

## Chapter 2

# Microscopic Formulation of Two-Phase Flows

**Abstract** The equations governing a two-phase flow at the microscopic level are presented. By microscopic level, we mean that no averaging operation is done. The equations at the microscopic level are local and instantaneous equations. We begin by presenting the topological equation, followed by the mass, momentum and energy balance equations. The entropy equation is also presented in order to examine the consequences of the second law of thermodynamics. We end this chapter by deriving the equations which will be useful for the derivation of the two-fluid models in Chap. 3. The balance equations governing a whole fluid particle (i.e. a bubble or a droplet) are also given for completeness.

## 2.1 Introduction

Two-phase flows are characterized by the presence of numerous interfaces separating two immiscible phases. These phases can be in a gaseous, liquid or solid state and an interface can be seen as a two-dimensional surface embedded in the three-dimensional Euclidean space. If more than two distinct phases are present, we speak of a *multiphase* flow (e.g. a three-phase gas-liquid-solid is sometimes encountered in chemical reactors or in the steam generator of a nuclear reactor). Even if multiphase flows are more complicated than two-phase flows, most of the fundamentals issues already appeared in a two-phase flow. Therefore, for the sake of simplicity, we will consider only a two-phase flow constituted of two pure phases.

In general, the equations that will be presented in this first chapter cannot be solved directly (except for very simple problems for which Direct Numerical Simulation is accessible). Instead, a kind of averaging is performed in order to reduce the (great) number of freedom degrees which characterize most of the two-phase flows of interest. In the following, we will make the distinction between microscopic and macroscopic equations. The microscopic equations characterize the two-phase flows at the microscopic level, i.e. when no averaging has been

performed yet. The macroscopic equations can be obtained from the microscopic ones by performing an averaging operation, as it will be shown in Chap. 3. In this Chapter, microscopic equations are derived for one fluid phase as well as for the interface. At the end of the chapter, the local equations will be integrated over a complete fluid particle in order to obtain the equations governing the evolution of one fluid particle (a bubble or a droplet) embedded in the flow.

The microscopic equations for a two-phase flow have been derived by many authors (e.g. Delhay [1974a, b, 2008](#); Ishii [1975](#); Kataoka [1986](#); Ishii and Hibiki [2006](#); Jakobsen [2008](#)). In order to include the interfaces at the local instantaneous level in a natural way, we will use generalized functions (e.g. Schwartz [1966](#); Bousquet [1990](#); Pope [2000](#)). The main tools we need are the Heaviside and Dirac generalized functions which properties are recalled in the Appendix B. The equations for a single phase flow are recalled in the Appendix A. Before presenting the balance equations governing each phase and the interface, we begin by the topological equations which deal with interface motion and functions of presence.

## 2.2 Topological Equations

Let:

$$F(\underline{x}, t) = 0 \quad (2.1)$$

be the geometrical equation defining the position of the interface in the flow at each time. Let the function  $F$  be positive in phase 1 and negative in phase 2. The Phase Indicator Function (PIF) is defined for each phase by the following equation:

$$\chi_1(\underline{x}, t) \equiv 1 - \chi_2(\underline{x}, t) \equiv H(F(\underline{x}, t)) \quad (2.2)$$

where  $H(x)$  is the Heaviside generalized function (Appendix B). An equivalent but more intuitive definition to Eq. (2.2) is the following one:

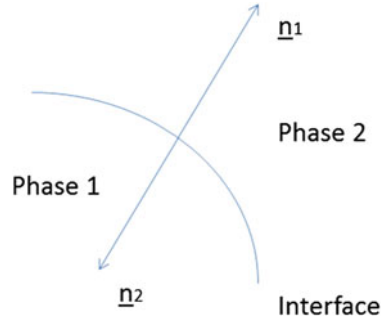
$$\chi_k(\underline{x}, t) \equiv \begin{cases} 1 & \text{if point } \underline{x} \text{ pertains to phase } k \text{ at time } t \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

The unit vector normal to the interface and outwardly directed from phase  $k$  ( $k = 1, 2$ ) is given by the normalized gradient of the function  $F$  (Aris [1962](#)):

$$\underline{n}_2 = -\underline{n}_1 = \nabla F / |\nabla F| \quad (2.4)$$

These normal vectors are sketched on Fig. [2.1](#).

**Fig. 2.1** Unit vectors normal to the interface



Let  $\underline{v}_I$  be a velocity field associated to the interface. As the function  $F$  is identically equal to zero at the interface points (Eq. 2.1), the convective derivative of  $F$  at the velocity  $\underline{v}_I$  is nil:

$$\frac{\partial F}{\partial t} + \underline{v}_I \cdot \nabla F = 0 \quad (2.5)$$

From Eqs. (2.4) and (2.5), the normal speed of displacement of the interface can be deduced (Delhay 2008):

$$\underline{v}_I \cdot \underline{n}_2 = -\underline{v}_I \cdot \underline{n}_1 = -\frac{\partial F / \partial t}{|\nabla F|} \quad (2.6)$$

The normal speed of displacement is only one component of the velocity vector (namely the one normal to the surface). The other two components constitute the tangential velocity vector which can be obtained in the following manner:

$$\underline{v}_{I,t} \equiv \underline{v}_I - (\underline{v}_I \cdot \underline{n})\underline{n} \quad (2.7)$$

In Eq. (2.7), the phase index  $k$  has been omitted on the unit normal vector. As the unit normal vector appears multiplied by itself in this expression, it is not useful to precise its sense. As remarked by Drew (1990), two velocity vectors  $\underline{v}_I$  differing only by their tangential components give rise to the same surface motion in space because the sole velocity component appearing in Eq. (2.5) is the component normal to the surface (2.6). As a consequence, a freedom degree is left for the choice of the tangential velocity components.

