y_2^n = y_1 \sin\beta_n + y_2 \cos\beta_n. \tag{2.6}$$

Hence, in light of (2.5) and (2.6), the coefficients on the right-hand side of (2.2) can be expressed as

$$2k_{11}^{(n)} = \kappa_1^{(n)} \cos^2\beta_n + \kappa_2^{(n)} \sin^2\beta_n, \quad 2k_{22}^{(n)} = \kappa_1^{(n)} \sin^2\beta_n + \kappa_2^{(n)} \cos^2\beta_n,$$

$$2k_{12}^{(n)} = \left(\kappa_2^{(n)} - \kappa_1^{(n)}\right) \sin\beta_n \cos\beta_n,$$

Fig. 2.2 Coordinate systems involved in determining the initial gap between the contacting surfaces



where $\kappa_1^{(n)}$ and $\kappa_2^{(n)}$ are the principal curvatures, i.e.,

$$\kappa_1^{(n)} = \frac{1}{R_1^{(n)}}, \quad \kappa_2^{(n)} = \frac{1}{R_2^{(n)}}.$$

Now, by a suitable choice of the coordinate axes (y_1, y_2), we can make k_{12} zero in (2.4). For instance, let us choose the angle β_1 in such a way that

$$\big(\kappa_2^{(1)} - \kappa_1^{(1)}\big) \sin 2\beta_1 + \big(\kappa_2^{(2)} - \kappa_1^{(2)}\big) \sin 2\beta_2 = 0. \tag{2.7}$$

Taking into account that (see Fig. 2.2)

$$\beta = \beta_1 - \beta_2, \tag{2.8}$$

from Eq. (2.7), we readily find

$$\tan 2\beta_1 = \frac{\big(\kappa_2^{(2)} - \kappa_1^{(2)}\big) \sin 2\beta}{\kappa_2^{(1)} - \kappa_1^{(1)} + \big(\kappa_2^{(2)} - \kappa_1^{(2)}\big) \cos 2\beta}. \tag{2.9}$$

Thus, the parabolic gap function (2.4) takes the simplest form

$$\varphi(\mathbf{y}) = k_1 y_1^2 + k_2 y_2^2, \tag{2.10}$$

where the coefficients k_1 and k_2 are evaluated by the formulas

$$\begin{aligned} 2k_1 &= \kappa_1^{(1)} \cos^2\beta_1 + \kappa_2^{(1)} \sin^2\beta_1 + \kappa_1^{(2)} \cos^2\beta_2 + \kappa_2^{(2)} \sin^2\beta_2, \\ 2k_2 &= \kappa_1^{(1)} \sin^2\beta_1 + \kappa_2^{(1)} \cos^2\beta_1 + \kappa_1^{(2)} \sin^2\beta_2 + \kappa_2^{(2)} \cos^2\beta_2. \end{aligned} \tag{2.11}$$

From the above equations, it follows that

$$\begin{aligned} 2(k_1 + k_2) &= \kappa_1^{(1)} + \kappa_2^{(1)} + \kappa_1^{(2)} + \kappa_2^{(2)}, \\ 2(k_1 - k_2) &= \big(\kappa_1^{(1)} - \kappa_2^{(1)}\big) \cos 2\beta_1 + \big(\kappa_1^{(2)} - \kappa_2^{(2)}\big) \cos 2\beta_2. \end{aligned} \tag{2.12}$$

Note [19, 20] that by taking into account (2.12), Eq. (2.11) can be rewritten as

$$2k_1 = \left(\kappa_1^{(1)} + \kappa_2^{(1)} + \kappa_1^{(2)} + \kappa_2^{(2)}\right) \sin^2 \frac{\tau}{2},$$
$$2k_2 = \left(\kappa_1^{(1)} + \kappa_2^{(1)} + \kappa_1^{(2)} + \kappa_2^{(2)}\right) \cos^2 \frac{\tau}{2}, \tag{2.13}$$

where τ is an auxiliary parameter given by $\cos\tau = (k_2 - k_1)/(k_2 + k_1)$. From Eq. (2.13), it follows that the coefficients k_1 and k_2 have the same sign, explaining why the equidistant curves $k_1 y_1^2 + k_2 y_2^2 = const$ are concentric ellipses.

Finally, it should be underlined that since the angle β, as well as the curvatures $\kappa_1^{(n)}$ and $\kappa_2^{(n)}$ $(n = 1, 2)$, are supposed to be known, we can evaluate the angle β_1 from Eq. (2.9), and after that Eq. (2.8) will yield $\beta_2 = \beta_1 - \beta$. Equation (2.12) will then give k_1 and k_2, the parameters of the gap function $\varphi(\mathbf{y})$ (see Eq. (2.10)).

2.1.2 Unilateral Contact Conditions

We consider the contact interaction of the elastic layers as a normal load F is applied to the substrates, producing their (vertical) contact approach δ_0 (see Fig. 2.3).

Before deformation, the gap between the layer surfaces was given by Eq. (2.10), which can be rewritten as

$$\varphi(\mathbf{y}) = \frac{y_1^2}{2R_1} + \frac{y_2^2}{2R_2}, \tag{2.14}$$

where $R_1 = 1/(2k_1)$ and $R_2 = 1/(2k_2)$, for k_1 and k_2 given by (2.13). In what follows, we may assume that $R_1 \geq R_2$.

If the surface point $\mathrm{M}'(\mathbf{y}', z')$ of the first layer (laying in the domain $z \leq -\varphi_1(\mathbf{y})$) and the surface point $\mathrm{M}''(\mathbf{y}'', z'')$ of the second layer (laying in the domain $z \geq \varphi_2(\mathbf{y})$) coincide after deformation (see Fig. 2.4), the following relations hold true [20]:

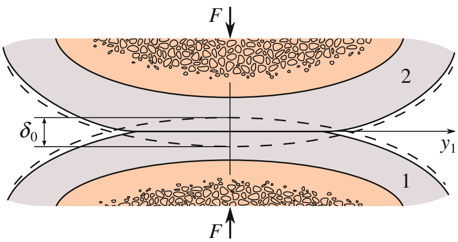


Fig. 2.3 Schematic diagram for the frictionless contact interaction of elastic layers 1 and 2 under an external load F, which implies the corresponding contact approach δ_0 along the axis of the force direction

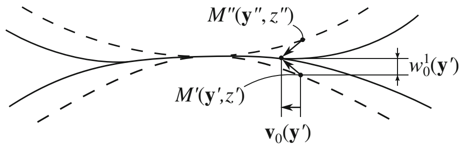


Fig. 2.4 Schematic diagram of the elastic contact interaction with allowance for tangential displacements on the contact interface

$$\mathbf{y}' + \mathbf{v}_0^{(1)}(\mathbf{y}') = \mathbf{y}'' + \mathbf{v}_0^{(2)}(\mathbf{y}''), \tag{2.15}$$

$$\varphi_2(\mathbf{y}'') + w_0^{(2)}(\mathbf{y}'') = -\big(\varphi_1(\mathbf{y}') + w_0^{(1)}(\mathbf{y}')\big) + \delta_0. \tag{2.16}$$

Here, $w_0^n(\mathbf{y})$ $(n = 1, 2)$ are the absolute values of the vertical displacements of the surface points of the n-th elastic layer (the normal displacements are assumed to be measured positive into each layer), $\mathbf{v}_0^n(\mathbf{y})$ $(n = 1, 2)$ are the tangential displacements of the surface points, and the relations $z' = -\varphi_1(\mathbf{y}')$ and $z'' = \varphi_2(\mathbf{y}'')$ were taken into account in writing Eq. (2.16).

From Eq. (2.15), it follows that

$$\mathbf{y}'' = \mathbf{y}' + \mathbf{v}_0^{(1)}(\mathbf{y}') - \mathbf{v}_0^{(2)}(\mathbf{y}''). \tag{2.17}$$

Now, identifying the coordinates $\mathbf{y}'$ and $\mathbf{y}''$ when evaluating the displacements of the contact points M$'$ and M$''$, we reduce Eq. (2.16) to a more simple form

$$\varphi_2\big(\mathbf{y} + \mathbf{v}_0^{(1)} - \mathbf{v}_0^{(2)}\big) + w_0^{(2)} + \varphi_1(\mathbf{y}) + w_0^{(1)} = \delta_0, \tag{2.18}$$

where $\mathbf{y} \in \omega$, and ω is the contact area. To simplify the notation, we dropped the primes on the left-hand side of Eq. (2.18).

The next simplifying step consists of linearizing Eq. (2.18). In this way, by taking Eq. (2.3) into account, we replace the above nonlinear contact condition by the linearized condition of frictionless contact

$$w_0^{(1)}(\mathbf{y}) + w_0^{(2)}(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}) - \nabla_y\varphi_2(\mathbf{y}) \cdot \big(\mathbf{v}_0^{(1)}(\mathbf{y}) - \mathbf{v}_0^{(2)}(\mathbf{y})\big), \tag{2.19}$$

where $\mathbf{y} \in \omega$, and the dot denotes the scalar product.

We note here that the choice of numbering of the elastic layers should assume that the modulus of the gradient $|\nabla_y\varphi_2(\mathbf{y})|$ is in a sense greater than $|\nabla_y\varphi_1(\mathbf{y})|$, or in other words, the surface of layer 1 (master surface) is assumed to be flatter than the surface of layer 2.

Observe that the refined contact condition (2.19) reduces to (1.40) in the case of a single elastic layer in contact with a punch, when $\varphi_1(\mathbf{y}) \equiv 0$ (the surface of the layer is flat) and $\varphi_2(\mathbf{y}) = \varphi(\mathbf{y})$, while $w_0^{(2)}(\mathbf{y}) \equiv 0$ and $\mathbf{v}_0^{(2)}(\mathbf{y}) \equiv 0$ (the punch is absolutely rigid).

Finally, by neglecting the effect of tangential displacements, we reduce Eq. (2.19) to the classical contact condition

$$w_0^{(1)}(\mathbf{y}) + w_0^{(2)}(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega. \tag{2.20}$$

As negative stresses are not allowed at the contact interface, the contour Γ of the contact area ω is determined from the positiveness condition that the contact pressure $p(\mathbf{y})$ is positive inside ω and vanishes at its boundary, that is

$$p(\mathbf{y}) > 0, \quad \mathbf{y} \in \omega, \quad p(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{2.21}$$

Consequently, according to Newton's third law we have

$$p(\mathbf{y}) = -\sigma_{33}^{(1)}(\mathbf{y}, 0) = -\sigma_{33}^{(2)}(\mathbf{y}, 0),$$

where $\sigma_{33}^{(n)}$ $(n = 1, 2)$ is the normal stress in the n-th layer.

2.1.3 Governing Integral Equation

According to the principle of superposition, the contact problem (2.20), (2.21) can be recast as a Fredholm integral equation of the first kind with the kernel being a combination of the layer surface vertical displacements resulting from a normal point force. The corresponding Green's function problem for an elastic layer subjected to a point force applied to its surface is conveniently treated by the two-dimensional Fourier transform technique [6, 24].

By applying the standard Fourier transformation, the local indentation of the n-th elastic layer $(n = 1, 2)$ can be expressed in the form

$$w_0^{(n)}(\mathbf{y}) = \frac{1}{2\pi\theta_n} \iint\limits_{-\infty}^{+\infty} \hat{p}(\alpha_1, \alpha_2) \frac{\mathscr{L}_n(\alpha h_n)}{\alpha} e^{-i(\alpha_1 y_1 + \alpha_2 y_2)} d\alpha_1 d\alpha_2, \tag{2.22}$$

where θ_n is a dimensional elastic constant, h_n is the thickness of the n-th layer, $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$, and $\hat{p}(\alpha_1, \alpha_2)$ denotes the transform of the contact pressure, i.e.,

$$\hat{p}(\alpha_1, \alpha_2) = \frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} p(y_1, y_2) e^{i(\alpha_1 y_1 + \alpha_2 y_2)} dy_1 dy_2. \tag{2.23}$$

In the case of a transversely isotropic elastic layer bonded to a flat rigid substrate, in accordance with the known solution [11], the kernel function is as follows:

$$\theta = \frac{A_{11}A_{33} - A_{13}^2}{(\gamma_1 + \gamma_2)A_{11}}, \tag{2.24}$$

$$\mathscr{L}(\lambda) = 1 + \frac{2\big[m_+(\gamma_1 e^{-2\lambda_1} + \gamma_2 e^{-2\lambda_2}) - \gamma_- m_- e^{-2\lambda_1 - 2\lambda_2} - 4\gamma_1\gamma_2 e^{-\lambda_1-\lambda_2}\big]}{8\gamma_1\gamma_2 e^{-\lambda_1-\lambda_2} + \gamma_- m_-(1 + e^{-2\lambda_1-2\lambda_2}) - \gamma_+ m_+(e^{-2\lambda_1} + e^{-2\lambda_2})}. \tag{2.25}$$

Here, γ_1 and γ_2 are the roots of the bi-quadratic equation

$$\gamma^4 A_{11}A_{44} - \gamma^2\big[A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})\big] + A_{33}A_{44} = 0, \tag{2.26}$$

and we have employed the following notation:

$$\lambda_1 = \frac{\lambda}{\gamma_1}, \quad \lambda_2 = \frac{\lambda}{\gamma_2}, \quad \gamma_+ = \gamma_1 + \gamma_2, \quad \gamma_- = \gamma_1 - \gamma_2, \tag{2.27}$$

$$m_+ = m_2\gamma_1 + m_1\gamma_2, \quad m_- = m_2\gamma_1 - m_1\gamma_2,$$

$$m_1 = \frac{A_{11}\gamma_1^2 - A_{44}}{A_{13} + A_{44}}, \quad m_2 = \frac{A_{11}\gamma_2^2 - A_{44}}{A_{13} + A_{44}}.$$

Note that θ_n and $\mathscr{L}_n(\lambda)$ are obtained from (2.24) and (2.25) by suitable selection of the n-th layer elastic parameters.

In the case of a bonded isotropic layer, we have $\gamma_1 = \gamma_2 = 1$ and Eqs. (2.24), (2.25) for the kernel function reduce as follows [25]:

$$\theta = \frac{E}{2(1-\nu^2)}, \tag{2.28}$$

$$\mathscr{L}(\lambda) = \frac{2\kappa \sinh 2\lambda - 4\lambda}{2\kappa \cosh 2\lambda + 1 + \kappa^2 + 4\lambda^2}. \tag{2.29}$$

Here, E is Young's modulus, ν is Poisson's ratio, $\kappa = 3 - 4\nu$ is Kolosov's constant.

