Chapter 2
General Principles for Fundamental Solutions

The correct definition of a *fundamental solution* of a linear differential operator was anticipated by N. Zeilon in 1911 and finally given in the framework of distribution theory by L. Schwartz in 1950, see Schwartz [246], pp. 135, 136. More generally, L. Schwartz defined *fundamental matrices* $E \in \mathcal{D}'(\mathbf{R}^n)^{l\times l}$ for systems $A(\partial) = \big(A_{ij}(\partial)\big)_{1\le i,j\le l}$ of differential operators by $A(\partial)E = I_l\delta$, or, more explicitly, $\sum_{k=1}^{l} A_{ik}(\partial)E_{kj} = \delta_{ij}\delta$ for $1 \le i,j \le l$. The reason for this more general definition lies in the importance of such systems in the natural sciences: Physical phenomena are in general described by vector or tensor fields (as, e.g., displacements, electric and magnetic fields etc.) instead of by single scalar quantities, as e.g., the temperature. Therefore we present three such systems in Examples 2.1.3 and 2.1.4 describing the displacements in isotropic, cubic and hexagonal elastic media, respectively.

The content of the Malgrange–Ehrenpreis Theorem is that every non-trivial linear differential operator with constant coefficients has a fundamental solution. We give a short new constructive proof of this fact in Proposition 2.2.1. Section 2.3 deals with the existence of *temperate* fundamental solutions, a problem which was solved first by S. Łojasiewicz and L. Hörmander.

Apart from the question of existence of fundamental solutions, the search for uniqueness criteria such as support or growth properties in dependence on the operator is essential. This question is investigated in Sect. 2.4. An existence and uniqueness theorem for homogeneous *elliptic* operators (for the definition of ellipticity, see Definition 2.4.7) is given in Proposition 2.4.8. On the other hand, if, for $N \in \mathbf{R}^n \setminus \{0\}$, there exists a tube domain $T = \{\mathrm{i}\xi + \sigma N;\ \xi \in \mathbf{R}^n, \sigma > \sigma_0\} \subset \mathbf{C}^n$ such that $\det A$ does not vanish on T, then a fundamental matrix E of the system $A(\partial)$ is uniquely determined by the condition $\exists \sigma > \sigma_0 : \mathrm{e}^{-\sigma x N}E \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}$, see Definition and Proposition 2.4.13. Furthermore, the support of E is contained in the half-space $H_N = \{x \in \mathbf{R}^n;\ xN \ge 0\}$. In the literature, such systems are called temperate evolution systems (B. Melrose) or systems correct in the sense of Petrovsky (S.G. Gindikin, L. Hörmander). For shortness and due to the many

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similarities with hyperbolic operators (see Definition 2.4.10), we prefer to call such systems *quasihyperbolic* (cf. Example 2.2.2 and Definition and Proposition 2.4.13).

In Sect. 2.5, the effect on the fundamental matrix of linear transformations of the coordinates is studied since this allows to reduce many operators and systems to simpler ones. Finally, in Sect. 2.6, the construction of fundamental solutions by invariance methods is explained.

2.1 Fundamental Matrices

As we have said above, it is more often *systems* of differential equations than scalar operators which originally occur when one sets up models to describe physical processes. Scalar operators of higher order then appear as the determinants of such systems. Let us therefore introduce the notion of *fundamental matrices* of linear square systems $A(\partial)$ of differential operators, and let us explain the connection with the fundamental solutions of their determinants $P(\partial) = \det A(\partial)$. In the following, $I_l \in \mathrm{Gl}_l(\mathbf{R})$ denotes the $l \times l$ unit matrix.

Definition 2.1.1 Let $A(\partial) = \big(A_{ij}(\partial)\big)_{1\le i,j\le l}$, where $A_{ij}(\partial) = \sum_{|\alpha|\le m} a_{ij,\alpha}\partial^\alpha$, be an $l\times l$ matrix of linear differential operators in $\mathbf{R}^n$ with constant coefficients $a_{ij,\alpha} \in \mathbf{C}$. A matrix $E \in \mathcal{D}'(\mathbf{R}^n)^{l\times l}$ is called a *right-sided* or a *left-sided fundamental matrix* of $A(\partial)$, respectively, iff the respective equation

$$A(\partial)E = I_l\delta, \text{ i.e., } \forall 1 \le i,j \le l: \sum_{k=1}^{l} A_{ik}(\partial)E_{kj} = \begin{cases} \delta, & \text{if } i = j, \\ 0, & \text{else,} \end{cases}$$

or

$$E * A(\partial)\delta = I_l\delta, \text{ i.e., } \forall 1 \le i,j \le l: \sum_{k=1}^{l} A_{ki}(\partial)E_{jk} = \begin{cases} \delta, & \text{if } i = j, \\ 0, & \text{else} \end{cases}$$

holds in $\mathcal{D}'(\mathbf{R}^n)^{l\times l}$. If E is a right-sided as well as left-sided fundamental matrix of $A(\partial)$, then it is called a *two-sided fundamental matrix* of $A(\partial)$.

Note that the term "fundamental matrix" is not generally in use. Instead, it is also called *Green's matrix, Green's tensor* or simply *fundamental solution*. For Definition 2.1.1, see Malgrange [174], pp. 298, 299; Schwartz [246], Eq. (V, 6; 30), p. 140; Petersen [228], pp. 56, 57; Hörmander [136], p. 94; Jones [152], p. 421.

Let us observe that the transposed matrix E^T of a right-sided fundamental matrix E of $A(\partial)$ is a left-sided fundamental matrix of the transposed system $A(\partial)^T$ and vice versa. As we shall see in Sect. 2.2, a system $A(\partial)$ has a right- or a left-sided fundamental matrix if and only if its determinant operator $\det A(\partial)$ does not vanish identically. In this case, we can construct a two-sided fundamental matrix E of

$A(\partial)$ in the following way: Take a fundamental solution F of $\det A(\partial)$ and set $E := A(\partial)^{\mathrm{ad}} F$, where $A(\partial)^{\mathrm{ad}}$ denotes the adjoint matrix to $A(\partial)$. In fact,

$$\begin{aligned} &P(\partial) = \det A(\partial), \quad P(\partial)F = \delta, \quad E = A(\partial)^{\mathrm{ad}} F \\ &\qquad \Longrightarrow \begin{cases} A(\partial)E = A(\partial)A(\partial)^{\mathrm{ad}} F = I_l P(\partial) F = I_l \delta \\ \text{and } E * A(\partial)\delta = A(\partial)^{\mathrm{ad}} A(\partial) F = I_l \delta. \end{cases} \end{aligned} \tag{2.1.1}$$

(This procedure is a classical one: see Weierstrass [300], pp. 287–288; Hörmander [136], Section 3.8, p. 94.) However, in general, a right-sided fundamental matrix is not necessarily a left-sided one, and conversely, see Example 2.1.2.

Example 2.1.2 For $A(\partial) = \begin{pmatrix} \mathrm{d}/\mathrm{d}x & 1 \\ 0 & 1 \end{pmatrix}$, a right-sided fundamental matrix is of the form $\begin{pmatrix} Y(x) + C_1 & -Y(x) + C_2 \\ 0 & \delta \end{pmatrix}$, whereas a left-sided fundamental matrix is given by $\begin{pmatrix} Y(x) + C_3 & -Y(x) - C_3 \\ C_4 & \delta - C_4 \end{pmatrix}$, $C_1, \dots, C_4 \in \mathbf{C}$ being arbitrary.

The two-sided fundamental matrices are of the form $E = \begin{pmatrix} Y(x)+C & -Y(x)-C \\ 0 & \delta \end{pmatrix}$, $C \in \mathbf{C}$. In fact, $E = A(\partial)^{\mathrm{ad}} F$, where $F = Y(x) + C$ is a fundamental solution of $\det A(\partial) = \frac{\mathrm{d}}{\mathrm{d}x}$. □

Example 2.1.3 Let us use formula (2.1.1) in order to calculate the fundamental matrix of the Lamé system $A(\partial)$ governing elastodynamics inside a homogeneous isotropic medium.

If $u = (u_1, u_2, u_3)^T$ denotes the displacement in an elastic medium, ρ, f the densities of mass and force, respectively, then $A(\partial)u = \rho f$ where $\partial = (\partial_t, \partial_1, \partial_2, \partial_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)^T$,

$$A(\partial) := \rho I_3 \partial_t^2 - B(\nabla), \qquad B(\nabla) := \mu \Delta_3 I_3 + (\lambda + \mu) \nabla \cdot \nabla^T, \tag{2.1.2}$$

and $\lambda, \mu > 0$ denote Lamé's constants.

Generally, the matrix $A = \alpha I_l + \beta \xi \cdot \xi^T \in \mathbf{C}^{l \times l}$ (with $\alpha, \beta \in \mathbf{C}$ and $\xi \in \mathbf{R}^l$) has the eigenvalues $\alpha + \beta |\xi|^2$ with multiplicity 1 and α with multiplicity $l - 1$ and hence $\det A = \alpha^{l-1}(\alpha + \beta |\xi|^2)$. Similarly, the ansatz $A^{\mathrm{ad}} = \gamma I_l + \epsilon \xi \cdot \xi^T$ yields

$$A^{\mathrm{ad}} = \alpha^{l-2} \big[(\alpha + \beta |\xi|^2) I_l - \beta \xi \cdot \xi^T \big].$$

Therefore,

$$\begin{aligned} P(\partial) = \det A(\partial) &= \det\big((\rho \partial_t^2 - \mu \Delta_3) I_3 - (\lambda + \mu) \nabla \cdot \nabla^T \big) \\ &= (\rho \partial_t^2 - \mu \Delta_3)^2 \big(\rho \partial_t^2 - (\lambda + 2\mu) \Delta_3 \big) \end{aligned} \tag{2.1.3}$$

and

$$A(\partial)^{\mathrm{ad}} = (\rho\partial_t^2 - \mu\Delta_3)\big[\big(\rho\partial_t^2 - (\lambda + 2\mu)\Delta_3\big)I_3 + (\lambda + \mu)\nabla \cdot \nabla^T\big].$$

For the two different irreducible factors of $P(\partial)$ in (2.1.3), let us introduce the abbreviations

$$W_s(\partial) = \rho\partial_t^2 - \mu\Delta_3, \qquad W_p(\partial) = \rho\partial_t^2 - (\lambda + 2\mu)\Delta_3.$$

These two wave operators account for the propagation of shear and of pressure waves, respectively, in the medium, see Achenbach [2], 4.1, pp. 122–124. By formula (1.4.11) and Proposition 1.3.19, their forward fundamental solutions F_s, F_p are given by

$$F_s = \frac{\delta\big(t - \frac{|x|}{c_s}\big)}{4\pi\mu|x|}, \qquad F_p = \frac{\delta\big(t - \frac{|x|}{c_p}\big)}{4\pi(\lambda + 2\mu)|x|},$$

where $c_s = \sqrt{\frac{\mu}{\rho}}$, $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$ are the velocities of the shear and pressure waves, respectively.

From the uniqueness (see Proposition 2.4.11 below) of the forward fundamental solution F of the hyperbolic operator $P(\partial) = W_s(\partial)^2 W_p(\partial)$ and the convolvability of F_s, F_p, we obtain that $F = F_s * F_s * F_p$. Hence we infer from (2.1.1) that the unique fundamental matrix E of $A(\partial)$ with support in the half-space $t \geq 0$ is given by

$$\begin{aligned} E = A(\partial)^{\mathrm{ad}} F &= W_s(\partial)\big[W_p(\partial)I_3 + (\lambda + \mu)\nabla \cdot \nabla^T\big](F_s * F_s * F_p) \\ &= \big[W_p(\partial)I_3 + (\lambda + \mu)\nabla \cdot \nabla^T\big](F_s * F_p) \qquad (2.1.4) \\ &= I_3 F_s + (\lambda + \mu)\nabla \cdot \nabla^T (F_s * F_p). \end{aligned}$$

From formula (2.1.4), we conclude that the fundamental matrix E can be expressed by means of the fundamental solution $F_s * F_p$ of the fourth-order operator $W_s(\partial)W_p(\partial)$; in particular, there is no need to calculate the fundamental solution of the sixth-order operator $P(\partial) = \det A(\partial) = W_s(\partial)^2 W_p(\partial)$, as was done in Piskorek [229], p. 95.

In order to derive Stokes's representation of the fundamental matrix E from the year 1849, see Stokes [265], let us apply the "difference device", which shall be developed in more generality in Sect. 3.3. Due to

$$\begin{aligned} W_s(\partial)W_p(\partial)\big[(\lambda + 2\mu)F_p - \mu F_s\big] &= \big[(\lambda + 2\mu)W_s(\partial) - \mu W_p(\partial)\big]\delta \\ &= (\lambda + \mu)\rho\partial_t^2\delta, \end{aligned}$$

we conclude, by convolution with the fundamental solution $tY(t) \otimes \delta(x)$ of the operator ∂_t^2, that the forward fundamental solution $F_s * F_p$ of $W_s(\partial)W_p(\partial)$ has the following representation:

$$F_s * F_p = \frac{1}{(\lambda+\mu)\rho}\big[tY(t)\otimes\delta(x)\big] * \big[(\lambda+2\mu)F_p - \mu F_s\big]. \tag{2.1.5}$$

Since $F_p \in \mathcal{C}\big(\mathbf{R}^3_x\setminus\{0\}, \mathcal{E}'(\mathbf{R}^1_t)\big)$ is given by $(\lambda+2\mu)F_p(x) = \frac{1}{4\pi|x|}\delta_{|x|/c_p}(t)$ for $x \neq 0$, we obtain

$$\big[tY(t)\otimes\delta(x)\big] * (\lambda+2\mu)F_p = \frac{\big(t-\frac{|x|}{c_p}\big)Y\big(t-\frac{|x|}{c_p}\big)}{4\pi|x|} \tag{2.1.6}$$

for $x \neq 0$, and (2.1.6) then holds in $\mathcal{D}'(\mathbf{R}^4)$ by homogeneity. If we insert (2.1.6) and the analogous equation for $\big[tY(t)\otimes\delta(x)\big] * \mu F_s$ into (2.1.5), we obtain

$$\begin{aligned}(\lambda+\mu)F_s * F_p &= \frac{1}{4\pi\rho|x|}\Big[\Big(t-\frac{|x|}{c_p}\Big)Y\Big(t-\frac{|x|}{c_p}\Big) - \Big(t-\frac{|x|}{c_s}\Big)Y\Big(t-\frac{|x|}{c_s}\Big)\Big]\\ &= \frac{1}{4\pi\rho}\Big[\Big(\frac{1}{c_s}-\frac{1}{c_p}\Big) + \Big(\frac{t}{|x|}-\frac{1}{c_s}\Big)Y(|x|-c_st) - \Big(\frac{t}{|x|}-\frac{1}{c_p}\Big)Y(|x|-c_pt)\Big].\end{aligned}$$

For the differentiation of $F_s * F_p$, we then employ the many-dimensional jump formula (1.3.13):

$$\begin{aligned}(\lambda+\mu)\nabla\cdot\nabla^T(F_s * F_p) = \frac{1}{4\pi\rho}\Big\{&\frac{3xx^T - I_3|x|^2}{|x|^5}\,t\big[Y(|x|-c_st) - Y(|x|-c_pt)\big]\\ &- xx^T\Big[\frac{1}{c_s^4t^3}\,\delta(|x|-c_st) - \frac{1}{c_p^4t^3}\,\delta(|x|-c_pt)\Big]\Big\}\end{aligned}$$

Inserting this into (2.1.4) finally yields Stokes's formula for the forward fundamental matrix E of Lamé's system $A(\partial)$ defined in (2.1.2):

$$\begin{aligned}E = \frac{I_3|x|^2 - xx^T}{4\pi\mu|x|^3}\,\delta\Big(t-\frac{|x|}{c_s}\Big) + \frac{xx^T}{4\pi(\lambda+2\mu)|x|^3}\,\delta\Big(t-\frac{|x|}{c_p}\Big)\\ + \frac{t}{4\pi\rho|x|^3}\Big(I_3 - \frac{3xx^T}{|x|^2}\Big)\Big[Y\Big(t-\frac{|x|}{c_s}\Big) - Y\Big(t-\frac{|x|}{c_p}\Big)\Big],\end{aligned} \tag{2.1.7}$$

cf. Achenbach [2], (3.95/96/98), pp. 99f.; Achenbach and Wang [3], (8.15), p. 282; Duff [62], pp. 270f.; [64], p. 79; Eringen and Şuhubi [69], (5.10.30), p. 400; Love [171], (36), p. 305; Mura [185], (9.34), p. 63; Willis [302], (34), p. 387; Wagner [293], p. 406. □

Example 2.1.4 Let us consider now the equations of *anisotropic* elastodynamics. The investigation of this 3 by 3 system will also show the importance of higher order partial differential operators in mathematical physics. We shall develop here only the algebraic part of the construction of the fundamental matrix, and we shall postpone the application of the Herglotz–Petrovsky formula to Chap. 4, where hyperbolic operators and systems will be analyzed.

(a) In a homogeneous anisotropic medium, the displacements u_p, the stresses σ_{pq} and the strains v_{pq} satisfy the following equations (where we use Einstein's summation convention):

$$\begin{aligned} \rho\partial_t^2 u_p &= \partial_q\sigma_{pq} + \rho f_p, \qquad \sigma_{pq} = \sigma_{qp}; \\ v_{pq} &= \frac{1}{2}(\partial_q u_p + \partial_p u_q); \\ \sigma_{pq} &= c_{pqrs}v_{rs} \text{ (Hooke's law)}, \qquad c_{pqrs} = c_{qprs} = c_{rspq}. \end{aligned} \tag{2.1.8}$$

Herein $p, q, r, s \in \{1, 2, 3\}$, c_{pqrs} are the elastic constants, and ρ, f denote the densities of mass and of force, respectively, cf. Achenbach and Wang [3], (2.1/2), (2.4), p. 274; Buchwald [31], (2.1–5), p. 564; Duff [62], (1.1), p. 249; Eringen and Şuhubi [69], (5.2.19), p. 346; Herglotz [127], (3.48), p. 75, and (6.9), p. 156; Musgrave [186], (6.1.4/6), p. 67; (3.11.1/2), p. 28; Payton [226], (1.1.1–5), p. 1; Poruchikov [231], (2.1.1–6), p. 4.

Abbreviating the symmetric matrix $(c_{pqrs}\partial_q\partial_s)_{r,s}$ by $B(\nabla)$ we derive the system

$$A(\partial)u = \big(\rho I_3\partial_t^2 - B(\nabla)\big)u = \rho f$$

(cf. Duff [62], (1.2), p. 250; Musgrave [186], (6.1.7), p. 68) by elimination of σ_{pq} and v_{pq}. In the sequel, we put $\rho = 1$.

The dimension of the linear space of tensors (c_{pqrs}) of rank 4 fulfilling the symmetry relations stated in (2.1.8) equals 21. This fact is exploited when the "contracted index notation" is used (cf. Musgrave [186], (3.13.4–6), p. 33; Payton [226], p. 3): The indices $11, 22, 33, 12, 13, 23$ are replaced by $1, 2, 3, 6, 5, 4$, respectively. Consequently, (2.1.8) takes the form $\sigma = C\tilde{v}$, where $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^T \in \mathbf{R}^6$, $\tilde{v} = (v_{11}, v_{22}, v_{33}, 2v_{23}, 2v_{13}, 2v_{12})^T \in \mathbf{R}^6$, and $C \in \mathbf{R}^{6\times 6}$.

Let us consider such particular cases of the elastic constants for which the 6×6-matrix C has the form $C = \begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix}$ with two symmetric 3×3-matrices H, L. This implies the equations

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = H \begin{pmatrix} v_{11} \\ v_{22} \\ v_{33} \end{pmatrix}, \quad \begin{pmatrix} \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = 2L \begin{pmatrix} v_{23} \\ v_{13} \\ v_{12} \end{pmatrix}, \quad c_{pqrs} = \begin{cases} h_{jk} & : p = q, r = s, \\ l_{j-3,k-3} & : p \neq q, r \neq s, \\ 0 & : \text{else}, \end{cases}$$

if j corresponds to pq or qp, and k to rs or sr, respectively ($1 \le p, q, r, s \le 3$, $1 \le j, k \le 6$). Hence the matrix $B(\xi)$ assumes the form

$$B(\xi) = \big(h_{jk}\xi_j\xi_k\big)_{j,k=1,2,3} + \begin{pmatrix} l_{22}\xi_3^2 + 2l_{23}\xi_2\xi_3 + l_{33}\xi_2^2 & l_{33}\xi_1\xi_2 + \xi_3\,\Xi & l_{22}\xi_1\xi_3 + \xi_2\,\Xi \\ l_{33}\xi_1\xi_2 + \xi_3\,\Xi & l_{11}\xi_3^2 + 2l_{13}\xi_1\xi_3 + l_{33}\xi_1^2 & l_{11}\xi_2\xi_3 + \xi_1\,\Xi \\ l_{22}\xi_1\xi_3 + \xi_2\,\Xi & l_{11}\xi_2\xi_3 + \xi_1\,\Xi & l_{11}\xi_2^2 + 2l_{12}\xi_1\xi_2 + l_{22}\xi_1^2 \end{pmatrix} \tag{2.1.9}$$

with $\Xi := l_{23}\xi_1 + l_{13}\xi_2 + l_{12}\xi_3$.

(b) Let us specify the above for *isotropic media.* In such media, the tensor (c_{pqrs}) is determined by *two* independent constants, the Lamé constants λ, μ, whereby

$$c_{pqrs} = \lambda\delta_{pq}\delta_{rs} + \mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})$$

or

$$C = \begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix}, \qquad H = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{pmatrix}, \qquad L = \mu I_3,$$

cf. Eringen and Şuhubi [69], (5.2.20), p. 346; Payton [226], (1.1.7), p. 2; Sommerfeld [259], p. 272.

Then $B(\xi) = \mu|\xi|^2 I_3 + (\lambda+\mu)\xi\cdot\xi^T$ as in (2.1.2) and the determinant operator of the system degenerates:

$$P(\partial) = \det\big(I_3\partial_t^2 - B(\nabla)\big) = (\partial_t^2 - \mu\Delta_3)^2\big(\partial_t^2 - (\lambda + 2\mu)\Delta_3\big),$$

cf. (2.1.3).

(c) *Cubic media* are characterized by the *three* independent constants $a = c_{11} - c_{44}$, $b = c_{12} + c_{44}$, $c = c_{44}$, whereby the tensor (c_{pqrs}) is given as

$$c_{pqrs} = (b - c)\delta_{pq}\delta_{rs} + c(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) + (a - b)\delta_{pj}\delta_{qj}\delta_{rj}\delta_{sj}$$

or

$$C = \begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix}, \qquad H = \begin{pmatrix} a + c & b - c & b - c \\ b - c & a + c & b - c \\ b - c & b - c & a + c \end{pmatrix}, \qquad L = cI_3, \tag{2.1.10}$$

cf. Chadwick and Smith [46], (6.1), p. 60; Dederichs and Leibfried [55], (13), p. 1176. If $a = b$, then the cubic medium is isotropic with $\lambda = b - c$, $\mu = c$. Thus the difference $b - a$ is a measure of the anisotropy of the cubic material, cf. Liess [165], p. 274.

From (2.1.9) and (2.1.10), we infer $\Xi = 0$ and

$$B(\xi) = c|\xi|^2 I_3 + b\xi \cdot \xi^T - (b-a)\begin{pmatrix} \xi_1^2 & 0 & 0 \\ 0 & \xi_2^2 & 0 \\ 0 & 0 & \xi_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} c|\xi|^2 + a\xi_1^2 & b\xi_1\xi_2 & b\xi_1\xi_3 \\ b\xi_1\xi_2 & c|\xi|^2 + a\xi_2^2 & b\xi_2\xi_3 \\ b\xi_1\xi_3 & b\xi_2\xi_3 & c|\xi|^2 + a\xi_3^2 \end{pmatrix},$$

cf. Duff [62], p. 271; Liess [165], p. 274; Sommerfeld [259], p. 272.