Starting from the definition (2.2), the time and spatial derivatives of the PIF can be calculated:

$$\begin{aligned} \nabla \chi_1 &= -\nabla \chi_2 = \delta(F) \nabla F \\ \frac{\partial \chi_1}{\partial t} &= -\frac{\partial \chi_2}{\partial t} = \delta(F) \frac{\partial F}{\partial t} \end{aligned} \quad (2.8)$$

In Eq. (2.8),  $\delta(x)$  represents the Dirac delta generalized function, which is the derivative of the Heaviside generalized function (Appendix B). From Eqs. (2.5) and (2.8), the following topological equation for phase  $k$  can be deduced:

$$\frac{\partial \chi_k}{\partial t} + \mathbf{v}_I \cdot \nabla \chi_k = 0 \quad k = 1, 2 \quad (2.9)$$

From Eqs. (2.4) and (2.8)<sub>1</sub>, the following useful relations can also be deduced:

$$-\mathbf{n}_k \cdot \nabla \chi_k = \delta(F) |\nabla F| \equiv \delta_I \Leftrightarrow \nabla \chi_k = -\mathbf{n}_k \delta_I \quad (2.10)$$

In the above equation,  $\delta_I$  is a generalized function having the different interfaces as its support. It is called *local instantaneous interfacial area concentration* by Kataoka (1986). The second relation (2.10) is obtained from the first by remarking that  $\mathbf{n}_k$  is a unit vector ( $\mathbf{n}_k \cdot \mathbf{n}_k = 1$ ). As the PIF can take only binary values (0 or 1), we can also remark the useful properties of the PIF:

$$\begin{aligned} \chi_k^n &= \chi_k \\ \chi_1 \chi_2 &= 0 \\ (\chi_k A_k)(\chi_k B_k) &= \chi_k A_k B_k \end{aligned} \quad (2.11)$$

where  $A_k$  and  $B_k$  are arbitrary quantities characterizing phase  $k$ .

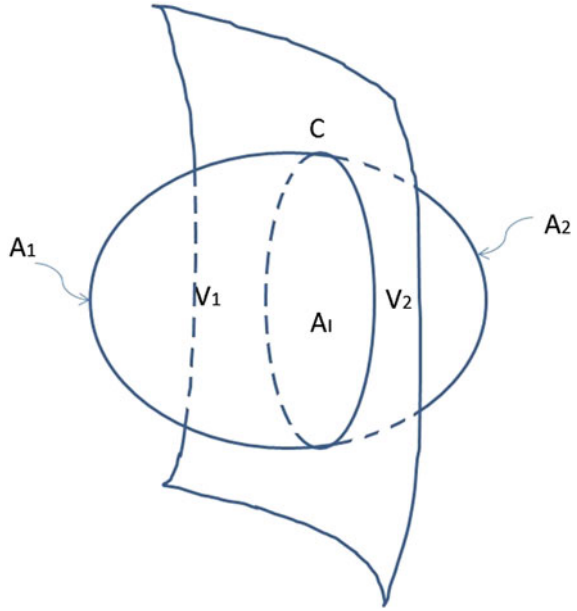
## 2.3 Mass Balance Equations

In this section and in the following ones, we begin by giving the balance into integral form on a two-phase material volume (Fig. 2.2). From the Leibniz rule for a volume, the Reynolds transport theorem for a surface and the Gauss theorems, we obtain a sum of two volume integrals and one surface integral. The volume integral furnishes the local balance equations valid in each phase and the surface integral furnishes the jump conditions which are valid at the interface only (Delhay 1974a, b; Ishii 1975).

As shown in Fig. 2.2, we consider a two-phase material volume  $V(t)$  divided into two sub-volumes  $V_1(t)$  and  $V_2(t)$  occupied by the two phases. The two phase volumes are separated by a two-dimensional interfacial surface and the portion  $A_I(t)$  of this surface contained into the volume  $V(t)$  is limited by a closed curve  $C(t)$ . The sub-volume  $V_1(t)$  is limited by the open surfaces  $A_1(t)$  and  $A_I(t)$  and the sub-volume  $V_2(t)$  is limited by the open surfaces  $A_2(t)$  and  $A_I(t)$ . Since the volume  $V(t)$  is material, also are the surfaces  $A_1(t)$  and  $A_2(t)$  but the interfacial surface  $A_I(t)$  is not, since it can be traversed by a mass transfer due to phase change (vaporization or condensation).

The volume depicted above being assumed material, it always contains the same mass during time. Mathematically, this can be written as:

**Fig. 2.2** A two-phase volume



$$\frac{d}{dt} \int_{V_1(t)} \rho_1 dv + \frac{d}{dt} \int_{V_2(t)} \rho_2 dv + \frac{d}{dt} \int_{A_1(t)} \rho_I dS = 0 \quad (2.12)$$

In this equation,  $\rho_k(\underline{x}, t)$  is the density field of phase  $k$  and  $\rho_I(\underline{x}, t)$  is the interface density field. The interface being considered as a *two-dimensional surface*, the interface density is a mass per unit surface hence it is expressed in  $\text{kg/m}^2$ . Using the Leibniz rules (B.8) for the volume integrals and (B.10) for the surface integral, Eq. (2.12) can equivalently be rewritten:

$$\begin{aligned} \int_{V_1} \frac{\partial \rho_1}{\partial t} dv + \int_{A_1 \cup A_1} \rho_1 \underline{v}_1 \cdot \underline{n}_1 dS + \int_{V_2} \frac{\partial \rho_2}{\partial t} dv + \int_{A_2 \cup A_1} \rho_2 \underline{v}_2 \cdot \underline{n}_2 da \\ + \int_{A_1} (\rho_1 (\underline{v}_I - \underline{v}_1) \cdot \underline{n}_1 + \rho_2 (\underline{v}_I - \underline{v}_2) \cdot \underline{n}_2) dS + \int_{A_1} \left( \frac{D_I \rho_I}{Dt} + \rho_I \nabla_s \cdot \underline{v}_I \right) dS = 0 \end{aligned} \quad (2.13)$$

Using the Gauss theorem (B.9) twice, Eq. (2.13) becomes:

$$\begin{aligned} \int_{V_1} \left[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \underline{v}_1) \right] dv + \int_{V_2} \left[ \frac{\partial \rho_2}{\partial t} + \nabla \cdot (\rho_2 \underline{v}_2) \right] dv \\ + \int_{A_1} \left[ (\rho_1 (\underline{v}_I - \underline{v}_1) \cdot \underline{n}_1 + \rho_2 (\underline{v}_I - \underline{v}_2) \cdot \underline{n}_2) + \left( \frac{D_I \rho_I}{Dt} + \rho_I \nabla_s \cdot \underline{v}_I \right) \right] dS = 0 \end{aligned} \quad (2.14)$$

The balance (2.14) must be satisfied for any  $V_1$ ,  $V_2$  and  $A_I$ , thus the arguments in the surface and volume integrals must all independently be equal to zero. We obtain for a point located in phase  $k$ :

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \underline{v}_k) = 0 \quad k = 1, 2 \quad (2.15)$$

which is the mass balance equation for a single fluid (Eq. A.7). The difference with the single phase case is that Eq. (2.15) is not valid everywhere but only in the portions of the flow occupied by phase  $k$ . Equation (2.15) will be extended by the means of generalized functions to obtain an equation valid everywhere in Sect. 2.6.