The function $\mathscr{L}(\lambda)$ defined by formula (2.29), being continuous and positive for $\lambda \in (0, +\infty)$, satisfies the following asymptotic relations:

$$\begin{aligned} &\mathscr{L}(\lambda) = \mathscr{A}\lambda + O(\lambda^3), \quad \lambda \to 0; \\ &\mathscr{L}(\lambda) = 1 + O(\lambda^2 e^{-2\lambda}), \quad \lambda \to \infty. \end{aligned} \tag{2.30}$$

For the function $\mathscr{L}(\lambda)$ given by (2.25), the first asymptotic relation (2.30) can be checked directly, while the asymptotic remainder in the second relation can be easily replaced with $O(e^{-c_1\lambda})$, where c_1 is some positive constant. On the basis of these properties for each function $\mathscr{L}_n(\lambda)$, it can be shown [25] that the kernel

$$K_n(\mathbf{y}) = \iint_0^{+\infty} \frac{\mathscr{L}_n(s)}{s} \cos\frac{s_1 y_1}{h_n} \cos\frac{s_2 y_2}{h_n}\, ds_1 ds_2, \tag{2.31}$$

where $s = \sqrt{s_1^2 + s_2^2}$, decreases at infinity as rapidly as $e^{-c_2 h_n^{-1}|\mathbf{y}|}$ for some $c_2 > 0$.

Note also that by changing the integration variables in (2.31), we obtain

$$K_n(\mathbf{y}) = h_n \iint_0^{+\infty} \frac{\mathscr{L}_n(h_n\alpha)}{\alpha} \cos\alpha_1 y_1 \cos\alpha_2 y_2\, d\alpha_1 d\alpha_2, \tag{2.32}$$

where $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$.

Now, using Eqs. (2.23) and (2.31), we rewrite Eq. (2.22) in the form

$$w_0^{(n)}(\mathbf{y}) = \frac{1}{\pi^2 h_n \theta_n} \iint_\omega p(\mathbf{y}') K_n(y_1 - y_1', y_2 - y_2')\, d\mathbf{y}', \tag{2.33}$$

where the contact pressure density $p(\mathbf{y})$ vanishing outside the contact area ω has already been taken into account.

Thus, substituting the expressions (2.33) ($n = 1, 2$) into the contact condition (2.20), and recalling (2.32), we arrive at the following integral equation:

$$\frac{1}{\pi^2 h\theta} \iint_\omega p(\mathbf{y}') K(y_1 - y_1', y_2 - y_2')\, d\mathbf{y}' = \delta_0 - \varphi(\mathbf{y}). \tag{2.34}$$

Here we have introduced the notation

$$h = h_1 + h_2, \quad \theta = \frac{\theta_1\theta_2}{\theta_1 + \theta_2}, \tag{2.35}$$

$$K(\mathbf{y}) = \iint_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos\frac{s_1 y_1}{h} \cos\frac{s_2 y_2}{h}\, ds_1 ds_2, \tag{2.36}$$

$$\mathscr{L}(s) = \frac{\theta}{\theta_1}\mathscr{L}_1\Big(\frac{h_1}{h}s\Big) + \frac{\theta}{\theta_2}\mathscr{L}_2\Big(\frac{h_2}{h}s\Big). \tag{2.37}$$

Equation (2.34) represents the governing integral equation of the frictionless contact problem for two elastic layers bonded to slightly curved rigid substrates, where the substrate shapes are taken into account through the gap function $\varphi(\mathbf{y})$, and the curvature effect on the integral operator on the right-hand side is neglected.

2.2 Distributional Asymptotic Analysis

In this section, we develop an alternative for the perturbation approach to the contact problem in the thin-layer approximation considered in Sects. 1.2 and 1.4. It is assumed that the joint layer thickness h is small compared to the characteristic length of the contact area ω. Correspondingly, we require that

$$h = \varepsilon h_*, \quad \delta_0 = \varepsilon \delta_0^*, \quad R_1 = \varepsilon^{-1} R_1^*, \quad R_2 = \varepsilon^{-1} R_2^*, \tag{2.38}$$

where δ_0^*, R_1^*, and R_2^* are comparable with h_*, all being independent of ε. Note that we follow the same notation as in the previous chapter, wherever possible.

2.2.1 Moment Asymptotic Expansion for the Integral Operator of the Frictionless Contact Problem for a Thin Elastic Layer

A key point of the distributional asymptotic analysis is to make use of a large positive dimensionless parameter

$$\Lambda = \frac{1}{\varepsilon} \tag{2.39}$$

contained in the kernel (2.36) as a consequence of (2.37) and $(2.38)_1$. For this purpose, it is convenient to introduce dimensionless variables

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_i = h_*^{-1} y_i, \quad i = 1, 2. \tag{2.40}$$

Substituting expressions (2.38) and (2.40) into Eq. (2.34), we readily obtain

$$\iint\limits_{\omega_*} p_*(\boldsymbol{\eta}') k\big(\Lambda(\eta_1 - \eta_1'), \Lambda(\eta_2 - \eta_2')\big)\, d\boldsymbol{\eta}' = \frac{\pi^2 \theta}{\Lambda^2 h_*} \big(\delta_0^* - \varphi^*(\boldsymbol{\eta})\big), \tag{2.41}$$

where we have introduced the notation

$$p_*(\boldsymbol{\eta}) = p(h_* \eta_1, h_* \eta_2), \tag{2.42}$$

$$k(\boldsymbol{\xi}) = \iint\limits_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos s_1 \xi_1 \cos s_2 \xi_2 \, ds_1 ds_2, \tag{2.43}$$

$$\varphi^*(\boldsymbol{\eta}) = h_*^2 \big((2R_1^*)^{-1} \eta_1^2 + (2R_2^*)^{-1} \eta_2^2\big). \tag{2.44}$$

Following Argatov [7], we apply the so-called distributional asymptotic approach developed by Estrada and Kanwal [10]. Before proceeding, let us clarify the notation used. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be a multi-index of nonnegative integers and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, then for any $\boldsymbol{\eta} \in \mathbb{R}^2$ we put $\boldsymbol{\eta}^\alpha = \eta_1^{\alpha_1}\eta_2^{\alpha_2}$ and define

$$\mathbf{D}^\alpha f(\boldsymbol{\eta}) = \frac{\partial^{|\alpha|} f(\eta_1, \eta_2)}{\partial \eta_1^{\alpha_1} \partial \eta_2^{\alpha_2}}, \quad \mathbf{D}^0 f(\boldsymbol{\eta}) = f(\boldsymbol{\eta}).$$

We also employ the standard notation $\boldsymbol{\alpha}! = \alpha_1!\alpha_2!$ for the multi-index α, where $\alpha_1!$ denotes the factorial of α_1.

The moment asymptotic expansion can be written as follows [10]:

$$k(\Lambda\boldsymbol{\eta}) \sim \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|}\mu_\alpha \mathbf{D}^\alpha \delta(\boldsymbol{\eta})}{\boldsymbol{\alpha}!\Lambda^{|\alpha|+2}}, \quad \Lambda \to \infty. \tag{2.45}$$

Here, $\mu_\alpha = \mu_{\alpha_1\alpha_2}$ are the moments of the generalized function $k(\boldsymbol{\xi})$ given by

$$\mu_\alpha = \langle k(\boldsymbol{\xi}), \boldsymbol{\xi}^\alpha \rangle = \iint\limits_{-\infty}^{+\infty} k(\boldsymbol{\xi}) \xi_1^{\alpha_1} \xi_2^{\alpha_2} d\xi_1 d\xi_2. \tag{2.46}$$

The asymptotic expansion (2.45) is valid in several important spaces of distributions (see, e.g., [10, 26]). In particular, it holds for distributions of rapid decay at infinity, and in particular, for the kernel (2.36).

The interpretation of the asymptotic relation (2.45) is in the distributional sense. This means that the asymptotic formula

$$\langle k(\Lambda\boldsymbol{\eta}), \phi(\boldsymbol{\eta}) \rangle = \sum_{|\alpha|=0}^{N} \frac{\mu_\alpha \mathbf{D}^\alpha \phi(\mathbf{0})}{\boldsymbol{\alpha}!\Lambda^{|\alpha|+2}} + O(\Lambda^{-N-3}), \quad \Lambda \to \infty, \tag{2.47}$$

holds true for any $\phi(\boldsymbol{\eta})$ from the corresponding space of test functions.

For the kernel function (2.36), the application of (2.46) yields the moments

$$\begin{aligned} \mu_\alpha &= \iint\limits_{-\infty}^{+\infty} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \iint\limits_{0}^{+\infty} \frac{\mathscr{L}(s)}{s} \cos s_1 \xi_1 \cos s_2 \xi_2 \, ds_1 ds_2 d\xi_1 d\xi_2 \\ &= \frac{1}{4} \iint\limits_{-\infty}^{+\infty} \frac{\mathscr{L}(s)}{s} \prod_{j=1}^{2} \int\limits_{-\infty}^{+\infty} \xi_j^{\alpha_j} \cos s_j \xi_j \, d\xi_j ds_1 ds_2. \end{aligned} \tag{2.48}$$

Using the well-known representation for Dirac's delta function

$$\delta(s_j) = \frac{1}{\pi} \int\limits_0^{+\infty} \cos s_j \xi_j \, d\xi_j, \tag{2.49}$$

we find from Eq. (2.48), after integration by parts, that

$$\mu_{2k,2n-2k} = (-1)^n \pi^2 \iint\limits_{-\infty}^{+\infty} \frac{\mathscr{L}(s)}{s} \delta^{(2k)}(s_1) \delta^{(2n-2k)}(s_2) \, ds_1 ds_2, \tag{2.50}$$

where $k = 0, 1, \ldots, n$ and $n \in \mathbb{N} \cup \{0\}$, and

$$\mu_\alpha = 0, \quad |\alpha| = 2n - 1, \quad n \in \mathbb{N};$$

$$\mu_\alpha = 0, \quad \alpha_1 = 2k - 1, \quad \alpha_2 = 2n - 2k + 1, \quad |\alpha| = 2n, \quad k = 1, 2, \ldots, n.$$

Now, by recalling the definition of the two-dimensional Dirac delta function $\delta(s_1, s_2) = \delta(s_1)\delta(s_2)$ and substituting the expansion

$$\frac{\mathscr{L}(s)}{s} = \mathscr{A}(1 + m_1 s^2 + m_2 s^4 + \ldots) \tag{2.51}$$

into Eq. (2.50), we find

$$\begin{aligned} \mu_{2k,2n-2k} &= (-1)^n \pi^2 \mathscr{A} m_n \iint\limits_{-\infty}^{+\infty} (s_1^2 + s_2^2) \delta^{(2k)}(s_1) \delta^{(2n-2k)}(s_2) \, ds_1 ds_2 \\ &= (-1)^n \pi^2 \mathscr{A} m_n C_n^k 2^n k!(n-k)!, \end{aligned} \tag{2.52}$$

where C_n^k are binomial coefficients given by

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Then, from relations (2.45) and (2.52), where we may set $m_0 = 1$, we find

$$k(\Lambda \boldsymbol{\eta}) \sim \sum_{n=0}^{\infty} (-1)^n \pi^2 \mathscr{A} \frac{m_n}{\Lambda^{2n+2}} \sum_{k=0}^{n} C_n^k \frac{\partial^{2n} \delta(\boldsymbol{\eta})}{\partial \eta_1^{2k} \partial \eta_2^{2n-2k}}. \tag{2.53}$$

Correspondingly, substituting the moment asymptotic expansion (2.53) into the left-hand side of Eq. (2.41), we obtain

$$\iint\limits_{\omega_*} p_*(\boldsymbol{\xi}) k\big(\Lambda(\boldsymbol{\eta}-\boldsymbol{\xi})\big)\, d\boldsymbol{\xi} \sim \sum_{n=0}^{\infty} (-1)^n \pi^2 \mathscr{A} \frac{m_n}{\Lambda^{2n+2}} \sum_{k=0}^{n} C_n^k \frac{\partial^{2n} p_*(\boldsymbol{\eta})}{\partial \eta_1^{2k} \partial \eta_2^{2n-2k}}. \quad (2.54)$$

To simplify the right-hand side of (2.54), we recall that

$$\Delta_\eta^n \equiv \left(\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2} \right)^n = \sum_{k=0}^{n} C_n^k \frac{\partial^{2n}}{\partial \eta_1^{2k} \partial \eta_2^{2n-2k}}.$$

Hence, by the above formula, we have

$$\frac{\Lambda^2 h_*}{\pi^2 \theta} \iint\limits_{\omega_*} p_*(\boldsymbol{\eta}') k\big(\Lambda(\boldsymbol{\eta}-\boldsymbol{\eta}')\big)\, d\boldsymbol{\eta}' \sim \frac{\mathscr{A} h_*}{\theta} \sum_{n=0}^{\infty} (-1)^n \frac{m_n}{\Lambda^{2n}} \Delta_\eta^n p_*(\boldsymbol{\eta}). \quad (2.55)$$

Note that the above integral operator is normalized by the factor $\Lambda^2 h_*/(\pi^2\theta)$, such that the left-hand side of (2.55) represents the local indentation normalized by Λ when stretching the normal coordinate (see, in particular, (2.38) and (2.41)).

2.2.2 Asymptotic Solution of the Contact Problem for Slightly Curved Thin Compressible Elastic Layers

Using the notation (2.39) and the asymptotic expansions (2.55), we rewrite the governing integral equation (2.41) in the form

$$\frac{\mathscr{A} h_*}{\theta} \sum_{n=0}^{\infty} (-1)^n \varepsilon^{2n} m_n \Delta_\eta^n p_*(\boldsymbol{\eta}) \sim \delta_0^* - \varphi^*(\boldsymbol{\eta}). \quad (2.56)$$

The solution to Eq. (2.56) can be represented in the form of an asymptotic series in powers of ε as follows:

$$p_*(\boldsymbol{\eta}) \sim p_*^0(\boldsymbol{\eta}) + \varepsilon^2 p_*^1(\boldsymbol{\eta}) + \varepsilon^4 p_*^2(\boldsymbol{\eta}) + \dots . \quad (2.57)$$

The substitution of the asymptotic expansion (2.57) into Eq. (2.56) yields the following system of equations for the successive evaluation of its coefficients:

$$\frac{\mathscr{A} h_*}{\theta} m_0 p_*^0(\boldsymbol{\eta}) = \delta_0^* - \varphi^*(\boldsymbol{\eta}),$$

$$\sum_{j=0}^{k} (-1)^{k-j} m_{k-j} \Delta_\eta^{k-j} p_*^j(\boldsymbol{\eta}) = 0, \quad k = 1, 2, \dots .$$

It then follows that

$$p_*^0(\boldsymbol{\eta}) = \frac{\theta}{\mathscr{A} h_*}\big(\delta_0^* - \varphi^*(\boldsymbol{\eta})\big), \tag{2.58}$$

$$p_*^k(\boldsymbol{\eta}) = -\sum_{j=0}^{k-1}(-1)^{k-j} m_{k-j} \Delta_\eta^{k-j} p_*^j(\boldsymbol{\eta}), \quad k = 1, 2, \ldots . \tag{2.59}$$

We further note that formula (2.59) can be rewritten in the form

$$p_*^k(\boldsymbol{\eta}) = (-1)^k \frac{\theta}{\mathscr{A} h_*} M_k \Delta_\eta^k f_*(\boldsymbol{\eta}), \quad k = 0, 1, 2, \ldots , \tag{2.60}$$

where in light of (2.58) we have $M_0 = 1$ and $f_*(\boldsymbol{\eta}) = \delta_0^* - \varphi^*(\boldsymbol{\eta})$.

