Chapter 2
Review of Classical Action Principles

This section grew out of lectures given by Schwinger at UCLA around 1974, which were substantially transformed into Chap. 8 of *Classical Electrodynamics* (Schwinger 1998). (Remarkably, considering his work on waveguide theory during World War II, now partially recorded in Ref. (Milton 2006), he never gave lectures on this subject at Harvard after 1947.)

We start by reviewing and generalizing the Lagrange-Hamilton principle for a single particle. The action, W_{12}, is defined as the time integral of the Lagrangian, L, where the integration extends from an initial configuration or state at time t_2 to a final state at time t_1:

$$W_{12} = \int_{t_2}^{t_1} dt\, L. \tag{2.1}$$

The integral refers to any path, any line of time development, from the initial to the final state, as shown in Fig. 2.1. The actual time evolution of the system is selected by the principle of stationary action: In response to infinitesimal variations of the integration path, the action W_{12} is stationary—does not have a corresponding infinitesimal change—for variations about the correct path, provided the initial and final configurations are held fixed,

$$\delta W_{12} = 0. \tag{2.2}$$

This means that, if we allow infinitesimal changes at the initial and final times, including alterations of those times, the only contribution to δW_{12} then comes from the endpoint variations, or

$$\delta W_{12} = G_1 - G_2, \tag{2.3}$$

where G_a, $a = 1$ or 2, is a function, called the generator, depending on dynamical variables only at time t_a. In the following, we will consider three different realizations

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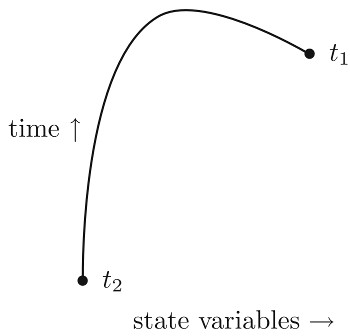

Fig. 2.1 A possible path from initial state to final state

of the action principle, where, for simplicity, we will restrict our attention to a single particle.

2.1 Lagrangian Viewpoint

The nonrelativistic motion of a particle of mass m moving in a potential $V(\mathbf{r}, t)$ is described by the Lagrangian

$$L = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 - V(\mathbf{r}, t). \tag{2.4}$$

Here, the independent variables are $\mathbf{r}$ and t, so that two kinds of variations can be considered. First, a particular motion is altered infinitesimally, that is, the path is changed by an amount $\delta\mathbf{r}$:

$$\mathbf{r}(t) \to \mathbf{r}(t) + \delta\mathbf{r}(t). \tag{2.5}$$

Second, the final and initial times can be altered infinitesimally, by δt_1 and δt_2, respectively. It is more convenient, however, to think of these time displacements as produced by a continuous variation of the time parameter, $\delta t(t)$,

$$t \to t + \delta t(t), \tag{2.6}$$

so chosen that, at the endpoints,

$$\delta t(t_1) = \delta t_1, \qquad \delta t(t_2) = \delta t_2. \tag{2.7}$$

The corresponding change in the time differential is

$$dt \to d(t + \delta t) = \left(1 + \frac{d\delta t}{dt}\right) dt, \tag{2.8}$$

which implies the transformation of the time derivative,

$$\frac{d}{dt} \to \left(1 - \frac{d\delta t}{dt}\right)\frac{d}{dt}. \tag{2.9}$$

Because of this redefinition of the time variable, the limits of integration in the action,

$$W_{12} = \int_2^1 \left[\frac{1}{2}m\frac{(d\mathbf{r})^2}{dt} - dt\, V\right], \tag{2.10}$$

are *not* changed, the time displacement being produced through $\delta t(t)$ subject to (2.7). The resulting variation in the action is now

$$\begin{aligned}
\delta W_{12} &= \int_2^1 dt \left\{ m\frac{d\mathbf{r}}{dt} \cdot \frac{d}{dt}\delta\mathbf{r} - \delta\mathbf{r}\cdot\nabla V - \frac{d\delta t}{dt}\left[\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right] - \delta t\frac{\partial}{\partial t}V \right\} \\
&= \int_2^1 dt \left\{ \frac{d}{dt}\left[m\frac{d\mathbf{r}}{dt}\cdot\delta\mathbf{r} - \left(\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right)\delta t\right] \right. \\
&\quad \left. + \delta\mathbf{r}\cdot\left[-m\frac{d^2}{dt^2}\mathbf{r} - \nabla V\right] + \delta t\left(\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right] - \frac{\partial}{\partial t}V\right)\right\},
\end{aligned} \tag{2.11}$$

where, in the last form, we have integrated by parts in order to isolate $\delta\mathbf{r}$ and δt.