For a point located on the surface, Eq. (2.14) also gives the following *mass interfacial balance*:

$$\frac{D_I \rho_I}{Dt} + \rho_I \nabla_s \cdot \underline{v}_I = \sum_{k=1,2} \rho_k (\underline{v}_k - \underline{v}_I) \cdot \underline{n}_k \quad (2.16)$$

Let us define:

$$\dot{m}_k \equiv \rho_k (\underline{v}_I - \underline{v}_k) \cdot \underline{n}_k \quad (2.17)$$

The above quantity is the mass *gain* for phase  $k$ , per unit surface and unit time due to phase change (vaporization or condensation). It is expressed in  $\text{kg/m}^2\text{s}$ . It should be noted the sense we have adopted for the quantity  $\dot{m}_k$  which is a gain for phase  $k$ . Some authors (Delhay 1974a, b; Ishii 1975) prefer to define  $\dot{m}_k$  as the mass loss for phase  $k$ , hence their  $\dot{m}_k$  has the opposite sign in comparison to Eq. (2.17). Using Eq. (2.17), Eq. (2.16) becomes:

$$\sum_{k=1,2} \dot{m}_k = - \left( \frac{D_I \rho_I}{Dt} + \rho_I \nabla_s \cdot \underline{v}_I \right) \quad (2.18)$$

In the major part of applications, the RHS of Eq. (2.18) can be considered to be nil:

$$\sum_{k=1,2} \dot{m}_k = 0 \quad (2.19)$$

Equation (2.18) or its simplified form Eq. (2.19) is the *mass jump condition at the interfaces*.

## 2.4 Momentum Balance Equations

We consider again the two-phase material volume depicted in Fig. 2.2. The balance of (linear) momentum for this volume can be stated in the following way (Delhay 1974a, b). The time rate of change of the linear momentum contained in the volume

$V(t)$  is equal to the sum of the external forces acting on it. These external forces are constituted of:

- the gravity force  $\underline{g}$  which is a force per unit mass,
- the stresses in each phase through the surfaces  $A_1$  and  $A_2$ ,
- the line surface tension force through the curve  $C$ .

Mathematically, this is written as:

$$\begin{aligned} & \frac{d}{dt} \int_{V_1} \rho_1 \underline{v}_1 dv + \frac{d}{dt} \int_{V_2} \rho_2 \underline{v}_2 dv + \frac{d}{dt} \int_{A_1} \rho_1 \underline{v}_1 dS \\ &= \left( \int_{V_1} \rho_1 dv + \int_{V_2} \rho_2 dv + \int_{A_1} \rho_1 dS \right) \underline{g} \\ &+ \int_{A_1} \underline{n}_1 \cdot \underline{\underline{\sigma}}_1 dS + \int_{A_2} \underline{n}_2 \cdot \underline{\underline{\sigma}}_2 dS + \oint_C \sigma \underline{N} dC \end{aligned} \quad (2.20)$$

where  $\underline{\underline{\sigma}}_k$  is the stress tensor in phase  $k$  ( $k = 1, 2$ ). For an inviscid interface, the stress tensor in the interface reduces to the surface tension coefficient  $\sigma$  multiplied by the identity tensor for the surface  $\underline{\underline{I}} - \underline{n} \underline{n}$ . The surface tension in the surface is analogous to the pressure in the phases except for its sign. Proceeding in the same manner than with the mass balance, and using the first Gauss theorem (B.12) to transform the line integral of the surface tension force into a surface integral:

$$\int_{C(t)} \sigma \underline{N} dC = \int_{A_i(t)} (\nabla_s \sigma - \sigma \underline{n} \nabla_s \cdot \underline{n}) dS \quad (2.21)$$

we obtain the local momentum balance equation for a point located in phase  $k$ :

$$\frac{\partial \rho_k \underline{v}_k}{\partial t} + \nabla \cdot (\rho_k \underline{v}_k \underline{v}_k) = \nabla \cdot \underline{\underline{\sigma}}_k + \rho_k \underline{g} \quad (2.22)$$

and the momentum balance equation for a point located on the interface:

$$\frac{D_I \rho_I \underline{v}_I}{Dt} + \rho_I \underline{v}_I \nabla_s \cdot \underline{v}_I - \rho_I \underline{g} - \nabla_s \sigma + \sigma \underline{n} \nabla_s \cdot \underline{n} = - \sum_{k=1,2} \left( \dot{m}_k \underline{v}_k + \underline{\underline{\sigma}}_k \cdot \underline{n}_k \right) \quad (2.23)$$

In the above equation,  $\nabla_s \cdot \underline{n}$  is twice the mean curvature of the surface (Aris 1962). For this very particular vector, the surface divergence  $\nabla_s \cdot \underline{n}$  can be replaced by the usual divergence calculated on the surface  $\nabla \cdot \underline{n}$  since we have (Nadim 1996):

$$\nabla_s \cdot \underline{n} = \left( \underline{I} - \underline{n} \underline{n} \right) : \underline{\nabla} \underline{n} = \underline{I} : \underline{\nabla} \underline{n} = \nabla \cdot \underline{n} \quad \text{since} \quad \underline{n} \underline{n} : \underline{\nabla} \underline{n} = n_i n_j \frac{\partial n_j}{\partial x_i} = 0 \quad (2.24)$$

The  $\nabla_s \sigma$  is a tangential stress due to the variations of the surface tension coefficient along the interface. It is called the Marangoni effect.