If, as before, $A(\tau,\xi) = \tau^2 I_3 - B(\xi)$, then the determinant operator $P(\partial) = \det A(\partial)$ is of degree 6 and, in general, irreducible. In fact, $A(\tau,\xi) = M - b\xi \cdot \xi^T$, where M is the diagonal matrix with the elements $\tau^2 - c|\xi|^2 + (b-a)\xi_j^2$, $j = 1, 2, 3$. For the determinant of the difference $M - b\xi \cdot \eta^T$ of the diagonal matrix M and the rank-one matrix $b\xi \cdot \eta^T$, we have the general formula

$$\det(M - b\xi \cdot \eta^T) = \det M - b\xi^T (M^{\mathrm{ad}})^T \eta = \det M - b\eta^T M^{\mathrm{ad}} \xi,$$

and this implies

$$P(\partial) = \det A(\partial) = \prod_{j=1}^{3} W_j(\partial) - b\sum_{j=1}^{3} \partial_j^2 W_{j+1}(\partial) W_{j+2}(\partial), \tag{2.1.11}$$

where

$$W_j(\partial) = \partial_t^2 - c\Delta_3 + (b-a)\partial_j^2,\ j = 1,2,3, \text{ and } W_4(\partial) = W_1(\partial), W_5(\partial) = W_2(\partial).$$

According to (2.1.11), the *slowness surface* $\{(\tau,\xi) \in \mathbf{R}^4;\ P(\tau,\xi) = 0\}$ is then given by $\sum_{j=1}^{3} \dfrac{b\xi_j^2}{\tau^2 - c|\xi|^2 + (b-a)\xi_j^2} = 1$ cf. Duff [62], p. 271; Mura [185], (3.35), p. 14; Liess [165], p. 274.

A similar calculation yields for the adjoint matrix of $A(\partial)$ the following:

$$A(\partial)_{jj}^{\mathrm{ad}} = W_{j+1}(\partial)W_{j+2}(\partial) - b\partial_{j+1}^2 W_{j+2}(\partial) - b\partial_{j+2}^2 W_{j+1}(\partial),$$

$$\text{and } A(\partial)_{j,j+1}^{\mathrm{ad}} = b\partial_j\partial_{j+1}W_{j+2}(\partial),\ j = 1,2,3,$$

Therefore, the forward fundamental matrix E of $A(\partial)$ is given by $E = A^{\mathrm{ad}}(\partial)F$ where F is the forward fundamental solution of $P(\partial) = \det A(\partial)$.

Let us finally determine for which values of a, b, c the determinant operator $\det A(\partial)$ is reducible. Due to the apparent symmetry in ξ_1, ξ_2, ξ_3, the polynomial

$P(\tau,\xi)$ splits into factors either if it has the form $\prod_{j=1}^{3}\big(\tau^2-\alpha|\xi|^2+\beta\xi_j^2\big)$, i.e., if $b=0$, or if there exists a factor $\tau^2-\alpha|\xi|^2$, which is symmetric in ξ_1,ξ_2,ξ_3. This second assumption implies that either $a=b$, i.e., the medium is isotropic, or else $a=-2b$. In this last case,

$$P(\tau,\xi)=(\tau^2-c|\xi|^2)\big[(\tau^2+(b-c)|\xi|^2)^2-b^2(\xi_1^4+\xi_2^4+\xi_3^4-\xi_1^2\xi_2^2-\xi_1^2\xi_3^2-\xi_2^2\xi_3^2)\big].$$

(Concerning the case $b=0$ cf. Chadwick and Norris [45] (4.2), p. 601; (1.3), p. 590: "For cubic media there is just one constraint on the elastic moduli, i.e., $b=0$, under which the slowness surface is composed of three spheroids." Cf. also Chadwick and Smith [46], (8.10), p. 74.)

(d) For media of *hexagonal symmetry*, the elastic constants fulfill

$$C=\begin{pmatrix}H&0\\0&L\end{pmatrix},\qquad H=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{12}&c_{11}&c_{13}\\c_{13}&c_{13}&c_{33}\end{pmatrix},\qquad L=\begin{pmatrix}c_{44}&0&0\\0&c_{44}&0\\0&0&\frac12(c_{11}-c_{12})\end{pmatrix},$$

cf. Fedorov [72], (9.22), p. 31; Musgrave [186], p. 94; Payton [226], (1.3.2), p. 3. In the tensor (c_{pqrs}), there thus remain 5 independent constants, which we will choose, in accordance with Buchwald [31] (6.7), (6.10), pp. 572, 573, as

$$a_1=c_{11},\quad a_2=c_{33},\quad a_3=c_{13}+c_{44},\quad a_4=\frac12(c_{11}-c_{12}),\quad a_5=c_{44}. \tag{2.1.12}$$

With this notation, we obtain

$$B(\xi)=\begin{pmatrix}a_1\xi_1^2+a_4\xi_2^2+a_5\xi_3^2&(a_1-a_4)\xi_1\xi_2&a_3\xi_1\xi_3\\(a_1-a_4)\xi_1\xi_2&a_4\xi_1^2+a_1\xi_2^2+a_5\xi_3^2&a_3\xi_2\xi_3\\a_3\xi_1\xi_3&a_3\xi_2\xi_3&a_5(\xi_1^2+\xi_2^2)+a_2\xi_3^2\end{pmatrix}, \tag{2.1.13}$$

cf. Kröner [158], (4), p. 404; Payton [226], (1.5.10), p. 6.

As observed already by Christoffel in 1877 (see Payton [226], p. 7), the determinant operator $P(\partial)=\det A(\partial)=\det\big(I_3\partial_t^2-B(\nabla)\big)$ splits. In fact, $\tau^2I_3-B(\xi)=M-\eta\cdot\eta^T$ with $\eta=(\sqrt{a_1-a_4}\xi_1,\sqrt{a_1-a_4}\xi_2,a_3\xi_3/\sqrt{a_1-a_4})^T$ and M the diagonal matrix with the elements

$$m_{11}=m_{22}=\tau^2-a_4(\xi_1^2+\xi_2^2)-a_5\xi_3^2,\quad m_{33}=\tau^2-a_5(\xi_1^2+\xi_2^2)-\Big(a_2-\frac{a_3^2}{a_1-a_4}\Big)\xi_3^2.$$

Putting $\rho^2 = \xi_1^2 + \xi_2^2$, we obtain

$$\begin{aligned}P(\tau,\xi) &= \det M - \eta^T M^{\mathrm{ad}}\eta = m_{11}\Big[m_{11}m_{33} - m_{33}(a_1-a_4)\rho^2 - m_{11}\frac{a_3^2\xi_3^2}{a_1-a_4}\Big]\\ &= (\tau^2 - a_4\rho^2 - a_5\xi_3^2)\Big[(\tau^2 - a_4\rho^2 - a_5\xi_3^2)(\tau^2 - a_5\rho^2 - a_2\xi_3^2)\\ &\qquad - (a_1-a_4)\rho^2\Big(\tau^2 - a_5\rho^2 - \big(a_2 - \frac{a_3^2}{a_1-a_4}\big)\xi_3^2\Big)\Big]\end{aligned}$$

Hence $P(\partial)$ is the product of the wave operator $\partial_t^2 - a_4\Delta_2 - a_5\partial_3^2$ and of the quartic operator

$$R(\partial) := \partial_t^4 - \partial_t^2(a_1\Delta_2 + a_2\partial_3^2 + a_5\Delta_3) + a_1a_5\Delta_2^2 + (a_1a_2 - a_3^2 + a_5^2)\Delta_2\partial_3^2 + a_2a_5\partial_3^4, \tag{2.1.14}$$

cf. Mura [185], (3.38), p. 14; Payton [226], (1.5.13), p. 6.

There are exactly two cases in which the operator $R(\partial)$ in (2.1.14) is a product of two wave operators:

$$\begin{aligned}R(\tau,\xi) = 0 &\iff 2\tau^2 = (a_1+a_5)\rho^2 + (a_2+a_5)\xi_3^2 \pm \sqrt{D},\\ D &:= \big[(a_1-a_5)\rho^2 - (a_2-a_5)\xi_3^2\big]^2 + 4a_3^2\rho^2\xi_3^2.\end{aligned}$$

$\sqrt{D}$ is a polynomial in ξ if and only if either $a_3 = 0$ or $a_3^2 = (a_1-a_5)(a_2-a_5)$, cf. Chadwick and Norris [45], (1.2), p. 589; Payton [226], p. 96. In these cases only, $P(\partial)$ splits into 3 wave operators. Explicitly, in the case of $a_3 = 0$, we have

$$P(\partial) = (\partial_t^2 - a_4\Delta_2 - a_5\partial_3^2)(\partial_t^2 - a_1\Delta_2 - a_5\partial_3^2)(\partial_t^2 - a_5\Delta_2 - a_2\partial_3^2),$$

and in the case of $a_3^2 = (a_1-a_5)(a_2-a_5)$, we obtain

$$P(\partial) = (\partial_t^2 - a_4\Delta_2 - a_5\partial_3^2)(\partial_t^2 - a_1\Delta_2 - a_2\partial_3^2)(\partial_t^2 - a_5\Delta_3).$$

□

2.2 The Malgrange–Ehrenpreis Theorem

The Malgrange–Ehrenpreis theorem states that every (not identically vanishing) partial differential operator with constant coefficients possesses a fundamental solution in the space of distributions, i.e.,

$$\forall P(\partial) \in \mathbf{C}[\partial_1,\dots,\partial_n] \setminus \{0\} : \exists E \in \mathcal{D}'(\mathbf{R}^n) : P(\partial)E = \delta, \tag{2.2.1}$$

see Malgrange [174], Thm. 1, p. 288; Ehrenpreis [68], Thm. 6, p. 892.

Let us first give a short historical account concerning the development of the concept of fundamental solution. Before 1950, in which year the first edition of the first part of Schwartz [246] appeared, not even the *question* about the existence of a fundamental solution did make sense, since there did not at all exist a generally adopted definition of a fundamental solution. The definitions before L. Schwartz usually referred to special types of operators and, correspondingly, to a special kind of the singularity of the fundamental solution, see, e.g., Courant and Hilbert [51], pp. 351, 363–365, 370; Levi [164], p. 276; Bureau [33], p. 15; Somigliana [258]; Fredholm [82]. Let us also mention the different definition of J. Hadamard, later used by F. Bureau, of a fundamental solution of a hyperbolic second order operator, which is not equivalent to Schwartz's definition, see Lützen [173], p. 103; Leray [163], p. 66; Hadamard [123]; Bureau [36, 37].

In 1950, L. Schwartz wrote: "Les définitions habituelles d'une solution élémentaire comme solution *usuelle* du système homogène ayant en un point une singularité d'un certain type, doivent, à notre avis, être totalement rejetées" (Schwartz [246], p. 135, 136).

In particular, the earlier definitions determined fundamental solutions only up to multiplicative constants. E.g., before 1950, both functions $E = -\frac{1}{4\pi|x|}$ and $F = \frac{1}{|x|}$ served as fundamental solutions for the three-dimensional Laplacean Δ_3. L. Schwartz's definition (i.e., that in Definition 1.3.5) excludes F, since $\Delta_3 F = -4\pi\delta$. Hence "…Schwartz clarifie la notion de solution élémentaire en la définissant comme une solution d'une équation ayant la mesure de Dirac δ pour second membre" (Malgrange [175], p. 29; cf. also Horváth [142], p. 236, 237; Dieudonné [60], p. 255).

Let us remark that L. Schwartz's definition was, for locally integrable fundamental solutions, anticipated by N. Zeilon in 1911 (see Schwartz [246], first ed., Vol. 1, (V, 6; 25), p. 135 and footnote (1)) : "*Es soll*:

jede Funktion $F(x,y,z)$ ein Fundamentalintegral der linearen Differentialgleichung

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)u = 0$$

genannt werden, die der Bedingung genügt, dass

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\int_D F(x-\lambda,y-\mu,z-\nu)\,d\lambda\,d\mu\,d\nu$$

gleich 1 ist, wenn das Integrationsgebiet D den Punkt x,y,z einschliesst, und gleich 0, wenn dieser Punkt ausserhalb D liegt. Oder, was auf dasselbe herauskommt: Wenn $\phi(x,y,z)$ eine willkürliche Funktion ist, so soll:

$$u = \int_D F(x-\lambda,y-\mu,z-\nu)\phi(\lambda,\mu,\nu)\,d\lambda\,d\mu\,d\nu$$

im Gebiete D eine Lösung geben der Gleichung:

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)u = \phi.$$

Dabei ist D als ganz willkürlich vorausgesetzt, namentlich muss es gestattet sein, es beliebig klein zu machen." (Zeilon [306], pp. 1, 2; Lützen [173], p. 103.)

Already in 1948, L. Schwartz posed the problem to show that every not identically vanishing linear partial operator with constant coefficients has a fundamental solution (see Treves, Pisier, and Yor [276], p. 1078; Gårding [93], p. 80). This problem was solved, independently, in Malgrange [174], Thm. 1, p. 288, and in Ehrenpreis [68], Thm. 6, p. 892. Both used the Hahn–Banach theorem in order to extend a certain linear functional. The key step in their proofs consisted in showing the continuity of this functional on a suitable subspace of the space of all test functions.

Immediately thereafter, the search for explicit general formulae yielding fundamental solutions began; in particular, since such formulae were known for several special classes of differential operators (e.g., for hyperbolic operators and, more generally, for operators correct in the sense of Petrovsky, see Hörmander [138], p. 120, (12.5.3) and p. 143; for elliptic and, more generally, hypoelliptic operators, see Hörmander [133], p. 223; Mizohata [182], p. 142–144). We owe the first explicit general formula to L. Hörmander, who generalized the procedure used for hypoelliptic operators in his thesis (Hörmander [133], p. 223). F. Trèves adapted this method (dubbed "Hörmander's staircase", see Gel'fand and Shilov [106], Ch. II, Section 3.3, p. 103) in order to obtain a fundamental solution depending continuously on the coefficients of the differential operator (see Treves [270, 272]). A detailed description is contained in Ortner and Wagner [215].

An inconvenience of the "staircase" construction consists in its use of partitions of unity based on the location of the zeroes of $P(z)$. In König [155], a new method of proof of the Malgrange–Ehrenpreis theorem (2.2.1) was given. It avoided the use of partitions of unity, but involved n parametric integrations over inverse Fourier transforms of modulus one functions. In Ortner and Wagner [213], a formula involving only *one* parametric integration was given; in the still simpler proof below, which is due to Wagner [296], we represent a fundamental solution by *sums* of inverse Fourier transforms of modulus one functions.

Proposition 2.2.1 *Let* $P(\xi) = \sum_{|\alpha|\le m} c_\alpha \xi^\alpha \in \mathbf{C}[\xi] \setminus \{0\}$ *be a not identically vanishing polynomial on* $\mathbf{R}^n$ *of degree m. If* $P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$ *and* $\eta \in \mathbf{R}^n$ *with* $P_m(\eta) \neq 0$, *the real numbers* $\lambda_0, \dots, \lambda_m$ *are pairwise different, and* $a_j = \prod_{k=0,k\neq j}^m (\lambda_j - \lambda_k)^{-1}$, *then*

$$E = \frac{1}{\overline{P_m(2\eta)}} \sum_{j=0}^{m} a_j \, \mathrm{e}^{\lambda_j \eta x} \mathcal{F}_\xi^{-1}\left(\frac{\overline{P(\mathrm{i}\xi + \lambda_j \eta)}}{P(\mathrm{i}\xi + \lambda_j \eta)}\right) \tag{2.2.2}$$

is a fundamental solution of $P(\partial)$, *i.e.,* $P(\partial)E = \delta$.

Proof

(1) Let us first observe that, for $\lambda \in \mathbf{R}$ fixed, $N = \{\xi \in \mathbf{R}^n;\ P(\mathrm{i}\xi + \lambda\eta) = 0\}$ is a set of Lebesgue measure zero. In fact, after a linear change of the coordinates, we can assume that $P_m(1, 0, \dots, 0) \neq 0$, and then $\int_N \mathrm{d}\xi = \int_{\mathbf{R}^{n-1}} \big(\int_{N_{\xi'}} \mathrm{d}\xi_1\big)\mathrm{d}\xi' = 0$ by Fubini's theorem and since the sets $N_{\xi'} = \{\xi_1 \in \mathbf{R};\ P\big(\mathrm{i}(\xi_1, \xi') + \lambda\eta\big) = 0\}$ are finite for $\xi' = (\xi_2, \dots, \xi_n) \in \mathbf{R}^{n-1}$. Hence

$$S(\xi) = \frac{\overline{P(\mathrm{i}\xi + \lambda\eta)}}{P(\mathrm{i}\xi + \lambda\eta)} \in L^\infty(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$$

and formula (2.2.2) is meaningful.

(2) For $S \in \mathcal{S}'(\mathbf{R}^n)$ and $\zeta \in \mathbf{C}^n$, we have

$$P(\partial)(\mathrm{e}^{\zeta x}\mathcal{F}^{-1}S) = \mathrm{e}^{\zeta x}P(\partial + \zeta)\mathcal{F}^{-1}S = \mathrm{e}^{\zeta x}\mathcal{F}_\xi^{-1}\big(P(\mathrm{i}\xi + \zeta)S\big).$$

Taking $S = \overline{P(\mathrm{i}\xi + \lambda\eta)}/P(\mathrm{i}\xi + \lambda\eta)$ with $\lambda \in \mathbf{R}$, this implies

$$P(\partial)\left(\mathrm{e}^{\lambda\eta x}\mathcal{F}^{-1}\left(\frac{\overline{P(\mathrm{i}\xi + \lambda\eta)}}{P(\mathrm{i}\xi + \lambda\eta)}\right)\right) = \mathrm{e}^{\lambda\eta x}\mathcal{F}_\xi^{-1}\big(\overline{P(\mathrm{i}\xi + \lambda\eta)}\big).$$

Furthermore,

$$\mathcal{F}_\xi^{-1}\big(\overline{P(\mathrm{i}\xi + \lambda\eta)}\big) = \mathcal{F}_\xi^{-1}\big(\overline{P}(-\mathrm{i}\xi + \lambda\eta)\big) = \overline{P}(-\partial + \lambda\eta)\delta,$$

and hence

$$\begin{aligned} P(\partial)\left(\mathrm{e}^{\lambda\eta x}\mathcal{F}^{-1}\left(\frac{\overline{P(\mathrm{i}\xi + \lambda\eta)}}{P(\mathrm{i}\xi + \lambda\eta)}\right)\right) &= \mathrm{e}^{\lambda\eta x}\overline{P}(-\partial + \lambda\eta)\delta = \overline{P}(-\partial + 2\lambda\eta)(\mathrm{e}^{\lambda\eta x}\delta) \\ &= \overline{P}(-\partial + 2\lambda\eta)\delta = \lambda^m\overline{P_m(2\eta)}\delta + \sum_{k=0}^{m-1}\lambda^k T_k, \end{aligned}$$

for certain distributions $T_k \in \mathcal{E}'(\mathbf{R}^n)$. (Note that $\mathrm{e}^{\lambda\eta x}\delta = \delta$.)

Since $a_0, \dots, a_m$ fulfill the system of linear equations

$$\sum_{j=0}^{m} a_j\lambda_j^k = \begin{cases} 0, & \text{if } k = 0, \dots, m-1, \\ 1, & \text{if } k = m, \end{cases}$$

cf. Proposition 1.3.7, we obtain

$$P(\partial)E = \frac{1}{\overline{P_m(2\eta)}} \sum_{j=0}^{m} a_j \big[\lambda_j^m \overline{P_m(2\eta)}\,\delta + \sum_{k=0}^{m-1} \lambda_j^k T_k\big] = \delta,$$

i.e., E is a fundamental solution of the operator $P(\partial)$. This completes the proof. □

Example 2.2.2 Let us illustrate the construction formula (2.2.2) for fundamental solutions in the case of *quasihyperbolic operators*, which will be studied more thoroughly in Chap. 4.

An operator $P(\partial)$ in $\mathbf{R}^n$ is called *quasihyperbolic* in the direction $N \in \mathbf{R}^n \setminus \{0\}$ iff the condition

$$\exists \sigma_0 \in \mathbf{R} : \forall \sigma > \sigma_0 : \forall \xi \in \mathbf{R}^n : P(\mathrm{i}\xi + \sigma N) \neq 0 \tag{2.2.3}$$

holds. Hence $P(\partial)$ is quasihyperbolic iff $P(z),\ z \in \mathbf{C}^n$, has no zeroes for large $\operatorname{Re} z$ in direction N, cf. Ortner and Wagner [207], Def. 2, p. 442. For quasihyperbolic operators, there exists one and only one fundamental solution F satisfying

$$\exists \sigma > \sigma_0 : \mathrm{e}^{-\sigma x N} F \in \mathcal{S}'(\mathbf{R}^n) \tag{2.2.4}$$

if σ_0 is as in (2.2.3). Furthermore, $\operatorname{supp} F \subset H_N := \{x \in \mathbf{R}^n; Nx \geq 0\}$, and $\mathrm{e}^{-\sigma x N} F \in \mathcal{S}'(\mathbf{R}^n)$ and the equation $F = \mathrm{e}^{\sigma x N} \mathcal{F}^{-1}\big(P(\mathrm{i}\xi + \sigma N)^{-1}\big)$ hold for each $\sigma > \sigma_0$ if σ_0 is as in (2.2.3), see Proposition 2.4.13 below or Ortner and Wagner [209], Prop. 1, p. 530.

Let us show now that the fundamental solution E in (2.2.2) coincides with F fulfilling (2.2.4) if $\eta = N$ and the real numbers $\lambda_0, \ldots, \lambda_m$ in Proposition 2.2.1 are chosen larger than σ_0. In fact, with these choices, we obtain

$$\begin{aligned}
\mathrm{e}^{\lambda_j N x} \mathcal{F}_\xi^{-1}\left(\frac{\overline{P(\mathrm{i}\xi + \lambda_j N)}}{P(\mathrm{i}\xi + \lambda_j N)}\right) &= \mathrm{e}^{\lambda_j N x} \bar{P}(-\partial + \lambda_j N) \mathcal{F}_\xi^{-1}\left(\frac{1}{P(\mathrm{i}\xi + \lambda_j N)}\right) \\
&= \bar{P}(-\partial + 2\lambda_j N) \mathrm{e}^{\lambda_j N x} \mathcal{F}_\xi^{-1}\left(\frac{1}{P(\mathrm{i}\xi + \lambda_j N)}\right) \\
&= \bar{P}(-\partial + 2\lambda_j N) F = \left[\overline{P_m(2N)} \lambda_j^m + \sum_{k=0}^{m-1} Q_k(\partial) \lambda_j^k\right] F
\end{aligned}$$

and hence

$$\begin{aligned}
E &= \frac{1}{\overline{P_m(2N)}} \sum_{j=0}^{m} a_j\, \mathrm{e}^{\lambda_j N x} \mathcal{F}_\xi^{-1}\left(\frac{\overline{P(\mathrm{i}\xi + \lambda_j N)}}{P(\mathrm{i}\xi + \lambda_j N)}\right) \\
&= \frac{1}{\overline{P_m(2N)}} \sum_{j=0}^{m} a_j \left[\overline{P_m(2N)} \lambda_j^m + \sum_{k=0}^{m-1} Q_k(\partial) \lambda_j^k\right] F = F.
\end{aligned}$$

□

By means of formula (2.1.1), we can infer from the Malgrange–Ehrenpreis theorem for scalar operators (i.e., (2.2.1)) the existence of fundamental matrices for square systems of linear partial differential operators with constant coefficients, cf. also Malgrange [174], Prop. 6, p. 299; Agranovich [4], pp. 37, 38; Hörmander [136], pp. 94, 95.