Because $\delta\mathbf{r}$ and δt are independent variations, the principle of stationary action implies that the actual motion is governed by

$$m\frac{d^2}{dt^2}\mathbf{r} = -\nabla V, \tag{2.12a}$$

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right] = \frac{\partial}{\partial t}V, \tag{2.12b}$$

while the total time derivative gives the change at the endpoints,

$$G = \mathbf{p}\cdot\delta\mathbf{r} - E\delta t, \tag{2.12c}$$

with

$$\text{momentum} = \mathbf{p} = m\frac{d\mathbf{r}}{dt}, \qquad \text{energy} = E = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V. \tag{2.12d}$$

Therefore, we have derived Newton's second law [the equation of motion in second-order form], (2.12a), and, for a static potential, $\partial V/\partial t = 0$, the conservation of energy, (2.12b). The significance of (2.12c) will be discussed later in Sect. 2.4.

2.2 Hamiltonian Viewpoint

Using the above definition of the momentum, we can rewrite the Lagrangian as

$$L = \mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - H(\mathbf{r}, \mathbf{p}, t), \tag{2.13}$$

where we have introduced the Hamiltonian

$$H = \frac{p^2}{2m} + V(\mathbf{r}, t). \tag{2.14}$$

We are here to regard $\mathbf{r}$, $\mathbf{p}$, and t as independent variables in

$$W_{12} = \int_2^1 [\mathbf{p} \cdot d\mathbf{r} - dt\, H]. \tag{2.15}$$

The change in the action, when $\mathbf{r}$, $\mathbf{p}$, and t are all varied, is

$$\begin{aligned}
\delta W_{12} &= \int_2^1 dt \left[\mathbf{p} \cdot \frac{d}{dt}\delta\mathbf{r} - \delta\mathbf{r} \cdot \frac{\partial H}{\partial \mathbf{r}} + \delta\mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - \delta\mathbf{p} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{d\delta t}{dt} H - \delta t \frac{\partial H}{\partial t} \right] \\
&= \int_2^1 dt \left[\frac{d}{dt}(\mathbf{p} \cdot \delta\mathbf{r} - H\delta t) + \delta\mathbf{r} \cdot \left(-\frac{d\mathbf{p}}{dt} - \frac{\partial H}{\partial \mathbf{r}} \right) \right. \\
&\qquad \left. + \delta\mathbf{p} \cdot \left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}} \right) + \delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right].
\end{aligned} \tag{2.16}$$

The action principle then implies

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \tag{2.17a}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = -\boldsymbol{\nabla} V, \tag{2.17b}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{2.17c}$$

$$G = \mathbf{p} \cdot \delta\mathbf{r} - H\delta t. \tag{2.17d}$$

In contrast with the Lagrangian differential equations of motion, which involve second derivatives, these Hamiltonian equations contain only first derivatives; they are called first-order equations. They describe the same physical system, because when (2.17a) is substituted into (2.17b), we recover the Lagrangian-Newtonian equation (2.12a). Furthermore, if we insert (2.17a) into the Hamiltonian (2.14), we identify H with E. The third equation (2.17c) is then identical with (2.12b). We also note the equivalence of the two versions of G.

But probably the most direct way of seeing that the same physical system is involved comes by writing the Lagrangian in the Hamiltonian viewpoint as

$$L = \frac{m}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - V - \frac{1}{2m}\left(\mathbf{p} - m\frac{d\mathbf{r}}{dt}\right)^2. \tag{2.18}$$

The result of varying $\mathbf{p}$ in the stationary action principle is to produce

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt}. \tag{2.19}$$

But, if we accept this as the *definition* of $\mathbf{p}$, the corresponding term in L disappears and we explicitly regain the Lagrangian description. We are justified in completely omitting the last term on the right side of (2.18), despite its dependence on the variables $\mathbf{r}$ and t, because of its quadratic structure. Its explicit contribution to δL is

$$-\frac{1}{m}\left(\mathbf{p} - m\frac{d\mathbf{r}}{dt}\right)\cdot\left(\delta\mathbf{p} - m\frac{d}{dt}\delta\mathbf{r} + m\frac{d\mathbf{r}}{dt}\frac{d\delta t}{dt}\right), \tag{2.20}$$

and the equation supplied by the stationary action principle for $\mathbf{p}$ variations, (2.19), also guarantees that there is no contribution here to the results of $\mathbf{r}$ and t variations.