To evaluate the coefficients M_k introduced in (2.60), we consider the expansion reciprocal to (2.51), that is

$$\frac{s}{\mathscr{L}(s)} = \frac{1}{\mathscr{A}}(1 + M_1 s^2 + M_2 s^4 + \ldots). \tag{2.61}$$

It is easily proved by induction that for any positive integer k the recurrence relation that facilitates the calculation of the coefficients M_k in (2.61) from the coefficients of the the expansion (2.51) has the form

$$M_k = -\sum_{j=0}^{k-1} m_{k-j} M_j, \quad k = 1, 2, \ldots .$$

Alternatively, the above formula can be directly recovered from (2.59) and (2.60).

Thus, the constructed inner asymptotic expansion (2.57)–(2.59) is similar to the solution obtained earlier by Vorovich et al. [25].

Finally, in the case of the parabolic punch (2.14), we get by simple calculation

$$p_*^1(\boldsymbol{\eta}) = \frac{\theta m_1 h_*}{\mathscr{A}}\Big(\frac{1}{R_1^*} + \frac{1}{R_2^*}\Big), \quad p_*^k(\boldsymbol{\eta}) \equiv 0, \quad k = 2, 3, \ldots . \tag{2.62}$$

Upon substituting (2.58) and (2.62) into (2.57), our final result is

$$p_*(\boldsymbol{\eta}) \sim \frac{\theta}{\mathscr{A} h_*}\Bigg(\delta_0^* + \varepsilon^2 2m_1 \frac{h_*^2}{R^*} - h_*^2\Big(\frac{\eta_1^2}{2R_1^*} + \frac{\eta_2^2}{2R_2^*}\Big)\Bigg), \tag{2.63}$$

where $R^* = 2R_1^* R_2^*/(R_1^* + R_2^*)$ is the harmonic mean of R_1^* and R_2^*.

2.2.3 Comparison of the Results Obtained by the Perturbation and Distributional Asymptotic Methods

Let us consider the case of a single layer in contact with a rigid punch, so that $\theta_2^{-1} = 0$, $\theta = \theta_1$, and $h = h_1$. According to the asymptotic expansion (2.55) (see also Eq. (2.56)), the local indentation of the elastic layer, i.e., the normal displacement of the surface points, is expressed by

$$w_0(\mathbf{y}) \sim \frac{\mathscr{A} h}{\theta} \sum_{n=0}^{\infty} (-1)^n m_n h^{2n} \Delta_y^n p(\mathbf{y}), \tag{2.64}$$

where $\mathscr{A}$ and m_n are dimensionless coefficients in the series expansion (2.51) for the kernel function $\mathscr{L}(s)/s$.

Formula (2.64) can be rewritten in the form

$$w_0(\mathbf{y}) \sim h \sum_{n=0}^{\infty} (-1)^n \mathscr{M}_n h^{2n} \Delta_y^n p(\mathbf{y}), \tag{2.65}$$

where $\mathscr{M}_n$ are dimensional coefficients in the expansion

$$\begin{aligned} \frac{\mathscr{L}(s)}{\theta s} &= \frac{\mathscr{A}}{\theta}(1 + m_1 s^2 + m_2 s^4 + \dots) \\ &= \mathscr{M}_0 + \mathscr{M}_1 s^2 + \mathscr{M}_2 s^4 + \dots . \end{aligned} \tag{2.66}$$

In particular, from (2.65), it follows that

$$w_0(\mathbf{y}) \simeq \mathscr{M}_0 h p(\mathbf{y}) - \mathscr{M}_1 h^3 \Delta_y p(\mathbf{y}). \tag{2.67}$$

On the other hand, based on the perturbation algorithm [14] in Sect. 1.2 (see formula (1.26)), we obtained the asymptotic expansion

$$w_0(\mathbf{y}) \simeq \frac{h}{A_{33}} p(\mathbf{y}) - \frac{h^3 A_{13}(A_{13} - A_{44})}{3A_{33}^2 A_{44}} \Delta_y p(\mathbf{y}). \tag{2.68}$$

By comparing (2.67) and (2.68), we arrive at the following relations, whose validity should be checked:

$$\mathscr{M}_0 = \frac{1}{A_{33}}, \quad \mathscr{M}_1 = \frac{A_{13}(A_{13} - A_{44})}{3A_{33}^2 A_{44}}. \tag{2.69}$$

Expanding the function (2.25) into a Maclaurin series, we find

$$\mathscr{A} = \frac{A_{44}(\gamma_1 - \gamma_2)^2(\gamma_1 + \gamma_2)}{\gamma_1^2\gamma_2^2\big[A_{11}(\gamma_1^2 + \gamma_2^2) - 2(A_{13} + 2A_{44})\big]}, \tag{2.70}$$

where γ_1 and γ_2 are the roots of the characteristic equation (2.26).

In order to establish the equality between A_{33}^{-1} and $\mathscr{M}_0 = \theta^{-1}\mathscr{A}$, where θ and $\mathscr{A}$ are given by (2.24) and (2.70), we make use of Vieta's theorem for the bi-quadratic equation (2.26) and the formulas

$$\gamma_1^2\gamma_2^2 = \frac{A_{33}}{A_{11}}, \quad \gamma_1^2 + \gamma_2^2 = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})}{A_{11}A_{44}},$$

$$\big(\gamma_1^2 - \gamma_2^2\big)^2 = \frac{(A_{11}A_{33} - A_{13}^2)(A_{11}A_{13} - 4A_{13}A_{44} - 4A_{44}^2 - A_{13}^2)}{A_{11}^2A_{44}^2}.$$

The check of the second equality in (2.69) is more tedious, because the expression for m_1, and correspondingly for $\mathscr{M}_1$, is much more cumbersome and is not written here for brevity.

2.3 Boundary-Layer Problem in the Compressible Case

Both the perturbation technique and the distributional asymptotic method provide approximate solutions, which are valid inside the contact area but do not describe the true solution near its contour, where a special approximate solution of the boundary-layer type should be constructed. In this section, the case of compressible layer materials is considered.

2.3.1 Variation of the Contact Area

The leading-order asymptotic approximation (2.58) for the contact pressure distribution density, i.e.,

$$p_*^0(\boldsymbol{\eta}) = \frac{\theta}{\mathscr{A}h_*}\big(\delta_0^* - \varphi^*(\boldsymbol{\eta})\big)_+, \tag{2.71}$$

determines the main approximation ω_*^0 to the sought-for contact area ω_* (in the dimensionless coordinates (2.40)).

It is obvious from Eq. (2.44) that the domain ω_*^0, which corresponds to the density (2.71), is elliptic. The major semiaxis and the eccentricity of the contour Γ_*^0 of the domain ω_*^0 will be denoted by a_* and e. By simple calculations we find

$$a_* = \frac{1}{h_*}\sqrt{2\delta_0^* R_1^*}, \quad e^2 = 1 - \frac{R_2^*}{R_1^*}. \tag{2.72}$$

Following the asymptotic procedure introduced by Aleksandrov [2], we consider the behavior of the integral (2.34) and its density in the neighborhood of the unknown contour Γ_* of the domain ω_*.

In light of (2.41), the integral equation (2.34) now takes the form

$$\iint\limits_{\omega_*} p_*(\boldsymbol{\xi})k\big(\varepsilon^{-1}(\boldsymbol{\eta} - \boldsymbol{\xi})\big)\,d\boldsymbol{\xi} = \varepsilon^2 \frac{\pi^2\theta}{h_*}\big(\delta_0^* - \varphi^*(\boldsymbol{\eta})\big). \tag{2.73}$$

Suppose $\eta_1 = f_1^*(s)$, $\eta_2 = f_2^*(s)$ is a natural parametrization of the contour Γ_*^0. We will assume that when traveling along Γ_*^0 in the direction of increasing s-coordinate, the region ω_*^0 enclosed by Γ_*^0 remains on the left. Then, the unit vector of the inward (with respect to the domain ω_*^0) normal to the contour Γ_*^0 is

$$\mathbf{n}^0(s) = -f_2^{*\prime}(s)\mathbf{e}_1 + f_1^{*\prime}(s)\mathbf{e}_2, \tag{2.74}$$

where the prime denotes differentiation with respect to s.

In a small neighborhood, $\Xi_\varepsilon^*(s)$, of the contour Γ_*^0, we introduce the local system of coordinates (s, n), associated with the Cartesian coordinates (η_1, η_2) by the formulas

$$\eta_1 = f_1^*(s) + nn_1^0(s), \quad \eta_2 = f_2^*(s) + nn_2^0(s), \tag{2.75}$$

where $\mathbf{n}^0(s) = \big(n_1^0(s), n_2^0(s)\big)$ is given by (2.74), and n is the distance (taking the sign into account) along the inward normal to the contour Γ_*^0.

Further, let us assume that the contour Γ_* of the contact area ω_* in the local coordinates is described by the equation

$$n = \Upsilon_\varepsilon^*(s), \tag{2.76}$$

where $\Upsilon_\varepsilon^*(s)$ is a function to be determined. We set

$$\Upsilon_\varepsilon^*(s) = \varepsilon\Upsilon^*(s). \tag{2.77}$$

In the neighborhood $\Xi_\varepsilon^*(s)$ of the point s, where $|\boldsymbol{\xi} - \boldsymbol{\eta}(s)| = O(\sqrt{\varepsilon}\rho^*(s))$ and $\rho^*(s) = \big[f_2^{*\prime\prime}(s)f_1^{*\prime}(s) - f_1^{*\prime\prime}(s)f_2^{*\prime}(s)\big]^{-1}$ is the radius of curvature of contour Γ_*^0 at the point s, we make in the integral (2.73) the following change of variables:

$$\xi_1 = f_1^{*\prime}(s) + n'n_1^0(s), \quad \xi_2 = f_2^{*\prime}(s) + n'n_2^0(s).$$

Next we introduce the so-called "fast" variables

$$\nu = \varepsilon^{-1} n, \quad \nu' = \varepsilon^{-1} n', \quad \sigma' = \varepsilon^{-1}(s' - s), \tag{2.78}$$

keeping the scale for the s-coordinate along Γ_*^0 unchanged. From now on, the "slow" variable s is considered to be fixed.

Thus, in the neighborhood $\Xi_\varepsilon^*(s)$, when $\varepsilon \to 0$, the following relations hold:

$$f_j^*(s') = f_j^*(s) + \varepsilon\sigma' f_j^{*\prime}(s) + O(\varepsilon^2), \quad n_j^0(s') = n_j^0(s) + O(\varepsilon), \quad j = 1, 2,$$

$$|\boldsymbol{\xi} - \boldsymbol{\eta}(s)| = \varepsilon\sqrt{(\sigma')^2 + (\nu - \nu')^2} + O(\varepsilon^2), \quad \rho^*(s') = \rho^*(s) + O(\varepsilon),$$

$$\Upsilon^*(s') = \Upsilon^*(s) + O(\varepsilon), \quad \frac{D(\xi_1, \xi_2)}{D(s', n')} = 1 - \frac{\varepsilon\nu'}{\rho^*(s + \varepsilon\sigma')} = 1 + O(\varepsilon).$$

Note, finally, that the above formulas are valid for any smooth contour Γ_*.

2.3.2 Boundary-Layer Integral Equation

Separating the principal asymptotic terms according to the previous formulas, we take the limit

$$\lim_{\varepsilon\to 0} k\big(\varepsilon^{-1}(\boldsymbol{\eta} - \boldsymbol{\xi})\big) = k\Big(\sigma' f_1^{*\prime}(s) + (\nu - \nu')n_1^0(s),$$
$$\sigma' f_2^{*\prime}(s) + (\nu - \nu')n_2^0(s)\Big). \tag{2.79}$$

By invoking the formulas (see Eq. (2.74))

$$f_1^{*\prime}(s) = \cos\psi, \quad f_2^{*\prime}(s) = \sin\psi, \quad n_1^0(s) = -\sin\psi, \quad n_2^0(s) = \cos\psi, \tag{2.80}$$

where ψ provides an angular parameterization to the ellipse Γ_*^0, it can be shown directly that the right-hand side of the relation (2.79) is equal to $k(\sigma', \nu' - \nu)$.

Indeed, in light of (2.43), we have

$$k(\boldsymbol{\xi}) = \frac{1}{4}\iint\limits_{-\infty}^{+\infty} \frac{\mathscr{L}(s)}{s} e^{\mathrm{i}(s_1\xi_1 + s_2\xi_2)} ds_1 ds_2, \tag{2.81}$$

and by making the substitutions

$$s_1 = t_1\cos\psi - t_2\sin\psi, \quad s_2 = t_1\sin\psi + t_2\cos\psi,$$

$$\frac{D(s_1, s_2)}{D(t_1, t_2)} = 1, \quad s \equiv \sqrt{s_1^2 + s_2^2} = \sqrt{t_1^2 + t_2^2} \equiv t,$$

we find that the representation (2.81) can be written in the form

$$k(\boldsymbol{\xi}) = \frac{1}{4} \iint\limits_{-\infty}^{+\infty} \frac{\mathscr{L}(t)}{t} \cos\bigl[\xi_1(t_1 \cos\psi - t_2 \sin\psi) + \xi_2(t_1 \sin\psi + t_2 \cos\psi)\bigr]\, dt_1 dt_2, \tag{2.82}$$

provided that $t^{-1}\mathscr{L}(t)$ is an even function for $t \in (-\infty, +\infty)$.