Proposition 2.2.3 *For the system $A(\partial) \in \mathbf{C}[\partial]^{l\times l}$, the following four assertions are equivalent:*

(1) $A(\partial)$ *has a right-sided fundamental matrix;*
(2) $A(\partial)$ *has a left-sided fundamental matrix;*
(3) $A(\partial)$ *has a two-sided fundamental matrix;*
(4) $\det A(\partial)$ *does not vanish identically.*

Proof Trivially, (3) implies (1) and (2). Furthermore, if (4) is satisfied, then the Malgrange–Ehrenpreis theorem in the scalar case (see (2.2.1) or (2.2.2)) implies the existence of a fundamental solution F of $P(\partial) = \det A(\partial)$, and formula (2.1.1) then yields the two-sided fundamental matrix $E = A(\partial)^{\mathrm{ad}}F$ of the system $A(\partial)$. Hence (4) implies (3). It thus remains to show only that (1) implies (4). (Then, by symmetry, i.e., using transposition, also (2) will imply (4).)

If E is a right-sided fundamental matrix of $A(\partial)$, i.e., if $A(\partial)E = I_l\delta$, then, evidently, $A(\partial)$ cannot be the zero matrix. Let $k > 0$ denote the rank of the matrix $A(\partial)$ and assume, contrary to (4), that $k < l$. After a possible renumbering of the coordinates, we can suppose that the operator $Q(\partial) = \det\bigl(A_{ij}(\partial)_{i,j=2,\dots,k+1}\bigr)$ does not vanish identically. If $C(\partial) \in \mathbf{C}[\partial]^{l\times l}$ contains the adjoint matrix of $A_{ij}(\partial)_{i,j=1,\dots,k+1}$ in the rows and columns corresponding to $i,j = 1,\dots,k+1$, and consists of zeroes in the remaining places, then $C_{11}(\partial) = Q(\partial) \not\equiv 0$. On the other hand, $C(\partial)\delta = C(\partial)A(\partial)E$ must have a zero in the upper left corner since

$$(C_{ij}(\partial))_{i,j=1,\dots,k+1} \cdot (A_{ij}(\partial))_{i,j=1,\dots,k+1} = I_{k+1} \det\bigl((A_{ij}(\partial))_{i,j=1,\dots,k+1}\bigr) = 0,$$

A having rank k. This contradicts the assumption $k < l$ and thus shows that (1) implies (4). Hence the proof is complete. □

2.3 Temperate Fundamental Solutions

We next investigate the problem of the existence of a *temperate* fundamental solution E of an operator $P(\partial)$. After Fourier transformation, this problem is equivalent to the *division problem* $P(\mathrm{i}\xi) \cdot \mathcal{F}E = 1$ in $\mathcal{S}'(\mathbf{R}^n)$ formulated by L. Schwartz in 1950, cf. Gårding [93], p. 80; Schwartz [248], p. 9. Obviously, in the dense open subset $\mathbf{R}^n \setminus Z,\ Z := \{\xi \in \mathbf{R}^n;\ P(\mathrm{i}\xi) = 0\}$ of $\mathbf{R}^n$, the distribution $\mathcal{F}E$ must coincide with $P(\mathrm{i}\xi)^{-1}$. Hence the division problem consists in extending $P(\mathrm{i}\xi)^{-1} \in \mathcal{D}'(\mathbf{R}^n \setminus Z)$ to a distribution in $\mathcal{S}'(\mathbf{R}^n)$. The historically first solutions in Hörmander [135] (see Thm. 3, p. 567) and in Łojasiewicz [168, 169] were based on an estimate of $|P(\mathrm{i}\xi)|$ from below by powers of the distance from ξ to Z. These methods of proof rely on the so-called Hörmander–Łojasiewicz inequalities, Whitney's extension theorem and partitions of unity and hence do not produce explicit formulae for fundamental solutions, cf. the elaborated presentations in Treves [271], pp. 221–242; Krantz and Parks [156], pp. 115–135.

In 1969, Bernstein and Gel'fand [13] presented a new method of proof of the division problem relying on the analytic continuation of the function $\lambda \mapsto P^\lambda$, which continuation was posed as problem by I.M. Gel'fand at the International Congress of Mathematicians in 1954 (see Gel'fand [102], p. 262):

"…the following two problems are of interest:

I. Let $P(x_1, x_2, \dots, x_n)$ be a polynomial. Consider the area in which $P > 0$. Let $\varphi(x_1, \dots, x_n)$ be an infinitely many differentiable function equal to zero outside a certain finite area. We shall examine the functional

$$(P^\lambda \cdot \varphi) = \int_{P>0} P^\lambda(x_1, \dots, x_n)\varphi(x_1, \dots, x_n)dx_1 \dots dx_n.$$

It is necessary to prove that this is a meromorphic function of λ (it would be natural to call it a ζ-function of the given polynomial), whose poles are located in points forming several arithmetic progressions, as well as to calculate the residues of this function."

Whereas the proof in Bernstein and Gel'fand [13] is based on Hironaka's theorem on the resolution of singularities, I.N. Bernstein succeeded later to perform the analytic continuation of P^λ by means of a functional equation, similarly as for the gamma function where one uses the equation $\Gamma(\lambda) = \frac{\Gamma(\lambda+1)}{\lambda}$, see Bernstein [11, 12]. We shall employ this approach to prove the existence of temperate fundamental solutions.

Proposition 2.3.1 *Let $P(\partial) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha \in \mathbf{C}[\partial_1, \dots, \partial_n] \setminus \{0\}$ be a not identically vanishing linear differential operator with constant coefficients. Then $P(\partial)$ possesses a temperate fundamental solution, i.e., $\exists E \in \mathcal{S}'(\mathbf{R}^n) : P(\partial)E = \delta$.*

Proof

(1) Let us first assume that $P(\mathrm{i}\xi)$ is real-valued and non-negative, i.e., $\forall \xi \in \mathbf{R}^n : P(\mathrm{i}\xi) \ge 0$. Then, obviously, the mapping

$$F : \{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda > 0\} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : \lambda \longmapsto P(\mathrm{i}\xi)^\lambda \tag{2.3.1}$$

is well-defined and holomorphic. Bernstein's functional equation (see Bernstein [12], p. 273, Thm. 1'; Björk [15], Ch. 1, 5.7, 5.8) stipulates the existence of a differential operator $Q(\lambda, \xi, \partial) \in \mathbf{C}[\lambda, \xi_1, \dots, \xi_n, \partial_1, \dots, \partial_n]$ with polynomial coefficients and of a polynomial $b(\lambda) \in \mathbf{C}[\lambda]$ such that

$$Q(\lambda, \xi, \partial)P(\mathrm{i}\xi)^{\lambda+1} = b(\lambda)P(\mathrm{i}\xi)^\lambda \tag{2.3.2}$$

holds for all complex λ with $\operatorname{Re}\lambda > 0$. (The normalized polynomial $b(\lambda)$ of minimal degree such that Eq. (2.3.2) is fulfilled with a suitable Q is called the *Bernstein–Sato polynomial* of $P(\mathrm{i}\xi)$. A systematic study of Bernstein–Sato polynomials is contained in Yano [304].)

By means of the functional equation (2.3.2), we can holomorphically continue the function F in (2.3.1) to the whole complex plane $\mathbf{C}$ with the exception of the points in the arithmetic progressions $\{\lambda - k;\ b(\lambda) = 0,\ k \in \mathbf{N}_0\}$. In these exceptional points λ_0, $F(\lambda)$ is meromorphic, and we set $F(\lambda_0) := \operatorname{Pf}_{\lambda=\lambda_0} F(\lambda)$, cf. Definition 1.4.6, Proposition 1.4.7, which holds for every Hausdorff, quasicomplete locally convex space, and in particular for $\mathcal{S}'(\mathbf{R}^n)$. Therefore, for $\operatorname{Re}\lambda > -k,\ k \in \mathbf{N}_0$, we have

$$F(\lambda) = \operatorname{Pf}\Big[\frac{Q(\lambda,\xi,\partial)}{b(\lambda)}\frac{Q(\lambda+1,\xi,\partial)}{b(\lambda+1)}\cdots\frac{Q(\lambda+k-1,\xi,\partial)}{b(\lambda+k-1)}P(\mathrm{i}\xi)^{\lambda+k}\Big] \in \mathcal{S}'(\mathbf{R}^n).$$

By analytic continuation, the equation $P(\mathrm{i}\xi)F(\lambda) = F(\lambda+1)$ holds for each $\lambda \in \mathbf{C}$. In particular, $F(-1)$ solves the division problem $P(\mathrm{i}\xi)F(-1) = 1$, and $E = \mathcal{F}^{-1}\big(F(-1)\big)$ is a temperate fundamental solution of $P(\partial)$. (It is sometimes called *Bernstein's fundamental solution.*)

(2) For general $P(\partial)$, we set $E = \overline{P}(-\partial)E_1$ where E_1 is the temperate fundamental solution constructed in (1) of the operator $Q(\partial) = P(\partial)\overline{P}(-\partial)$. Note that $Q(\partial)$ has the symbol

$$Q(\mathrm{i}\xi) = P(\mathrm{i}\xi)\overline{P}(-\mathrm{i}\xi) = |P(\mathrm{i}\xi)|^2,$$

which is real-valued and non-negative. The proof is complete. □

Let us explain Bernstein's method of construction of temperate fundamental solutions by several explicit examples.

Example 2.3.2 Let us first consider the negative *Laplace operator* $P(\partial) = -\Delta_n$, similarly as in Dieudonné [59], 17.9.2; Horváth [146], Ex. 1, p. 176; Wagner [287], Bsp. 1, p. 413.

(a) In this case, $P(\mathrm{i}\xi) = |\xi|^2$ is non-negative, and $F(\lambda) = P(\mathrm{i}\xi)^\lambda = |\xi|^{2\lambda}$ is holomorphic in $\mathbf{C}\setminus\{-\frac{n}{2}-j;\ j \in \mathbf{N}_0\}$ and has simple poles in $\lambda = -\frac{n}{2}-j,\ j \in \mathbf{N}_0$. In fact, from Example 1.4.9, we obtain, for $k \in \mathbf{N}_0$,

$$\operatorname*{Res}_{\lambda=-(n+k)/2}|\xi|^{2\lambda} = \frac{1}{2}\operatorname*{Res}_{\lambda=-n-k}|\xi|^{\lambda} = \frac{1}{2}(-1)^k\sum_{|\alpha|=k}\frac{\langle\omega^\alpha,1\rangle}{\alpha!}\,\partial^\alpha\delta$$
$$= \begin{cases} 0 & : k \text{ odd},\\ \frac{\pi^{n/2}\Delta_n^j\delta}{2^{2j}j!\,\Gamma(\frac{n}{2}+j)} & : k \text{ even}, k = 2j.\end{cases}$$

For the last equation see Gel'fand and Shilov [104], Ch. I, 3.9, (5'), p. 73; Ortner and Wagner [219], Ex. 2.3.1, p. 41.

(b) In this case, the functional equation (2.3.2) is simple and reads as

$$\frac{1}{4}\Delta_n|\xi|^{2\lambda+2} = (\lambda+1)(\lambda+\tfrac{n}{2})|\xi|^{2\lambda}, \tag{2.3.3}$$

cf. Yano [304], (2), p. 112. Hence $Q(\lambda,\xi,\partial) = \frac{1}{4}\Delta_n$ is here independent of ξ and λ, and $b(\lambda) = (\lambda+1)(\lambda+\frac{n}{2})$.

As in the proof of Proposition 2.3.1, we set $F(\lambda_0) = \mathrm{Pf}_{\lambda=\lambda_0} F(\lambda)$ in the poles $\lambda_0 = -\frac{n}{2} - j,\ j \in \mathbf{N}_0$. Then the equation $P(\mathrm{i}\xi)F(\lambda) = F(\lambda+1)$ holds for each $\lambda \in \mathbf{C}$, and, therefore, $E(\lambda) = \mathcal{F}^{-1}\big(F(\lambda)\big)$ fulfills $-\Delta_n E(\lambda) = E(\lambda+1)$. On the other hand, if we apply the inverse Fourier transform to (2.3.2), then we obtain the *Bernstein–Sato recursion formula*

$$Q(\lambda, -\mathrm{i}\partial, -\mathrm{i}x)E(\lambda+1) = b(\lambda)E(\lambda). \tag{2.3.4}$$

(Equations (2.3.2) and (2.3.4) hold in all points $\lambda \in \mathbf{C}$ where $F(\lambda)$, the analytic continuation of $P(\mathrm{i}\xi)^\lambda$, is holomorphic.) In our case, (2.3.4) reads

$$-\tfrac{1}{4}|x|^2 \cdot E(\lambda+1) = (\lambda+1)(\lambda+\tfrac{n}{2})E(\lambda), \qquad \lambda \in \mathbf{C} \setminus \{-\tfrac{n}{2} - j;\, j \in \mathbf{N}_0\}. \tag{2.3.5}$$

(c) The distributions $E(\lambda)$ coincide with the elliptic M. Riesz kernels introduced and investigated in Example 1.6.11(b), i.e., $E(\lambda) = R_{-2\lambda}$. Note that Eq. (2.3.5) furnishes a new method to derive the fundamental solutions $E(-k)$ of the iterated operator $(-\Delta_n)^k,\ k \in \mathbf{N}$, from the fundamental solution $E(-1)$ of $-\Delta_n$ if $b(-2), \dots, b(-k)$ do not vanish, namely

$$E(-k) = \frac{Q(-k, -\mathrm{i}\partial, -\mathrm{i}x)}{b(-k)} \cdots \frac{Q(-2, -\mathrm{i}\partial, -\mathrm{i}x)}{b(-2)}\, E(-1). \tag{2.3.6}$$

Hence, if n is odd, or $k \in \mathbf{N},\ k < \frac{n}{2}$, then Example 1.3.14(a) and (2.3.6) imply that

$$E(-k) = \frac{\Gamma(\frac{n}{2} - k)}{2^{2k}(k-1)!\pi^{n/2}}\, |x|^{2k-n}$$

is a fundamental solution of $(-\Delta_n)^k$. This result agrees with (1.6.19).

Similarly, if the numbers $b(-j)$ do not vanish for $l+1 \le j \le k$, then

$$E(-k) = \frac{Q(-k, -\mathrm{i}\partial, -\mathrm{i}x)}{b(-k)} \cdots \frac{Q(-l-1, -\mathrm{i}\partial, -\mathrm{i}x)}{b(-l-1)}\, E(-l),$$

and this yields for $P(\partial) = -\Delta_n$, n even, $l = \frac{n}{2},\ k > l$,

$$E(-k) = \frac{(\frac{n}{2}-1)!}{(k-\frac{n}{2})!(k-1)!}\, \big(-\tfrac{1}{4}|x|^2\big)^{-n/2+k} E(-\tfrac{n}{2})$$

in agreement with (1.6.20). Note that—in both cases—formula (2.3.5) allows to derive the fundamental solutions $E(-k)$ of the iterated operators $P(\partial)^k$ from $E(-1)$ respectively from $E(-\frac{n}{2})$ simply by multiplications with powers of $|x|$. □

Let us consider now, more generally as in Proposition 2.3.1 and Example 2.3.2, powers of *complex-valued* polynomials.

Proposition 2.3.3 *Given a polynomial* $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha,\ a_\alpha \in \mathbf{C},\ \xi \in \mathbf{R}^n$, *of degree* m *and a measurable bounded function* $k : \{\xi \in \mathbf{R}^n;\ P(\xi) \neq 0\} \longrightarrow \mathbf{Z}$, *we set*

$$P(\xi)^\lambda := \begin{cases} 0 & : P(\xi) = 0, \\ \exp\bigl(\lambda[2\pi \mathrm{i} k(\xi) + \mathrm{i}\arg(P(\xi)) + \log|P(\xi)|]\bigr) & : P(\xi) \neq 0, \end{cases}$$

where $z = |z| \cdot \mathrm{e}^{\mathrm{i}\arg z},\ z \in \mathbf{C}$, *with* $\arg z \in (-\pi, \pi]$.

Then $P(\xi)^\lambda \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ *for all* $\lambda \in \mathbf{C}$ *with* $\operatorname{Re}\lambda > -\frac{1}{m}$, *and the mapping*

$$\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > -\tfrac{1}{m}\} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : \lambda \longmapsto P(\xi)^\lambda$$

is well-defined and holomorphic.

Proof Obviously, for $\operatorname{Re}\lambda > 0$, the function

$$|P(\xi)^\lambda| = \exp\bigl(-\operatorname{Im}\lambda[2\pi k(\xi) + \arg P(\xi)]\bigr) \cdot |P(\xi)|^{\operatorname{Re}\lambda}$$

is polynomially bounded and thus yields a temperate distribution. Furthermore, if $\phi \in \mathcal{S}(\mathbf{R}^n)$, then

$$\langle \phi, P(\xi)^\lambda \rangle = \int_{\mathbf{R}^n} \phi(\xi)\, P(\xi)^\lambda \, \mathrm{d}\xi$$

analytically depends on $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > 0$.

For negative real values of λ, we use the estimate

$$\exists C > 0 : \forall N > 0 : \forall \epsilon > 0 : \int_{|\xi|<N} Y(\epsilon - |P(\xi|)\, \mathrm{d}\xi \le C N^{n-1} \epsilon^{1/m}$$

to conclude that $\int_{|\xi|<N} |P(\xi)^\lambda|\, \mathrm{d}\xi$ is finite for $\operatorname{Re}\lambda > -\frac{1}{m}$ and grows at most polynomially if $N \to \infty$, cf. also Ricci and E.M. Stein [233], Prop., p. 182. □

Example 2.3.4 Let us first observe that Bernstein [12], p. 273, Thm. 1'; Björk [15], Ch. 1, 5.7, 5.8, prove the existence of polynomials Q, b for arbitrary complex-valued polynomials $P(\mathrm{i}\xi)$ such that the functional equation

$$Q(\lambda, \xi, \partial) P(\mathrm{i}\xi)^{\lambda+1} = b(\lambda) P(\mathrm{i}\xi)^\lambda \tag{2.3.2}$$

holds in the *algebraic sense*, i.e., if $P(\mathrm{i}\xi)^\lambda$ is considered as a symbol which is subject to the relations $P(\mathrm{i}\xi)P(\mathrm{i}\xi)^\lambda = P(\mathrm{i}\xi)^{\lambda+1}$ and $\partial_j P(\mathrm{i}\xi)^{\lambda+1} = (\lambda+1)P(\mathrm{i}\xi)^\lambda \cdot \frac{\partial P(\mathrm{i}\xi)}{\partial \xi_j}$. For *non-negative* polynomials $P(\mathrm{i}\xi)$, we set $k(\xi) = 0$ in Proposition 2.3.3. Then the algebraic validity of (2.3.2) implies that (2.3.2) also holds in $\mathcal{S}'(\mathbf{R}^n)$, first for large $\operatorname{Re}\lambda$ since there $P(\mathrm{i}\xi)^\lambda$ is sufficiently often differentiable, and then, by analytic continuation, for all complex λ unless λ is one of the poles of $F(\lambda) = P(\mathrm{i}\xi)^\lambda$. For *complex-valued* polynomials, the relation between the algebraic and the distributional validity of (2.3.2) is more complicated. Let us illustrate this fact in the simple case of the Cauchy–Riemann operator $P(\partial) = \partial_1 + \mathrm{i}\partial_2$.

For $P(\mathrm{i}\xi) = \mathrm{i}\xi_1 - \xi_2$, the equations

$$\partial_1(\mathrm{i}\xi_1-\xi_2)^{\lambda+1} = \mathrm{i}(\lambda+1)(\mathrm{i}\xi_1-\xi_2)^\lambda, \qquad \partial_2(\mathrm{i}\xi_1-\xi_2)^{\lambda+1} = -(\lambda+1)(\mathrm{i}\xi_1-\xi_2)^\lambda$$

hold in the algebraic sense. Let us define the distribution-valued function

$$F : \{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > -1\} \longrightarrow \mathcal{S}'(\mathbf{R}^2) : \lambda \longmapsto (\mathrm{i}\xi_1-\xi_2)^\lambda = \mathrm{e}^{\lambda\log(\mathrm{i}\xi_1-\xi_2)},$$

where $\log(\mathrm{i}\xi_1-\xi_2) = \log|\xi| + \mathrm{i}\arg(\mathrm{i}\xi_1-\xi_2)$, i.e., we choose $k \equiv 0$ in Proposition 2.3.3. Then, due to the discontinuity of $F(\lambda)$ along the half-line $\xi_1 = 0,\ \xi_2 > 0$, the jump formula (1.3.9) yields

$$\begin{aligned} &\partial_1\big(F(\lambda+1)\big) = \mathrm{i}(\lambda+1)F(\lambda) - 2\mathrm{i}\sin(\lambda\pi)\delta(\xi_1)\otimes Y(\xi_2)\xi_2^{\lambda+1},\\ \text{and } &\partial_2\big(F(\lambda+1)\big) = -(\lambda+1)F(\lambda) \end{aligned} \tag{2.3.7}$$

for $\operatorname{Re}\lambda > -1$. Since $F(0) = 1$ and thus $\partial_2\big(F(0)\big) = 0$, the second equation in (2.3.7) shows that $F(\lambda)$ can analytically be extended to the whole complex plane. Therefore (2.3.7) remains valid for all complex λ if $Y(\xi_2)\xi_2^{\lambda+1}$ is replaced by $\xi_{2+}^{\lambda+1}$. (Note that $\sin(\lambda\pi)\xi_{2+}^{\lambda+1}$ also depends holomorphically on λ.)

In particular, for $\lambda = -2$, we obtain

$$\partial_1\big(F(-1)\big) = -\mathrm{i}F(-2) - 2\pi\mathrm{i}\delta, \qquad \partial_2\big(F(-1)\big) = F(-2),$$

and hence $(\partial_1 + \mathrm{i}\partial_2)\big(F(-1)\big) = -2\pi\mathrm{i}\delta$, i.e.,

$$(\partial_1 + \mathrm{i}\partial_2)\frac{1}{2\pi(\xi_1 + \mathrm{i}\xi_2)} = \delta,$$

which is in accordance with Example 1.3.14(b). □

We consider next quasihyperbolic operators, for which the continuation of $\lambda \mapsto P(\mathrm{i}\xi)^\lambda$ to the whole complex plane can be achieved without poles, and without the use of Bernstein's equation (2.3.2).