2.3 A Third, Schwingerian, Viewpoint

Here we take $\mathbf{r}$, $\mathbf{p}$, and the velocity, $\mathbf{v}$, as independent variables, so that the Lagrangian is written in the form

$$L = \mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \mathbf{v}\right) + \frac{1}{2}mv^2 - V(\mathbf{r}, t) \equiv \mathbf{p}\cdot\frac{d\mathbf{r}}{dt} - H(\mathbf{r}, \mathbf{p}, \mathbf{v}, t), \tag{2.21}$$

where

$$H(\mathbf{r}, \mathbf{p}, \mathbf{v}, t) = \mathbf{p}\cdot\mathbf{v} - \frac{1}{2}mv^2 + V(\mathbf{r}, t). \tag{2.22}$$

The variation of the action is now

$$\begin{aligned}\delta W_{12} =& \delta\int_2^1 [\mathbf{p}\cdot d\mathbf{r} - H\,dt] \\ =& \int_2^1 dt\left[\delta\mathbf{p}\cdot\frac{d\mathbf{r}}{dt} + \mathbf{p}\cdot\frac{d}{dt}\delta\mathbf{r} - \delta\mathbf{r}\cdot\frac{\partial H}{\partial\mathbf{r}} - \delta\mathbf{p}\cdot\frac{\partial H}{\partial\mathbf{p}} - \delta\mathbf{v}\cdot\frac{\partial H}{\partial\mathbf{v}}\right. \\ &\left. - \delta t\frac{\partial H}{\partial t} - H\frac{d\delta t}{dt}\right]\end{aligned}$$

$$= \int_2^1 dt \left[\frac{d}{dt}(\mathbf{p} \cdot \delta\mathbf{r} - H\delta t) - \delta\mathbf{r} \cdot \left(\frac{d\mathbf{p}}{dt} + \frac{\partial H}{\partial \mathbf{r}} \right) \right.$$
$$\left. + \delta\mathbf{p} \cdot \left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}} \right) - \delta\mathbf{v} \cdot \frac{\partial H}{\partial \mathbf{v}} + \delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right], \tag{2.23}$$

so that the action principle implies

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = -\boldsymbol{\nabla} V, \tag{2.24a}$$
$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}, \tag{2.24b}$$
$$\mathbf{0} = -\frac{\partial H}{\partial \mathbf{v}} = -\mathbf{p} + m\mathbf{v}, \tag{2.24c}$$
$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{2.24d}$$
$$G = \mathbf{p} \cdot \delta\mathbf{r} - H\delta t. \tag{2.24e}$$

Notice that there is no equation of motion for $\mathbf{v}$ since $d\mathbf{v}/dt$ does not occur in the Lagrangian, nor is it multiplied by a time derivative. Consequently, (2.24c) refers to a single time and is an equation of constraint.

From this third approach, we have the option of returning to either of the other two viewpoints by imposing an appropriate restriction. Thus, if we write (2.22) as

$$H(\mathbf{r}, \mathbf{p}, \mathbf{v}, t) = \frac{p^2}{2m} + V(\mathbf{r}, t) - \frac{1}{2m}(\mathbf{p} - m\mathbf{v})^2, \tag{2.25}$$

and we adopt

$$\mathbf{v} = \frac{1}{m}\mathbf{p} \tag{2.26}$$

as the *definition* of $\mathbf{v}$, we recover the Hamiltonian description, (2.13) and (2.14). Alternatively, we can present the Lagrangian (2.21) as

$$L = \frac{m}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - V + (\mathbf{p} - m\mathbf{v}) \cdot \left(\frac{d\mathbf{r}}{dt} - \mathbf{v}\right) - \frac{m}{2}\left(\frac{d\mathbf{r}}{dt} - \mathbf{v}\right)^2. \tag{2.27}$$

Then, if we adopt the following as *definitions*,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{p} = m\mathbf{v}, \tag{2.28}$$

the resultant form of L is that of the Lagrangian viewpoint, (2.4). It might seem that only the definition $\mathbf{v} = d\mathbf{r}/dt$, inserted in (2.27), suffices to regain the Lagrangian description. But then the next to last term in (2.27) would give the following additional contribution to δL, associated with the variation $\delta\mathbf{r}$:

$$(\mathbf{p} - m\mathbf{v}) \cdot \frac{d}{dt}\delta\mathbf{r}. \tag{2.29}$$

In the next Chapter, where the action formulation of electrodynamics is considered, we will see the advantage of adopting this third approach, which is characterized by the introduction of additional variables, similar to $\mathbf{v}$, for which there are no equations of motion.