Equation (2.22) is nothing but the momentum equation for a pure fluid (Eq. A.8). The stress tensor  $\underline{\underline{\sigma}}_k$  can be decomposed into a pressure and a viscous stress tensor according to:

$$\underline{\underline{\sigma}}_k = -p_k \underline{I} + \underline{\underline{\tau}}_k \quad (2.25)$$

Substituting Eq. (2.25) into Eq. (2.22), the following form of the momentum equation is obtained:

$$\frac{\partial \rho_k \underline{v}_k}{\partial t} + \nabla \cdot (\rho_k \underline{v}_k \underline{v}_k) = -\nabla p_k + \nabla \cdot \underline{\underline{\tau}}_k + \rho_k \underline{g} \quad (2.26)$$

Multiplying the mass balance equation (2.15) by the velocity field and subtracting the resulting equation from Eq. (2.26), the following *non conservative form* of the momentum equation is obtained:

$$\rho_k \frac{D_k \underline{v}_k}{Dt} = -\nabla p_k + \nabla \cdot \underline{\underline{\tau}}_k + \rho_k \underline{g} \quad (2.27)$$

In this equation, the following definition of the material derivative has been used:

$$\frac{D_k}{Dt} \equiv \frac{\partial}{\partial t} + \underline{v}_k \cdot \nabla \quad (2.28)$$

If we neglect the surface material properties, we must also neglect the surface tension (Delhaye 1974a, b). If all the surface material properties are neglected, Eq. (2.23) reduces to:

$$\sum_{k=1,2} \left( \dot{m}_k \underline{v}_k + \underline{\underline{\sigma}}_k \cdot \underline{n}_k \right) = 0 \quad (2.29)$$

Equation (2.29) is the *momentum jump condition* at the interface.

## 2.5 Energy Balance Equations

The first law of thermodynamics states that the time rate of change of the total energy (kinetic and internal energy) contained into the volume  $V(t)$  (Fig. 2.2) is equal to the sum of:



- The power of the external forces
- The heat fluxes entering the volume  $V$

Mathematically, this law reads:

$$\begin{aligned}
 & \frac{d}{dt} \int_{V_1} \rho_1 \left( e_1 + \frac{v_1^2}{2} \right) dv + \frac{d}{dt} \int_{V_2} \rho_2 \left( e_2 + \frac{v_2^2}{2} \right) dv + \frac{d}{dt} \int_{A_I} \rho_I \left( e_I + \frac{v_I^2}{2} \right) dS \\
 &= \left( \int_{V_1} \rho_1 \underline{v}_1 dv + \int_{V_2} \rho_2 \underline{v}_2 dv + \int_{A_I} \rho_I \underline{v}_I dS \right) \cdot \underline{g} \\
 &+ \int_{A_1} \underline{n}_1 \cdot \underline{\underline{\sigma}}_1 \cdot \underline{v}_1 dS + \int_{A_2} \underline{n}_2 \cdot \underline{\underline{\sigma}}_2 \cdot \underline{v}_2 dS + \oint_C \underline{\sigma}_{v_I} \cdot \underline{N} dC \\
 &- \int_{A_1} \underline{q}_1 \cdot \underline{n}_1 dS - \int_{A_2} \underline{q}_2 \cdot \underline{n}_2 dS - \int_C \underline{q}_I \cdot \underline{N} dC
 \end{aligned} \tag{2.30}$$

In the above equation,  $e_k$  ( $k = 1, 2$ ) is the internal energy per unit mass of phase  $k$  and  $e_I$  is the internal energy per unit mass of the interface. The vectors  $\underline{q}_k$  ( $k = 1, 2$ ) and  $\underline{q}_I$  represent the thermal energy flux through the surfaces  $A_k$  ( $k = 1, 2$ ) and through the line  $C$  respectively. Proceeding in the same manner than for the mass and momentum balances and using the second Gauss theorem (B.12) to transform the line integral of the heat flux into a surface integral:

$$\int_C \underline{q}_I \cdot \underline{N} dC = \int_{A_I} \nabla_s \cdot \underline{q}_I dS \tag{2.31}$$

we obtain the total energy balance equation for a point located in phase  $k$ :

$$\frac{\partial}{\partial t} \left[ \rho_k \left( e_k + \frac{v_k^2}{2} \right) \right] + \nabla \cdot \left[ \rho_k \left( e_k + \frac{v_k^2}{2} \right) \underline{v}_k \right] = \rho_k \underline{g} \cdot \underline{v}_k + \nabla \cdot \left( \underline{\underline{\sigma}}_k \cdot \underline{v}_k \right) - \nabla \cdot \underline{q}_k \tag{2.32}$$

and the total energy balance equation for a point located on the interface:

$$\begin{aligned}
 & \frac{D_I \rho_I \left( e_I + \frac{v_I^2}{2} \right)}{Dt} + \rho_I \left( e_I + \frac{v_I^2}{2} \right) \nabla_s \cdot \underline{v}_I - \rho_I \underline{v}_I \cdot \underline{g} + \nabla_s \cdot \underline{q}_I - \nabla_s \cdot (\underline{\sigma}_{v_I, I}) \\
 &= \sum_{k=1}^2 \left[ \underline{q}_k \cdot \underline{n}_k - \left( \underline{\underline{\sigma}}_k \cdot \underline{n}_k \right) \cdot \underline{v}_k - \dot{m}_k \left( e_k + \frac{v_k^2}{2} \right) \right]
 \end{aligned} \tag{2.33}$$

If we consider the simplified case of an interface without any material properties, the above equation reduces to:

$$\sum_{k=1}^2 \left[ \underline{q}_k \cdot \underline{n}_k - \left( \underline{\sigma}_k \cdot \underline{n}_k \right) \cdot \underline{v}_k - \dot{m}_k \left( e_k + \frac{v_k^2}{2} \right) \right] = 0 \quad (2.34)$$

Equation (2.34) is the jump condition for total energy at the interface.

The second law of thermodynamics is formulated in terms of an inequality since this second law is an evolution law. This inequality is transformed into an equation by introducing entropy generation terms which must be positive for an irreversible evolution or equal to zero for a reversible one. The second law states that the time rate of change of the entropy contained in the volume  $V(t)$  (Fig. 2.2) is equal to the sum of the entropy entering the volume  $V$  due to conduction and to the sources. Mathematically, this reads:

$$\begin{aligned} & \frac{d}{dt} \int_{V_1} \rho_1 s_1 dv + \frac{d}{dt} \int_{V_2} \rho_2 s_2 dv + \frac{d}{dt} \int_{A_I} \rho_I s_I dS + \int_{A_I} \frac{q_I}{T_I} \cdot \underline{n}_I dS \\ & + \int_{A_2} \frac{q_2}{T_2} \cdot \underline{n}_2 dS + \int_C \frac{q_I}{T_I} \cdot \underline{N} dC \\ & = \int_{V_1} \Delta_1 dv + \int_{V_2} \Delta_2 dv + \int_{A_I} \Delta_I dS \end{aligned} \quad (2.35)$$