Proposition 2.3.5 *For* $P(\partial) \in \mathbf{C}[\partial_1, \ldots, \partial_n]$ *and* $N \in \mathbf{R}^n \setminus \{0\}$ *suppose that*

$$\forall \sigma > 0 : \forall \xi \in \mathbf{R}^n : P(\mathrm{i}\xi + \sigma N) \neq 0, \tag{2.3.8}$$

i.e., $P(\partial)$ *is quasihyperbolic in the direction* N *with* $\sigma_0 = 0$*, see* (2.2.3)*. Furthermore, let the function*

$$X := \mathbf{R}^n \times (0, \infty) \longrightarrow \mathbf{C} : (\xi, \sigma) \longmapsto \log\big(P(\mathrm{i}\xi + \sigma N)\big)$$

be determined by the choice of a value at $(\xi, \sigma) = (0, 1)$ *and continuous extension in the simply connected space* X*.*

Then the distribution-valued function

$$F : \mathbf{C} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : \lambda \longmapsto F(\lambda) = \lim_{\sigma \searrow 0} P(\mathrm{i}\xi + \sigma N)^\lambda$$

is well-defined and entire. Furthermore, $E(\lambda) := \mathcal{F}^{-1}\big(F(\lambda)\big) \in \mathcal{S}'(\mathbf{R}^n)$ *fulfills*

$$\forall \lambda \in \mathbf{C} : \operatorname{supp} E(\lambda) \subset \{x \in \mathbf{R}^n;\ x \cdot N \geq 0\}$$

and $\forall k \in \mathbf{N}_0 : P(\partial)^k E(-k) = \delta$*, i.e.,* $E(-k)$ *is a temperate fundamental solution of* $P(\partial)^k$*.*

Proof

(a) An appeal to the Seidenberg–Tarski lemma, i.e., Lemma 2 in Hörmander [135], p. 557, furnishes that

$$|P(\mathrm{i}\xi + \sigma N)| \geq c\sigma^k (1 + |\xi|^2 + \sigma^2)^{-k} \tag{2.3.9}$$

for some positive constants c, k and all $\xi \in \mathbf{R}^n$ and $\sigma > 0$. This implies that $P(\mathrm{i}\xi + \sigma N)^\lambda \in \mathcal{S}'(\mathbf{R}^n_\xi)$ for all $\lambda \in \mathbf{C}$ and $\sigma > 0$. Furthermore, also the boundary value for $\sigma \searrow 0$ exists in $\mathcal{S}'(\mathbf{R}^n_\xi)$ due to Atiyah, Bott and Gårding [5], pp. 121–122; Hörmander [139], Thm. 3.1.15; Zuily [309], Exercise 52, p. 93. Hence F is well-defined.

Let U be the open complex right half-plane $U = \{z \in \mathbf{C};\ \operatorname{Re} z > 0\}$ and consider

$$T : U \times \mathbf{C} \longrightarrow \mathcal{S}'(\mathbf{R}^n) : (z, \lambda) \longmapsto P(\mathrm{i}\xi + zN)^\lambda.$$

Then T is holomorphic since this holds for the integrals

$$\int \phi(\xi) P(\mathrm{i}\xi + zN)^\lambda \, \mathrm{d}\xi, \qquad \phi \in \mathcal{S},$$

due to the estimate (2.3.9) and Lebesgue's theorem. By Morera's theorem, we conclude that $F(\lambda) = \lim_{\sigma \searrow 0} T(\sigma, \lambda)$ is entire.

(b) Let us next show that $E(\lambda)|_H = 0$ if $H := \{x \in \mathbf{R}^n;\ Nx < 0\}$. For that reason, let us define

$$G(z,\lambda) := \mathrm{e}^{zNx}\mathcal{F}_\xi^{-1}\big(P(\mathrm{i}\xi + zN)^\lambda\big),$$

which is a holomorphic function on $U \times \mathbf{C}$ with values in $\mathcal{D}'(\mathbf{R}^n)$. Since $G(z,\lambda) = G(\operatorname{Re} z, \lambda)$, this implies that G is independent of z. For $\lambda \in \mathbf{C}$ fixed, we then obtain

$$\begin{aligned} E(\lambda) = \mathcal{F}^{-1}(F(\lambda)) &= \lim_{\sigma\searrow 0} \mathcal{F}_\xi^{-1}\big(P(\mathrm{i}\xi + \sigma N)^\lambda\big) \\ &= \lim_{\sigma\searrow 0} \mathrm{e}^{\sigma Nx}\mathcal{F}_\xi^{-1}\big(P(\mathrm{i}\xi + \sigma N)^\lambda\big) = G(1,\lambda). \end{aligned}$$

This implies

$$E(\lambda) = \lim_{\sigma\to\infty} G(\sigma,\lambda) = \lim_{\sigma\to\infty}\Big[\mathrm{e}^{\sigma Nx}\sigma^l \cdot \sigma^{-l}\mathcal{F}^{-1}\big(P(\mathrm{i}\xi + \sigma N)^\lambda\big)\Big] = 0 \text{ in } \mathcal{D}'(H)$$

since $\mathrm{e}^{\sigma Nx}\sigma^l$ converges to 0 in $\mathcal{E}(H)$ for each $l \in \mathbf{N}$ and the set $\{\sigma^{-l}P(\mathrm{i}\xi + \sigma N)^\lambda;\ \sigma \geq 1\}$ is bounded in $\mathcal{S}'(\mathbf{R}^n_\xi)$ for suitable $l \in \mathbf{N}$. (More precisely, if $\operatorname{Re}\lambda \geq 0$, then we can choose any $l \geq m \cdot \operatorname{Re}\lambda$ if $m = \deg P$; for $\operatorname{Re}\lambda < 0$, one can either use the estimate (2.3.9) or argue by analytic continuation from the case $\operatorname{Re}\lambda > 0$.)

Finally, $P(\mathrm{i}\xi)^k \cdot F(-k) = 1$ implies $P(\partial)^k E(-k) = \delta$, and hence the proof is complete. □

Example 2.3.6 Let us investigate the entire function $\lambda \mapsto E(\lambda) = \mathcal{F}^{-1}\big(F(\lambda)\big)$ in Proposition 2.3.5 in the particular case of the *wave operator* $P(\partial) = \partial_t^2 - \Delta_n$. (Note that $E(-1)$ has already been calculated in Example 1.4.12 for $n = 2, 3$ and in Example 1.6.17 for general n.)

$P(\partial)$ is (quasi)hyperbolic in the direction $N = (1,0)$ since

$$P\big(\mathrm{i}(\tau,\xi) + \sigma N\big) = P(\mathrm{i}\tau + \sigma, \mathrm{i}\xi) = (\mathrm{i}\tau + \sigma)^2 + |\xi|^2 \neq 0$$

for $\sigma > 0$ and $(\tau,\xi) \in \mathbf{R}^{n+1}$. We observe that $(\mathrm{i}\tau + \sigma)^2 + |\xi|^2 \in \mathbf{C} \setminus (-\infty, 0]$ for $\sigma > 0$ and therefore

$$\log P\big(\mathrm{i}(\tau,\xi) + \sigma N\big) = \log\big|P\big(\mathrm{i}(\tau,\xi) + \sigma N\big)\big| + \mathrm{i}\arg P\big(\mathrm{i}(\tau,\xi) + \sigma N\big)$$

is a continuous function on $X = \mathbf{R}^{n+1}_{\tau,\xi} \times (0,\infty)$ if we take the argument of $P\big(\mathrm{i}(\tau,\xi) + \sigma N\big)$ in the interval $(-\pi,\pi)$. This yields

$$\lim_{\sigma\searrow 0}\log\big[(\mathrm{i}\tau + \sigma)^2 + |\xi|^2\big] = \log\big|\tau^2 - |\xi|^2\big| + \mathrm{i}\pi Y(\tau^2 - |\xi|^2)\operatorname{sign}\tau.$$

Hence, for $\operatorname{Re}\lambda > -1$, the distributions $F(\lambda) = \lim_{\sigma\searrow 0} P\big(\mathrm{i}(\tau,\xi)+\sigma N\big)^{\lambda}$ are locally integrable and given by

$$F(\lambda) = \big|\tau^2 - |\xi|^2\big|^{\lambda} \cdot \Big[Y(|\xi|^2-\tau^2) + Y(\tau^2-|\xi|^2)\cdot \mathrm{e}^{\mathrm{i}\pi\lambda\,\mathrm{sign}\,\tau}\Big].$$

(a) Let us calculate $E(\lambda) = \mathcal{F}^{-1}F(\lambda)$ first by partial Fourier transform, see Definition 1.6.15.

For $\sigma > 0$, $\operatorname{Re}\lambda < -\frac{1}{2}$ and fixed $\xi \in \mathbf{R}^n$, the function $\big[(\mathrm{i}\tau+\sigma)^2 + |\xi|^2\big]^{\lambda}$ is absolutely integrable with respect to τ, and its inverse Fourier transform is given by

$$\begin{aligned}\mathcal{F}_\tau^{-1}\Big(\big[(\mathrm{i}\tau+\sigma)^2+|\xi|^2\big]^{\lambda}\Big) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\tau}\big[(\mathrm{i}\tau+\sigma)^2+|\xi|^2\big]^{\lambda}\,\mathrm{d}\tau\\ &= \frac{\mathrm{e}^{-\sigma t}}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{e}^{pt}(p^2+|\xi|^2)^{\lambda}\,\mathrm{d}p.\end{aligned}$$

The last integral is a well-known inverse Laplace transform, see Badii and Oberhettinger [7], Part II, Eq. 4.27, p. 240, which gives

$$\mathcal{F}_\tau^{-1}\Big(\big[(\mathrm{i}\tau+\sigma)^2+|\xi|^2\big]^{\lambda}\Big) = \mathrm{e}^{-\sigma t}Y(t)\frac{\sqrt{\pi}}{\Gamma(-\lambda)}\Big(\frac{2|\xi|}{t}\Big)^{1/2+\lambda} J_{-1/2-\lambda}(|\xi|t).$$

Hence

$$E(\lambda) = \frac{\sqrt{\pi}}{\Gamma(-\lambda)}\,\mathcal{F}_\xi^{-1}\Big[Y(t)\Big(\frac{2|\xi|}{t}\Big)^{1/2+\lambda} J_{-1/2-\lambda}(|\xi|t)\Big]. \tag{2.3.10}$$

For $\operatorname{Re}\lambda < -\frac{1}{2}$, the right-hand side in (2.3.10) continuously depends on t, and we can fix t in order to perform the inverse Fourier transform with respect to ξ by means of the Poisson–Bochner formula (1.6.14). This yields, for $\operatorname{Re}\lambda < -n$,

$$\begin{aligned}E(\lambda) &= \frac{\sqrt{\pi}\,2^{1/2+\lambda}Y(t)|x|^{-n/2+1}}{\Gamma(-\lambda)(2\pi)^{n/2}t^{1/2+\lambda}}\int_0^\infty \rho^{\lambda+(n+1)/2}J_{n/2-1}(\rho|x|)\cdot J_{-1/2-\lambda}(\rho t)\,\mathrm{d}\rho\\ &= \frac{2^{2\lambda+1}Y(t-|x|)(t^2-|x|^2)^{-\lambda-(n+1)/2}}{\pi^{(n-1)/2}\Gamma(-\lambda)\Gamma(-\lambda-\frac{n-1}{2})}\end{aligned} \tag{2.3.11}$$

by Gradshteyn and Ryzhik [113], Eq. 6.575.1, p. 692. Note that the right-hand side in (2.3.11) is locally integrable for $\operatorname{Re}\lambda < -\frac{n-1}{2}$, and hence the same is true for $E(\lambda)$ and (2.3.11) holds for $\operatorname{Re}\lambda < -\frac{n-1}{2}$ by analytic continuation.

Let us remark that $Z_\lambda = E(-\frac{\lambda}{2})$ is traditionally called *hyperbolic Marcel Riesz kernel*, and that (2.3.11) is also given in Schwartz [246], Eq. (I, 3; 31),

p. 50; Atiyah, Bott and Gårding [5], (4.20), p. 147; Dieudonné [59], (17.9.4.5), p. 267; Riesz [234], p. 156; Riesz [235], p. 4.

As observed already by M. Riesz, the "composition law" $Z_\lambda * Z_\mu = Z_{\lambda+\mu}$ holds for all $\lambda, \mu \in \mathbf{C}$. In fact, $Z_\lambda, Z_\mu \in \mathcal{D}'_\Gamma = \{T \in \mathcal{D}'(\mathbf{R}^{n+1});\ \operatorname{supp} T \subset \Gamma\}$ if Γ is the forward wave cone $\Gamma = \{(t,x) \in \mathbf{R}^{n+1};\ t \geq |x|\}$, and hence Z_λ, Z_μ are convolvable by support, see Example 1.5.11. Furthermore, for $\operatorname{Re}\lambda < 1$, we have $Z_\lambda \in \mathcal{D}'_{L^2}$ (since $F(-\frac{\lambda}{2})$ is a polynomial times an L^2-function), and hence $Z_\lambda * Z_\mu = Z_{\lambda+\mu}$ holds by the exchange theorem Proposition 1.6.6 (5). This relation then persists for all complex λ, μ by analytic continuation.

Note that $E(-k),\ k \in \mathbf{N}$, is the only fundamental solution of the hyperbolic operator $(\partial_t^2 - \Delta_n)^k$ with support in the half-space $t \geq 0$, see Hörmander [138], Thm. 12.5.1, p. 120. If $k > \frac{n-1}{2}$, then $E(-k)$ is locally integrable and, according to (2.3.11), given by

$$E(-k) = \frac{2^{1-2k} Y(t-|x|)(t^2-|x|^2)^{k-(n+1)/2}}{(k-1)!\,\pi^{(n-1)/2}\Gamma(k-\frac{n-1}{2})}. \tag{2.3.12}$$

For $k \leq \frac{n-1}{2}$, we obtain the fundamental solution $E(-k)$ by analytically continuing $E(\lambda)$, $\operatorname{Re}\lambda < -\frac{n-1}{2}$, in (2.3.11) since $\lambda \mapsto E(\lambda)$ is entire. For $t \neq |x|$ and $k = 1$, the result coincides with the formulas in (1.6.26), (1.6.27).

(b) A second evaluation of the inverse Fourier transform of $F(\lambda)$ employs the Lorentz invariance and the homogeneity of $F(\lambda)$, which properties are passed on to $E(\lambda)$. Since, furthermore, $\operatorname{supp} E(\lambda)$ is contained in the half-space $\{(t,x) \in \mathbf{R}^{n+1};\ t \geq 0\}$ by Proposition 2.3.5, we conclude that

$$\forall \lambda \in \mathbf{C} \text{ with } \operatorname{Re}\lambda < -\tfrac{n-1}{2} : \exists c \in \mathbf{C} : E(\lambda) = cY(t-|x|)(t^2-|x|^2)^{-\lambda-(n+1)/2}.$$

Finally, in order to determine the constant $c = c(\lambda)$, we use formula (2.3.10) for $\operatorname{Re}\lambda < -n$, and we obtain

$$\begin{aligned} c = E(\lambda)(1,0) &= \frac{\sqrt{\pi}}{\Gamma(-\lambda)(2\pi)^n} \int_{\mathbf{R}^n} (2|\xi|)^{1/2+\lambda} J_{-1/2-\lambda}(|\xi|)\,\mathrm{d}\xi \\ &= \frac{2^{3/2+\lambda-n}}{\pi^{(n-1)/2}\Gamma(-\lambda)\Gamma(\frac{n}{2})} \int_0^\infty r^{-1/2+\lambda+n} J_{-1/2-\lambda}(r)\,\mathrm{d}r \\ &= \frac{2^{2\lambda+1}}{\pi^{(n-1)/2}\Gamma(-\lambda)\Gamma(-\lambda-\frac{n-1}{2})}. \end{aligned}$$

Therefrom (2.3.11) follows for $\operatorname{Re}\lambda < -\frac{n-1}{2}$ by analytic continuation.

(c) Let us eventually calculate $E(\lambda)$ by means of an improved version of the so-called *Cagniard–de Hoop method,* see de Hoop [132], Achenbach [2].

If $\operatorname{Re}\lambda < -\frac{n+1}{2}$ and $\sigma > 0$, then $\left[(\mathrm{i}\tau+\sigma)^2 + |\xi|^2\right]^\lambda \in L^1(\mathbf{R}^{n+1}_{\tau,\xi})$ and hence $E(\lambda)$ is given by the following absolutely convergent Fourier integral (see the proof of Proposition 2.3.5):

$$E(\lambda) = G(\sigma, \lambda) = \frac{e^{\sigma t}}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} e^{i t \tau} \left(\int_{\mathbf{R}^n} e^{i x \xi} \big[(i\tau + \sigma)^2 + |\xi|^2 \big]^{\lambda} \, d\xi \right) d\tau.$$

The substitution $p = i\tau + \sigma$ then yields

$$E(\lambda) = \frac{1}{2\pi i} \int_{\operatorname{Re} p = \sigma} e^{pt} \, U(p) \, dp,$$

where U denotes the analytic function

$$U : \{p \in \mathbf{C};\ \operatorname{Re} p > 0\} \longrightarrow \mathcal{S}'(\mathbf{R}^n_x) : p \longmapsto \mathcal{F}_{\xi}^{-1}\big((p^2 + |\xi|^2)^{\lambda} \big).$$

Let us represent $U(p)$ by a one-fold integral. If $p > 0$, then the scale transformations $\xi = p \cdot \eta,\ \eta = (s, \eta') \in \mathbf{R}^n$ and $\eta' = \sqrt{1 + s^2}\, \zeta \in \mathbf{R}^{n-1}$ yield, due to the rotational symmetry of $U(p)$, the following:

$$\begin{aligned} U(p) &= \frac{p^{2\lambda+n}}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i p x \eta} [1 + |\eta|^2]^{\lambda} \, d\eta \\ &= \frac{p^{2\lambda+n}}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i p |x| s} (1 + s^2)^{\lambda + (n-1)/2} \, ds \cdot \int_{\mathbf{R}^{n-1}} (1 + |\zeta|^2)^{\lambda} \, d\zeta. \end{aligned}$$

The inner integral can easily be evaluated:

$$\int_{\mathbf{R}^{n-1}} (1 + |\zeta|^2)^{\lambda} \, d\zeta = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^{\infty} (1 + t^2)^{\lambda} t^{n-2} \, dt = \frac{\pi^{(n-1)/2} \Gamma(-\lambda - \frac{n-1}{2})}{\Gamma(-\lambda)}.$$

Applying Cauchy's integral theorem we next deform the integration contour for s from the real axis to one along $s = iv \pm 0,\ 1 \le v < \infty$. This yields

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{i p |x| s} (1 + s^2)^{\lambda + (n-1)/2} \, ds \\ &\quad = -2 \sin\big(\pi(\lambda + \tfrac{n-1}{2})\big) \int_1^{\infty} e^{-p|x|v} (v^2 - 1)^{\lambda + (n-1)/2} \, dv \\ &\quad = -\frac{2}{|x|} \sin\big(\pi(\lambda + \tfrac{n-1}{2})\big) \int_{|x|}^{\infty} e^{-pt} \Big(\frac{t^2}{|x|^2} - 1 \Big)^{\lambda + (n-1)/2} \, dt. \end{aligned}$$

Making use of the complement formula for the gamma function we obtain

$$U(p) = \frac{p^{2\lambda+n}}{2^{n-1} \pi^{(n-1)/2} \Gamma(-\lambda) \Gamma(\lambda + \frac{n+1}{2}) |x|} \cdot \mathcal{L}_t \left(Y(t - |x|) \Big(\frac{t^2}{|x|^2} - 1 \Big)^{\lambda + (n-1)/2} \right)$$

where $\mathcal{L}_t(f) = \int_0^\infty f(t)\mathrm{e}^{-pt}\,\mathrm{d}t$ denotes the Laplace transform of f. Due to

$$p^{2\lambda+n} = \frac{1}{\Gamma(-2\lambda-n)}\,\mathcal{L}_t\big(Y(t)t^{-2\lambda-n-1}\big),$$

the convolution theorem for the Laplace transformation then yields

$$U(p) = \frac{1}{2^{n-1}\pi^{(n-1)/2}\Gamma(-\lambda)|x|}\times$$
$$\times\,\mathcal{L}_t\bigg(\frac{Y(t-|x|)}{\Gamma(-2\lambda-n)\Gamma(\lambda+\frac{n+1}{2})}\int_{|x|}^t (t-s)^{-2\lambda-n-1}\Big(\frac{s^2}{|x|^2}-1\Big)^{\lambda+(n-1)/2}\,\mathrm{d}s\bigg).$$

The definite integral therein is evaluated by means of Gröbner and Hofreiter [115], 421.4, p. 175:

$$\frac{Y(t-|x|)}{\Gamma(-2\lambda-n)\Gamma(\lambda+\frac{n+1}{2})}\int_{|x|}^t (t-s)^{-2\lambda-n-1}\Big(\frac{s^2}{|x|^2}-1\Big)^{\lambda+(n-1)/2}\,\mathrm{d}s$$
$$= \frac{2^{2\lambda+n}Y(t-|x|)|x|}{\Gamma(-\lambda-\frac{n-1}{2})}\,(t^2-|x|^2)^{-\lambda-(n+1)/2}.$$

Hence

$$U(p) = \frac{2^{2\lambda+1}\mathcal{L}_t\big(Y(t-|x|)(t^2-|x|^2)^{-\lambda-(n+1)/2}\big)}{\pi^{(n-1)/2}\Gamma(-\lambda)\Gamma(-\lambda-\frac{n-1}{2})},$$

and, due to $U = \mathcal{L}\big(E(\lambda)\big)$, we infer that

$$E(\lambda) = \frac{2^{2\lambda+1}Y(t-|x|)(t^2-|x|^2)^{-\lambda-(n+1)/2}}{\pi^{(n-1)/2}\Gamma(-\lambda)\Gamma(-\lambda-\frac{n-1}{2})}$$

for $\operatorname{Re}\lambda < -\frac{n+1}{2}$ in accordance with (2.3.11). □

Example 2.3.7 Let us generalize the last example so as also to cover the convolution group $E(\lambda)$ of the *Klein–Gordon operator* $P(\partial) = \partial_t^2 - \Delta_n + m^2,\ m > 0$.

Similarly as in Example 2.3.6 (a), we use partial Fourier transformation with respect to t and x. As above, we have $E(\lambda) = \mathcal{F}^{-1}\big(F(\lambda)\big)$ where

$$F(\lambda) = \big|\tau^2 - |\xi|^2 + m^2\big|^\lambda\cdot\Big[Y(|\xi|^2-\tau^2+m^2) + Y(\tau^2-|\xi|^2-m^2)\cdot\mathrm{e}^{\mathrm{i}\pi\lambda\,\mathrm{sign}\,\tau}\Big]$$

and

$$\mathcal{F}_\tau^{-1}\big(F(\lambda)\big) = \frac{\sqrt{\pi}\,Y(t)}{\Gamma(-\lambda)}\left(\frac{2\sqrt{|\xi|^2+m^2}}{t}\right)^{1/2+\lambda} J_{-1/2-\lambda}\big(t\sqrt{|\xi|^2+m^2}\big), \quad \operatorname{Re}\lambda < 0. \tag{2.3.13}$$

Hence, applying the Poisson–Bochner formula (1.6.14) yields, for $\operatorname{Re}\lambda < -n$,

$$E(\lambda) = \frac{\sqrt{\pi}\,2^{1/2+\lambda}Y(t)|x|^{-n/2+1}}{\Gamma(-\lambda)(2\pi)^{n/2}t^{1/2+\lambda}}\times \\ \times\int_0^\infty \rho^{n/2}(\rho^2+m^2)^{\lambda/2+1/4}J_{n/2-1}(\rho|x|)\cdot J_{-1/2-\lambda}\big(t\sqrt{\rho^2+m^2}\big)\,\mathrm{d}\rho.$$

Finally, Gradshteyn and Ryzhik [113], Eq. 6.596.6, p. 706, furnishes

$$E(\lambda) = \frac{2^{\lambda-(n-1)/2}Y(t-|x|)m^{\lambda+(n+1)/2}}{\pi^{(n-1)/2}\Gamma(-\lambda)}\,(t^2-|x|^2)^{-\lambda/2-(n+1)/4}\times \\ \times J_{-\lambda-(n+1)/2}\big(m\sqrt{t^2-|x|^2}\big). \tag{2.3.14}$$

As in Example 2.3.6, formula (2.3.14) holds for $\operatorname{Re}\lambda < -\frac{n-1}{2}$ by analytic continuation.

In particular, for $n = 2$ and $\lambda = -1$, we obtain the fundamental solution of the Klein–Gordon operator $\partial_t^2 - \Delta_2 + m^2$ in two space dimensions:

$$E(-1) = \frac{Y(t-|x|)}{2\pi\sqrt{t^2-|x|^2}}\,\cos\big(m\sqrt{t^2-|x|^2}\big) \in L^1_{\mathrm{loc}}(\mathbf{R}^3_{t,x}). \tag{2.3.15}$$

(As always for hyperbolic operators, this is the only fundamental solution with support in the half-space $t \geq 0$.)