2.4 Invariance and Conservation Laws

There is more content to the principle of stationary action than equations of motion. Suppose one considers a variation such that

$$\delta W_{12} = 0, \tag{2.30}$$

independently of the choice of initial and final times. We say that the action, which is left unchanged, is *invariant* under this alteration of path. Then the stationary action principle (2.3) asserts that

$$\delta W_{12} = G_1 - G_2 = 0, \tag{2.31}$$

or, there is a quantity $G(t)$ that has the same value for any choice of time t; it is conserved in time. A differential statement of that is

$$\frac{d}{dt}G(t) = 0. \tag{2.32}$$

The G functions, which are usually referred to as generators, express the interrelation between conservation laws and invariances of the system.

Invariance implies conservation, and vice versa. A more precise statement is the following:

> If there is a conservation law, the action is stationary under an infinitesimal transformation in an appropriate variable.

The converse of this statement is also true.

> If the action W is invariant under an infinitesimal transformation (that is, $\delta W = 0$), then there is a corresponding conservation law.

This is the celebrated theorem of Amalie Emmy Noether (Noether 1918).

Here are some examples. Suppose the Hamiltonian of (2.13) does not depend explicitly on time, or

$$W_{12} = \int_2^1 [\mathbf{p} \cdot d\mathbf{r} - H(\mathbf{r}, \mathbf{p})dt]. \tag{2.33}$$

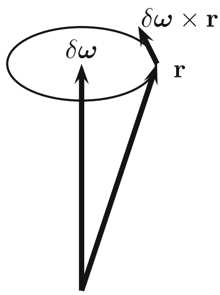

Fig. 2.2 $\delta\boldsymbol{\omega} \times \mathbf{r}$ is perpendicular to $\delta\boldsymbol{\omega}$ and $\mathbf{r}$, and represents an infinitesimal rotation of $\mathbf{r}$ about the $\delta\boldsymbol{\omega}$ axis

Then the variation (which as a rigid displacement in time, amounts to a shift in the time origin)

$$\delta t = \text{constant} \tag{2.34}$$

will give $\delta W_{12} = 0$ [see the first line of (2.16), with $\delta\mathbf{r} = 0$, $\delta\mathbf{p} = 0$, $d\delta t/dt = 0$, $\partial H/\partial t = 0$]. The conclusion is that G in (2.17d), which here is just

$$G_t = -H\delta t, \tag{2.35}$$

is a conserved quantity, or that

$$\frac{dH}{dt} = 0. \tag{2.36}$$

This inference, that the Hamiltonian—the energy—is conserved, if there is no explicit time dependence in H, is already present in (2.17c). But now a more general principle is at work.

Next, consider an infinitesimal, rigid rotation, one that maintains the lengths and scalar products of all vectors. Written explicitly for the position vector $\mathbf{r}$, it is

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \tag{2.37}$$

where the constant vector $\delta\boldsymbol{\omega}$ gives the direction and magnitude of the rotation (see Fig. 2.2). Now specialize (2.14) to

$$H = \frac{p^2}{2m} + V(r), \tag{2.38}$$

where $r = |\mathbf{r}|$, a rotationally invariant structure. Then

$$W_{12} = \int_2^1 [\mathbf{p} \cdot d\mathbf{r} - H\, dt] \tag{2.39}$$

is also invariant under the rigid rotation, implying the conservation of

$$G_{\delta\omega} = \mathbf{p} \cdot \delta\mathbf{r} = \delta\boldsymbol{\omega} \cdot \mathbf{r} \times \mathbf{p}. \tag{2.40}$$

This is the conservation of angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \frac{d}{dt}\mathbf{L} = \mathbf{0}. \tag{2.41}$$

Of course, this is also contained within the equation of motion,

$$\frac{d}{dt}\mathbf{L} = -\mathbf{r} \times \nabla V = -\mathbf{r} \times \hat{\mathbf{r}}\frac{\partial V}{\partial r} = \mathbf{0}, \tag{2.42}$$

since V depends only on $|\mathbf{r}|$.