In the above equation,  $s_k$  ( $k = 1, 2$ ) is the entropy per unit mass of phase  $k$  and  $s_I$  is the entropy per unit mass of the interface. The entropy sources in the two sub-volumes  $V_1$  and  $V_2$  and on the interface  $A_I$  are denoted  $\Delta_k$  ( $k = 1, 2$ ) and  $\Delta_I$  respectively. Proceeding in the same manner than for the mass, momentum and total energy balances, we obtain the local entropy equation for a point located in phase  $k$ :

$$\frac{\partial \rho_k s_k}{\partial t} + \nabla \cdot (\rho_k s_k \underline{v}_k) + \nabla \cdot \left( \frac{q_k}{T_k} \right) = \Delta_k \geq 0 \quad (2.36)$$

as well as the entropy equation for a point located on the interface:

$$\frac{D_I \rho_I s_I}{Dt} + \rho_I s_I \nabla_s \cdot \underline{v}_I + \nabla_s \cdot \left( \frac{q_I}{T_I} \right) + \sum_{k=1}^2 \left( \dot{m}_k s_k - \frac{q_k}{T_k} \cdot \underline{n}_k \right) = \Delta_I \geq 0 \quad (2.37)$$

Neglecting the interface material properties and assuming no entropy creation at the interface, the above equation reduces to:

$$\sum_{k=1}^2 \left( \dot{m}_k s_k - \frac{q_k}{T_k} \cdot \underline{n}_k \right) = 0 \quad (2.38)$$

Equation (2.38) is the jump condition for entropy at the interface.

Now we can develop several other forms of the energy equations from the ones we have previously presented. First of all, let us develop the *kinetic energy equation* which will be very useful in the study of turbulent flows. In order to obtain this equation, we make the dot product of the momentum balance equation (2.27) by the velocity field  $\underline{v}_k$ . The result is:

$$\rho_k \frac{D_k}{Dt} \left( \frac{v_k^2}{2} \right) = -\nabla \cdot (p_k \underline{v}_k) + p_k \nabla \cdot \underline{v}_k + \nabla \cdot (\underline{\tau}_k \cdot \underline{v}_k) - \underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k + \rho_k \underline{v}_k \cdot \underline{g} \quad (2.39)$$

which, by using the mass equation, can be rewritten into the following conservative form:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho_k \frac{v_k^2}{2} \right) + \nabla \cdot \left( \rho_k \frac{v_k^2}{2} \underline{v}_k \right) &= -\nabla \cdot (p_k \underline{v}_k) + p_k \nabla \cdot \underline{v}_k \\ &+ \nabla \cdot (\underline{\tau}_k \cdot \underline{v}_k) - \underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k + \rho_k \underline{v}_k \cdot \underline{g} \end{aligned} \quad (2.40)$$

Subtracting Eq. (2.40) from the total energy equation (2.32), the following *internal energy equation* is obtained:

$$\frac{\partial}{\partial t} (\rho_k e_k) + \nabla \cdot (\rho_k e_k \underline{v}_k) = -\nabla \cdot \underline{q}_k - p_k \nabla \cdot \underline{v}_k + \underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k \quad (2.41)$$

The two terms  $p_k \nabla \cdot \underline{v}_k$  and  $\underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k$  appear with opposite signs in the two Eqs. (2.40) and (2.41). This signifies that these two terms represent energy exchanges between internal and kinetic energies. The pressure exchange term  $p_k \nabla \cdot \underline{v}_k$  is due to compressibility and is reversible but the viscous exchange term  $\underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k$  is not, as we will see later. The second law of thermodynamics implies that this last term is necessary positive; hence it corresponds to a kinetic energy loss and to an internal energy gain.

Introducing the enthalpy defined by:

$$h_k \equiv e_k + \frac{p_k}{\rho_k} \quad (2.42)$$

the internal energy equation becomes:

$$\frac{\partial}{\partial t} (\rho_k h_k) + \nabla \cdot (\rho_k h_k \underline{v}_k) = -\nabla \cdot \underline{q}_k + \frac{D_k p_k}{Dt} + \underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k \quad (2.43)$$

The fundamental equation of state for a thermodynamically homogeneous fluid is given by a function relating the internal energy to the entropy and density (e.g. Ishii 1975):

$$e_k = e_k(s_k, \rho_k) \quad (2.44)$$

The temperature and the thermodynamic pressure are defined by:

$$T_k \equiv \frac{\partial e_k}{\partial s_k} \quad p_k \equiv -\frac{\partial e_k}{\partial (1/\rho_k)} \quad (2.45)$$

Into differential form, the fundamental equation of state becomes:

$$de_k = \frac{\partial e_k}{\partial s_k} ds_k + \frac{\partial e_k}{\partial \rho_k} d\rho_k = T_k ds_k + \frac{p_k}{\rho_k^2} d\rho_k \quad (2.46)$$

The above equation is called the *Gibbs equation*. We can also define the *Gibbs free energy* by the following equation:

$$g_k \equiv e_k - T_k s_k + \frac{p_k}{\rho_k} \quad (2.47)$$

The Gibbs relation (2.46) can be rewritten in terms of the corresponding material derivatives:

$$\frac{D_k e_k}{Dt} = T_k \frac{D_k s_k}{Dt} + \frac{p_k}{\rho_k^2} \frac{D_k \rho_k}{Dt} \quad (2.48)$$

Introducing Eq. (2.48) into Eq. (2.41) and using the mass balance equation (2.15), the following equation is obtained for the entropy:

$$\rho_k T_k \frac{D_k s_k}{Dt} = -\nabla \cdot \underline{q}_k + \underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k \quad (2.49)$$

which can be rewritten into the following conservative form:

$$\frac{\partial \rho_k s_k}{\partial t} + \nabla \cdot (\rho_k s_k \underline{v}_k) = -\frac{1}{T_k} \nabla \cdot \underline{q}_k + \frac{\underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k}{T_k} \quad (2.50)$$

The comparison of Eqs. (2.36) and (2.50) gives the following expression for the source term of entropy:

$$\Delta_k \equiv \frac{\underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k}{T_k} + \underline{q}_k \cdot \nabla \left( \frac{1}{T_k} \right) \geq 0 \quad (2.51)$$