For $n = 3$, the calculation of $E(-1)$ is slightly more difficult since the assumption $\operatorname{Re}\lambda < -\frac{n-1}{2}$ is not satisfied for $\lambda = -1$. From the equations

$$\partial_\tau F(\lambda+1) = -2(\lambda+1)\tau F(\lambda) \quad \text{and} \quad E(\lambda) = \mathcal{F}^{-1}\big(F(\lambda)\big),$$

we deduce $tE(\lambda+1) = -2(\lambda+1)\partial_t E(\lambda)$. This implies, for $n = 3$ and $t \neq 0$,

$$E(-1) = \frac{2}{t}\frac{\partial}{\partial t}E(-2) = \frac{2}{t}\frac{\partial}{\partial t}\Big[\frac{Y(t-|x|)}{8\pi}\,J_0\big(m\sqrt{t^2-|x|^2}\big)\Big] \\ = \frac{\delta(t-|x|)}{4\pi t} - \frac{mY(t-|x|)}{4\pi\sqrt{t^2-|x|^2}}\,J_1\big(m\sqrt{t^2-|x|^2}\big). \tag{2.3.16}$$

(For the definition of $\frac{1}{t}\,\delta(t-|x|)$, see Example 1.4.12.) Since $E(\lambda) \in \mathcal{C}\big(\mathbf{R}^1_t, \mathcal{D}'(\mathbf{R}^n)\big)$ for $\operatorname{Re}\lambda < -\frac{1}{2}$ by Eq. (2.3.13), formula (2.3.16) yields a representation of the fundamental solution $E(-1)$ of the Klein–Gordon operator $\partial_t^2 - \Delta_3 + m^2$ which is valid in $\mathcal{D}'(\mathbf{R}^4)$.

In general, $E(-k)$ is the unique fundamental solution of the iterated Klein–Gordon operator $(\partial_t^2 - \Delta_3 + m^2)^k,\ k \in \mathbf{N}$, with support in the half-space $t \geq 0$ according to Proposition 2.3.5. □

Example 2.3.8 Let us next consider quasihyperbolic operators in $\mathbf{R}^{n+1}$ of the form $P(\partial) = \partial_t + R(\partial_1, \dots, \partial_n)$, which contain as particular cases the *heat operator* and the *Schrödinger operator*.

(a) If $N = (1, 0, \dots, 0)$, then the condition (2.3.8) of quasihyperbolicity takes the form

$$\forall \sigma > 0 : \forall (\tau, \xi) \in \mathbf{R}^{n+1} : \sigma + \mathrm{i}\tau + R(\mathrm{i}\xi) \neq 0,$$

which is equivalent to

$$\inf\{\operatorname{Re} R(\mathrm{i}\xi);\ \xi \in \mathbf{R}^n\} \geq 0. \tag{2.3.17}$$

If condition (2.3.17) is satisfied, then the numbers $z = \sigma + \mathrm{i}\tau + R(\mathrm{i}\xi)$ belong to the complex half-plane $\operatorname{Re} z > 0$ for $\sigma > 0,\ (\tau, \xi) \in \mathbf{R}^{n+1}$, and we can take the usual determination of z^λ for $\lambda \in \mathbf{C}$. Thus, by Proposition 2.3.5, the convolution group of $P(\partial)$ is defined by

$$E(\lambda) = \mathcal{F}^{-1}\big(F(\lambda)\big) = \lim_{\sigma \searrow 0} \mathcal{F}^{-1}\Big[\big(\sigma + \mathrm{i}\tau + R(\mathrm{i}\xi)\big)^\lambda\Big].$$

For $\operatorname{Re}\lambda < 0$ and $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$, the function $t \mapsto Y(t)\mathrm{e}^{-zt}t^{-\lambda-1} = \mathrm{e}^{-zt}t_+^{-\lambda-1}$ is integrable and its Fourier transform is

$$\mathcal{F}_t\big(\mathrm{e}^{-zt}t_+^{-\lambda-1}\big) = \int_0^\infty \mathrm{e}^{-(z+\mathrm{i}\tau)t}t^{-\lambda-1}\,\mathrm{d}t = \Gamma(-\lambda)(z+\mathrm{i}\tau)^\lambda.$$

Hence

$$\mathcal{F}_\tau^{-1}\big((z+\mathrm{i}\tau)^\lambda\big) = \frac{\mathrm{e}^{-zt}}{\Gamma(-\lambda)}\,t_+^{-\lambda-1}. \tag{2.3.18}$$

Note that (2.3.18) holds for each $\lambda \in \mathbf{C}$ since the left-hand side is obviously entire in λ, and so is the right-hand side if one takes into account that $t_+^{-\lambda-1}$ and $\Gamma(-\lambda)$ have both simple poles at $\lambda \in \mathbf{N}_0$, cf. Example 1.4.8.

By partial Fourier transformation, we conclude that

$$
\begin{aligned}
E(\lambda) &= \mathcal{F}_\xi^{-1}\mathcal{F}_\tau^{-1}\big(F(\lambda)\big) = \mathcal{F}_\xi^{-1}\bigg(\lim_{\sigma\searrow 0}\bigg[\frac{\mathrm{e}^{-(\sigma+R(\mathrm{i}\xi))t}}{\Gamma(-\lambda)}\, t_+^{-\lambda-1}\bigg]\bigg) \\
&= \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-R(\mathrm{i}\xi)t}\big)\cdot\frac{t_+^{-\lambda-1}}{\Gamma(-\lambda)}.
\end{aligned}
\tag{2.3.19}
$$

We observe that the function

$$
\mathbf{R} \longrightarrow \mathcal{S}'(\mathbf{R}^n_\xi) : t \longmapsto \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-R(\mathrm{i}\xi)t}\big)
$$

is infinitely differentiable and hence can be multiplied with $t_+^{-\lambda-1}/\,\Gamma(-\lambda)$.

Let us mention that the convolution equation $E(\lambda) * E(\mu) = E(\lambda+\mu)$ holds generally for all complex λ, μ if $P(\partial)$ is quasihyperbolic, see Ortner and Wagner [218], Prop., p. 147. However, in the non-hyperbolic case, the convolvability of $E(\lambda), E(\mu)$ is more difficult to establish; it relies on the fact that $E(\lambda) \in \mathcal{D}'_{[0,\infty)}\hat{\otimes}\,\mathcal{O}'_C(\mathbf{R}^n_x)$, where $\mathcal{D}'_{[0,\infty)} = \{T \in \mathcal{D}'(\mathbf{R}^1);\, \operatorname{supp} T \subset [0,\infty)\}$, cf. Ortner and Wagner [219], Section 3.7, p. 114, and $\mathcal{O}'_C(\mathbf{R}^n) = \cap_{k\in\mathbf{N}}(1+|x|^2)^{-k}\mathcal{D}'_{L^\infty}(\mathbf{R}^n)$, see Schwartz [246], p. 244.

In particular, the equation $E(k) * E(-k) = \delta,\ k \in \mathbf{N}$, shows that

$$
E(-k) = \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-R(\mathrm{i}\xi)t}\big)\cdot\frac{t_+^{k-1}}{(k-1)!}
$$

is a fundamental solution of $\big(\partial_t + R(\partial)\big)^k$. As we will see in Proposition 2.4.13 below, this fundamental solution is the only one which is temperate and vanishes for $t < 0$.

(b) Let us specialize the above now to the heat and the Schrödinger operator, respectively.

If we set $R(\xi) = -|\xi|^2$, then we obtain the convolution group of the *heat operator* $\partial_t - \Delta_n$. From (2.3.19) and (1.6.24), we obtain

$$
E(\lambda) = \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-|\xi|^2 t}\big)\cdot\frac{t_+^{-\lambda-1}}{\Gamma(-\lambda)} = \frac{\mathrm{e}^{-|x|^2/(4t)}\cdot t_+^{-n/2-1-\lambda}}{\Gamma(-\lambda)(4\pi)^{n/2}}.
$$

Hence the locally integrable functions

$$
E(-k) = \frac{Y(t)t^{-n/2-1+k}\mathrm{e}^{-|x|^2/(4t)}}{(k-1)!(4\pi)^{n/2}}
$$

are the fundamental solutions of $(\partial_t - \Delta_n)^k,\ k \in \mathbf{N}$, cf. (1.3.14) and Example 1.6.16 for the case $k = 1$.

For the *Schrödinger operator* $\partial_t - \mathrm{i}\Delta_n$, we have $R(\mathrm{i}\xi) = \mathrm{i}|\xi|^2$ and

$$E(\lambda) = \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-\mathrm{i}|\xi|^2 t}\big) \cdot \frac{t_+^{-\lambda-1}}{\Gamma(-\lambda)} = \frac{\mathrm{e}^{\mathrm{i}|x|^2/(4t)-\mathrm{i}n\pi/4} \cdot t_+^{-n/2-1-\lambda}}{\Gamma(-\lambda)(4\pi)^{n/2}}, \tag{2.3.20}$$

see Example 1.6.14.

The multiplication in these products is understood as explained in (a), i.e., for $\phi \in \mathcal{D}(\mathbf{R}^{n+1}_{t,x})$, we set

$$\langle \phi, E(\lambda)\rangle = (4\pi)^{-n/2}\mathrm{e}^{-\mathrm{i}n\pi/4} \cdot \langle t^{-n/2} \int \phi(t,x)\mathrm{e}^{\mathrm{i}|x|^2/(4t)}\,\mathrm{d}x, \frac{t_+^{-\lambda-1}}{\Gamma(-\lambda)}\rangle.$$

Note that $E(\lambda)$ is not locally integrable for $\operatorname{Re}\lambda \geq -\frac{n}{2}$. In particular, the fundamental solution $E(-1)$ of the Schrödinger operator $\partial_t - \mathrm{i}\Delta_n$ is locally integrable only for $n = 1$ and else is given by the iterated, not absolutely convergent integral

$$\langle \phi, E(-1)\rangle = \frac{\mathrm{e}^{-\mathrm{i}n\pi/4}}{(4\pi)^{n/2}} \int_0^\infty \Big(\int_{\mathbf{R}^n} \phi(t,x)\mathrm{e}^{\mathrm{i}|x|^2/(4t)}\,\mathrm{d}x\Big)\frac{\mathrm{d}t}{t^{n/2}},$$

cf. Treves [274], 6.2, p. 45.

(c) Following S.L. Sobolev, let us finally consider the quasihyperbolic operator $\partial_t - \partial_1\partial_2\partial_3$ and construct its fundamental solution E. By (2.3.19), we have

$$E = \mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-\mathrm{i}\xi_1\xi_2\xi_3 t}\big) \cdot Y(t).$$

Since the mapping

$$\mathbf{R}^2_{\xi_1,\xi_2} \longrightarrow \mathcal{S}'(\mathbf{R}^1_{\xi_3}) : (\xi_1,\xi_2) \longmapsto \mathrm{e}^{-\mathrm{i}\xi_1\xi_2\xi_3 t}$$

is continuous, we can apply the partial Fourier transform (see Definition 1.6.15), and we conclude that

$$\mathcal{F}_\xi^{-1}\big(\mathrm{e}^{-\mathrm{i}\xi_1\xi_2\xi_3 t}\big) = \mathcal{F}^{-1}_{\xi_1,\xi_2}\mathcal{F}^{-1}_{\xi_3}\big(\mathrm{e}^{-\mathrm{i}\xi_1\xi_2\xi_3 t}\big) = \mathcal{F}^{-1}_{\xi_1,\xi_2}\big[\delta(x_3 - t\xi_1\xi_2)\big].$$

Note that, for fixed $t > 0$,

$$\delta(x_3 - t\xi_1\xi_2) \in \mathcal{C}\big(\mathbf{R}^2_{\xi_1,\xi_2}, \mathcal{S}'(\mathbf{R}^1_{x_3})\big) \cap \mathcal{S}'(\mathbf{R}^3_{\xi_1,\xi_2,x_3})$$

and also

$$\delta(x_3 - t\xi_1\xi_2) \in \mathcal{C}\big(\mathbf{R}^1_{x_3}, \mathcal{S}'(\mathbf{R}^2_{\xi_1,\xi_2})\big) \cap \mathcal{S}'(\mathbf{R}^3_{\xi_1,\xi_2,x_3}).$$

Therefore, we can fix now x_3 in order to evaluate the partial Fourier transform $\mathcal{F}^{-1}_{\xi_1,\xi_2}\big[\delta(x_3 - t\xi_1\xi_2)\big]$. This inverse Fourier transform of a delta distribution along the hyperbola $\xi_1\xi_2 = \frac{x_3}{t}$ has already been calculated in Example 1.6.18 up to a linear transformation. There we have shown that

$$\mathcal{F}^{-1}_{\eta}\big(Y(\eta_1)\delta(\eta_1^2 - \eta_2^2 - m^2)\big) = \frac{Y(|y_2| - |y_1|)}{4\pi^2} K_0\big(m\sqrt{y_2^2 - y_1^2}\big)$$
$$+ \frac{Y(|y_1| - |y_2|)}{8\pi}\Big[-N_0\big(m\sqrt{y_1^2 - y_2^2}\big) + \mathrm{i}\,\mathrm{sign}\, y_1 \cdot J_0\big(m\sqrt{y_1^2 - y_2^2}\big)\Big] \in L^1_{\mathrm{loc}}(\mathbf{R}^2_y),$$

see (1.6.28).

By Proposition 1.6.6 (1), $\mathcal{F}^{-1}(T \circ A) = \frac{1}{|\det A|}(\mathcal{F}^{-1}T) \circ A^{-1T}$ for $T \in \mathcal{S}'(\mathbf{R}^n)$, $A \in \mathrm{Gl}_n(\mathbf{R})$. Hence, if we set

$$T = Y(\eta_1)\delta(\eta_1^2 - \eta_2^2 - m^2) \qquad \text{and} \qquad A\xi = \frac{1}{2}(\xi_1 + \xi_2, \xi_1 - \xi_2)^T = \eta,$$

we infer

$$\mathcal{F}^{-1}\big(Y(\xi_1 + \xi_2)\delta(\xi_1\xi_2 - m^2)\big) = \frac{Y(-x_1x_2)}{2\pi^2} K_0\big(2m\sqrt{-x_1x_2}\big)$$
$$+ \frac{Y(x_1x_2)}{4\pi}\big[-N_0\big(2m\sqrt{x_1x_2}\big) + \mathrm{i}\,\mathrm{sign}(x_1 + x_2) \cdot J_0\big(2m\sqrt{x_1x_2}\big)\big].$$

Adding this with the distribution reflected at the origin yields

$$\mathcal{F}^{-1}\big(\delta(\xi_1\xi_2 - m^2)\big) = \frac{Y(-x_1x_2)}{\pi^2} K_0\big(2m\sqrt{-x_1x_2}\big) - \frac{Y(x_1x_2)}{2\pi} N_0\big(2m\sqrt{x_1x_2}\big).$$

Finally, upon distinguishing the cases $x_3 > 0$ and $x_3 < 0$, we arrive at

$$E = Y(t)\mathcal{F}^{-1}_{\xi_1,\xi_2}\big[\delta(x_3 - t\xi_1\xi_2)\big]$$
$$= Y(t)\Big[\frac{Y(-x_1x_2x_3)}{\pi^2 t} K_0\big(2\sqrt{-x_1x_2x_3/t}\big) - \frac{Y(x_1x_2x_3)}{2\pi t} N_0\big(2\sqrt{x_1x_2x_3/t}\big)\Big]. \tag{2.3.21}$$

Note that—in contrast to the Schrödinger operator—the fundamental solution E in (2.3.21) is locally integrable. This was observed already in Sobolev [255], p. 1247, where E is derived by introducing the similarity variable $\frac{x_1x_2x_3}{t}$ and by performing the "ansatz" $E = \frac{Y(t)}{t}\Lambda(\frac{x_1x_2x_3}{t})$. The ensuing third-order ordinary differential equation for Λ splits and yields Bessel functions, see Example 2.6.4 below. Note that two errors with respect to signs should be corrected in Sobolev's final result, see Sobolev [255], (8), p. 1247. A further derivation of the formula in (2.3.21) is given in Ortner [205], see Prop. 6, p. 158.

We also remark that Sobolev's operator $\partial_t - \partial_1\partial_2\partial_3$ serves as a prototype of q-hyperbolic operators (here $q = \frac{3}{2}$) introduced and studied in Gindikin [107], pp. 6, 71. □

Let us next investigate fundamental solutions of *homogeneous* differential operators. For the particular case of elliptic homogeneous operators, we refer to Hörmander [139], Thm. 7.1.20, p. 169.

Proposition 2.3.9 *If* $P(\partial) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$ *is a linear differential operator which is homogeneous of degree* $m \in \mathbf{N}$, *then there exists a fundamental solution* E *which is associated homogeneous of degree* $m-n$. *More precisely, if* $m < n$, *then* $E = F \cdot |x|^{m-n}$, *and if* $m \geq n$, *then* $E = F \cdot |x|^{m-n} + Q(x) \log|x|$, *where* $F \in \mathcal{D}'(\mathbf{S}^{n-1})$ *and* Q *is a homogeneous polynomial of degree* $m-n$. (*Recall that* $\langle \phi, F \cdot |x|^\lambda \rangle = \langle \langle \phi(t\omega), F(\omega) \rangle, t_+^{\lambda+n-1} \rangle$, $\lambda \in \mathbf{C}$, $\phi \in \mathcal{D}(\mathbf{R}^n)$, *see* Example 1.4.9.)

Proof

(a) Let us first consider the following division problem on the sphere:

$$P(\mathrm{i}\omega) \cdot U = 1, \quad U \in \mathcal{D}'(\mathbf{S}^{n-1}),$$

cf. Gårding [89], p. 407.

Without restriction, we may assume that $P(N) \neq 0$ for $N = (0, \ldots, 0, 1)$. We employ the stereographic projection

$$p : \mathbf{R}^{n-1} \longrightarrow \mathbf{S}^{n-1} \setminus \{N\} : \eta \longmapsto \frac{1}{1+|\eta|^2}(2\eta, |\eta|^2 - 1)$$

in order to transform the equation $P(\mathrm{i}\omega) \cdot U = 1$ into

$$P(2\eta, |\eta|^2 - 1) \cdot \mathrm{i}^m (1 + |\eta|^2)^{-m} p^*(U) = 1.$$

By Proposition 2.3.1 and by the identification of temperate distributions on $\mathbf{R}^{n-1}$ with distributions on $\mathbf{S}^{n-1}$ (see Schwartz [246], Ch. VII, Thm. V, p. 238), we obtain $U \in \mathcal{D}'(\mathbf{S}^{n-1})$ which solves $P(\mathrm{i}\omega) \cdot U = 1$ on $\mathbf{S}^{n-1} \setminus \{N\}$. Finally, near N, U is uniquely determined by $P(\mathrm{i}\omega) \cdot U = 1$ due to $P(N) \neq 0$.

(b) If we define, as in (1.4.3), $V = U \cdot |\xi|^{-m} \in \mathcal{S}'(\mathbf{R}^n)$ by

$$\langle \phi, V \rangle = \langle \langle \phi(t\omega), U(\omega) \rangle, t_+^{-m+n-1} \rangle, \qquad \phi \in \mathcal{S}'(\mathbf{R}^n),$$

then $P(\mathrm{i}\xi) \cdot V = 1$ holds in $\mathcal{S}'(\mathbf{R}^n)$. Hence $E = \mathcal{F}^{-1}V$ is a fundamental solution of $P(\partial)$.

(c) If $m < n$, then V is homogeneous in $\mathbf{R}^n$ of degree $-m$ and hence E is homogeneous of degree $m-n$. By Gårding [89], Lemmes 1.5, 4.1, pp. 393, 400, or Ortner and Wagner [219], Thm. 2.5.1, p. 58, $E = \mathcal{F}^{-1}V$ can be cast in the form $E = F \cdot |x|^{m-n}$ for some $F \in \mathcal{D}'(\mathbf{S}^{n-1})$.

(d) If $m \geq n$, then V is still homogeneous in $\mathbf{R}^n \setminus \{0\}$, but can cease to be homogeneous in $\mathbf{R}^n$, cf. Examples 1.2.10, 1.4.10 for the case of $m = n$.

Generally, by analytic continuation, $U \cdot |\xi|^\lambda$ is homogeneous of degree λ where this function of λ is analytic, i.e. for $\lambda \in \mathbf{C} \setminus \{-n, -n-1, \dots\}$. In the possible poles $\lambda = -m,\ m \geq n$, we set

$$R := \operatorname*{Res}_{\lambda=-m} (U \cdot |\xi|^\lambda) = (-1)^{m-n} \sum_{\substack{\alpha\in\mathbf{N}_0^n \\ |\alpha|=m-n}} \frac{1}{\alpha!} \langle \omega^\alpha, U \rangle\, \partial^\alpha \delta, \tag{2.3.22}$$

see Example 1.4.9, and we have

$$V = \operatorname*{Pf}_{\lambda=-m} (U \cdot |\xi|^\lambda) = \lim_{\lambda\to-m} \Big[U \cdot |\xi|^\lambda - \frac{R}{\lambda+m} \Big].$$

From this we conclude that

$$\begin{aligned} V(c\xi) &= \lim_{\lambda\to-m} \Big[(U \cdot |\xi|^\lambda)(c\xi) - c^{-m} \frac{R}{\lambda+m} \Big] \\ &= c^{-m} \lim_{\lambda\to-m} \Big[U \cdot |\xi|^\lambda - \frac{R}{\lambda+m} \Big] + \lim_{\lambda\to-m} (c^\lambda - c^{-m}) U \cdot |\xi|^\lambda \\ &= c^{-m} V + c^{-m} (\log c) R \end{aligned}$$

for $c > 0$, cf. Ortner and Wagner [219], (2.5.1), p. 59.

Hence V is *associated homogeneous* of order m in $\mathbf{R}^n$, i.e., $\forall c > 0 : V(c\xi) = c^{-m} V + c^{-m} (\log c) R$ with R homogeneous. Due to Proposition 1.6.6 (1), this implies

$$E(cx) = c^{m-n} E - c^{m-n} (\log c) \cdot \mathcal{F}^{-1} R, \qquad c > 0, \tag{2.3.23}$$

where $Q := -\mathcal{F}^{-1} R$ is a homogeneous polynomial of degree $m - n$. By the structure theorem Prop. 2.5.3 in Ortner and Wagner [219] (cf. also Grudzinski [119], Thm. 4.25', p. 178), we conclude that E has the representation $E = F \cdot |x|^{m-n} + Q(x) \log |x|$ for some $F \in \mathcal{D}'(\mathbf{S}^{n-1})$. This completes the proof. $\square$

Example 2.3.10

(a) Reconsidering the case of $P(\partial) = (-\Delta_n)^k$, let us comment on the structure of the fundamental solution E given explicitly in (1.6.19) and (1.6.20). Indeed, E is homogeneous if the degree $m = 2k$ of $P(\partial)$ satisfies $m < n$. On the other hand, if $m \geq n$ and n is even, then E is equal to a polynomial times a logarithm, see (1.6.20). However, if n is odd, then there is no logarithmic term present since the residue R in (2.3.22) vanishes due to $\langle \omega^\alpha, 1 \rangle = 0$ for $\alpha \in \mathbf{N}_0^n$ of degree $|\alpha| = m - n = 2k - n$, this degree being odd.