Conservation of linear momentum appears analogously when there is invariance under a rigid translation. For a single particle, (2.17b) tells us immediately that $\mathbf{p}$ is conserved if V is a constant, say zero. Then, indeed, the action

$$W_{12} = \int_2^1 \left[\mathbf{p} \cdot d\mathbf{r} - \frac{p^2}{2m}dt\right] \tag{2.43}$$

is invariant under the displacement

$$\delta\mathbf{r} = \delta\boldsymbol{\epsilon} = \text{constant}, \tag{2.44}$$

and

$$G_{\delta} = \mathbf{p} \cdot \delta\boldsymbol{\epsilon} \tag{2.45}$$

is conserved. But the general principle acts just as easily for, say, a system of two particles, a and b, with Hamiltonian

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + V(\mathbf{r}_a - \mathbf{r}_b). \tag{2.46}$$

This Hamiltonian and the associated action

$$W_{12} = \int_2^1 [\mathbf{p}_a \cdot d\mathbf{r}_a + \mathbf{p}_b \cdot d\mathbf{r}_b - H\,dt] \tag{2.47}$$

are invariant under the rigid translation

$$\delta\mathbf{r}_a = \delta\mathbf{r}_b = \delta\boldsymbol{\epsilon}, \tag{2.48}$$

with the implication that

$$G_{\delta\epsilon} = \mathbf{p}_a \cdot \delta\mathbf{r}_a + \mathbf{p}_b \cdot \delta\mathbf{r}_b = (\mathbf{p}_a + \mathbf{p}_b) \cdot \delta\boldsymbol{\epsilon} \tag{2.49}$$

is conserved. This is the conservation of the total linear momentum,

$$\mathbf{P} = \mathbf{p}_a + \mathbf{p}_b, \quad \frac{d}{dt}\mathbf{P} = \mathbf{0}. \tag{2.50}$$

Something a bit more general appears when we consider a rigid translation that grows linearly in time:

$$\delta\mathbf{r}_a = \delta\mathbf{r}_b = \delta\mathbf{v}\, t, \tag{2.51}$$

using the example of two particles. This gives each particle the common additional velocity $\delta\mathbf{v}$, and therefore must also change their momenta,

$$\delta\mathbf{p}_a = m_a\delta\mathbf{v}, \quad \delta\mathbf{p}_b = m_b\delta\mathbf{v}. \tag{2.52}$$

The response of the action (2.47) to this variation is

$$\begin{aligned}\delta W_{12} &= \int_2^1 [(\mathbf{p}_a + \mathbf{p}_b) \cdot \delta\mathbf{v}\, dt + \delta\mathbf{v} \cdot (m_a d\mathbf{r}_a + m_b d\mathbf{r}_b) - (\mathbf{p}_a + \mathbf{p}_b) \cdot \delta\mathbf{v}\, dt] \\ &= \int_2^1 d[(m_a\mathbf{r}_a + m_b\mathbf{r}_b) \cdot \delta\mathbf{v}]. \end{aligned} \tag{2.53}$$

The action is *not* invariant; its variation has end-point contributions. But there is still a conservation law, not of $G = \mathbf{P} \cdot \delta\mathbf{v}t$, but of $\mathbf{N} \cdot \delta\mathbf{v}$, where

$$\mathbf{N} = \mathbf{P}t - (m_a\mathbf{r}_a + m_b\mathbf{r}_b). \tag{2.54}$$

Written in terms of the center-of-mass position vector

$$\mathbf{R} = \frac{m_a\mathbf{r}_a + m_b\mathbf{r}_b}{M}, \quad M = m_a + m_b, \tag{2.55}$$

the statement of conservation of

$$\mathbf{N} = \mathbf{P}t - M\mathbf{R}, \tag{2.56}$$

namely

$$\mathbf{0} = \frac{d\mathbf{N}}{dt} = \mathbf{P} - M\frac{d\mathbf{R}}{dt}, \tag{2.57}$$

is the familiar fact that the center of mass of an isolated system moves at the constant velocity given by the ratio of the total momentum to the total mass of that system.