Thus, from Eqs. (2.79), (2.80), and (2.82) it follows immediately that

$$\lim_{\varepsilon \to 0} k(\varepsilon^{-1}(\boldsymbol{\eta} - \boldsymbol{\xi})) = k(\sigma', \nu' - \nu), \tag{2.83}$$

while on the other hand, we see on the right-hand side of Eq. (2.73) that

$$\delta_0^* - \varphi^*(\boldsymbol{\eta}) = h_*^2\bigl[\varepsilon\nu b_1^*(s) + \varepsilon^2\nu^2 b_2^*(s)\bigr], \tag{2.84}$$

where

$$\begin{aligned} b_1^*(s) &= -\frac{f_1^*(s) n_1^0(s)}{R_1^*} - \frac{f_2^*(s) n_2^0(s)}{R_2^*}, \\ 2b_2^*(s) &= -\frac{n_1^0(s)^2}{R_1^*} - \frac{n_2^0(s)^2}{R_2^*}. \end{aligned} \tag{2.85}$$

Hence, by approximating the contact pressure density as $p_\varepsilon^*(\boldsymbol{\xi}) \sim q_\varepsilon^*(s, \nu')$ in the neighborhood $\varXi_\varepsilon^*(s)$ of the boundary of the contact area (see, for example, [3]) and letting $\varepsilon \to 0$ in light of Eqs. (2.73), (2.75)–(2.78), (2.83) and (2.84), we arrive at the following integral equation:

$$\int\limits_{\Upsilon^*(s)}^{+\infty} q^{**}(s, \nu') M(\nu' - \nu)\, d\nu' = \pi\theta h_* b_1^*(s)\nu. \tag{2.86}$$

Here, $q^{**}(s, \nu) = \varepsilon^{-1} q_\varepsilon^*(s, \nu)$, and we have introduced the notation

$$M(t) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} k(\sigma', t)\, d\sigma'.$$

Observe that the s-coordinate is present in Eq. (2.86) as a parameter.

Making use of formula (2.49), we represent the above formula in the form

$$M(t) = \int_0^{+\infty} \frac{\mathscr{L}(u)}{u} \cos ut\, du. \tag{2.87}$$

Finally, in addition to Eq. (2.86), which the boundary-layer solution $q^{**}(s, \nu)$ must satisfy, it is necessary to obey the contact pressure positivity condition (2.21). Hence, the function $\Upsilon^*(s)$ satisfies the equation

$$q^{**}(s, \Upsilon^*(s)) = 0. \tag{2.88}$$

Otherwise, we would contradict the assumption that the contact must be made over the whole domain ω_*.

2.3.3 Aleksandrov's Approximation

We note that the integral equation (2.86) is of the Wiener–Hopf type [23] and can be solved in closed form (see, in particular, [1, 5]). However, since a simple factorization for the function $w^{-1}\mathscr{L}(w)$ of the complex variable $w = u + \mathrm{i}v$ is not available, it is not possible to obtain the exact solution of Eq. (2.86) in a simple form. Thus, the approximate version of the Wiener–Hopf method has to be used. Confining our considerations to the first-order approximation, we replace the function $\mathscr{L}(u)$ by the following simple algebraic approximation [3]:

$$\tilde{\mathscr{L}}(u) = u\frac{\sqrt{u^2 + B^2}}{u^2 + C}. \tag{2.89}$$

It can easily be shown that the functions $u^{-1}\mathscr{L}(u)$ and $u^{-1}\tilde{\mathscr{L}}(u)$ satisfy Koiter's conditions [18] that they should have the same limits for s tending to zero and infinity, provided that the following relation holds:

$$\frac{B}{C} = \mathscr{A}. \tag{2.90}$$

In addition, following Aleksandrov [3], we select the constants B and C in such a manner that

$$\lim_{u \to 0} \frac{d^2}{du^2}\left(\frac{u}{\mathscr{L}(u)} - \frac{u}{\tilde{\mathscr{L}}(u)}\right) = 0.$$

From here it immediately follows that

$$\frac{m_1}{\mathscr{A}} = \frac{C}{2B^3} - \frac{1}{B}, \tag{2.91}$$

where (see formula (2.51))

$$\mathscr{A} = \lim_{u\to 0} \frac{\mathscr{L}(u)}{u}, \quad m_1 = \frac{1}{2\mathscr{A}} \lim_{u\to 0} \frac{d^2}{du^2} \frac{\mathscr{L}(u)}{u}. \tag{2.92}$$

From (2.24), (2.66), and (2.70), we have

$$\mathscr{A} = \frac{A_{11}A_{33} - A_{13}^2}{(\gamma_1 + \gamma_2)A_{11}A_{33}}, \quad m_1 = \frac{A_{13}(A_{13} - A_{44})}{3A_{33}A_{44}},$$

where γ_1 and γ_2 are the roots of the characteristic equation (2.26).

At the same time, Eqs. (2.90) and (2.91) yield the following formulas (cf. [12]):

$$B = \frac{1}{\mathscr{A} + \sqrt{\mathscr{A}^2 + 2m_1}}, \quad C = \frac{1}{\mathscr{A}\big[\mathscr{A} + \sqrt{\mathscr{A}^2 + 2m_1}\big]}.$$

Recall that $\mathscr{A}$ and m_1 are defined by (2.92).

In the case of isotropic layer bonded to a rigid foundation (see formula (2.29)), we have

$$\mathscr{A} = \frac{1 - 2\nu}{2(1 - \nu)^2},$$

where ν is Poisson's ratio of the layer material. From the above, it is readily seen (see also Fig. 2.5) that the asymptotic constant $\mathscr{A}$ approaches zero when the material becomes incompressible.

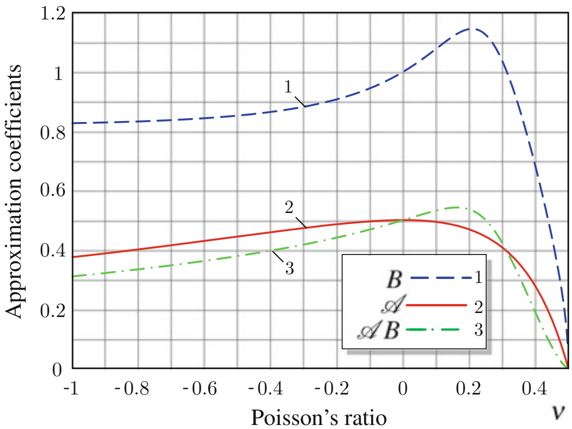


Fig. 2.5 Variation of the approximation coefficients in the isotropic case

Let us consider a Wiener–Hopf integral equation of the first kind

$$\int_0^{+\infty} \varphi(\tau')\tilde{M}(\tau' - \tau)\,d\tau' = \psi(\tau), \quad 0 \le \tau < \infty, \tag{2.93}$$

with the kernel

$$\tilde{M}(\tau) = \int_0^{+\infty} \frac{\tilde{\mathscr{L}}(u)}{u} \cos u\tau\, du, \tag{2.94}$$

where the kernel function $\tilde{\mathscr{L}}(u)$ is given by (2.89).

It can be shown [5, 25] that for the right-hand sides of Eq. (2.93)

$$\psi_0(\tau) = 1, \quad \psi_1(\tau) = \tau, \quad \psi_2(\tau) = \tau^2,$$

the corresponding special solutions are, respectively,

$$\varphi_0(\tau) = \frac{1}{\mathscr{A}} \operatorname{erf}\sqrt{B\tau} + \frac{e^{-B\tau}}{\sqrt{\pi \mathscr{A} \tau}}, \tag{2.95}$$

$$\varphi_1(\tau) = \frac{\tau}{\mathscr{A}} \operatorname{erf}\sqrt{B\tau} - \frac{e^{-B\tau}}{\sqrt{\pi B\tau}}\left(1 - \frac{\tau}{\mathscr{A}} - \frac{1}{2\sqrt{\mathscr{A}B}}\right), \tag{2.96}$$

$$\begin{aligned}\varphi_2(\tau) = {} & \left(\frac{\tau^2}{\mathscr{A}} + \frac{1}{\mathscr{A}B^2} - \frac{2}{B}\right)\operatorname{erf}\sqrt{B\tau} \\ & - \frac{e^{-B\tau}}{\sqrt{\pi B\tau}}\left(\frac{1}{B} - \frac{3}{4B\sqrt{\mathscr{A}B}} + \frac{\tau}{2\mathscr{A}B} - \frac{\tau^2}{\mathscr{A}}\right).\end{aligned} \tag{2.97}$$

Here, $\operatorname{erf}(x)$ is the error function, which is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The above solutions are illustrated in Fig. 2.6. Observe that after the normalization these functions depend on the dimensionless elastic constants $\mathscr{A}$ and B via their product $\mathscr{A}B$, while the dependence on the coordinate τ comes as $B\tau$.

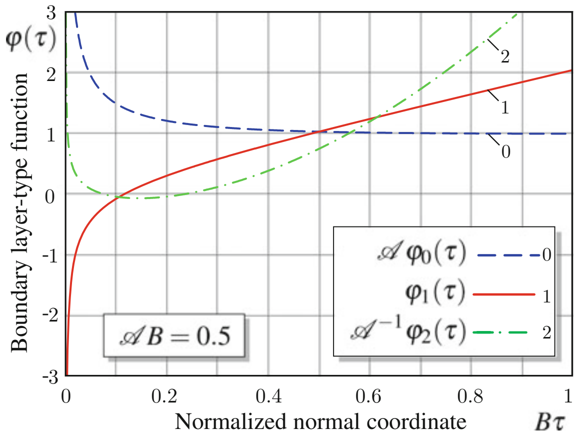


Fig. 2.6 Aleksandrov's approximate boundary layer-type solutions (2.95)–(2.97)

2.3.4 Boundary-Layer in the Compressible Case

We replace $\mathscr{L}(s)$ in the integral (2.87) by $\tilde{\mathscr{L}}(s)$, substitute the corresponding kernel $\tilde{M}(t)$ defined by (2.94) into Eq. (2.86), and implement in the resulting equation a change of variables

$$\nu = \Upsilon^*(s) + \tau, \quad \nu' = \Upsilon^*(s) + \tau'.$$

In this way the integral equation (2.86) can be transformed into the form

$$\int_0^{+\infty} \tilde{q}^{**}(s, \Upsilon^*(s) + \tau')\tilde{N}(\tau' - \tau)\, d\tau' = \pi\theta h_* b_1^*(s)(\Upsilon^*(s) + \tau). \tag{2.98}$$

In the case (2.89), make use of Aleksandrov's results [3] (see, in particular, formulas (2.95) and (2.96)), thus producing the solution to Eq. (2.98) in the form

$$\frac{\tilde{q}^{**}(s, \Upsilon^*(s) + \tau)}{\theta h_* b_1^*(s)} = \frac{\tau}{\mathscr{A}} \operatorname{erf}\sqrt{B\tau} - \frac{1}{\sqrt{\pi B\tau}} e^{-B\tau}\Big(1 - \frac{\sqrt{C}}{2B} - \frac{\tau}{\mathscr{A}}\Big) + \frac{\Upsilon^*(s)}{\mathscr{A}} \operatorname{erf}\sqrt{B\tau} + \frac{\Upsilon^*(s)}{\sqrt{\pi\mathscr{A}\tau}} e^{-B\tau}. \tag{2.99}$$

Since, in light of (2.88), the function (2.99) must satisfy Eq. (2.88), the boundary layer is found to be

$$\tilde{q}^{**}(s, \nu) = \frac{\theta h_* b_1^*(s)}{\mathscr{A}} \Big\{ \nu \operatorname{erf}\sqrt{B(\nu - \Upsilon^*(s))}$$

$$+\sqrt{\frac{\nu-\Upsilon^*(s)}{\pi B}}\exp\bigl(-B[\nu-\Upsilon^*(s)]\bigr)\Biggr\}, \tag{2.100}$$

provided that

$$\Upsilon^*(s)=\sqrt{\frac{\mathscr{A}}{B}}-\frac{1}{2B}. \tag{2.101}$$

Note that in the axisymmetric case the obtained boundary layer (2.100) is essentially similar to the leading term of the asymptotics for the contact pressure density constructed in [25] (see formula (49.11)). The resultant of the contact pressure (with the boundary layer taken into account) was evaluated in [8]. It is also worth noting [3] that the relations (2.90) and (2.91) are necessary for the correct matching between the boundary-layer solution (2.100) and the inner asymptotic solution (2.71).

2.4 Incompressible Transversely Isotropic Elastic Material

The generalized Hooke's law (1.1), establishing a linear relationship between components of the stress tensor $\boldsymbol{\sigma}$, and components of the tensor of infinitesimal strains $\boldsymbol{\varepsilon}$, can be written in the tensor form

$$\boldsymbol{\varepsilon}=\mathbf{S}:\boldsymbol{\sigma} \tag{2.102}$$

or alternatively in the matrix form as follows [21]:

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{pmatrix}=\begin{bmatrix}S_{1111} & S_{1221} & S_{1331} & 0 & 0 & 0\\ S_{1221} & S_{1111} & S_{1331} & 0 & 0 & 0\\ S_{1331} & S_{1331} & S_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & 2S_{2233} & 0 & 0\\ 0 & 0 & 0 & 0 & 2S_{2233} & 0\\ 0 & 0 & 0 & 0 & 0 & 2S_{1122}\end{bmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix}. \tag{2.103}$$

Here, S_{ijkl} are the components of the fourth-rank compliance tensor. For a transversely isotropic material the compliance matrix involves only five independent entries, so that $2S_{1122}=S_{1111}-S_{1221}$, while S_{1111}, S_{1221}, S_{1331}, S_{3333}, and S_{2233} are expressed in terms of engineering elastic constants by the formulas

$$S_{1111}=\frac{1}{E},\quad S_{1221}=-\frac{\nu}{E},\quad S_{1331}=-\frac{\nu'}{E'},\quad S_{3333}=\frac{1}{E'},\quad S_{2233}=\frac{1}{4G'}, \tag{2.104}$$

where E and E' are Young's moduli in the plane of transverse isotropy and in the direction normal to it, ν and ν' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, G' is the shear modulus in planes normal to the plane of transverse isotropy. We note also that the in-plane shear modulus is given by $G = E/[2(1+\nu)]$.

2.4.1 Stress-Strain Relations for Incompressible Material

Recall that a solid's resistance to all-round compression is characterized by the bulk modulus, K. For a homogeneous anisotropic material occupying a volume V, the bulk modulus is defined as the ratio of the small hydrostatic pressure increase, Δp, to the resulting relative decrease of the volume, $\Delta V/V$.