In order to guarantee the positivity of  $\Delta_k$ , each of the two terms in the RHS of Eq. (2.51) should be positive. As the temperature (expressed in K) is a positive quantity, the positivity of the first term implies the positivity of the exchange term  $\underline{\tau}_k : \underline{\underline{\nabla}} \underline{v}_k$  [see our remark after Eq. (2.41)]. This term is called the *dissipation*

*function*, or simply *dissipation*. In order to examine the consequence of the second law of thermodynamics and to close the system of local instantaneous equations for phase  $k$ , we consider a *Newtonian-Stokes fluid*. For such a fluid, the viscous stress tensor is given by the following relation [Eqs. A.14 and A.17]:

$$\begin{aligned}\underline{\underline{\tau}}_k &= -\frac{2}{3}\mu_k \nabla \cdot \underline{\underline{v}}_k \underline{\underline{I}} + 2\mu_k \underline{\underline{D}}_k \\ \underline{\underline{D}}_k &\equiv \frac{1}{2} \left( \underline{\underline{\nabla}} \underline{\underline{v}}_k + \underline{\underline{\nabla}}^T \underline{\underline{v}}_k \right)\end{aligned}\quad (2.52)$$

where  $\mu_k$  is the dynamic viscosity and  $\underline{\underline{D}}_k$  is the deformation rate tensor. Most fluids obey the Fourier's law of heat conduction:

$$\underline{\underline{q}}_k = -\lambda_k \nabla T_k \quad (2.53)$$

where  $\lambda_k$  is the thermal conductivity. The second law of thermodynamics (2.51) implies the positivity of the two coefficients  $\lambda_k$  and  $\mu_k$ .

## 2.6 Two-Fluid Formulation

In this section, we give a presentation of the balance equations which will be useful in the derivation of the two-fluid model (Chap. 3). In order to do this, we multiply the balance equations written for phase  $k$  by the PIF  $\chi_k$  and we introduce  $\chi_k$  into the derivatives by using Eqs. (2.9) and (2.10). For example, the mass balance equation (2.15) becomes:

$$\frac{\partial \chi_k \rho_k}{\partial t} + \nabla \cdot (\chi_k \rho_k \underline{\underline{v}}_k) = \rho_k \frac{\partial \chi_k}{\partial t} + \rho_k \underline{\underline{v}}_k \cdot \nabla \chi_k \quad (2.54)$$

Using Eqs. (2.9) and (2.10) allows to transform the above equation into the following one:

$$\frac{\partial \chi_k \rho_k}{\partial t} + \nabla \cdot (\chi_k \rho_k \underline{\underline{v}}_k) = \rho_k (\underline{\underline{v}}_I - \underline{\underline{v}}_k) \cdot \underline{\underline{n}}_k \delta_I \equiv \dot{m}_k \delta_I \quad (2.55)$$

Doing the same procedure, the following momentum equation is obtained:

$$\begin{aligned}\frac{\partial \chi_k \rho_k \underline{\underline{v}}_k}{\partial t} + \nabla \cdot (\chi_k \rho_k \underline{\underline{v}}_k \underline{\underline{v}}_k) &= -\nabla (\chi_k p_k) + \nabla \cdot \left( \chi_k \underline{\underline{\tau}}_k \right) + \chi_k \rho_k \underline{\underline{g}} \\ &\quad + \dot{m}_k \underline{\underline{v}}_k \delta_I - p_k \underline{\underline{n}}_k \delta_I + \underline{\underline{\tau}}_k \cdot \underline{\underline{n}}_k \delta_I\end{aligned}\quad (2.56)$$

The total energy equation becomes:

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[ \chi_k \rho_k \left( e_k + \frac{v_k^2}{2} \right) \right] + \nabla \cdot \left[ \chi_k \rho_k \left( e_k + \frac{v_k^2}{2} \right) \underline{v}_k \right] \\
 &= -\nabla \cdot \left( \chi_k \underline{q}_k \right) - \nabla \cdot \left( \chi_k p_k \underline{v}_k \right) + \nabla \cdot \left( \chi_k \underline{\tau}_k \cdot \underline{v}_k \right) + \chi_k \rho_k \underline{v}_k \cdot \underline{g} \\
 &+ \dot{m}_k \left( e_k + \frac{v_k^2}{2} \right) \delta_I - \underline{q}_k \cdot \underline{n}_k \delta_I - p_k \underline{v}_k \cdot \underline{n}_k \delta_I + \underline{\tau}_k \cdot \underline{v}_k \cdot \underline{n}_k \delta_I
 \end{aligned} \quad (2.57)$$

and the entropy equation becomes:

$$\begin{aligned}
 & \frac{\partial \chi_k \rho_k s_k}{\partial t} + \nabla \cdot (\chi_k \rho_k s_k \underline{v}_k) = -\nabla \cdot \left( \chi_k \frac{\underline{q}_k}{T_k} \right) + \chi_k \Delta_k \\
 &+ \dot{m}_k s_k \delta_I - \frac{\underline{q}_k \cdot \underline{n}_k}{T_k} \delta_I
 \end{aligned} \quad (2.58)$$

The same operation can be done on the other forms of the energy equations. This is left as an exercise for the reader.

## 2.7 Single-Fluid Formulation

In some situations, it can be useful to sum the Eqs. (2.55)–(2.58) on the two phases to obtain a *single fluid formulation*. This is done for example in Direct Numerical Simulation models (e.g. Tryggvason et al. 2001; Toutant et al. 2009). In order to obtain the single fluid-formulation, we must introduce the following “mixture” quantities:

$$\begin{aligned}
 \rho &\equiv \sum_{k=1,2} \chi_k \rho_k & \rho \underline{v} &\equiv \sum_{k=1,2} \chi_k \rho_k \underline{v}_k \\
 \rho e &\equiv \sum_{k=1,2} \chi_k \rho_k e_k & \rho s &\equiv \sum_{k=1,2} \chi_k \rho_k s_k \\
 p &\equiv \sum_{k=1,2} \chi_k p_k & \underline{\tau} &\equiv \sum_{k=1,2} \chi_k \underline{\tau}_k \\
 \Delta &\equiv \sum_{k=1,2} \chi_k \Delta_k \quad \dots
 \end{aligned} \quad (2.59)$$