(b) Similarly, for a homogeneous *quasihyperbolic* operator, the logarithmic term disappears if the solution U of the division problem $P(\mathrm{i}\omega) \cdot U = 1$ is chosen as in Proposition 2.3.5, i.e., $U = \lim_{\sigma\searrow 0} P(\mathrm{i}\omega + \sigma N)^{-1}$. Then $V = U \cdot |\xi|^{-m}$ must be homogeneous. In fact, as we have seen in the proof of Proposition 2.3.5,

$\lim_{\sigma\searrow 0} P(\mathrm{i}\xi + \sigma N)^\lambda$ is entire and coincides with $U \cdot |\xi|^\lambda$ for $\operatorname{Re}\lambda > 0$. Thus also $\lambda \mapsto U \cdot |\xi|^\lambda$ is entire and $\operatorname{Res}_{\lambda=-m} U \cdot |\xi|^\lambda = 0$. □

2.4 Uniqueness and Representations of Fundamental Solutions

In the following, we investigate properties of distributions which imply uniqueness for fundamental solutions, namely

(i) growth and decay properties,

(ii) support properties.

We first consider systems with non-vanishing symbol, cf. Petersen [228], Lemma 8.6 B, p. 296; Szmydt and Ziemian [267], Prop. 2, p. 219; Gindikin and Volevich [108], p. 16, [109], p. 58, for the scalar case.

Proposition 2.4.1 *Let $A(\partial) \in \mathbf{C}[\partial]^{l\times l}$ be a quadratic system of linear partial differential operators in $\mathbf{R}^n$. Then the following conditions are equivalent:*

(1) *$A(\partial)$ has one and only one two-sided fundamental matrix in $\mathcal{S}'(\mathbf{R}^n)^{l\times l}$;*
(2) *$A(\partial)$ has a two-sided fundamental matrix in $\mathcal{O}_C'(\mathbf{R}^n)^{l\times l}$;*
(3) *$A(\mathrm{i}\xi)$ is invertible for each $\xi \in \mathbf{R}^n$, i.e., $\forall \xi \in \mathbf{R}^n : \det A(\mathrm{i}\xi) \neq 0$.*

Proof

(1) $\Rightarrow$ (3) : This follows from the fact that

$$T := B \cdot \mathrm{e}^{\mathrm{i}x\xi_0} \in \mathcal{S}'(\mathbf{R}^n)^{l\times l} \setminus \{0\}$$

solves the homogeneous equation $A(\partial)T = 0$ if $\det A(\mathrm{i}\xi_0) = 0$ and $B \in \mathbf{C}^{l\times l} \setminus \{0\}$ satisfies $A(\mathrm{i}\xi_0)B = 0$.

(3) $\Rightarrow$ (2) : Due to the Hörmander–Łojasiewicz inequality (see Hörmander [135], Lemma 2, (2.5), p. 557; Hörmander [138], Ex. A.2.7, (A.2.6), p. 368), the assumption in (3) implies that $A(\mathrm{i}\xi)^{-1}$ and its derivatives have at most polynomial growth for $|\xi| \to \infty$, i.e.,

$$A(\mathrm{i}\xi)^{-1} \in \mathcal{O}_M(\mathbf{R}^n)^{l\times l},$$

cf. Schwartz [246], p. 243, for the definition of the spaces $\mathcal{O}_M(\mathbf{R}^n)$, $\mathcal{O}_C'(\mathbf{R}^n)$ and the equation $\mathcal{O}_M(\mathbf{R}^n) = \mathcal{F}\mathcal{O}_C'(\mathbf{R}^n)$. Hence

$$E = A(\partial)^{\mathrm{ad}} \mathcal{F}^{-1}\big(\det(A(\mathrm{i}\xi))^{-1}\big) \in \mathcal{O}_C'(\mathbf{R}^n)^{l\times l}$$

is a two-sided fundamental matrix of $A(\partial)$.

(2) ⇒ (1) : If $E \in \mathcal{O}'_C(\mathbf{R}^n)^{l\times l}$ and $F \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}$ are two-sided fundamental matrices of $A(\partial)$, then they are convolvable and hence

$$E = E * A(\partial)F = \big(E * A(\partial)\delta\big) * F = F.$$

The proof is complete. □

Example 2.4.2 As in Example 1.4.11, let us consider the metaharmonic operator $P(\partial) = \Delta_n + \lambda,\ \lambda \in \mathbf{C} \setminus [0,\infty)$. Then condition (3) in Proposition 2.4.1 is satisfied since $P(\mathrm{i}\xi) = \lambda - |\xi|^2 \neq 0$ for $\xi \in \mathbf{R}^n$. The fast decreasing fundamental solution E of $P(\partial)$ is then given by

$$E = -\mathrm{i}d_n(\lambda)|x|^{-n/2+1}H^{(1)}_{n/2-1}\big(\sqrt{\lambda}|x|\big), \qquad d_n(\lambda) = \frac{\lambda^{n/4-1/2}}{2^{n/2+1}\pi^{n/2-1}},$$

where $\sqrt{\lambda}$ is defined in the slit plane $\mathbf{C} \setminus [0,\infty)$ by $0 < \arg\sqrt{\lambda} < \pi$, see Example 1.4.11. (Note that $H^{(1)}_\nu(z)$ decreases exponentially if $|z| \to \infty$ with $\operatorname{Im} z > 0$.)

In particular, for $\lambda = -\mu,\ \mu > 0$, the unique temperate fundamental solution of $(\Delta_n - \mu)^k,\ k \in \mathbf{N}$, is given in terms of MacDonald's function, see (1.4.9) and Example 1.6.11(a). □

Example 2.4.3 As another application of Proposition 2.4.1, we consider the *time-harmonic Lamé system*

$$A(\nabla) = -\rho\tau^2 I_3 - B(\nabla), \qquad B(\nabla) := \mu\Delta_3 I_3 + (\lambda+\mu)\nabla\cdot\nabla^T.$$

As in Example 2.1.3, $\lambda, \mu > 0$ denote Lamé's constants. Then the system $A(\nabla)$ arises from the one in formula (2.1.2) by partial Fourier transform with respect to the time variable t. Hence we obtain, for fixed $\tau > 0$, the following locally integrable fundamental matrix F of $A(\nabla)$ by partial Fourier transform applied to the fundamental matrix E in Stokes's formula (2.1.7):

$$\begin{aligned} F = \mathcal{F}_t E &= \frac{I_3|x|^2 - xx^T}{4\pi\mu|x|^3}\,\mathrm{e}^{-\mathrm{i}\tau|x|/c_s} + \frac{xx^T}{4\pi(\lambda+2\mu)|x|^3}\,\mathrm{e}^{-\mathrm{i}\tau|x|/c_p} \\ &+ \frac{1}{4\pi\rho|x|^3\tau^2}\Big(I_3 - \frac{3xx^T}{|x|^2}\Big)\Big[\mathrm{e}^{-\mathrm{i}\tau|x|/c_p}\Big(1 + \frac{\mathrm{i}\tau|x|}{c_p}\Big) - \mathrm{e}^{-\mathrm{i}\tau|x|/c_s}\Big(1 + \frac{\mathrm{i}\tau|x|}{c_s}\Big)\Big]. \end{aligned} \tag{2.4.1}$$

Herein, $c_s = \sqrt{\frac{\mu}{\rho}}$, $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$ are the velocities of the shear and pressure waves, respectively.

By analytic continuation, with respect to ρ, (2.4.1) yields a fundamental matrix of $A(\nabla)$ for all $\rho \in \mathbf{C} \setminus \{0\}$. For $\rho \in \mathbf{C} \setminus [0,\infty)$, let us choose $\sqrt{\rho}$ such that $\operatorname{Im}\sqrt{\rho} < 0$, i.e., $\operatorname{Im}(c_s^{-1}),\ \operatorname{Im}(c_p^{-1}) < 0$. This implies that $F \in \mathcal{O}'_C(\mathbf{R}^3)^{3\times 3}$ and, by Proposition 2.4.1, F is the only temperate fundamental matrix if $\rho \in \mathbf{C} \setminus [0,\infty)$ and $\sqrt{\rho}$ as above.

For $\rho > 0$, which is the physically relevant case, there exist many temperate fundamental matrices. If we take the real part in (2.4.1), we obtain

$$\begin{aligned}\operatorname{Re} F(x) &= \frac{I_3|x|^2 - xx^T}{4\pi\mu|x|^3}\cos\Big(\frac{\tau|x|}{c_s}\Big) + \frac{xx^T\cos(\tau|x|/c_p)}{4\pi(\lambda+2\mu)|x|^3}\\ &+ \frac{1}{4\pi\rho|x|^3\tau^2}\Big(I_3 - \frac{3xx^T}{|x|^2}\Big)\Big[\cos\Big(\frac{\tau|x|}{c_p}\Big) - \cos\Big(\frac{\tau|x|}{c_s}\Big)\\ &+ \tau|x|\Big(c_p^{-1}\sin\Big(\frac{\tau|x|}{c_p}\Big) - c_s^{-1}\sin\Big(\frac{\tau|x|}{c_s}\Big)\Big)\Big].\end{aligned} \tag{2.4.2}$$

For (2.4.2), see Mura [185], (9.40), p. 65; Norris [190], (B3), p. 187; Ortner and Wagner [217], p. 331. □

Let us generalize now Proposition 2.4.1 to symbols with finitely many real zeroes, cf. Zuily [309], Ex. 82, p. 147; Gel'fand and Shilov [106], Ch. III, Section 2.4, p. 135; Friedman [85], p. 98.

Proposition 2.4.4 *Let $P(\partial)$ be a linear differential operator such that the set $Z = \{\xi \in \mathbf{R}^n;\ P(\mathrm{i}\xi) = 0\}$ is finite. Then two temperate fundamental solutions of $P(\partial)$ differ only by an exponential polynomial $\sum_{\xi\in Z} Q_\xi(x)\mathrm{e}^{\mathrm{i}x\xi}$, where $Q_\xi(x) = \sum_{|\alpha|\le m} a_{\xi\alpha}x^\alpha$ with $m \in \mathbf{N}_0$ and $a_{\xi\alpha} \in \mathbf{C}$.*

Proof This is evident by Fourier transformation: If $T \in \mathcal{S}'$ and $P(\partial)T = 0$, then $P(\mathrm{i}\xi)(\mathcal{F}T) = 0$ and thus $\operatorname{supp}(\mathcal{F}T) \subset Z$. Therefore, by Proposition 1.3.15,

$$\mathcal{F}T = (2\pi)^n \sum_{\xi\in Z} Q_\xi(\mathrm{i}\partial)\delta_\xi,$$

and hence $T = \sum_{\xi\in Z} Q_\xi(x)\mathrm{e}^{\mathrm{i}x\xi}$. □

As an example, let us first investigate temperate fundamental solutions of ordinary differential operators, cf. Proposition 1.3.7, Example 1.3.8.

Proposition 2.4.5 *Let $m \in \mathbf{N}$, $\alpha \in \mathbf{N}_0^m$ and $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$ be pairwise different. Let us set* $\operatorname{sign} t = \left\{\begin{array}{rl} 1 & : t \ge 0, \\ -1 & : t < 0 \end{array}\right\}$. *Then the ordinary differential operator*

$$P_{\lambda,\alpha}(\tfrac{\mathrm{d}}{\mathrm{d}x}) = \prod_{j=1}^m \Big(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda_j\Big)^{\alpha_j+1}$$

has the following temperate fundamental solution:

$$E = \sum_{j=1}^m Y\big(-x\operatorname{sign}(\operatorname{Re}\lambda_j)\big)\frac{\operatorname{sign}(-\operatorname{Re}\lambda_j)}{\alpha_j!}\Big(\frac{\partial}{\partial\lambda_j}\Big)^{\alpha_j}\Big(\mathrm{e}^{\lambda_j x}\prod_{k\ne j}(\lambda_j-\lambda_k)^{-\alpha_k-1}\Big). \tag{2.4.3}$$

E is uniquely determined in $\mathcal{S}'(\mathbf{R}^1)$ up to an exponential polynomial of the form

$$\sum_{\operatorname{Re}\lambda_j=0}\sum_{k=0}^{\alpha_j} c_{jk}x^k e^{\lambda_j x}, \qquad c_{jk}\in\mathbf{C}.$$

Proof Similarly as in the proof of Proposition 1.3.7, we first assume that $\alpha = 0$ and set

$$E = \sum_{j=1}^{m} Y\big(-x\operatorname{sign}(\operatorname{Re}\lambda_j)\big)a_j e^{\lambda_j x}, \qquad a_j\in\mathbf{C}.$$

Considering the jump conditions for E as in the proof of Proposition 1.3.7 yields

$$\sum_{j=1}^{m} a_j\lambda_j^k \operatorname{sign}(-\operatorname{Re}\lambda_j) = \begin{cases} 0 & : k = 0,\dots,m-2,\\ 1 & : k = m-1,\end{cases}$$

and hence $a_j = \operatorname{sign}(-\operatorname{Re}\lambda_j)\prod_{k\neq j}(\lambda_j-\lambda_k)^{-1}$.

If $\alpha \neq 0$ and $\forall j = 1,\dots,m : \operatorname{Re}\lambda_j \neq 0$, then differentiation with respect to λ as in the second part of the proof of Proposition 1.3.7 furnishes formula (2.4.3). Finally, if the real part of some of the roots λ_j vanishes, then one uses a limit process. The uniqueness statement of Proposition 2.4.5 is a consequence of Proposition 2.4.4. This completes the proof. □

Example 2.4.6 In the simple case of $P(\frac{d}{dx}) = \frac{d^2}{dx^2} - \lambda^2$, $\lambda\in\mathbf{C}\setminus\{0\}$, formula (2.4.3) yields the temperate fundamental solution

$$E = -\frac{\operatorname{sign}(\operatorname{Re}\lambda)}{2\lambda}\Big[Y\big(-x\operatorname{sign}(\operatorname{Re}\lambda)\big)e^{\lambda x} + Y\big(x\operatorname{sign}(\operatorname{Re}\lambda)\big)e^{-\lambda x}\Big],$$

which is unique if $\operatorname{Re}\lambda\neq 0$ and else is unique up to $c_1e^{\lambda x} + c_2e^{-\lambda x}$. □

Definition 2.4.7 The operator $P(\partial) = \sum_{|\alpha|\le m} a_\alpha\partial^\alpha$ of order m with the principal part $P_m(\partial) = \sum_{|\alpha|=m} a_\alpha\partial^\alpha$ is called *elliptic* if and only if $\forall\xi\in\mathbf{R}^n\setminus\{0\} : P_m(\xi)\neq 0$.

For this definition, cf. Hörmander [139], Def. 7.1.19, p. 169. Note that elliptic operators are "hypoelliptic", i.e., each solution $u\in\mathcal{D}'(\Omega)$ of $P(\partial)u = 0$ in some open set $\Omega\subset\mathbf{R}^n$ is necessarily $\mathcal{C}^\infty$ in Ω, see Hörmander [138], Thms. 11.1.1, 11.1.10, pp. 61, 67; Zuily [309], Exercise 97, p. 187. Hence each fundamental solution E of an elliptic operator $P(\partial)$ is $\mathcal{C}^\infty$ outside the origin.

Proposition 2.4.8 *Let $P(\partial)$ be an elliptic operator in $\mathbf{R}^n$ which is homogeneous of degree m.*

(1) *$P(\partial)$ has a fundamental solution E such that $E(x)\cdot|x|^{-m+n}/\log|x|$ is bounded for $|x|\to\infty$, i.e.,*

$$\exists C>0 : \forall x\in\mathbf{R}^n \text{ with } |x|\ge 2 : |E(x)|\le C|x|^{m-n}\log|x|. \qquad (2.4.4)$$

If $m < n$ then there exists precisely one fundamental solution E satisfying (2.4.4); *if $m \geq n$, then E is uniquely determined by condition* (2.4.4) *up to polynomials of degree $m - n$.*

(2) *For odd dimensions $n \geq 3$, there exists one and only one fundamental solution E which is homogeneous and even.*

Proof

(1) We have already shown in Proposition 2.3.9 that $P(\partial)$ has an associated homogeneous fundamental solution of the form

$$E(x) = F \cdot |x|^{m-n} + Q(x) \log |x|$$

for some $F \in \mathcal{D}'(\mathbf{S}^{n-1})$ and a homogeneous polynomial Q of degree m. Due to the ellipticity of $P(\partial)$, the distribution F is $\mathcal{C}^\infty$, and hence the estimate (2.4.4) follows.

On the other hand, a fundamental solution E_1 satisfying (2.4.4) is necessarily temperate, and, because of $P(\partial)(E - E_1) = 0$, we conclude that $P(\mathrm{i}\xi) \cdot \mathcal{F}(E - E_1) = 0$ and that the support of $\mathcal{F}(E - E_1)$ is contained in $\{0\}$. Therefore, $\mathcal{F}(E - E_1)$ is a sum of derivatives of δ (see Proposition 1.3.15) and $E - E_1 = R$ for a polynomial R. Due to (2.4.4), R must vanish if $m < n$, and R is of degree at most $m - n$ if $m \geq n$.

(2) If $P(\partial)$ is elliptic in $\mathbf{R}^n$ and $n \geq 3$, then the order m of the symbol $P(\mathrm{i}\xi)$ is necessarily even, see Lions and Magenes [167], Prop. 1.1, p. 121. Let us recall now some steps in the construction of the fundamental solution E in the course of the proof of Proposition 2.3.9.

First, the division problem $P(\mathrm{i}\omega) \cdot U = 1$ is solved on the sphere $\mathbf{S}^{n-1}$. Due to the ellipticity of $P(\partial)$, the solution $U = P(\mathrm{i}\omega)^{-1} \in \mathcal{C}^\infty(\mathbf{S}^{n-1})$ is uniquely determined. Note that U is an even function on $\mathbf{S}^{n-1}$ since m is even. This implies that the distribution $V = U \cdot |\xi|^{-m} \in \mathcal{D}'(\mathbf{R}^n)$ is even and homogeneous of degree $-m$. In fact, $U \cdot |\xi|^\lambda$ is analytic in λ and yields homogeneous distributions for $\lambda \in \mathbf{C} \setminus \{-n - k;\, k \in \mathbf{N}_0\}$, see Example 1.4.9. Hence V is clearly homogenous if $m < n$. In the case $m \geq n$, this is also true since then $R = \operatorname{Res}_{\lambda=-m} U \cdot |\xi|^\lambda$ vanishes due to the fact that U is even:

$$\operatorname*{Res}_{\lambda=-m} U \cdot |\xi|^\lambda = \frac{(-1)^{m-n}}{(m-n)!} \langle (\omega^T \cdot \nabla)^{m-n}, U(\omega) \rangle \delta = 0$$

for $m \geq n$, m even, n odd. Thus $E = \mathcal{F}^{-1} V$ is an even and homogeneous fundamental solution of $P(\partial)$.

If E_1, E_2 are two even, homogeneous fundamental solutions of $P(\partial)$, then E_i, $i = 1, 2$, are both homogeneous of degree $m - n$ and $E_1 - E_2$ is a polynomial by part (1). Hence $E_1 - E_2$ is an even polynomial of the odd degree $m - n$ and consequently vanishes. This shows that an even, homogeneous fundamental solution of $P(\partial)$ is uniquely determined and completes the proof. □

The second part of Proposition 2.4.8 goes back to Wagner [292], Prop. 1, p. 1193.

Example 2.4.9 Let us illustrate the uniqueness assertions in Proposition 2.4.8 for homogeneous elliptic operators of degree 2, i.e.,

$$P(\partial) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \partial_i \partial_j = \nabla^T A \nabla, \qquad A = A^T \in \mathbf{C}^{n \times n},$$

by calculating explicitly the associated homogeneous fundamental solutions of $P(\partial)$.

(a) Let us consider first the two-dimensional case $n = 2$. Then the set

$$\begin{aligned} M_2 &= \{A \in \mathbf{C}^{2\times 2};\ A = A^T,\ \nabla^T A \nabla \text{ is elliptic}\} \\ &= \{A \in \mathbf{C}^{2\times 2};\ A = A^T \text{ and } \forall x \in \mathbf{R}^2 \setminus \{0\} : x^T A x \neq 0\} \end{aligned}$$

consists of three connectivity components. In fact, $x^T Ax$ does not vanish for $x \neq 0$ if and only if $x^T Ax = a(x_1 - \lambda_1 x_2)(x_1 - \lambda_2 x_2)$ with $a \in \mathbf{C} \setminus \{0\},\ \lambda_1, \lambda_2 \in \mathbf{C} \setminus \mathbf{R}$. Therefore, M_2 is the disjoint union of the components M_2^0, M_2^+, M_2^- given by the sets of matrices of the form

$$a \begin{pmatrix} 1 & -(\lambda_1 + \lambda_2)/2 \\ -(\lambda_1 + \lambda_2)/2 & \lambda_1 \lambda_2 \end{pmatrix}, \qquad a \in \mathbf{C} \setminus \{0\},\ \lambda_1, \lambda_2 \in \mathbf{C} \setminus \mathbf{R},$$

where either $\operatorname{Im} \lambda_1 \cdot \operatorname{Im} \lambda_2 < 0$ or $\operatorname{Im} \lambda_i > 0$ $(i = 1, 2)$ or $\operatorname{Im} \lambda_i < 0$ $(i = 1, 2)$, respectively. Hence M_2^0, M_2^+, M_2^- are the components which contain the matrices corresponding to the operators Δ_2, $(\partial_1 - \mathrm{i}\partial_2)^2$, $(\partial_1 + \mathrm{i}\partial_2)^2$, respectively. (Note that the set Γ_2 in Example 1.4.12 corresponds to the diagonal matrices in M_2, and that $\Gamma_2 \subset M_2^0$.)

According to Proposition 2.3.9, we obtain a fundamental solution E of $P(\partial) = \nabla^T A \nabla,\ A \in M_2$, in the form

$$E = \mathcal{F}^{-1} V, \quad V = U(\omega) \cdot |\xi|^{-2} \in \mathcal{S}'(\mathbf{R}^2), \quad U(\omega) = -\frac{1}{\omega^T A \omega} \in \mathcal{C}^\infty(\mathbf{S}^1).$$

Hence

$$\langle \phi, V \rangle = -\langle \langle \phi(t\omega), (\omega^T A \omega)^{-1} \rangle, t_+^{-1} \rangle, \qquad \phi \in \mathcal{S}(\mathbf{R}^2).$$

Furthermore, according to Proposition 2.4.8, E is uniquely determined up to a constant by the condition

$$\exists C > 0 : \forall \in \mathbf{R}^2 \text{ with } |x| \geq 2 : |E(x)| \leq C \log |x|.$$

As in the proof of Proposition 2.3.9, we observe that E and V are homogeneous if and only if

$$R = \operatorname*{Res}_{\lambda=-2} U \cdot |\xi|^\lambda = \Big(\int_{\mathbf{S}^1} U(\omega)\, \mathrm{d}\sigma(\omega) \Big) \delta$$

vanishes. In order to evaluate this integral, let us use the *gnomonian projection* $\omega = \pm \frac{1}{\sqrt{1+t^2}} \binom{t}{1}$, $\mathrm{d}\sigma(\omega) = \frac{\mathrm{d}t}{1+t^2}$, which projects both the upper and the lower semicircle onto the real axis $\mathbf{R}^1_t$. This yields

$$\int_{\mathbf{S}^1} \frac{\mathrm{d}\sigma(\omega)}{(\omega_1 - \lambda_1\omega_2)(\omega_1 - \lambda_2\omega_2)} = 2\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(t-\lambda_1)(t-\lambda_2)}$$
$$= \begin{cases} 0 & : \operatorname{Im}\lambda_1 \cdot \operatorname{Im}\lambda_2 > 0, \\ \frac{4\pi \mathrm{i}\, \operatorname{sign}(\operatorname{Im}\lambda_1)}{\lambda_1 - \lambda_2} & : \operatorname{Im}\lambda_1 \cdot \operatorname{Im}\lambda_2 < 0. \end{cases} \tag{2.4.5}$$

Therefore, E and V are homogeneous iff $A \in M_2^+ \cup M_2^-$.