Since under the assumption of small deformations, $\Delta V/V = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, by substituting $\sigma_{11} = \sigma_{22} = \sigma_{33} = -\Delta p$ into Eq. (2.103), we readily obtain

$$\frac{1}{K} = 2S_{1111} + 2S_{1221} + 4S_{1331} + S_{3333}.$$

In terms of the stiffnesses, the bulk modulus is given by

$$K = \frac{A_{33}(A_{11} + A_{12}) - 2A_{13}^2}{A_{11} + A_{12} - 4A_{13} + 2A_{33}},$$

where A_{ij} are determined by formulas (1.3).

According to (2.104), the above formula can be recast in terms of the engineering constants as

$$K = \left(\frac{2(1-\nu)}{E} + \frac{1-4\nu'}{E'}\right)^{-1}, \tag{2.105}$$

which in the isotropic case reduces to

$$K = \frac{E}{3(1-2\nu)}.$$

If the material is incompressible, then the bulk modulus is infinite. At the same time, the incompressibility condition

$$\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 \tag{2.106}$$

must be satisfied for arbitrary stress $\boldsymbol{\sigma}$ satisfying the equilibrium equations.

As was shown by Itskov and Aksel [15], the incompressibility condition (2.106) written in the form $\mathbf{I} : \boldsymbol{\varepsilon} = 0$, where $\mathbf{I}$ is the second-order identity tensor, implies that $\mathbf{S} : \mathbf{I} = \mathbf{0}$. The last tensor equation, in the general case, imposes 6 additional constrains on the compliance tensor $\mathbf{S}$. In the case of transverse isotropy, only two of these incompressibility conditions are not identically satisfied.

For the compliance matrix to be positive definite, it is required that

$$E > 0, \quad E' > 0, \quad G' > 0, \quad \nu^2 \leq 1,$$

and

$$\nu'^2 \leq \frac{E'(1-\nu)}{2E}. \tag{2.107}$$

By considering the special case of a hydrostatic loading, when $\sigma_{11} = \sigma_{22} = \sigma_{33} = p$ and $\sigma_{23} = \sigma_{13} = \sigma_{12} = 0$, it can be easily verified that the incompressibility condition (2.106) is achieved if

$$\frac{2}{E} - \frac{2\nu}{E} - \frac{4\nu'}{E'} + \frac{1}{E'} = 0. \tag{2.108}$$

Following [13], we eliminate the Poisson's ratio ν between (2.107) and (2.108) to get $(\nu' - 0.5)^2 \leq 0$, from which it immediately follows that $\nu' = 0.5$. Now, substituting this value into Eq. (2.108), we obtain $\nu = 1 - 0.5(E/E')$.

Thus, for an incompressible transversely isotropic material only 3 material constants remain independent and the following relations hold [13, 15]:

$$\nu' = \frac{1}{2}, \quad \nu = 1 - \frac{E}{2E'}. \tag{2.109}$$

These relations determine the incompressibility limit. Consequently, the condition of positive definiteness of the compliance tensor reduces to

$$E < 4E'. \tag{2.110}$$

It should be emphasized that whereas the stress-strain relations for compressible materials can be obtained through the direct inversion of Hooke's law (2.103), leading to $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$ with the stiffness tensor $\mathbf{C} = \mathbf{S}^{-1}$, for incompressible materials the constitutive relation is given by

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} - p\mathbf{I}, \tag{2.111}$$

where evaluation of the super-symmetric fourth-order elasticity tensor $\mathbf{C}$ requires a special procedure developed in [15].

In the matrix form, Eq. (2.111) can be rewritten as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix} - \begin{pmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.112}$$

where, according to Itskov and Aksel [15], we have

$$a_{11} = \frac{4E'(2E + E')}{9(4E' - E)}, \quad a_{33} = \frac{4}{9}E', \quad a_{12} = \frac{2E'(2E' - 5E)}{9(4E' - E)},$$

$$a_{13} = -\frac{2}{9}E', \quad a_{44} = G', \quad a_{66} = G. \tag{2.113}$$

As a consequence of $(2.109)_2$, the in-plane shear modulus is given by

$$G = \frac{EE'}{4E' - E}. \tag{2.114}$$

Thus, for an incompressible transversely isotropic material, the number of independent material constants is equal to 3, and E, E', and G' can be taken, for instance.

Note also that according to $(2.109)_2$ and (2.110), the in-plane Poisson's ratio satisfies the inequality $\nu > -1$.

In the isotropic case, when $E = E'$, $G = G'$, and $\nu = \nu'$, Eqs. (2.109) and (2.114) imply that $\nu = 0.5$ and $G = E/3$. Consequently, Eq. (2.113) yield $a_{11} = a_{33} = 4E/9$ and $a_{12} = a_{13} = -2E/9$. Taking into account the incompressibility condition (2.106), we represent the stress-strain relations as follows, where no summation is implied by the repeated index:

$$\sigma_{ii} = 2G\varepsilon_{ii} - p, \quad \sigma_{ij} = 2G\varepsilon_{ij}, \quad i \neq j, \quad i, j = 1, 2, 3. \tag{2.115}$$

Notice that Eq. (2.115) represent a usual form of the constitutive relations for isotropic incompressible material [9, 28].

Finally, the unknown hydrostatic pressure parameter p in the constitutive law (2.112) should be determined from the incompressibility condition (2.106).

2.4.2 Isotropically Compressible Transversely Isotropic Materials

Note that for an incompressible material under a uniform hydrostatic pressure not only the trace of the strain tensor is equal to zero, but also the deviatoric part of the strain tensor vanishes. Following [15], we consider anisotropic materials which under a uniform hydrostatic pressure exhibit strictly isotropic volumetric response. In this case, the bulk modulus, K, of such an isotropically compressible material is independent of the stress state and represents its intrinsic property.

For an isotropically compressible transversely isotropic material, the number of independent material constants is equal to 4, while E, E', G', and K can be taken, in this instance. As a result, the following relations hold [15]:

$$\nu' = \frac{1}{2} - \frac{E'}{6K}, \quad \nu = 1 - \frac{E}{2}\Big(\frac{1}{E'} + \frac{1}{3K}\Big), \quad G = \frac{E'E}{4E' - E - \dfrac{E'E}{3K}}.$$

Correspondingly, the constitutive relations can be written in the form

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix} - \begin{pmatrix} p \\ p \\ p \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (2.116)$$

$$p = -K(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}), \qquad (2.117)$$

where, according to Itskov and Aksel [15], c_{ij} are given by

$$c_{11} = \frac{1}{3D}\Big(\frac{2}{E'} + \frac{1}{E} - \frac{1}{3K}\Big), \quad c_{33} = \frac{1}{3D}\Big(\frac{4}{E} - \frac{1}{E'} - \frac{1}{3K}\Big),$$
$$c_{12} = \frac{1}{6D}\Big(\frac{2}{E} - \frac{5}{E'} + \frac{1}{3K}\Big), \quad c_{13} = \frac{1}{6D}\Big(\frac{1}{E'} - \frac{4}{E} + \frac{1}{3K}\Big), \qquad (2.118)$$

$$D = \frac{3}{4E'}\Big(\frac{4}{E} - \frac{1}{E'}\Big) - \frac{1}{6K}\Big(\frac{1}{E'} + \frac{2}{E}\Big) + \frac{1}{36K^2}.$$

Thus, for isotropically compressible materials, the case of weak compressibility can be treated as the limit as $E'/K \to 0$.

2.5 Deformation of a Thin Incompressible Transversely Isotropic Elastic Layer Bonded to a Rigid Substrate

Let us consider a thin transversely isotropic elastic layer of uniform thickness, h, ideally bonded to a rigid substrate and loaded by a normal load, q, (see Fig. 2.7).

According to the perturbation analysis performed in Sect. 1.2, the leading-order asymptotic approximation for the displacement field is given by the following formulas (see Eqs. (1.18), (1.19), (1.22)–(1.24)):

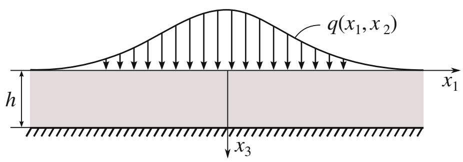


Fig. 2.7 An elastic layer of uniform thickness bonded to a rigid substrate and carrying a normal load

$$\mathbf{v} \simeq \varepsilon^2 \left(\frac{A_{13} + A_{44}}{2A_{33}} (1-\zeta)^2 - \frac{A_{13}}{A_{33}} (1-\zeta) \right) \frac{h_*}{A_{44}} \nabla_\eta q, \tag{2.119}$$

$$\begin{aligned} w \simeq \varepsilon \frac{h_* q}{A_{33}} (1-\zeta) + \varepsilon^3 \bigg\{ & \frac{A_{13}(A_{13} + 2A_{44})}{6A_{33}^2} (1-\zeta)^3 \\ & - \frac{A_{13}(A_{13} + A_{44})}{2A_{33}^2} (1-\zeta)^2 + \frac{A_{13} A_{44}}{2A_{33}^2} (1-\zeta) \bigg\} \frac{h_*}{A_{44}} \Delta_\eta q. \end{aligned} \tag{2.120}$$

For any transversely isotropic material, we have

$$\frac{A_{13}}{A_{33}} = \frac{E\nu'}{E'(1-\nu)}.$$

When the material approaches the incompressible limit, the right-hand side of the above relation tends to 1 (see Eq. (2.109)), while the ratio A_{44}/A_{33} vanishes. Hence, formulas (2.119) and (2.120) reduce to the following:

$$\mathbf{v} \simeq \varepsilon^2 \left(\frac{1}{2} (1-\zeta)^2 - (1-\zeta) \right) \frac{h_*}{a_{44}} \nabla_\eta q, \tag{2.121}$$

$$w \simeq \varepsilon^3 \left\{ \frac{1}{6} (1-\zeta)^3 - \frac{1}{2} (1-\zeta)^2 \right\} \frac{h_*}{a_{44}} \Delta_\eta q. \tag{2.122}$$

Note that $a_{44} = A_{44}$ is the out-of-plane shear modulus.

2.5.1 Perturbation Analysis of the Deformation Problem for a Thin Incompressible Elastic Layer

From another point of view, the problem under consideration can be formulated based on the constitutive relation (2.112). Indeed, the equilibrium equations yield

$$a_{66} \Delta_y \mathbf{v} + (a_{11} - a_{66}) \nabla_y \nabla_y \cdot \mathbf{v} + a_{44} \frac{\partial^2 \mathbf{v}}{\partial z^2} + (a_{13} + a_{44}) \frac{\partial}{\partial z} \nabla_y w - \nabla_y p = \mathbf{0},$$

$$a_{44} \Delta_y w + a_{33} \frac{\partial^2 w}{\partial z^2} + (a_{13} + a_{44}) \frac{\partial}{\partial z} \nabla_y \cdot \mathbf{v} - \frac{\partial p}{\partial z} = 0, \tag{2.123}$$

and the incompressibility condition is

$$\nabla_y \cdot \mathbf{v} + \frac{\partial w}{\partial z} = 0. \tag{2.124}$$

In the case of the elastic layer bonded to a rigid substrate, the boundary conditions at the layer bottom surface are

$$\mathbf{v}\big|_{z=h} = \mathbf{0}, \quad w\big|_{z=h} = 0, \tag{2.125}$$

while at the upper surface, under the assumption of normal loading, we have

$$\sigma_{13}\big|_{z=0} = \sigma_{23}\big|_{z=0} = 0, \quad \sigma_{33}\big|_{z=0} = -q.$$

The last boundary conditions can be rewritten as

$$\nabla_y w + \frac{\partial \mathbf{v}}{\partial z}\bigg|_{z=0} = \mathbf{0}, \tag{2.126}$$

$$a_{13}\nabla_y \cdot \mathbf{v} + a_{33}\frac{\partial w}{\partial z} - p\bigg|_{z=0} = -q. \tag{2.127}$$

Assuming that the elastic layer is relatively thin and $h = \varepsilon h_*$, we introduce the stretched normal coordinate

$$\zeta = \varepsilon^{-1} h_*^{-1} z$$

and the dimensionless in-plane coordinates

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_i = h_*^{-1} y_i, \quad i = 1, 2.$$

Correspondingly, the system of equations (2.123), (2.124) with the boundary conditions (2.125)–(2.127) takes the form

$$\varepsilon^{-2} a_{44}\frac{\partial^2 \mathbf{v}}{\partial \zeta^2} + \varepsilon^{-1}(a_{13} + a_{44})\nabla_\eta \frac{\partial w}{\partial \zeta} \\ + a_{66}\Delta_\eta \mathbf{v} + (a_{11} - a_{66})\nabla_\eta \nabla_\eta \cdot \mathbf{v} - h_*\nabla_\eta p = \mathbf{0}, \tag{2.128}$$

$$\varepsilon^{-2} a_{33}\frac{\partial^2 w}{\partial \zeta^2} + \varepsilon^{-1}\biggl((a_{13} + a_{44})\nabla_\eta \cdot \frac{\partial \mathbf{v}}{\partial \zeta} - h_*\frac{\partial p}{\partial \zeta}\biggr) + a_{44}\Delta_\eta w = 0, \tag{2.129}$$

$$\varepsilon^{-1}\frac{\partial w}{\partial \zeta} + \nabla_\eta \cdot \mathbf{v} = 0, \tag{2.130}$$

$$\mathbf{v}\big|_{\zeta=1} = \mathbf{0}, \quad w\big|_{\zeta=1} = 0, \tag{2.131}$$

$$\varepsilon^{-1}\frac{\partial \mathbf{v}}{\partial \zeta} + \nabla_\eta w\bigg|_{\zeta=0} = \mathbf{0}, \tag{2.132}$$

$$\frac{1}{h_*}\left(\varepsilon^{-1}a_{33}\frac{\partial w}{\partial \zeta}+a_{13}\nabla_\eta\cdot\mathbf{v}\right)-p\Big|_{\zeta=0}=-q. \tag{2.133}$$

In light of (2.119), (2.120), and (2.133), the asymptotic ansatz for the solution to the system (2.128)–(2.133) is represented in the form

$$\mathbf{v}\simeq\varepsilon^2\mathbf{v}^1(\boldsymbol{\eta},\zeta), \tag{2.134}$$

$$w\simeq\varepsilon^3 w^2(\boldsymbol{\eta},\zeta), \tag{2.135}$$

$$p\simeq q+\varepsilon^2 p^2(\boldsymbol{\eta},\zeta). \tag{2.136}$$

Substituting (2.134)–(2.136) into Eq. (2.128)–(2.133), we arrive at the problem

$$a_{44}\frac{\partial^2\mathbf{v}^1}{\partial\zeta^2}=h_*\nabla_\eta q,\quad \zeta\in(0,1),\quad \mathbf{v}^1\big|_{\zeta=1}=\mathbf{0},\quad \frac{\partial\mathbf{v}^1}{\partial\zeta}\Big|_{\zeta=0}=\mathbf{0}; \tag{2.137}$$

$$a_{33}\frac{\partial^2 w^2}{\partial\zeta^2}=-(a_{13}+a_{44})\nabla_\eta\cdot\frac{\partial\mathbf{v}^1}{\partial\zeta}+h_*\frac{\partial p^2}{\partial\zeta},\quad \zeta\in(0,1), \tag{2.138}$$

$$w^2\big|_{\zeta=1}=0,\quad a_{33}\frac{\partial w^2}{\partial\zeta}+a_{13}\nabla_\eta\cdot\mathbf{v}^1-h_*p^2\Big|_{\zeta=0}=0, \tag{2.139}$$

$$\frac{\partial w^2}{\partial\zeta}+\nabla_\eta\cdot\mathbf{v}^1=0,\quad \zeta\in(0,1). \tag{2.140}$$

From (2.137), it immediately follows that

$$\mathbf{v}^1(\boldsymbol{\eta},\zeta)=-\frac{h_*}{2a_{44}}(1-\zeta^2)\nabla_\eta q(\boldsymbol{\eta}). \tag{2.141}$$

Now, integrating Eq. (2.138), we get

$$a_{33}\frac{\partial w^2}{\partial\zeta}=-(a_{13}+a_{44})\nabla_\eta\cdot\mathbf{v}^1+h_*p^2+C_2, \tag{2.142}$$

where C_2 is an integration constant, which may depend on η_1 and η_2.