Now we can sum the Eqs. (2.55)–(2.58) on the two phases by using Eq. (2.11) as well as the jump conditions. We obtain for the mass balance:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad (2.60)$$

where we have assumed that there is no mass accumulation at the interface (Eq. 2.19). The momentum balance equation for the mixture reads:

$$\begin{aligned} \frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v} \underline{v}) &= -\nabla p + \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g} + \underline{F}_s \delta_I \\ \underline{F}_s \delta_I &\equiv \sum_{k=1,2} \left[ \dot{m}_k \underline{v}_k \delta_I - p_k \underline{n}_k \delta_I + \underline{\underline{\tau}}_k \cdot \underline{n}_k \delta_I \right] \end{aligned} \quad (2.61)$$

where  $\underline{F}_s \delta_I$  has a quite complicated expression which can be obtained from the momentum jump condition (2.23). However, in practical applications,  $\underline{F}_s \delta_I$  is often assumed to be given by the sole interfacial tension force:

$$\underline{F}_s \delta_I \cong (\nabla_s \sigma - \sigma \underline{n} \nabla_s \cdot \underline{n}) \delta_I \quad (2.62)$$

The total energy balance equation for the mixture is given by:

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \rho \left( e + \frac{v^2}{2} \right) \right] + \nabla \cdot \left[ \rho \left( e + \frac{v^2}{2} \right) \underline{v} \right] \\ = -\nabla \cdot \underline{q} - \nabla \cdot (\rho \underline{v}) + \nabla \cdot \left( \underline{\underline{\tau}} \cdot \underline{v} \right) + \rho \underline{v} \cdot \underline{g} + E_I \\ E_I \equiv \sum_{k=1,2} \left[ \dot{m}_k \left( e_k + \frac{v_k^2}{2} \right) \delta_I - \underline{q}_k \cdot \underline{n}_k \delta_I - p_k \underline{v}_k \cdot \underline{n}_k \delta_I + \underline{\underline{\tau}}_k \cdot \underline{v}_k \cdot \underline{n}_k \delta_I \right] \end{aligned} \quad (2.63)$$

where  $E_I$  is the interfacial source of mixture energy, which can be transformed according to the jump condition (2.33).

The entropy balance for the mixture is given by:

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \underline{v}) = -\nabla \cdot \left( \frac{\underline{q}}{T} \right) + \Delta + \sum_{k=1,2} \left( \dot{m}_k s_k \delta_I - \frac{\underline{q}_k \cdot \underline{n}_k}{T_k} \delta_I \right) \quad (2.64)$$

## 2.8 Balance Equations for a Whole Fluid Particle

We end this chapter by deriving the equations for a whole fluid particle. This particle can be a bubble or a droplet. The equations derived in this section will be useful for the derivation of the *hybrid two-fluid model* (Chap. 3). In this derivation, all the surface properties including surface tension will be neglected. Following Lhuillier et al. (2010), Zaepffel (2011) and Zaepffel et al. (2012), we define the following properties for a whole particle labeled with the index  $j$ :

$$\begin{aligned}
m_j(t) &\equiv \int_{V_j} \rho_d dv \\
m_j(t) \underline{w}_j(t) &\equiv \int_{V_j} \rho_d \underline{v}_d dv \\
m_j(t) E_j(t) &\equiv \int_{V_j} \rho_d \left( e_d + \frac{v_d^2}{2} \right) dv \\
m_j(t) s_j(t) &\equiv \int_{V_j} \rho_d s_d dv
\end{aligned} \tag{2.65}$$

In these definitions, the index  $d$  denotes the disperse phase material. The particle mass, center of mass velocity, total energy and entropy are denoted by  $m_j(t)$ ,  $\underline{w}_j(t)$ ,  $E_j(t)$  and  $s_j(t)$  respectively. The volume of the particle is denoted by  $V_j$  and its surface area by  $S_j$ . The equations governing the quantities (2.65) can be obtained by integration of Eqs. (2.15), (2.22), (2.32) and (2.36) written for the disperse phase ( $k = d$ ) on the whole particle volume. For example, for the mass balance equation, using the Leibniz rule (B.8), we obtain:

$$\frac{dm_j}{dt} = \frac{d}{dt} \int_{V_j} \rho_d dv = \int_{V_j} \frac{\partial \rho_d}{\partial t} dv + \oint_{S_j} \rho_d \underline{v}_d \cdot \underline{n}_d dS + \oint_{S_j} \rho_d (\underline{v}_I - \underline{v}_d) \cdot \underline{n}_d dS \tag{2.66}$$

Using the Gauss theorem (B.9) on the second term, the mass balance equation (2.15) and the definition (2.17), the above equation reduces to:

$$\frac{dm_j}{dt} = \oint_{S_j} \dot{m}_d dS \tag{2.67}$$

Equation (2.67) states that the time rate of change of the particle mass is only due to phase change (vaporization or condensation).

The time derivative of the momentum gives, using the Leibniz rule:

$$\begin{aligned}
\frac{dm_j \underline{w}_j}{dt} &= \frac{d}{dt} \int_{V_j} \rho_d \underline{v}_d dv \\
&= \int_{V_j} \frac{\partial \rho_d \underline{v}_d}{\partial t} dv + \oint_{S_j} \rho_d \underline{v}_d \underline{v}_d \cdot \underline{n}_d dS + \oint_{S_j} \rho_d \underline{v}_d (\underline{v}_I - \underline{v}_d) \cdot \underline{n}_d dS
\end{aligned} \tag{2.68}$$



Using the Gauss theorem (B.9) on the second term, the momentum balance equation (2.22) and the definition (2.17), the above equation becomes:

$$\frac{dm_j \underline{w}_j}{dt} = m_j \underline{g} + \oint_{S_j} \left( \dot{m}_d \underline{v}_d + \underline{\underline{\sigma}}_d \cdot \underline{n}_d \right) dS \quad (2.69)$$

Now using the simplified momentum jump condition (2.29) together with the mass jump condition (2.19) and the fact that (Eq. 2.4):

$$\underline{n}_c = -\underline{n}_d \quad (2.70)$$

where the index  $k = c$  denotes the continuous phase, Eq. (2.69) can be rewritten:

$$\frac{dm_j \underline{w}_j}{dt} = m_j \underline{g} + \oint_{S_j} \left( \dot{m}_d \underline{v}_c + \underline{\underline{\sigma}}_c \cdot \underline{n}_d \right) dS \quad (2.71)$$

Equation (2.71) enlighten the action of the continuous phase on the disperse one through  $\underline{\underline{\sigma}}_c \cdot \underline{n}_d$ .