2.5 Nonconservation Laws: The Virial Theorem

The action principle also supplies useful nonconservation laws. Consider, for constant $\delta\lambda$,

$$\delta\mathbf{r} = \delta\lambda\mathbf{r}, \quad \delta\mathbf{p} = -\delta\lambda\mathbf{p}, \tag{2.58}$$

which leaves $\mathbf{p}\cdot d\mathbf{r}$ invariant,

$$\delta(\mathbf{p}\cdot d\mathbf{r}) = (-\delta\lambda\mathbf{p})\cdot d\mathbf{r} + \mathbf{p}\cdot(\delta\lambda d\mathbf{r}) = 0. \tag{2.59}$$

But the response of the Hamiltonian

$$H = T(p) + V(\mathbf{r}), \quad T(p) = \frac{p^2}{2m}, \tag{2.60}$$

is given by the noninvariant form

$$\delta H = \delta\lambda(-2T + \mathbf{r}\cdot\boldsymbol{\nabla}V). \tag{2.61}$$

Therefore we have, for an arbitrary time interval, for the variation of the action (2.15),

$$\delta W_{12} = \int_2^1 dt[\delta\lambda(2T - \mathbf{r}\cdot\boldsymbol{\nabla}V)] = G_1 - G_2 = \int_2^1 dt\frac{d}{dt}(\mathbf{p}\cdot\delta\lambda\mathbf{r}) \tag{2.62}$$

or, the theorem

$$\frac{d}{dt}\mathbf{r}\cdot\mathbf{p} = 2T - \mathbf{r}\cdot\boldsymbol{\nabla}V. \tag{2.63}$$

For the particular situation of the Coulomb potential between charges, $V = \text{constant}/r$, where

$$\mathbf{r}\cdot\boldsymbol{\nabla}V = r\frac{d}{dr}V = -V, \tag{2.64}$$

the virial theorem asserts that

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{p}) = 2T + V. \tag{2.65}$$

We apply this to a *bound* system produced by a force of attraction. On taking the time average of (2.65) the time derivative term disappears. That is because, over an arbitrarily long time interval $\tau = t_1 - t_2$, the value of $\mathbf{r}\cdot\mathbf{p}(t_1)$ can differ by only a finite amount from $\mathbf{r}\cdot\mathbf{p}(t_2)$, and

$$\overline{\frac{d}{dt}(\mathbf{r}\cdot\mathbf{p})} \equiv \frac{1}{\tau}\int_{t_2}^{t_1} dt\frac{d}{dt}\mathbf{r}\cdot\mathbf{p} = \frac{\mathbf{r}\cdot\mathbf{p}(t_1) - \mathbf{r}\cdot\mathbf{p}(t_2)}{\tau} \to 0, \tag{2.66}$$

as $\tau \to \infty$. The conclusion, for time averages,

$$2\overline{T} = -\overline{V}, \tag{2.67}$$

is familiar in elementary discussions of motion in a $1/r$ potential.

Here is one more example of a nonconservation law: Consider the variations

$$\delta\mathbf{r} = \delta\lambda\frac{\mathbf{r}}{r}, \tag{2.68a}$$

$$\delta\mathbf{p} = -\delta\lambda\left(\frac{\mathbf{p}}{r} - \frac{\mathbf{r}\,\mathbf{p}\cdot\mathbf{r}}{r^3}\right) = \delta\lambda\frac{\mathbf{r}\times(\mathbf{r}\times\mathbf{p})}{r^3}. \tag{2.68b}$$

Again $\mathbf{p}\cdot d\mathbf{r}$ is invariant:

$$\delta(\mathbf{p}\cdot d\mathbf{r}) = -\delta\lambda\left(\frac{\mathbf{p}}{r} - \frac{\mathbf{r}\,\mathbf{p}\cdot\mathbf{r}}{r^3}\right)\cdot d\mathbf{r} + \mathbf{p}\cdot\left(\delta\lambda\frac{d\mathbf{r}}{r} - \delta\lambda\mathbf{r}\frac{\mathbf{r}\cdot d\mathbf{r}}{r^3}\right) = 0, \tag{2.69}$$

and the change of the Hamiltonian (2.60) is now

$$\delta H = \delta\lambda\left[-\frac{\mathbf{L}^2}{mr^3} + \frac{\mathbf{r}}{r}\cdot\boldsymbol{\nabla}V\right]. \tag{2.70}$$

The resulting theorem, for $V = V(r)$, is

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\cdot\mathbf{p}\right) = \frac{\mathbf{L}^2}{mr^3} - \frac{dV}{dr}, \tag{2.71}$$

which, when applied to the Coulomb potential, gives the bound-state time average relation

$$\frac{L^2}{m}\overline{\left(\frac{1}{r^3}\right)} = -\overline{\left(\frac{V}{r}\right)}. \tag{2.72}$$

This relation is significant in hydrogen fine-structure calculations (for example, see (Schwinger 2001)).

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