By taking into account (2.140) and (2.141), we rewrite Eq. (2.142) in the form

$$\begin{aligned} h_*p^2&=-C_2+(a_{13}+a_{44}-a_{33})\nabla_\eta\cdot\mathbf{v}^1\\ &=-C_2-\frac{a_{13}+a_{44}-a_{33}}{2a_{44}}(1-\zeta^2)h_*\Delta_\eta q(\boldsymbol{\eta}). \end{aligned} \tag{2.143}$$

Again, making use of Eqs. (2.140) and (2.141), we transform the boundary condition $(2.139)_2$ into

$$h_* p^2\big|_{\zeta=0} = (a_{13} - a_{33})\nabla_\eta \cdot \mathbf{v}^1\big|_{\zeta=0} \\ = -\frac{a_{13} - a_{33}}{2a_{44}} h_* \Delta_\eta q(\boldsymbol{\eta}). \tag{2.144}$$

From (2.143) and (2.144), it follows that

$$C_2 = a_{44}\nabla_\eta \cdot \mathbf{v}^1\big|_{\zeta=0} = -\frac{h_*}{2}\Delta_\eta q(\boldsymbol{\eta}). \tag{2.145}$$

Hence, the substitution of (2.145) into Eq. (2.143) yields

$$p^2 = \frac{\Delta_\eta q(\boldsymbol{\eta})}{2a_{44}}\big(a_{44} - (a_{44} + a_{13} - a_{33})(1 - \zeta^2)\big). \tag{2.146}$$

Finally, in light of (2.141), (2.145), and (2.146), Eq. (2.142) takes the form

$$\frac{\partial w^2}{\partial \zeta} = \frac{h_*}{2a_{44}}(1 - \zeta^2)\Delta_\eta q(\boldsymbol{\eta}), \tag{2.147}$$

which after integration (with the boundary condition $(2.139)_1$ at the bottom layer surface taken into account) becomes

$$w^2(\boldsymbol{\eta}, \zeta) = -\frac{h_*}{6a_{44}}(1 - \zeta)^2(2 + \zeta)\Delta_\eta q(\boldsymbol{\eta}). \tag{2.148}$$

It is easy to see that formulas (2.141) and (2.148) completely agree with (2.121) and (2.122), respectively, which are obtained from (2.119) and (2.120) for the compressible layer by passing to the incompressible limit.

Thus, the displacements of the surface points of the bonded incompressible elastic layer can be approximated by the following leading-order asymptotic formulas:

$$\mathbf{v}\big|_{\zeta=0} \simeq -\varepsilon^2\frac{h_*}{2a_{44}}\nabla_\eta q(\boldsymbol{\eta}), \tag{2.149}$$

$$w\big|_{\zeta=0} \simeq -\varepsilon^3\frac{h_*}{3a_{44}}\Delta_\eta q(\boldsymbol{\eta}). \tag{2.150}$$

After recovering the dimensional coordinates, these relations take the form

$$\mathbf{v}\big|_{z=0} \simeq -\frac{h^2}{2a_{44}}\nabla_y q(\mathbf{y}), \tag{2.151}$$

$$w\big|_{z=0} \simeq -\frac{h^3}{3a_{44}}\Delta_y q(\mathbf{y}). \tag{2.152}$$

We note that the two-term asymptotic approximation for the hydrostatic pressure is given by formulas (2.136) and (2.146).

Finally, we observe that the case of a practically incompressible elastic layer (weakly compressible layer) requires special consideration [22].

2.5.2 *Local Indentation of a Thin Weakly Compressible Elastic Layer*

Following Mishuris [22], we briefly consider the case of a weakly compressible material, where

$$\frac{G'}{K} \ll 1. \tag{2.153}$$

Excluding the hydrostatic pressure p from Eqs. (2.116) and (2.117), we obtain

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}, \tag{2.154}$$

where we have introduced the notation (with the coefficients c_{ij} given by (2.118))

$$C_{1j} = c_{1j} + K, \quad j = 1, 2, 3, \quad C_{33} = c_{33} + K, \quad C_{44} = G', \quad C_{66} = G.$$

Observe that the form of constitutive equation (2.154) for isotropically compressible materials is similar to that of Eq. (1.2) for compressible materials. Therefore, the following asymptotic approximations hold (see Sect. 1.2):

$$\mathbf{v}_0 \simeq -\varepsilon^2 \frac{h_*(C_{13} - C_{44})}{2C_{33}C_{44}} \nabla_\eta p, \tag{2.155}$$

$$w_0 \simeq \varepsilon \frac{h_* p}{C_{33}} - \varepsilon^3 \frac{h_* C_{13}(C_{13} - C_{44})}{3C_{33}^2 C_{44}} \Delta_\eta p. \tag{2.156}$$

Recall that the in-plane coordinates (η_1, η_2) are dimensionless (see Eq. (1.13)).

Thus, in light of (2.153), the elastic constants entering formulas (2.155), (2.156) can be expanded as

$$\frac{C_{13} - C_{44}}{C_{33}C_{44}} = \frac{1}{G'} - \frac{2E' + 3G'}{3G'K} + O\left(\frac{E'^2}{G'K^2}\right),$$

$$\frac{1}{C_{33}} = \frac{1}{K} - \frac{4E'}{9K^2} + O\left(\frac{E'^2}{K^3}\right), \tag{2.157}$$

$$\frac{C_{13}(C_{13} - C_{44})}{C_{33}^2 C_{44}} = \frac{1}{G'} - \frac{4E' + 3G'}{3G'K} + O\left(\frac{E'^2}{G'K^2}\right). \tag{2.158}$$

Let L be a characteristic length of the contact area, so that $\varepsilon = h/L$ and $h = \varepsilon h^*$. Thus, following three cases can then occur:

(1) Compressible layer when $(h/L)^2 \ll G'/K$ or, equivalently, $K \ll G'(L/h)^2$;
(2) Weakly compressible layer when $G'/K \sim (h/L)^2$;
(3) Practically incompressible layer when $G'/K \ll (h/L)^2$.

In the first and third cases, respectively, the first or second term in the asymptotic formula (2.156) becomes dominant. In the second case, both terms on the right-hand side of (2.156) are equally important.

Finally, we would like to emphasize that in the case of a thin elastic layer, the effect of incompressibility depends on the degree of its thinness.

2.6 Boundary-Layer Problem in the Incompressible Case

The case of incompressible materials requires a special consideration. In this section, the Wiener–Hopf method will be applied to the corresponding boundary-layer integral equation with a polynomial right-hand side.

2.6.1 Transformation of the Governing Integral Equation

In light of (2.22), Eq. (2.34) can be rewritten in the form

$$\frac{h}{\pi^2 \theta} \iint\limits_{\omega} p(\mathbf{y}') K_*(y_1 - y_1', y_2 - y_2')\, d\mathbf{y}' = \mathscr{W}_0(\mathbf{y}), \tag{2.159}$$

where we have introduced the notation

$$K_*(\mathbf{y}) = \iint\limits_{0}^{+\infty} \frac{\mathscr{L}(s)}{s^3} \cos\frac{s_1 y_1}{h} \cos\frac{s_2 y_2}{h}\, ds_1 ds_2,$$

$$\mathscr{W}_0(\mathbf{y}) = \frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{1}{\alpha^2} \sum_{n=1}^{2} \hat{w}_0^{(n)}(\alpha_1, \alpha_2) e^{-i(\alpha_1 y_1 + \alpha_2 y_2)} d\alpha_1 d\alpha_2. \tag{2.160}$$

Recall that the kernel function $\mathscr{L}(s)$ is given by (2.37), and $\hat{w}_0^{(n)}(\alpha_1, \alpha_2)$ denotes the Fourier transform of the local indentation of the n-th elastic layer.

In the contact problem, the sum of the local indentation functions $w_0^{(1)}(\mathbf{y})$ and $w_0^{(2)}(\mathbf{y})$, whose Fourier transforms appear in (2.160), is known only inside the contact area ω according to the contact condition (2.20), whereas outside ω the normal surface displacements $w_0^{(n)}(\mathbf{y})$ are determined by the contact pressure density $p(\mathbf{y})$, which is not given a priori, even in the case of fixed contact area. Thus, the right-hand side of Eq. (2.159) is also unknown.

By definition (see formula (2.160)), we can represent the right-hand side of the governing integral equation (2.159) in the form

$$\mathscr{W}_0(\mathbf{y}) = \mathscr{W}_0^1(\mathbf{y}) + \mathscr{W}_0^0(\mathbf{y}), \tag{2.161}$$

where $\mathscr{W}_0^0(\mathbf{y})$ is a harmonic function, and $\mathscr{W}_0^1(\mathbf{y})$ satisfies the problem

$$-\Delta_y \mathscr{W}_0^1(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega; \quad \mathscr{W}_0^1(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{2.162}$$

Denoting by $C_0(s)$ an arbitrary function of the arc length coordinate s along the contour Γ_*, we have

$$\Delta_\eta \mathscr{W}_0^0(h_* \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \omega_*; \quad \mathscr{W}_0^0(h_* \boldsymbol{\eta}) = C_0(s), \quad s \in \Gamma_*, \tag{2.163}$$

where ω_* is the contact area in the dimensionless coordinates (2.40).

Note that as

$$-\iint\limits_{-\infty}^{+\infty} \left(\frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2} \right) e^{i(\alpha_1 y_1 + \alpha_2 y_2)} dy_1 dy_2 = (\alpha_1^2 + \alpha_2) \hat{v}(\alpha_1, \alpha_2),$$

Equation (2.159) can be regarded as the result of application of the inverse Laplace operator $-\Delta_\eta^{-1}$ to the original governing integral equation (2.34).

2.6.2 Boundary-Layer Integral Equation

Performing the same analysis as in Sects. 2.3.1 and 2.3.2, but differing in that the local coordinates (s, n) are introduced in a small neighborhood of the contour Γ_*, we arrive at the following integral equation:

$$\int\limits_0^{+\infty} q^{**}(s, \nu') M_*(\nu' - \nu) \, d\nu' = \pi \theta h_* C_0(s). \tag{2.164}$$

Here, $q^{**}(s, \nu) = \varepsilon^{-1} q^*_\varepsilon(s, \nu)$, and the kernel is given by

$$M_*(t) = \int\limits_0^{+\infty} \frac{\mathscr{L}(u)}{u^3} \cos ut \, du. \tag{2.165}$$

Let us consider Eq. (2.164) with a special right-hand side

$$\int\limits_0^{+\infty} \phi_j(\tau) M_*(\tau - t) \, dt = t^j, \quad 0 < t < \infty, \tag{2.166}$$

where $j = 0, 1, 2, \ldots$

Recall (in particular, see formula (2.66)) that in the incompressible case, the function $\mathscr{L}(u)$ satisfies the following asymptotic relationships:

$$\mathscr{L}(u) = \mu_1 u^3 + \mu_2 u^5 + O(u^7), \quad u \to 0, \tag{2.167}$$

$$\mathscr{L}(u) = 1 + O(e^{-c_1 u}), \quad u \to \infty. \tag{2.168}$$

The dimensionless coefficients μ_k are given by

$$\mu_k = \theta \mathscr{M}_k, \tag{2.169}$$

where the dimensional quantities $\mathscr{M}_k$ are introduced in (2.66).

Therefore, Eq. (2.166) possesses a solution such that

$$\phi_j(t) \sim \omega_j t^{-3/2}, \quad t \to 0, \tag{2.170}$$

$$\phi_j(t) = O(t^j), \quad t \to \infty. \tag{2.171}$$

Let us reformulate the problem (2.166) into the distribution type formulation. Specifically, by introducing a small parameter $\varepsilon > 0$, we consider the problem

$$\int_0^\infty \phi_j^\varepsilon(\tau) M_*(\tau - t) \, dt = t^j e^{-\varepsilon t}, \quad 0 < t < \infty, \tag{2.172}$$

and will pass to the limit $\varepsilon \to 0$ later. We omit, where clarity permits, the subscript and superscript in ϕ_j^ε, using the common notation ϕ and clarifying where necessary.

Consequently,, we will look for a solution to Eq. (2.172) satisfying the following assumption at infinity:

$$\phi(t) = O(t^j) e^{-\varepsilon t}, \quad t \to \infty. \tag{2.173}$$

Under the assumptions made, we extend by zero the definition of the function $\phi(t)$ for $t \in (-\infty, 0)$ and apply the Fourier transformation to Eq. (2.173), obtaining the following classical Wiener–Hopf type equation as a result:

$$\overline{\phi}_+(s)\overline{M}_*(s) = r_-(s) + b_j \frac{1}{(\mathrm{i}s - \varepsilon)^{1+j}}. \tag{2.174}$$

Here an overbar denotes the Fourier transform, $b_0 = -1$, $b_1 = 1$, $b_2 = -2$, and

$$\overline{\phi}_+(s) = \int_{-\infty}^{+\infty} \phi(t)e^{\mathrm{i}st}dt = \int_0^{\infty} \phi(t)e^{\mathrm{i}st}dt$$

is the analytic function in the upper half plane behaving at infinity as

$$\overline{\phi}_+(\mathrm{i}\xi) = O(\xi^{1/2}), \quad \xi \to \infty, \tag{2.175}$$

Note that, when writing (2.175), we take the condition (2.170) into account.