Proceeding in the same manner, and by using the simplified total energy jump condition (2.34), we obtain the following total energy equation for a whole particle:

$$\frac{dm_j E_j}{dt} = m_j \underline{w}_j \cdot \underline{g} + \oint_{S_j} \left[ \dot{m}_d \left( e_c + \frac{v_c^2}{2} \right) - \underline{q}_c \cdot \underline{n}_d + \underline{\underline{\sigma}}_c \cdot \underline{v}_c \cdot \underline{n}_d \right] dS \quad (2.72)$$

In the same manner, we obtain the following entropy equation for the whole particle:

$$\frac{dm_j s_j}{dt} = \Delta_j + \oint_{S_j} \left( \dot{m}_d s_c - \frac{\underline{q}_c}{T_c} \cdot \underline{n}_d \right) dS \quad (2.73)$$

where the entropy source for the whole particle volume is defined by:

$$\Delta_j(t) \equiv \int_{V_j} \Delta_d dv \quad (2.74)$$

Now, in order to prepare the derivation of the hybrid two-fluid model, we will introduce a function of presence of the particles centers. If  $\underline{X}_j(t)$  is the position vector of the  $j$ th particle center, a Dirac delta function having  $\underline{X}_j(t)$  as its support can be defined in the following manner (Eq. B.7):

$$\delta_j(\underline{x}, t) \equiv \delta(\underline{x} - \underline{X}_j(t)) = \delta(x_1 - X_{j,1}) \delta(x_2 - X_{j,2}) \delta(x_3 - X_{j,3}) \quad (2.75)$$

This Dirac generalized function makes the link between the Lagrangian description, where the particle is identified by the position vector  $\underline{X}_j(t)$  and the Eulerian description characterized by the fixed position  $\underline{x}$ . The generalized function defined by Eq. (2.75) follows the particle at the velocity  $\underline{w}_j(t)$ , which depends on time only, hence we can write:

$$\frac{\partial \delta_j}{\partial t} + \nabla \cdot (\delta_j \underline{w}_j) = 0 \quad (2.76)$$

Now, assuming that the total number of particles in the flow is  $N$ , we can sum the Dirac delta function (2.75) on the  $N$  particles:

$$\delta_d(\underline{x}, t) \equiv \sum_{j=1}^N \delta_j(\underline{x}, t) \quad (2.77)$$

The above quantity gives the distribution of the particle centers in space at a given time  $t$ . Now, we will take each of Eqs. (2.67), (2.71)–(2.73), multiply it by the generalized function  $\delta_j$ , use Eq. (2.76) and sum the resulting equations on the  $N$  particles. We thus obtain the following set of equations:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \sum_{j=1}^N \delta_j m_j \right) + \nabla \cdot \left( \sum_{j=1}^N \delta_j m_j \underline{w}_j \right) &= \sum_{j=1}^N \delta_j \oint_{S_j} \dot{m}_d dS \\ \frac{\partial}{\partial t} \left( \sum_{j=1}^N \delta_j m_j \underline{w}_j \right) + \nabla \cdot \left( \sum_{j=1}^N \delta_j m_j \underline{w}_j \underline{w}_j \right) &= \sum_{j=1}^N \delta_j \left( m_j \underline{g} + \oint_{S_j} (\dot{m}_d \underline{v}_c + \underline{\sigma}_c \cdot \underline{n}_d) dS \right) \\ \frac{\partial}{\partial t} \left( \sum_{j=1}^N \delta_j m_j E_j \right) + \nabla \cdot \left( \sum_{j=1}^N \delta_j m_j E_j \underline{w}_j \right) &= \sum_{j=1}^N \delta_j \left( m_j \underline{w}_j \cdot \underline{g} \right. \\ &\quad \left. + \oint_{S_j} \left[ \dot{m}_d \left( \underline{e}_c + \frac{\underline{v}_c^2}{2} \right) - \underline{q}_c \cdot \underline{n}_d + \underline{\sigma}_c \cdot \underline{v}_c \cdot \underline{n}_d \right] dS \right) \\ \frac{\partial}{\partial t} \left( \sum_{j=1}^N \delta_j m_j s_j \right) + \nabla \cdot \left( \sum_{j=1}^N \delta_j m_j s_j \underline{w}_j \right) &= \sum_{j=1}^N \delta_j \Delta_j + \sum_{j=1}^N \delta_j \oint_{S_j} \left( \dot{m}_d s_c - \frac{q_c}{T_c} \cdot \underline{n}_d \right) dS \end{aligned} \quad (2.78)$$

Introducing the following simplified notations:

$$\delta_d m = \sum_{j=1}^N \delta_j m_j \quad \delta_d m \underline{w} = \sum_{j=1}^N \delta_j m_j \underline{w}_j \quad \delta_d m E = \sum_{j=1}^N \delta_j m_j E_j \quad \dots \quad (2.79)$$

Equation (2.78) can be rewritten into the following ones:

$$\begin{aligned}
 \frac{\partial}{\partial t}(\delta_d \underline{m}) + \nabla \cdot (\delta_d \underline{m} \underline{w}) &= \delta_d \oint_S \dot{m}_d dS \\
 \frac{\partial}{\partial t}(\delta_d \underline{m} \underline{w}) + \nabla \cdot (\delta_d \underline{m} \underline{w} \underline{w}) &= \delta_d \underline{m} \underline{g} + \delta_d \oint_S \dot{m}_d \underline{v}_c dS + \delta_d \oint_S \underline{\underline{c}}_c \cdot \underline{n}_d dS \\
 \frac{\partial}{\partial t}(\delta_d \underline{m} \underline{E}) + \nabla \cdot (\delta_d \underline{m} \underline{E} \underline{w}) &= \delta_d \underline{m} \underline{w} \cdot \underline{g} + \delta_d \oint_S \dot{m}_d \left( \underline{e}_c + \frac{v_c^2}{2} \right) dS \\
 &\quad - \delta_d \oint_S \underline{q}_c \cdot \underline{n}_d dS + \delta_d \oint_S \underline{\underline{c}}_c \cdot \underline{v}_c \cdot \underline{n}_d dS \\
 \frac{\partial}{\partial t}(\delta_d \underline{m} \underline{s}) + \nabla \cdot (\delta_d \underline{m} \underline{s} \underline{w}) &= \delta_d \Delta_d + \delta_d \oint_S \left( \dot{m}_d s_c - \frac{q_c}{T_c} \cdot \underline{n}_d \right) dS
 \end{aligned} \tag{2.80}$$

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