Let us finally calculate E. For $A \in M_2^0$, we can find E by analytic continuation from the real-valued case. Starting from the fundamental solution $E = \frac{1}{4\pi} \log |x|^2$ of Δ_2, see Example 1.3.14(a), the linear transformation formula in Proposition 1.3.19 yields, upon addition of a constant,

$$E = \frac{1}{4\pi\sqrt{\det A}} \log(x^T A^{\mathrm{ad}} x) \tag{2.4.6}$$

as fundamental solution of $\nabla^T A \nabla$ for positive definite $A \in \mathbf{R}^{2\times 2}$. (Note that $x^T A x = |\sqrt{A}x|^2$.) Formula (2.4.6) is then generally valid for $A \in M_2^0$ if we take into account that $x^T A^{\mathrm{ad}} x \neq t a_{11}$ for $t < 0$ and $x \in \mathbf{R}^2$. (On $\mathbf{C} \setminus a_{11} \cdot (-\infty, 0]$, the logarithm can be defined continuously.) Furthermore, $\sqrt{\det A}$ is uniquely determined if it is chosen positive for positive definite A and continuously extended for $A \in M_2^0$.

If $P(\partial)$ is expressed in the form $P(\partial) = (\partial_1 - \lambda_1\partial_2)(\partial_1 - \lambda_2\partial_2)$, $\operatorname{Im}\lambda_1 \cdot \operatorname{Im}\lambda_2 < 0$, (and hence $A \in M_2^0$), then we obtain

$$E = -\frac{\operatorname{sign}(\operatorname{Im}\lambda_1)}{2\pi\mathrm{i}(\lambda_1 - \lambda_2)} \log[(x_2 + \lambda_1 x_1)(x_2 + \lambda_2 x_1)]. \tag{2.4.7}$$

We observe that the multiplicative constant $d(\lambda_1, \lambda_2)$ in (2.4.7) preceding the logarithm is connected with the residue $R = \operatorname{Res}_{\lambda=-2}(U \cdot |\xi|^\lambda)$ in the proof of Proposition 2.3.9. In fact, (2.3.23) yields $\mathcal{F}^{-1}R = -2d(\lambda_1, \lambda_2)$ in accordance with formula (2.4.5) which furnished

$$R = -\frac{4\pi\mathrm{i}\, \operatorname{sign}(\operatorname{Im}\lambda_1)}{\lambda_1 - \lambda_2}\, \delta, \qquad \operatorname{Im}\lambda_1 \cdot \operatorname{Im}\lambda_2 < 0.$$

E.g., if $P(\partial) = \partial_1^2 + \partial_2^2 + 2\mathrm{i}\partial_1\partial_2$, then we obtain for a fundamental solution of $P(\partial)$

$$E = \frac{1}{4\sqrt{2}\,\pi}\,\log(|x|^2 - 2\mathrm{i}x_1x_2) = \frac{1}{4\sqrt{2}\,\pi}\Big[\frac{1}{2}\log(|x|^4 + 4x_1^2x_2^2) - \mathrm{i}\arctan\Big(\frac{2x_1x_2}{|x|^2}\Big)\Big].$$

Let us yet give an explicit formula for a homogeneous fundamental solution E of $P(\partial) = \nabla^T A \nabla$ if $A \in M_2^{\pm}$. We set, without loss of generality, $P(\partial) = (\partial_1 - \lambda_1\partial_2)(\partial_1 - \lambda_2\partial_2)$ with $\epsilon = \operatorname{sign}(\operatorname{Im}\lambda_1) = \operatorname{sign}(\operatorname{Im}\lambda_2) \in \{\pm 1\}$ and $\lambda_1 \neq \lambda_2$, and

$$E = -\frac{\epsilon}{2\pi\mathrm{i}(\lambda_1 - \lambda_2)} \cdot \log\Big(\frac{x_2 + \lambda_1 x_1}{x_2 + \lambda_2 x_1}\Big). \tag{2.4.8}$$

Note that $\frac{x_2+\lambda_1x_1}{x_2+\lambda_2x_1} \in \mathbf{C} \setminus (-\infty, 0]$ for $x \in \mathbf{R}^2 \setminus \{0\}$. We define the logarithm in the usual way in the slit plane $\mathbf{C} \setminus (-\infty, 0]$. In order to verify (2.4.8), we use the jump formula (1.3.9) and obtain

$$(\partial_1 - \lambda_j\partial_2)\log(x_2 + \lambda_j x_1) = 2\pi\mathrm{i}\epsilon\delta(x_1) \otimes Y(-x_2),\ j = 1, 2,$$

and hence

$$P(\partial)E = -\frac{\epsilon}{2\pi\mathrm{i}(\lambda_1 - \lambda_2)} \cdot \big[(\partial_1 - \lambda_2\partial_2) - (\partial_1 - \lambda_1\partial_2)\big] 2\pi\mathrm{i}\epsilon\delta(x_1) \otimes Y(-x_2) = \delta.$$

The case of $\lambda_1 = \lambda_2 \in \mathbf{C} \setminus \mathbf{R}$ follows by a limit procedure. We obtain

$$(\partial_1 - \lambda\partial_2)^2\Big[-\frac{\operatorname{sign}(\operatorname{Im}\lambda)}{2\pi\mathrm{i}} \cdot \frac{x_1}{x_2 + \lambda x_1}\Big] = \delta, \qquad \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

The formulas (2.4.6–2.4.8) will be generalized in Proposition 3.3.2 below, where we shall consider $R(\partial) = \prod_{j=1}^{l}(\partial_1 - \lambda_j\partial_2)^{\alpha_j+1}$, see also Somigliana [258], Wagner [285], Ch. III, Satz 4, p. 40.

(b) In contrast to the two-dimensional case $n = 2$, the set

$$\begin{aligned} M_n &= \{A \in \mathbf{C}^{n\times n};\ A = A^T,\ \nabla^T A \nabla \text{ is elliptic}\} \\ &= \{A \in \mathbf{C}^{n\times n};\ A = A^T \text{ and } \forall x \in \mathbf{R}^n \setminus \{0\} : x^T A x \neq 0\} \end{aligned}$$

is *connected* if $n \geq 3$. Let us prove this fact analogously to the proof of Prop. 1.1 in Ch. II of Lions and Magenes [167], p. 121.

For $\xi, \eta \in \mathbf{R}^n$ linearly independent, we first show that

$$B := \begin{pmatrix} \xi^T A \xi & \xi^T A \eta \\ \xi^T A \eta & \eta^T A \eta \end{pmatrix} \in M_2^0.$$

In fact, if $\eta \in \mathbf{R}^n \setminus \{0\}$ is fixed and $\xi \in \mathbf{R}^n \setminus \mathbf{R}\eta$, then the zeroes λ_1, λ_2 of the parabola $\tau \mapsto (\xi + \tau\eta)^T A(\xi + \tau\eta)$ belong to $\mathbf{C} \setminus \mathbf{R}$ and depend continuously on ξ. If ξ is replaced by $-\xi$, then λ_1, λ_2 change their signs, and hence, since $\mathbf{R}^n \setminus \mathbf{R}\eta$ is connected for $n \geq 3$, we must have $\operatorname{Im} \lambda_1 \cdot \operatorname{Im} \lambda_2 < 0$, i.e., $B \in M_2^0$.

If $A \in M_n$ and, without loss of generality, $a_{11} = 1$, then this implies $\xi^T A\xi \in \mathbf{C} \setminus (-\infty, 0]$ by part (a). Hence A can be joined in M_n to I_n by means of the path $tA + (1-t)I_n,\ 0 \leq t \leq 1$, and thus M_n is connected.

If $A = I_n$, then the unique homogeneous fundamental solution of $P(\partial) = \nabla^T A\nabla = \Delta_n$ is given by

$$E = \frac{\Gamma(\frac{n}{2})}{(2-n)2\pi^{n/2}}\, |x|^{2-n}$$

according to Example 1.3.14(a). A linear transformation in $\mathrm{Gl}_n(\mathbf{R})$ then yields, by Proposition 1.3.19, the homogeneous fundamental solution E of $P(\partial) = \nabla^T A\nabla$ for real-valued positive definite A:

$$E = \frac{\Gamma(\frac{n}{2})}{(2-n)2\pi^{n/2}}\, (\det A)^{(n-3)/2}(x^T A^{\mathrm{ad}} x)^{-n/2+1}. \tag{2.4.9}$$

Finally, (2.4.9) remains valid by analytic continuation for each $A \in M_n$ if the values of $\sqrt{\det A}$ and of $\sqrt{x^T A^{\mathrm{ad}} x}$ are determined by continuity.

To illustrate formula (2.4.9) for a concrete example, let us consider the operator

$$P(\partial) = \partial_1^2 + \partial_2^2 + 2\mathrm{i}\partial_1\partial_2 + \partial_3^2 = \nabla^T A\nabla, \qquad A = \begin{pmatrix} 1 & \mathrm{i} & 0 \\ \mathrm{i} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then (2.4.9) yields

$$E = -\frac{1}{4\pi\sqrt{x_1^2 + x_2^2 - 2\mathrm{i}x_1x_2 + 2x_3^2}}$$

in agreement with Hörmander [139], Exercise 7.1.39, p. 388.

(c) In order to deduce the associated homogeneous fundamental solutions E of Proposition 2.4.8 for the powers $(\nabla^T A\nabla)^k,\ k \in \mathbf{N}$, we can apply the same procedure as above. Passing from the fundamental solutions of Δ_n^k in (1.6.19) and (1.6.20) by a linear transformation to $(\nabla^T A\nabla)^k$ for A real-valued and positive definite, we obtain

$$E = \begin{cases} \frac{(-1)^k\Gamma(\frac{n}{2}-k)(\det A)^{(n-1)/2-k}}{2^{2k}(k-1)!\,\pi^{n/2}}(x^T A^{\mathrm{ad}} x)^{-n/2+k} & : n \text{ odd or } k < \frac{n}{2}, \\ \frac{(-1)^{n/2-1}(\det A)^{(n-1)/2-k}}{2^{2k}(k-1)!(k-\frac{n}{2})!\,\pi^{n/2}}(x^T A^{\mathrm{ad}} x)^{-n/2+k}\log(x^T A^{\mathrm{ad}} x) & : n \text{ even and } k \geq \frac{n}{2}. \end{cases} \tag{2.4.10}$$

Again, (2.4.10) remains valid by analytic continuation if $n = 2$ and $A \in M_2^0$ or if $n \geq 3$ and $A \in M_n$. (Therein, the functions $A \mapsto (\det A)^{(n-1)/2-k}$ and $A \mapsto \log(x^T A^{\mathrm{ad}} x)$ have to be determined by continuity.) Furthermore, E is the only homogeneous fundamental solution if n is odd or $k < \frac{n}{2}$ and else is uniquely determined by the condition (2.4.4) up to polynomials of degree $2k - n$. □

We next consider hyperbolic systems of differential operators. For these, the uniqueness of the fundamental matrix is implied by support properties.

Definition 2.4.10 Let $N \in \mathbf{R}^n \setminus \{0\}$.

(1) A linear differential operator $P(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$, $a_\alpha \in \mathbf{C}$, in $\mathbf{R}^n$ with principal part $P_m(\partial) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$ is called *hyperbolic* in the direction N if and only if

$$\text{(i) } P_m(N) \neq 0 \text{ and (ii) } \exists \sigma_0 \in \mathbf{R} : \forall \sigma > \sigma_0 : \forall \xi \in \mathbf{R}^n : P(\mathrm{i}\xi + \sigma N) \neq 0.$$

(2) A quadratic system $A(\partial) \in \mathbf{C}[\partial]^{l \times l}$ of linear differential operators in $\mathbf{R}^n$ is called *hyperbolic* in the direction N if and only if $P(\partial) = \det A(\partial)$ is hyperbolic in the direction N.

These definitions go back to Gårding [88]; see also Hörmander [138], Def. 12.3.3, p. 112; Atiyah, Bott and Gårding [5], Section 3, p. 126; Gårding [93], Section 8, p. 55. In the cited literature, it is also shown that $A(\partial)$ is hyperbolic in the direction N if and only if $A(\partial)$ possesses a two-sided fundamental matrix E with support in a cone contained in $\{x \in \mathbf{R}^n; xN > 0\} \cup \{0\}$, see also Ch. IV.

Let us next show that, if $A(\partial)$ is hyperbolic in the direction N, then there exists only one fundamental matrix with support in the closed half-space $H_N = \{x \in \mathbf{R}^n; xN \geq 0\}$.

Proposition 2.4.11 *Let $A(\partial)$ be a system which is hyperbolic in the direction N and let E be a two-sided fundamental matrix with support in a cone K contained in $\{x \in \mathbf{R}^n; xN > 0\} \cup \{0\}$. Then each right-sided and each left-sided fundamental matrix of $A(\partial)$ with support in $H_N = \{x \in \mathbf{R}^n; xN \geq 0\}$ coincides with E.*

Proof If F is a fundamental matrix satisfying $\operatorname{supp} F \subset H_N$, then E and F are convolvable by support (see Example 1.5.11), and we conclude that

$$E = E * A(\partial) F = \big(E * A(\partial)\delta\big) * F = (I_l \delta) * F = F.$$

□

In particular, Proposition 2.4.11 applies to the system of elastodynamics considered in Examples 2.1.3, 2.1.4.

Definition 2.4.12 Let $P(\partial)$ be an operator in $\mathbf{R}^n$ which is hyperbolic in the direction N. Then the only fundamental solution E of $P(\partial)$ with support in $H_N = \{x \in \mathbf{R}^n; xN \geq 0\}$ is called the *forward fundamental solution of $P(\partial)$ with respect* *to* N. Similarly, if $A(\partial)$ is a system which is hyperbolic in the direction N, then the only fundamental matrix with support in H_N is called the *forward fundamental matrix of* $A(\partial)$ *with respect to* N. In particular, if $P(\partial) = P(\partial_t, \partial_1, \dots, \partial_n)$ and $N = (1, 0, \dots, 0)$, we will just speak of the *forward fundamental solution* without mentioning N, and similarly for a matrix A.

If an operator is hyperbolic with respect to N, then it is also hyperbolic with respect to the direction $-N$ (see, e.g., Hörmander [136], Thm. 5.5.1, p. 132), and the forward fundamental solution with respect to $-N$ is called the *backward fundamental solution with respect to* N. We also note that, in a physical context, in particular in connection with the wave equation or the Klein–Gordon equation, the forward and backward fundamental solutions are also called "retarded" and "advanced fundamental solutions", respectively, or "retarded and advanced potentials", see, e.g., Friedlander [83], p. 117; Zeidler [305], 12.5.3, p. 715; Komech [154], Ch. V, 6.2. (Let us mention that the forward fundamental solution of the wave equation is called "advanced" in Hörmander [138], p. 195.)

If $A(\partial)$ is only *quasihyperbolic* instead of hyperbolic, i.e., $P(\partial) = \det A(\partial)$ fulfills condition (2.2.3), then support conditions alone do not suffice to ensure the uniqueness of the fundamental matrix.

Definition and Proposition 2.4.13 $A(\partial) \in \mathbf{C}[\partial]^{l\times l}$ *is called quasihyperbolic in the direction* $N \in \mathbf{R}^n \setminus \{0\}$ *iff* $P(\partial) = \det A(\partial)$ *fulfills the condition*

$$\exists \sigma_0 \in \mathbf{R} : \forall \sigma > \sigma_0 : \forall \xi \in \mathbf{R}^n : P(\mathrm{i}\xi + \sigma N) \neq 0. \tag{2.4.11}$$

If $A(\partial)$ *is quasihyperbolic and* σ_0 *is as in* (2.4.11), *then there exists a two-sided fundamental matrix* E *satisfying*

$$\operatorname{supp} E \subset H_N = \{x \in \mathbf{R}^n;\ xN \geq 0\} \text{ and } \forall \sigma \geq \sigma_0 : \mathrm{e}^{-\sigma x N} E \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}. \tag{2.4.12}$$

Furthermore, each right-sided and each left-sided fundamental matrix F *of* $A(\partial)$ *satisfying* $\exists \sigma > \sigma_0 : \mathrm{e}^{-\sigma x N} F \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}$ *coincides with* E. *For* $\sigma > \sigma_0$, σ_0 *as in* (2.4.11), E *has the representation*

$$E = \mathrm{e}^{\sigma N x} \cdot \mathcal{F}^{-1}\big(A(\mathrm{i}\xi + \sigma N)^{-1}\big). \tag{2.4.13}$$

If E_1 *is the fundamental solution of* $P(\partial)$ *satisfying* $\exists \sigma > \sigma_0 : \mathrm{e}^{-\sigma x N} E_1 \in \mathcal{S}'(\mathbf{R}^n)$, *then* $E = A^{\mathrm{ad}}(\partial) E_1$.

Proof In order to reduce the assertion to Proposition 2.3.5, we set $Q(\partial) = P(\partial + \sigma_0 N)$. Then $Q(\partial)$ satisfies condition (2.3.8) in Proposition 2.3.5, i.e.,

$$\forall \sigma > 0 : \forall \xi \in \mathbf{R}^n : Q(\mathrm{i}\xi + \sigma N) = P\big(\mathrm{i}\xi + (\sigma + \sigma_0) N\big) \neq 0.$$

Hence, by Proposition 2.3.5, $Q(\partial)$ possesses a fundamental solution $G \in \mathcal{S}'(\mathbf{R}^n)$ such that $\operatorname{supp} G \subset H_N$. Then $E_1 = \mathrm{e}^{\sigma_0 N x} G$ is a fundamental solution of $P(\partial)$

satisfying $\mathrm{e}^{-\sigma_0 Nx}E_1 \in \mathcal{S}'(\mathbf{R}^n)$ and $\operatorname{supp} E_1 \subset H_N$. For $\sigma \geq \sigma_0$, this implies

$$\mathrm{e}^{-\sigma Nx}E_1 = \chi(Nx)\mathrm{e}^{-(\sigma-\sigma_0)Nx}\mathrm{e}^{-\sigma_0 Nx}E_1 \in \mathcal{S}'(\mathbf{R}^n)$$

if we take $\chi \in \mathcal{E}(\mathbf{R}^1_t)$, $\chi(t) = 1$ for $t > 0$ and $\chi(t) = 0$ for $t < -1$, and observe that $\chi(Nx)\mathrm{e}^{-(\sigma-\sigma_0)Nx} \in \mathcal{D}_{L^\infty}(\mathbf{R}^n)$. Therefore, if $E = A^{\mathrm{ad}}(\partial)E_1$, then E is a two-sided fundamental matrix of $P(\partial)$ fulfilling $\operatorname{supp} E \subset H_N$ and $\mathrm{e}^{-\sigma Nx}E \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}$ for each $\sigma \geq \sigma_0$.

For a further right-sided fundamental matrix F of $A(\partial)$ which fulfills $G = \mathrm{e}^{-\sigma Nx}F \in \mathcal{S}'(\mathbf{R}^n)^{l\times l}$ for some $\sigma > \sigma_0$, we conclude that

$$A(\partial + \sigma N)(G - \mathrm{e}^{-\sigma Nx}E) = \mathrm{e}^{-\sigma Nx}A(\partial)(F - E) = 0$$

and hence $A(\mathrm{i}\xi + \sigma N)\mathcal{F}(G - \mathrm{e}^{-\sigma Nx}E) = 0$. Because of $\det A(\mathrm{i}\xi + \sigma N) = P(\mathrm{i}\xi + \sigma N) \neq 0$ for $\xi \in \mathbf{R}^n$, we have $G = \mathrm{e}^{-\sigma Nx}E$, i.e., $F = E$. An analogous reasoning applies if F is a left-sided fundamental solution. This completes the proof. □

For scalar hyperbolic operators, the formula in (2.4.13) coincides with (12.5.3) in Hörmander [138], p. 120; for the generalization to quasihyperbolic operators and systems, see Ortner and Wagner [207], Prop. 1, p. 442, Ortner and Wagner [209], Prop. 1, p. 530, and Ortner and Wagner [218], Prop. 9, p. 147.

Example 2.4.14 Let us apply Proposition 2.4.13 to a less known quasihyperbolic but non-hyperbolic system arising in elasticity, namely *Rayleigh's system.*

According to S. Timoshenko, the transverse vibrations in a homogeneous bar are governed by the following 2×2 system of linear partial differential equations (cf. Graff [114], pp. 181–183, in particular Eqs. (3.4.11/12); Timoshenko and Young [269], pp. 330, 331):

$$\rho A\partial_t^2 u - \kappa AG(\partial_x^2 u - \partial_x\psi) = q, \tag{2.4.14}$$

$$EI\partial_x^2\psi + \kappa AG(\partial_x u - \psi) - \rho I\partial_t^2\psi = 0. \tag{2.4.15}$$

In the above equations, $u(t, x)$ denotes the displacement of the bar at the co-ordinate x and at time t, and $\psi(t, x)$ is the slope of the deflection curve diminished by the angle of shear at the neutral axis. The positive parameters A, I, ρ, E, G and κ stand for the cross-section area, the moment of inertia, the mass density, Young's modulus, the shear modulus, and Timoshenko's shear coefficient, respectively.

If we neglect the inertia term $-\rho I\partial_t^2\psi$ in (2.4.15), this equation becomes

$$EI\partial_x^2\psi + \kappa AG(\partial_x u - \psi) = 0,$$

cf. Flügge [79], (6a/b), p. 313; Love [171], Ch. XX, § 280, (7), p. 431, "Rayleigh's equation". With the abbreviations $\alpha = \sqrt{\frac{\rho}{\kappa G}}$ and $\beta = \sqrt{\frac{\kappa AG}{EI}}$, we can rewrite this system as

$$B(\partial)\begin{pmatrix}u\\ \psi\end{pmatrix}=\begin{pmatrix}q/(\kappa AG)\\ 0\end{pmatrix},\qquad B(\partial)=\begin{pmatrix}\alpha^2\partial_t^2-\partial_x^2 & \partial_x\\ \beta^2\partial_x & \partial_x^2-\beta^2\end{pmatrix}.$$

Let us show that $B(\partial)$ is quasihyperbolic in the direction $N=(1,0)$. In fact, for $\sigma>0=\sigma_0$ and for $(\tau,\xi)\in\mathbf{R}^2$, the determinant

$$P(\partial)=\det B(\partial)=(\alpha^2\partial_t^2-\partial_x^2)\partial_x^2-\alpha^2\beta^2\partial_t^2,\qquad \alpha,\beta>0, \tag{2.4.16}$$

fulfills

$$P(\sigma+\mathrm{i}\tau,\mathrm{i}\xi)=-\alpha^2(\sigma+\mathrm{i}\tau)^2(\xi^2+\beta^2)-\xi^4\neq 0.$$

On the other hand, $\deg P=4$ and $P_4(N)=0$ imply that $B(\partial)$ and $P(\partial)$ are not hyperbolic in the direction N.

By Proposition 2.4.13, there exists a temperate two-sided fundamental matrix E of $B(\partial)$ with support in $\{(t,x)\in\mathbf{R}^2;\ t\geq 0\}$, and E is uniquely determined by the condition $\mathrm{e}^{-\sigma t}E\in\mathcal{S}'(\mathbf{R}^2)^{2\times 2}$ for some $\sigma>0$. As explained in Sect. 2.1, E has the representation $E=B(\partial)^{\mathrm{ad}}F$ where F is the fundamental solution of $P(\partial)=\det B(\partial)$ (see (2.4.16)) with $\mathrm{e}^{-\sigma t}F\in\mathcal{S}'(\mathbf{R}^2),\ \sigma\geq 0$. An explicit expression for F was given in Ortner and Wagner [214], Prop. 2, p. 226. In Example 4.1.9, we shall come back to Rayleigh's operator $P(\partial)$ in (2.4.16). □

2.5 Linear Transformations

We recall that Proposition 1.3.19 describes the effect of linear transformations of the independent variables x on the fundamental solution of a scalar operator $P(\partial)$, i.e., $|\det B|^{-1}\cdot E\circ B^{-1T}$ is a fundamental solution of $(P\circ B)(\partial)$ if $P(\partial)E=\delta$ and $B\in\mathrm{Gl}_n(\mathbf{R})$. Let us slightly generalize this formula and then provide some examples.