From (2.165), we directly compute

$$\overline{M}_*(s) = \frac{\pi}{s^3}\mathscr{L}(s), \tag{2.176}$$

and thus $\overline{M}_*(s)$ is bounded at zero, while from (2.173), we only conclude that

$$\overline{\phi}_+(s) = O\left(\frac{1}{(s + \mathrm{i}\varepsilon)^{1+j}}\right), \quad s \to -\mathrm{i}\varepsilon. \tag{2.177}$$

We now assume that $\varepsilon \to 0$ and transform Eq. (2.174) into the equivalent form

$$\overline{\phi}_+(s)L_*(s)N(s) = r_-(s) + \frac{\tilde{b}_j}{(s + \mathrm{i}0)^{1+j}}, \tag{2.178}$$

where $\tilde{b}_0 = \mathrm{i}$, $\tilde{b}_1 = -1$, $\tilde{b}_2 = -2\mathrm{i}$, and

$$L_*(s) = \frac{\pi}{(s^2 + A^2)\sqrt{s^2 + B^2}}, \tag{2.179}$$

$$N(s) = \frac{\mathscr{L}(s)(s^2 + A^2)\sqrt{s^2 + B^2}}{s^3}. \tag{2.180}$$

It can easily be checked that $N(s)$ is an even function, and is always positive along the real axis, while for any positive values of A and B it satisfies the condition

$$N(s) = 1 + O(s^{-2}), \quad s \to \pm\infty. \tag{2.181}$$

Thus, the function (2.180) allows for factorization along the real axis in the form

$$N(s) = \frac{N_+(s)}{N_-(s)},$$

where

$$N_\pm(s) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log N(t)}{t-s}dt\right\}, \quad \pm\Im s > 0.$$

The functions $N_\pm(s)$ possess the following asymptotic representations:

$$N_\pm(s) = 1 - \frac{1}{\mathrm{i}\pi s}\int_0^\infty \log N(t)dt + o(s^{-1}), \quad \pm\mathrm{i}s \to \infty, \tag{2.182}$$

$$N_\pm(s) = \exp\left\{\frac{1}{\pi}\int_0^\infty \arg N(t)\frac{dt}{t}\right\} + O(s), \quad s \to 0. \tag{2.183}$$

We now introduce the notation

$$L_*^\pm(s) = \frac{\sqrt{\pi}}{(A \mp \mathrm{i}s)\sqrt{B \mp \mathrm{i}s}}, \tag{2.184}$$

so that the Wiener–Hopf equation (2.178) can be rewritten in the equivalent form

$$\overline{\phi}_+(s)L_*^+(s)N_+(s) = F_-(s)\left(r_-(s) + \frac{\tilde{b}_j}{(s+\mathrm{i}0)^{1+j}}\right), \tag{2.185}$$

where

$$F_-(s) = \frac{N_-(s)}{L_*^-(s)}. \tag{2.186}$$

In light of (2.182)–(2.184), we have

$$F_-(s) = O(1), \quad s \to 0; \quad F_-(s) \sim (\mathrm{i}s)^{3/2}, \quad s \to -\mathrm{i}\infty. \tag{2.187}$$

Note that the branch cuts in the respective square roots are taken along the imaginary axis from the points $-\mathrm{i}$ to $+\infty$ and from $+\mathrm{i}$ to $-\infty$ for the functions $L_*^+(s)$ and $L_*^-(s)$, respectively. Thus, the function $F_-(s)$ is analytic in the neighbourhood of the zero point and can be expanded in the Taylor series as follows:

$$F_-(s) = \sum_{k=0}^{j} F_-^{(k)}(0)\frac{s^k}{k!} + O(s^{k+1}), \quad s \to 0. \tag{2.188}$$

Finally, we transform Eq. (2.185) to the form

$$\overline{\Phi}_+(s)L_*^+(s)N_+(s) = F_-(s)\overline{r}_-(s) + \frac{\tilde{b}_j}{(s+\mathrm{i}0)^{1+j}}\left(F_-(s) - \sum_{k=0}^{j} F_-^{(k)}(0)\frac{s^k}{k!}\right) + \frac{\tilde{b}_j}{(s+\mathrm{i}0)^{1+j}}\sum_{k=0}^{j} F_-^{(k)}(0)\frac{s^k}{k!}. \tag{2.189}$$

The left-hand side of this equation is an analytic function in the upper half-plane having a pole at the point $s = 0$. The right-hand side of (2.189) consists of three terms where the first two are analytic functions in the lower half-plane. Finally, the third term is a plus function having a pole at the point $s = 0$.

Taking into account the assumed behavior of the sought for solution, we can write it in the form

$$\overline{\Phi}_+(s) = \frac{\tilde{b}_j}{(s+\mathrm{i}0)^{1+j}L_*^+(s)N_+(s)}\sum_{k=0}^{j} F_-^{(k)}(0)\frac{s^k}{k!}, \tag{2.190}$$

$$\overline{r}_-(s) = -\frac{\tilde{b}_j}{(s+\mathrm{i}0)^{1+j}F_-(s)}\left(F_-(s) - \sum_{k=0}^{j} F_-^{(k)}(0)\frac{s^k}{k!}\right). \tag{2.191}$$

Note that $\overline{r}_-(s)$ is analytic in the lower half plane containing the real axis and the common analyticity strip $0 < \operatorname{Im} s < \chi$, for some positive χ.

2.6.3 *Special Solutions of the Boundary-Layer Integral Equation*

For further analysis, it is crucial to consider an auxiliary function

$$\Phi_j(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-\mathrm{i}st}\,ds}{(s+\mathrm{i}0)^{j+1}L_*^+(s)N_+(s)}. \tag{2.192}$$

Here it is worth mentioning that the functions $\Phi_j(t)$, $j = 0, 1, 2, \ldots$, enjoy a recurrent property

$$\Phi_j'(t) = -\mathrm{i}\Phi_{j-1}(t), \quad \Phi_j(t) = \mathrm{i}\Phi_{j+1}'(t). \tag{2.193}$$

We continue by analyzing the asymptotic behavior of $\Phi_j(t)$ at zero and infinity. Note that the function $1/[L_*^+(s)N_+(s)]$ is analytic near the zero point, and thus

$$\frac{1}{L_*^+(s)N_+(s)} = \sum_{k=0}^{j} C_k^{(0)} s^k + O(s^{j+1}), \quad s \to 0.$$

This allows us to consider the following representation of the function $\Phi_j(t)$, suitable for large values of the variable t:

$$\Phi_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{L_*^+(s)N_+(s)} - \sum_{k=0}^{j} C_k^{(0)} s^k \right) \frac{e^{-\mathrm{i}st}\,ds}{(s+\mathrm{i}0)^{j+1}} + \sum_{k=0}^{j} C_k^{(0)} P_{j-k}(t). \tag{2.194}$$

The polynomials $P_m(t)$ are given by the formula

$$P_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}st}\,ds}{(s+\mathrm{i}0)^{m+1}}, \quad m = 0, 1, 2, \ldots. \tag{2.195}$$

In particular, it immediately follows that

$$P_0(t) = -\mathrm{i}, \quad P_1(t) = -t, \quad P_2(t) = \frac{\mathrm{i}t^2}{2},$$

and thus formula (2.194) yields

$$\Phi_j(t) = \sum_{k=0}^{j} C_k^{(0)} P_{j-k}(t) + o(1), \quad t \to \infty. \tag{2.196}$$

To investigate the asymptotic behavior of the function $\Phi_j(t)$ near the point $t = 0$, we observe that

$$\frac{1}{L_*^+(s)N_+(s)} = -\mathrm{i}s \left(C_0^{(\infty)}(-\mathrm{i}s)^{1/2} + C_1^{(\infty)}(-\mathrm{i}s)^{-1/2} \right) + O\big((-\mathrm{i}s)^{-1/2}\big), \tag{2.197}$$

as $-\mathrm{i}s \to +\infty$. Correspondingly, as $t \to 0$, we arrive at the following estimate:

$$\Phi_j(t) = \frac{-\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{(C_0^{(\infty)}(-\mathrm{i}s)^{1/2} + C_1^{(\infty)}(-\mathrm{i}s)^{-1/2})e^{-\mathrm{i}st}}{(s+\mathrm{i}0)^j} ds + O(1). \tag{2.198}$$

Here the integration pass should be further deformed to impart a classical sense to the Fourier transform.

As a result, we obtain the asymptotic relationships

$$\Phi_0(t) = \frac{\mathrm{i}}{2\sqrt{\pi}} C_0^{(\infty)} t^{-3/2} - \frac{\mathrm{i}}{\sqrt{\pi}} C_1^{(\infty)} t^{-1/2} + O(1), \quad t \to 0, \tag{2.199}$$

$$\Phi_1(t) = -\frac{1}{\sqrt{\pi}} C_0^{(\infty)} t^{-1/2} + O(1), \quad t \to 0, \tag{2.200}$$

$$\Phi_2(t) = O(1), \quad t \to 0, \tag{2.201}$$

where we have used the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-\mathrm{i}s)^{-1/2} e^{-\mathrm{i}st} ds = \frac{1}{\sqrt{\pi}} t^{-1/2}. \tag{2.202}$$

We are now ready to return to the general solution (2.190) and consider three cases $j = 0, 1, 2$ separately.

The case $j = 0$. Due to the fact that $\tilde{b}_0 = \mathrm{i}$, the respective solution takes the form

$$\phi_0(t) = \mathrm{i} F_-(0) \Phi_0(t), \tag{2.203}$$

and based on the results presented above for the asymptotic behavior of the function $\Phi_j(t)$ at infinity (see (2.194)), we conclude that

$$\phi_0(t) = \frac{F_-(0)}{N_+(0) L_*^+(0)} + o(1), \quad t \to \infty. \tag{2.204}$$

In light of (2.167), (2.179), and (2.180), formula (2.204) can be recast as

$$\phi_0(t) = \frac{1}{\mu_1} + o(1), \quad t \to \infty. \tag{2.205}$$

Note that in the isotropic incompressible case (2.29), we have

$$\mathscr{L}(u) = \frac{\sinh 2u - 2u}{\cosh 2u + 1 + 2u^2} \tag{2.206}$$

and (see also formula (2.167))

$$\mathscr{L}(u) = \frac{2}{3} u^3 - \frac{6}{5} u^5 + O(u^7), \quad u \to 0.$$

Thus, $\mu_1 = 2/3$, and the asymptotic formula (2.205) agrees with the corresponding result obtained by Aleksandrov [4].

On the other hand, taking into account the asymptotic behavior (2.199) of the function $\Phi_0(t)$ at the zero point, we readily see that

$$\phi_0(t) \sim -\frac{F_-(0)}{2\sqrt{\pi}} C_0^{(\infty)} t^{-3/2}, \quad t \to 0.$$

Considering (2.180), (2.186) and (2.197), the above formula takes the form

$$\phi_0(t) \sim -\frac{A\sqrt{B}N_-(0)}{2\pi^{3/2}} t^{-3/2}, \quad t \to 0. \tag{2.207}$$

The case $j = 1$. Now, since $\tilde{b}_1 = -1$, Eq. (2.189) has the following solution:

$$\overline{\phi}_+(s) = -\frac{F_-(0) + sF'_-(0)}{(s + \mathrm{i}0)^2 L_*^+(s) N_+(s)}. \tag{2.208}$$

As a result, we obtain

$$\phi_1(t) = -F_-(0)\Phi_1(t) - F'_-(0)\Phi_0(t). \tag{2.209}$$

The case $j = 2$. Finally, we have $\tilde{b}_2 = -2\mathrm{i}$ and the solution to the problem takes the form

$$\overline{\phi}_+(s) = -\mathrm{i}\frac{2F_-(0) + 2sF'_-(0) + s^2 F''_-(0)}{(s + \mathrm{i}0)^3 L_*^+(s) N_+(s)}.$$

As a result, we see that

$$\phi_2(t) = -2\mathrm{i}F_-(0)\Phi_2(t) - 2\mathrm{i}F'_-(0)\Phi_1(t) - \mathrm{i}F''_-(0)\Phi_0(t). \tag{2.210}$$

Formulas (2.203), (2.209), and (2.210) provide solutions to Eq. (2.166) with special right-hand sides $\psi^j(t) = t^j$ for $j = 0, 1, 2$, respectively.

2.6.4 Solution of the Boundary-Layer Integral Equation with a Polynomial Right-Hand Side

We now consider Eq. (2.164), where the right-hand side is a linear combination of the polynomials t^0, t^1, and t^2. That is

$$\int_0^{+\infty} \phi(\tau) M_*(\tau - t)\, dt = \sum_{k=0}^{2} c_k t^k, \quad 0 < t < \infty. \tag{2.211}$$

According to (2.190), a solution to Eq. (2.211) can be represented in the form

$$\phi(t) = \Omega(t)t^{-3/2}, \quad 0 < t < \infty. \tag{2.212}$$

Formula (2.212) will give a finite-energy solution, if $\Omega(0) = 0$ and $\Omega(t)$ has a local derivative (near zero) belonging to the Holder class. In other words, $\Omega \sim \omega_0 t^{1+\alpha}$ as $t \to 0$, where $0 < \alpha < 1$ defines the Holder class, or equivalently

$$\phi(t) \sim \omega_0 t^{\alpha-1/2}, \quad t \to 0. \tag{2.213}$$

Therefore, in order to satisfy the asymptotic condition (2.213), one of the constants c_0, c_1, and c_2 in (2.211) should be a linear combination of the other two. In light of (2.211), the regularity condition (2.213) takes form

$$c_0\phi_0(t) + c_1\phi_1(t) + c_2\phi_2(t) = O(t^{-1/2}), \quad t \to 0. \tag{2.214}$$

Now taking into account Eqs. (2.203), (2.209), and (2.210), we conclude that the asymptotic condition (2.214) is equivalent to the following:

$$c_0 F_-(0) + ic_1 F'_-(0) - c_2 F''_-(0) = 0. \tag{2.215}$$

Here, $F_-(0)$, $F'_-(0)$ and $F''_-(0)$ are the values of the function (2.186) and its two successive derivatives at zero (see formula (2.188)).

Thus, it is clear that Eq. (2.164) has a finite-energy solution if

$$C_0(s) \equiv 0, \tag{2.216}$$

where s is the arc length coordinate along the contour Γ_*.

2.6.5 *Approximate Solution of the Boundary-Layer Integral Equation*

Following Aleksandrov [4], we look for a solution in its approximate form, assuming $N(s) \equiv 1$. The function $\Phi_j(t)$ simplifies as follows:

$$\Phi_j(t) = \frac{1}{2\pi\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(A - is)\sqrt{B - is}}{(s + i0)^{j+1}} e^{-ist} ds. \tag{2.217}$$

Note that it is enough to construct the function $\Phi_2(t)$, while $\Phi_0(t)$ and $\Phi_1(t)$ could be evaluated using the recurrence relationships between them (2.193).

We apply Cauchy's residue theorem to evaluate the Bromwich contour integral of (2.217). It is readily seen that the integrand function has a pole at $s = 0$ of the order $j + 1$ and a branch point at $s = -\mathrm{i}B$. In this way, we find

$$
\begin{aligned}
\mathrm{i}\sqrt{\pi}\,\Phi_0(t) &= A\sqrt{B} + \frac{e^{-Bt}}{\pi}\big(A\psi_1(t) - \psi_0(t)\big),\\
-\sqrt{\pi}\,\Phi_1(t) &= \sqrt{B} + \frac{A}{2\sqrt{B}} + A\sqrt{B}t - \frac{e^{-Bt}}{\pi}\big(A\psi_2(t) - \psi_1(t)\big),\\
\mathrm{i}\sqrt{\pi}\,\Phi_2(t) &= \sqrt{B}\left(\frac{A-4B}{8B^2} - \left(1 + \frac{A}{2B}\right)t - \frac{At^2}{2}\right) - \frac{e^{-Bt}}{\pi}\big(A\psi_3(t) - \psi_2(t)\big),
\end{aligned}
\tag{2.218}
$$

where we have introduced the notation

$$
\psi_n(t) = \int_0^\infty \frac{e^{-t\xi}\sqrt{\xi}}{(\xi + B)^n}\, d\xi .
$$

In particular, we have

$$
\begin{aligned}
\psi_0(t) &= \frac{\sqrt{\pi}}{2t^{3/2}},\\
\psi_1(t) &= \frac{\sqrt{\pi}}{\sqrt{t}} - \pi\sqrt{B}e^{Bt}\operatorname{erfc}(\sqrt{Bt}),\\
\psi_2(t) &= -\sqrt{\pi}\sqrt{t} + \frac{\pi(1+2Bt)}{2\sqrt{B}}e^{Bt}\operatorname{erfc}(\sqrt{Bt}),\\
\psi_3(t) &= \frac{\sqrt{\pi}\sqrt{t}(2Bt+1)}{4B} - \frac{\pi}{8B^{3/2}}e^{Bt}(4B^2t^2 + 4Bt - 1)\operatorname{erfc}(\sqrt{Bt}).
\end{aligned}
\tag{2.219}
$$

Furthermore, in light of (2.184), the formula

$$
F_-(s) = \frac{1}{L_*^-(s)}
$$

simplifies to

$$
F_-(s) = \frac{1}{\sqrt{\pi}}(A + is)\sqrt{B + is}.
$$

It now follows that

$$
\begin{aligned}
F_-(0) &= \frac{1}{\sqrt{\pi}} A\sqrt{B}, \\
F'_-(0) &= \frac{i}{2\sqrt{\pi}\sqrt{B}}(A+2B), \\
F''_-(0) &= \frac{1}{4\sqrt{\pi}B\sqrt{B}}(A-4B),
\end{aligned}
\tag{2.220}
$$

while correspondingly, the regularity condition (2.215) takes the form

$$
Ac_0 - c_1\frac{A+2B}{2B} + c_2\frac{4B-A}{4B^2} = 0. \tag{2.221}
$$

Further, taking into account formulas (2.203), (2.209), (2.210) and (2.218)–(2.220), we obtain

$$
\begin{aligned}
\phi_0(t) &= \frac{A^2B}{\pi}\,\mathrm{erf}\,(\sqrt{Bt}) + \frac{A\sqrt{B}e^{-Bt}}{2\pi^{3/2}t^{3/2}}(2At-1), \\
\phi_1(t) &= \frac{A^2Bt}{\pi}\,\mathrm{erf}\,(\sqrt{Bt}) + \frac{e^{-Bt}}{4\pi^{3/2}\sqrt{B}t^{3/2}}(4A^2Bt^2 - 2A^2t + A + 2B), \\
\phi_2(t) &= \frac{(A^2B^2t - A^2 - 2B^2)}{\pi B}\,\mathrm{erf}\,(\sqrt{Bt}) \\
&\quad + \frac{e^{-Bt}}{8\pi^{3/2}B^{3/2}t^{3/2}}\big(8A^2B^2t^3 - 4A^2Bt^2 - 2A^2t + A - 16B^2t - 4B\big).
\end{aligned}
\tag{2.222}
$$

Observe that following Aleksandrov [4], we can select the coefficients A and B in the approximate kernel function

$$
\tilde{\mathscr{L}}(s) = \frac{s^3}{(s^2+A^2)\sqrt{s^2+B^2}} \tag{2.223}
$$

in such a way that (cf. formula (2.167))

$$
\tilde{\mathscr{L}}(s) = \mu_1 s^3 + \mu_2 s^5 + O(s^7), \quad s \to 0. \tag{2.224}
$$

By simple calculations, we find

$$
A^2 = \frac{1}{\mu_1 B}, \tag{2.225}
$$

where the coefficient B is determined as a positive root of the cubic equation

$$2\mu_1^2 B^3 + 2\mu_2 B^2 + \mu_1 = 0. \tag{2.226}$$

Therefore, in light of (2.223) and (2.224), for the function $N(s)$ defined by (2.180), the following asymptotic formula holds true:

$$N(s) = 1 + O(s^4), \quad s \to 0. \tag{2.227}$$

Note that in the isotropic incompressible case (2.206), Eq. (2.226) is equivalent to $20B^3 - 54B^2 + 15 = 0$, which has one negative and two positive roots $B_1 = 0.597219$ and $B_2 = 2.588024$. By formula (2.225), we calculate $A_1 = 1.584816$ and $A_2 = 0.761310$. The errors of the corresponding approximations (2.223) to the kernel function (2.206) do not exceed 14 % and 20 %, respectively, for all $0 \leq s < +\infty$.

2.7 Leading-Order Asymptotic Solution of the Contact Problem for Incompressible Layers

In the incompressible case, the process of solving the contact problem by asymptotic methods reduces to that of solving the so-called resulting problem for the leading-order asymptotic solution. In this section, the resulting boundary value problem (later called asymptotic model) is formulated, including the governing differential equation and the corresponding boundary condition.

2.7.1 Governing Differential Equation

According to (2.67) and (2.152), the local indentation of a thin bonded incompressible elastic layer can be approximated by

$$w_0^{(n)}(\mathbf{y}) \simeq -\mathscr{M}_1^{(n)} h_n^3 \Delta_y p(\mathbf{y}), \tag{2.228}$$

where

$$\mathscr{M}_1^{(n)} = \frac{1}{3a_{44}^{(n)}}$$

and $a_{44}^{(n)} = G'_n$ is the out-of-plane shear modulus of the n-th elastic layer.

Substituting the asymptotic representations (2.228), $n = 1, 2$, into the contact condition (2.20), we arrive at the equation

$$-\big(\mathscr{M}_1^{(1)} h_1^3 + \mathscr{M}_1^{(2)} h_2^3\big)\Delta_y p(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega. \tag{2.229}$$

Let us introduce the notation

$$\mathscr{M}_1 = \mathscr{M}_1^{(1)} \frac{h_1^3}{h^3} + \mathscr{M}_1^{(2)} \frac{h_2^3}{h^3}, \tag{2.230}$$

where $h = h_1 + h_2$ is the joint thickness. Then, we can rewrite Eq. (2.229) as follows:

$$-\mathscr{M}_1 h^3 \Delta_y p(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega. \tag{2.231}$$

Recall that the elastic constant $\mathscr{M}_1^{(n)}$ is introduced by the expansion

$$\frac{\mathscr{L}_n(s)}{\theta_n s} = \mathscr{M}_1^{(n)} s^2 + O(s^4), \quad s \to 0,$$

where $\mathscr{L}_n(s)$ is the kernel function of the n-th layer. From here it follows that

$$\frac{\theta}{\theta_n} \mathscr{L}_n\Bigl(\frac{h_n}{h} s\Bigr) = \theta \mathscr{M}_1^{(n)} \frac{h_n^3}{h^3} s^3 + O(s^5), \quad s \to 0.$$

By substituting the above expansion into formula (2.37), we derive the following asymptotic expansion for the compound kernel function:

$$\frac{\mathscr{L}(s)}{\theta s} = \biggl(\mathscr{M}_1^{(1)} \frac{h_1^3}{h^3} + \mathscr{M}_1^{(2)} \frac{h_2^3}{h^3}\biggr) s^2 + O(s^4), \quad s \to 0.$$

It is readily seen that the latter formula is consistent with the definition (2.230) of the compound elastic constant $\mathscr{M}_1$.

The governing equation (2.231) should be supplemented by the appropriate boundary conditions at the contour Γ of the contact area ω.

2.7.2 Boundary Condition in the Case of Fixed Contact Area

By introducing the dimensionless coordinates (2.40) into the governing integral equation (2.159), and recollecting the notation $\Lambda = \varepsilon^{-1}$, we have

$$\frac{h_*^3}{\pi^2 \theta \Lambda} \iint\limits_{\omega_*} p_*(\boldsymbol{\xi}) k_*\bigl(\Lambda(\boldsymbol{\eta} - \boldsymbol{\xi})\bigr)\, d\boldsymbol{\xi} = \mathscr{W}_0^*(\boldsymbol{\eta}), \tag{2.232}$$

where $p_*(\boldsymbol{\eta}) = p(h_* \boldsymbol{\eta})$ and $\mathscr{W}_0^*(\boldsymbol{\eta}) = \mathscr{W}_0(h_* \boldsymbol{\eta})$.

Applying the distributional asymptotic analysis (see Sect. 2.2.1), in light of the asymptotic expansion (2.167) for the kernel function $\mathscr{L}(s)$, we find that

$$k_*(\Lambda\boldsymbol{\eta}) \sim \frac{\pi^2\mu_1}{\Lambda^2}\delta(\boldsymbol{\eta}) - \frac{\pi^2\mu_2}{\Lambda^4}\Delta_\eta\delta(\boldsymbol{\eta}) + \ldots, \quad \Lambda \to \infty, \tag{2.233}$$

where $\mu_k = \theta\mathscr{M}_k$, and $\delta(\boldsymbol{\eta})$ is the two-dimensional Dirac delta function.

Therefore, substituting the asymptotic approximation (2.233) into Eq. (2.232), we obtain

$$h^3\mathscr{M}_1 p(\mathbf{y}) - h^5\mathscr{M}_2\Delta_y p(\mathbf{y}) + \ldots = \mathscr{W}_0(\mathbf{y}). \tag{2.234}$$

To leading asymptotic order we have

$$h^3\mathscr{M}_1 p(\mathbf{y}) \simeq \mathscr{W}_0(\mathbf{y}). \tag{2.235}$$

We are now in a position to derive the leading order asymptotic model in the case of fixed contact area. Taking into account relations (2.161)–(2.163) and (2.216), we conclude that the function $\mathscr{W}_0(\mathbf{y})$ should satisfy the problem

$$-\Delta_y\mathscr{W}_0(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega; \quad \mathscr{W}_0(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{2.236}$$

On the other hand, by substituting the representation $\mathscr{W}_0(\mathbf{y}) \simeq h^3\mathscr{M}_1 p(\mathbf{y})$, which is none other than (2.235), into Eq. (2.236), we arrive at the following problem:

$$-h^3\mathscr{M}_1\Delta_y p(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{2.237}$$

$$p(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{2.238}$$

It is readily seen that Eq. (2.237) is identical to the governing differential equation (2.231), while Eq. (2.238) represents the sought for boundary condition.

It is interesting to note that the leading order asymptotic model (2.237), (2.238) provides the so-called [4] degenerate solution, which vanishes at the boundary of the contact area even for a punch with a sharp edge, whereas, generally speaking, the original solution has a square root singularity.

2.7.3 Boundary Conditions in the Case of Unilateral Contact

We now revisit the boundary-layer problem considered in Sect. 2.6.2, taking into account the fact that in light of (2.216), the right-hand side of the governing integral equation (2.159) now satisfies the boundary-value problem (2.236).

Considering the behavior of the function $\mathscr{W}_0(\mathbf{y})$ near the contour Γ in the fast dimensionless coordinates (see, in particular, Eqs. (2.40) and (2.78)), we have

$$\mathscr{W}_0(h_*\boldsymbol{\eta}) = \varepsilon\nu B_1^*(s) + \varepsilon^2\nu^2 B_2^*(s) + \ldots, \quad \nu \to 0^+, \quad s \in \Gamma, \tag{2.239}$$

where we have introduced the notation

$$B_1^*(s) = h_* \frac{\partial \mathscr{W}_0}{\partial n}(\mathbf{y}), \quad B_2^*(s) = \frac{h_*^2}{2} \frac{\partial^2 \mathscr{W}_0}{\partial n^2}(\mathbf{y}), \quad \mathbf{y} \in \Gamma.$$

Taking into account only the leading term in (2.239), we arrive at the boundary-layer integral equation

$$\int_0^{+\infty} q^{**}(s, \nu') M_*(\nu' - \nu)\, d\nu' = \pi \theta h_* B_1^*(s) \nu. \tag{2.240}$$

In the unilateral contact problem (see (2.21)), the solution of Eq. (2.240) should vanish at the contact area contour, i.e.,

$$q^{**}(s, 0) = 0, \quad s \in \Gamma.$$

On the other hand, according to (2.166), we have $q^{**}(s, \nu) = \pi \theta h_* B_1^*(s) \phi_1(\nu)$, while, due to (2.200) and (2.209), this function possesses a singularity at $\nu = 0$ unless $B_1^*(s) = 0$ for all $s \in \Gamma$. In other words, as a consequence of (2.235) and (2.239), the obtained result represents the second boundary condition

$$\frac{\partial p}{\partial n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma, \tag{2.241}$$

that should be added to the problem (2.237), (2.238) in the case of unilateral contact with unknown contact area ω. Note that Eq. (2.241) is called [17, 27] the zero-pressure-gradient condition.

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