Proposition 2.5.1 *Let $A(\partial)=\big(A_{ij}(\partial)\big)_{1\leq i,j\leq l}\in\mathbf{C}[\partial]^{l\times l}$ be an $l\times l$ matrix of linear differential operators in $\mathbf{R}^n$ with constant coefficients, $S,T\in\mathrm{Gl}_l(\mathbf{C}),\ B\in\mathrm{Gl}_n(\mathbf{R}),\ c\in\mathbf{C}^n$, and denote by $\tilde{A}(\partial)$ the following transformed system of constant coefficient operators:*

$$\tilde{A}(\partial)=S\cdot A(B\partial+c)\cdot T,$$

i.e.,

$$\tilde{A}_{ij}(\partial)=\sum_{k=1}^{l}\sum_{m=1}^{l}s_{ik}t_{mj}A_{km}\Big(c_1+\sum_{s=1}^{n}b_{1s}\partial_s,\ldots,c_n+\sum_{s=1}^{n}b_{ns}\partial_s\Big).$$

Then

$$\tilde{E} = |\det B|^{-1} \cdot T^{-1} \cdot \big((\mathrm{e}^{-cx}E) \circ B^{-1T}\big) \cdot S^{-1} \tag{2.5.1}$$

yields a right-sided fundamental matrix of $\tilde{A}(\partial)$ *if* E *is a right-sided fundamental matrix of* $A(\partial)$.

Proof For $U \in \mathcal{D}'(\mathbf{R}^n)$ and $c \in \mathbf{R}^n$, we generally have $P(\partial + c)(\mathrm{e}^{-cx}U) = \mathrm{e}^{-cx}P(\partial)U$ if $P(\partial)$ is a linear differential operator with constant coefficients. This implies that $A(\partial + c)(\mathrm{e}^{-cx}E) = I_l\delta$ if $A(\partial)E = I_l\delta$, i.e., if E is a right-sided fundamental matrix of $A(\partial)$, see Definition 2.1.1. Obviously, we then obtain $A^1(\partial)E^1 = I_l\delta$ for $A^1(\partial) = S \cdot A(\partial + c) \cdot T$ and $E^1 = \mathrm{e}^{-cx}T^{-1} \cdot E \cdot S^{-1}$. Finally, analogously to the proof of Proposition 1.3.19,

$$\begin{aligned}
\big(\tilde{A}(\partial)\tilde{E}\big)_{ij} &= |\det B|^{-1} \cdot \big[(A^1 \circ B)(\partial)(E^1 \circ B^{-1T})\big]_{ij} \\
&= |\det B|^{-1} \sum_{k=1}^{l} (A^1_{ik} \circ B)(\partial)(E^1_{kj} \circ B^{-1T}) \\
&= |\det B|^{-1} \delta_{ij} \cdot \delta \circ B^{-1T} = \delta_{ij} \cdot \delta(x),
\end{aligned}$$

and hence $\tilde{A}(\partial)\tilde{E} = I_l\delta$. □

Example 2.5.2

(a) The *iterated transport operator* in $\mathbf{R}^n$ has the form $\tilde{P}(\partial) = \big(\lambda + \sum_{j=1}^n a_j\partial_j\big)^m$, $a \in \mathbf{R}^n \setminus \{0\}$, $\lambda \in \mathbf{C}$, $m \in \mathbf{N}$. Since $P(\partial) = \partial_1^m$ has the fundamental solution

$$E = \frac{Y(x_1)x_1^{m-1}}{(m-1)!} \otimes \delta(x'), \qquad x' = (x_2, \dots, x_n),$$

and

$$\tilde{P}(\partial) = P(B\partial + c) = (B\partial + c)_1^m, \quad B = \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \; c = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

(where we suppose without restriction of generality that $a_1 \neq 0$), we infer that $\tilde{E} = \frac{1}{|a_1|}(\mathrm{e}^{-cx}E) \circ B^{-1T}$ is a fundamental solution of $\tilde{P}(\partial)$.

For a test function $\phi \in \mathcal{D}(\mathbf{R}^n)$, we obtain

$$\begin{aligned}
\langle \phi, \tilde{E} \rangle = \langle \phi \circ B^T, \mathrm{e}^{-cx}E \rangle &= \frac{1}{(m-1)!} \langle \phi \circ B^T, Y(x_1)\mathrm{e}^{-\lambda x_1}x_1^{m-1} \otimes \delta(x') \rangle \\
&= \frac{1}{(m-1)!} \int_0^\infty \phi(at)\mathrm{e}^{-\lambda t}t^{m-1}\,\mathrm{d}t.
\end{aligned} \tag{2.5.2}$$

For formula (2.5.2), cf. Garnir [97], Vladimirov [279], § 10, Section 11, p. 154.

(b) More generally, let us consider now the powers of a linear first-order operator with *complex* constant coefficients, i.e., $\tilde{P}(\partial) = \big(\lambda + \sum_{j=1}^{n} a_j \partial_j\big)^m$, $\lambda \in \mathbf{C}$, $a \in \mathbf{C}^n$, $m \in \mathbf{N}$, with $\alpha = \operatorname{Re} a, \beta = \operatorname{Im} a$ linearly independent in $\mathbf{R}^n$. Without restriction of generality, we can assume that $\det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \neq 0$. We then have $\tilde{P}(\partial) = P(B\partial + c)$ where $P(\partial) = (\partial_1 + \mathrm{i}\partial_2)^m$ is the iterated Cauchy–Riemann operator and

$$B = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots\dots\dots & & & \alpha_n \\ \beta_1 & \beta_2 & \dots\dots\dots & & & \beta_n \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \dots\dots & & 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Example 1.3.14(b) and Lemma 2.5.3 below,

$$E = \frac{x_1^{m-1} \otimes \delta(x'')}{(m-1)! \cdot 2\pi(x_1 + \mathrm{i}x_2)}, \qquad x'' = (x_3, \dots, x_n),$$

is a fundamental solution of $P(\partial)$ in $\mathbf{R}^n$. Hence Proposition 2.5.1 yields the following representation of a fundamental solution $\tilde{E}$ of $\tilde{P}(\partial) = \big(\lambda + \sum_{j=1}^{n} a_j \partial_j\big)^m$:

$$\begin{aligned} \phi \in \mathcal{D}(\mathbf{R}^n) \Longrightarrow \langle \phi, \tilde{E} \rangle &= \langle \phi \circ B^T, \mathrm{e}^{-cx} E \rangle \\ &= \frac{1}{2\pi(m-1)!} \langle \phi \circ B^T, \frac{\mathrm{e}^{-\lambda x_1} x_1^{m-1}}{x_1 + \mathrm{i}x_2} \otimes \delta(x'') \rangle \\ &= \frac{1}{2\pi(m-1)!} \int_{\mathbf{R}^2} \phi(s\alpha + t\beta) \mathrm{e}^{-\lambda s} \frac{s^{m-1}}{s + \mathrm{i}t} \, \mathrm{d}s \mathrm{d}t. \end{aligned}$$

□

Lemma 2.5.3 *If E_1 is a fundamental solution of the operator $P_1(\partial) = \partial_1 + R(\partial')$, $\partial' = (\partial_2, \dots, \partial_n)$, then the iterated operators $P_m(\partial) = P_1(\partial)^m = \big(\partial_1 + R(\partial')\big)^m$, $m \in \mathbf{N}$, have the fundamental solutions*

$$E_m = \frac{x_1^{m-1}}{(m-1)!} E_1.$$

Proof In fact, for $m \geq 2$, we have

$$\begin{aligned} P_1(\partial)E_m &= \frac{x_1^{m-2}}{(m-2)!}\,E_1 + \frac{x_1^{m-1}}{(m-1)!}\,P_1(\partial)E_1 \\ &= E_{m-1} + \frac{x_1^{m-1}}{(m-1)!}\,\delta = E_{m-1}. \end{aligned}$$

□

Let us point out that the formula in Lemma 2.5.3 can be conceived as a special case of the Bernstein–Sato recursion formula $Q(\lambda, -\mathrm{i}\partial, -\mathrm{i}x)E(\lambda+1) = b(\lambda)E(\lambda)$, see (2.3.4), when taking $Q(\lambda, \xi, \partial) = \partial_1,\ b(\lambda) = \mathrm{i}(\lambda+1)$ and $E_m = E(-m)$, i.e., $-\mathrm{i}x_1E_m = -\mathrm{i}mE_{m+1}$.

Example 2.5.4 The general *iterated anisotropic metaharmonic operator* has the form

$$\tilde{P}(\partial) = (\nabla^T A\nabla + b^T\nabla - \lambda)^m = \Big(\sum_{j=1}^n\sum_{k=1}^n a_{jk}\partial_j\partial_k + \sum_{j=1}^n b_j\partial_j - \lambda\Big)^m,$$

where $A = (a_{jk})_{1\leq j,k\leq n}$ is a real, symmetric, positive definite matrix, $b \in \mathbf{C}^n,\ \lambda \in \mathbf{C},\ \mu = \lambda + \frac{1}{4}b^TA^{-1}b \in \mathbf{C}\setminus(-\infty, 0],\ m \in \mathbf{N}$.

For $\mu > 0$, the uniquely determined temperate fundamental solution E of $P(\partial) = (\Delta_n - \mu)^m$ was derived in Example 1.4.11, see (1.4.9), and also in Example 1.6.11 (a):

$$E = \frac{(-1)^m|x|^{m-n/2}\mu^{n/4-m/2}}{2^{n/2+m-1}\pi^{n/2}(m-1)!}\,K_{n/2-m}(\sqrt{\mu}\,|x|).$$

By analytic continuation, this expression continues to yield the only temperate fundamental solution as long as $\mu \in \mathbf{C}\setminus(-\infty, 0]$, cf. also Example 2.4.2.

If we define the matrix B as the square root of the positive definite real matrix A, i.e., $B = \sqrt{A}$, and set $c = \frac{1}{2}B^{-1}b$, then

$$\begin{aligned} P(B\partial + c) &= \big((c^T + \nabla^TB^T)(B\nabla + c) - \mu\big)^m \\ &= \big(\nabla^TB^TB\nabla + 2c^TB\nabla + c^Tc - \mu\big)^m = (\nabla^TA\nabla + b^T\nabla - \lambda)^m = \tilde{P}(\partial) \end{aligned}$$

because of $\mu = \lambda + \frac{1}{4}b^TA^{-1}b$. Hence the temperate fundamental solution of $\tilde{P}(\partial) = (\nabla^TA\nabla + b^T\nabla - \lambda)^m$ is given by

$$\begin{aligned} \tilde{E} &= |\det B|^{-1}(\mathrm{e}^{-cx}E)\circ B^{-1T} = \frac{\exp(-\frac{1}{2}b^TA^{-1}x)}{\sqrt{\det A}}\,(E\circ A^{-1/2}) \\ &= \frac{(-1)^m\exp(-\frac{1}{2}b^TA^{-1}x)}{2^{n/2+m-1}\pi^{n/2}(m-1)!\sqrt{\det A}}\cdot\left(\frac{x^TA^{-1}x}{\mu}\right)^{m/2-n/4}\cdot K_{n/2-m}\big(\sqrt{\mu\cdot x^TA^{-1}x}\big). \end{aligned} \tag{2.5.3}$$

A direct verification of this fundamental solution by Fourier transformation is given in Lorenzi [170], pp 841–844. □

Example 2.5.5 Similarly as in the last example, let us deduce now a fundamental solution $\tilde{E}$ of the *iterated anisotropic heat and Schrödinger operators*

$$\tilde{P}(\partial) = (\partial_t - \nabla^T A \nabla + b^T \nabla - \lambda)^m = \Big(\partial_t - \sum_{j=1}^n \sum_{k=1}^n a_{jk}\partial_j\partial_k + \sum_{j=1}^n b_j\partial_j - \lambda\Big)^m,$$

where $A = A^T \in \mathbf{C}^{n\times n}$ is non-singular (i.e., $\det A \neq 0$) and with positive semi-definite real part, $b \in \mathbf{C}^n,\ \lambda \in \mathbf{C}$.

By Example 1.4.13 and Lemma 2.5.3, a fundamental solution E of $P(\partial) = (\partial_t - \nabla^T A \nabla)^m$ is given by

$$E = \frac{Y(t)t^{m-1}}{(4\pi t)^{n/2}\sqrt{\det A}(m-1)!}\, \mathrm{e}^{-x^T A^{-1}x/(4t)} \in \mathcal{C}\big([0,\infty), \mathcal{D}'(\mathbf{R}^n_x)\big),$$

see (1.4.14). Setting $\mu = \lambda - \frac14 b^T A^{-1} b,\ c = (-\mu, -\frac12 b^T A^{-1})^T \in \mathbf{C}^{n+1}$, we obtain

$$\begin{aligned} P(\partial + c) &= \big[\partial_t - \mu - (\nabla^T - \tfrac12 b^T A^{-1})A(\nabla - \tfrac12 A^{-1}b)\big]^m \\ &= (\partial_t - \nabla^T A \nabla + b^T\nabla - \lambda)^m = \tilde{P}(\partial) \end{aligned}$$

and hence

$$\begin{aligned} \tilde{E} &= \mathrm{e}^{\mu t} \cdot \mathrm{e}^{b^T A^{-1}x/2} \cdot E \\ &= \frac{Y(t)t^{m-1}\mathrm{e}^{\mu t}}{(4\pi t)^{n/2}\sqrt{\det A}(m-1)!}\, \mathrm{e}^{-x^T A^{-1}x/(4t) + b^T A^{-1}x/2} \in \mathcal{C}\big([0,\infty), \mathcal{D}'(\mathbf{R}^n_x)\big). \end{aligned}$$

□

Example 2.5.6 As our final example, we consider an iterated *anisotropic Klein–Gordon operator* of the form

$$\tilde{P}(\partial) = (\partial_t^2 + \beta\partial_t - \nabla^T A \nabla + b^T\nabla - \lambda)^m = \Big(\partial_t^2 + \beta\partial_t - \sum_{j=1}^n \sum_{k=1}^n a_{jk}\partial_j\partial_k + \sum_{j=1}^n b_j\partial_j - \lambda\Big)^m,$$

where $A = (a_{jk})_{1\le j,k\le n}$ is a real, symmetric, positive definite matrix, $b \in \mathbf{C}^n,\ \beta, \lambda \in \mathbf{C},\ \mu^2 = \frac14(b^T A^{-1} b - \beta^2) - \lambda \neq 0,\ m \in \mathbf{N}$.

By formula (2.3.14) in Example 2.3.7, the forward fundamental solution of the operator $P(\partial) = (\partial_t^2 - \Delta_n + \mu^2)^m$ is locally integrable if $m > \frac{n-1}{2}$ and given by

$$E = \frac{2^{-m-(n-1)/2} Y(t-|x|)\mu^{-m+(n+1)/2}}{\pi^{(n-1)/2}(m-1)!}\,(t^2-|x|^2)^{m/2-(n+1)/4} \times \\ \times J_{m-(n+1)/2}\big(\mu\sqrt{t^2-|x|^2}\big).$$

As in Example 2.5.4, we set $B = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{A} \end{pmatrix} \in \mathrm{Gl}_{n+1}(\mathbf{R})$, $c = \frac{1}{2}\binom{\beta}{-A^{-1/2}b}$ and conclude that

$$P(B\partial + c) = (\partial_t^2 + \beta\partial_t + \tfrac{\beta^2}{4} - \nabla^T A \nabla + b^T\nabla - \tfrac{1}{4} b^T A^{-1} b + \mu^2)^m = \tilde{P}(\partial).$$

Hence the forward fundamental solution of $\tilde{P}(\partial)$ is

$$\begin{aligned} \tilde{E} &= |\det B|^{-1}(\mathrm{e}^{-\beta t/2 + b^T A^{-1/2} x/2} E) \circ B^{-1} \\ &= \frac{2^{-m-(n-1)/2} Y(t - \sqrt{x^T A^{-1} x})\mu^{-m+(n+1)/2}}{\pi^{(n-1)/2}(m-1)!\sqrt{\det A}}\, \mathrm{e}^{-\beta t/2 + b^T A^{-1} x/2} \times \\ &\qquad \times (t^2 - x^T A^{-1} x)^{m/2-(n+1)/4} J_{m-(n+1)/2}\big(\mu\sqrt{t^2 - x^T A^{-1}x}\big). \qquad \square \end{aligned}$$

2.6 Invariance with Respect to Transformation Groups

In some cases, the invariance of a differential operator $P(\partial)$ under a group of linear transformations offers a means for the calculation of a fundamental solution. More precisely, such an invariance leads from the constant coefficient operator $P(\partial)$ in n dimensions to a linear operator in fewer variables having however, in general, non-constant coefficients. In order to exploit the invariance of $P(\partial)$, we have to employ some uniqueness class for the fundamental solution (see Sect. 2.4), and we use the following proposition, cf. Wagner [285], Satz 6, p. 11.

Proposition 2.6.1 *Let $P(\partial) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ be a linear constant coefficient operator in $\mathbf{R}^n$ and $A \in \mathrm{Gl}_n(\mathbf{R})$ such that $P \circ A = \lambda \cdot P$ for some $\lambda \in \mathbf{C} \setminus \{0\}$. Furthermore, assume that the subspace $\mathcal{H} \subset \mathcal{D}'(\mathbf{R}^n)$ is an A^T-invariant "set of uniqueness" for $P(\partial)$, i.e.,*

$$\text{(i)}\forall T \in \mathcal{H} : T \circ A^T \in \mathcal{H}; \qquad \text{(ii)}\exists_1 E \in \mathcal{H} : P(\partial)E = \delta.$$

Then the unique fundamental solution $E \in \mathcal{H}$ of $P(\partial)$ fulfills

$$E \circ A^T = \frac{\lambda}{|\det A|} E. \tag{2.6.1}$$

Proof By Proposition 1.3.19, we have

$$(P \circ A^{-1})(\partial)\big(|\det A| \cdot E \circ A^T\big) = \delta.$$

Due to $P \circ A^{-1} = \lambda^{-1} P$, we infer that $\lambda^{-1}|\det A| \cdot E \circ A^T$ is a fundamental solution of $P(\partial)$ belonging to $\mathcal{H}$ by assumption (i). From the uniqueness property (ii), we then conclude the equation in (2.6.1). □

Example 2.6.2 Let us first consider the *iterated Laplacean* $P(\partial) = \Delta_n^m$, $m \in \mathbf{N}$. Then $P \circ A = P$ if A belongs to the orthogonal group $\mathrm{O}_n(\mathbf{R})$, and $P \circ A = c^{2m} P$ if $A = cI$, $c \in \mathbf{R}$. On the other hand, the subspace $\mathcal{H} = \mathcal{E}' + \mathcal{C}_0 \subset \mathcal{S}'$ is invariant under orthogonal linear transformations and under dilatations. Furthermore, there exists at most one fundamental solution of $P(\partial)$ in $\mathcal{H}$. In fact, if $T \in \mathcal{H}$ and $P(\partial)T = 0$, then $|x|^{2m}\mathcal{F}T = 0$ and hence $\mathcal{F}T = \sum_{|\alpha|\le 2m} c_\alpha \partial^\alpha \delta$, $c_\alpha \in \mathbf{C}$, see Proposition 1.3.15. Therefore, $T = (2\pi)^{-n} \sum_{|\alpha|\le 2m} c_\alpha (-\mathrm{i}x)^\alpha$ is a polynomial, which must vanish due to $T \in \mathcal{H}$.

If there exists at all a fundamental solution E of $P(\partial)$ in $\mathcal{H}$, then Proposition 2.6.1 implies

$$\forall A \in \mathrm{O}_n(\mathbf{R}) : E \circ A^T = E \quad \text{and} \quad \forall c \in \mathbf{R} : E(cx) = c^{2m-n} E,$$

i.e., E is radially symmetric and homogeneous of degree $2m-n$. By Example 1.6.10, E then must have the form $E = c|x|^{2m-n} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $c \in \mathbf{C}$. Note that such distributions belong to $\mathcal{H}$ only if $m < \frac{n}{2}$.

If $m < \frac{n}{2}$, then there exists a fundamental solution in $\mathcal{H}$, which necessarily has the form $E = c|x|^{2m-n}$, see formula (1.6.19), and E is unique in $\mathcal{H}$ by the above reasoning. In contrast, for $m \le \frac{n}{2}$ and n even, $\Delta_n^m |x|^{2m-n} = 0$. □

Example 2.6.3 Let us next consider the *heat operator* $P(\partial) = \partial_t - \Delta_n$. This operator has the following invariance properties: $P \circ A = P$ for $A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, $B \in \mathrm{O}_n(\mathbf{R})$, and $P \circ A = c^2 P$ for $A = \begin{pmatrix} c^2 & 0 \\ 0 & cI_n \end{pmatrix}$, $c \in \mathbf{R}$. By Proposition 2.4.13, there exists a unique fundamental solution E in

$$\mathcal{H} = \{T \in \mathcal{S}'(\mathbf{R}^{n+1});\ T = 0 \text{ for } t < 0\}.$$

Furthermore, $\mathcal{H}$ is invariant under the linear transformations A considered above, and hence Proposition 2.6.1 implies that $E(t, Bx) = E$ for each $B \in \mathrm{O}_n(\mathbf{R})$ and $E(c^2 t, cx) = c^{-n} E$ for $c > 0$.

Since the heat operator $\partial_t - \Delta_n$ is hypoelliptic, we know that E is infinitely differentiable outside the origin. Therefore, $f(s) := E(1, \sqrt{s}, 0, \dots, 0) \in \mathcal{C}^\infty((0, \infty))$

and E can be represented by f in $\mathbf{R}^{n+1} \setminus \{0\}$:

$$t > 0,\ c = \frac{1}{\sqrt{t}} \Longrightarrow E(t,x) = c^n E(c^2 t, cx) = t^{-n/2} E\Big(1, \frac{x}{\sqrt{t}}\Big) = t^{-n/2} f\Big(\frac{|x|^2}{t}\Big),$$

i.e., $E|_{\mathbf{R}^{n+1}\setminus\{0\}} = Y(t)t^{-n/2} f(|x|^2/t)$.

In order to determine the explicit form of the function f, we use the differential equation $(\partial_t - \Delta_n)E = 0$ in $\mathbf{R}^{n+1} \setminus \{0\}$. Setting $s = |x|^2/t$ and $r = |x|$ we obtain, for $t > 0$,

$$\begin{aligned}(\partial_t - \Delta_n)E &= (\partial_t - \partial_r^2 - \tfrac{n-1}{r}\,\partial_r)E = -t^{-n/2-1}\big[4sf'' + (2n+s)f' + \tfrac{n}{2}f\big] \\ &= -t^{-n/2-1}\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \frac{n}{2}\Big)\Big(4\frac{\mathrm{d}}{\mathrm{d}s} + 1\Big)f.\end{aligned}$$

Hence f must fulfill

$$\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + \frac{n}{2}\Big)\Big(4\frac{\mathrm{d}}{\mathrm{d}s} + 1\Big)f(s) = 0, \qquad s > 0.$$

If $g = 4f' + f$, then $sg' + \frac{n}{2}g = 0$ yields $g(s) = Ds^{-n/2}$, and $4f' + f = Ds^{-n/2}$ implies

$$f(s) = \mathrm{e}^{-s/4}\Big(D\int_1^s \sigma^{-n/2}\mathrm{e}^{\sigma/4}\,\mathrm{d}\sigma + C\Big), \quad C, D \in \mathbf{C}.$$

Thus we obtain that

$$E(t,x) = Y(t)t^{-n/2}\mathrm{e}^{-|x|^2/(4t)}\Big(D\int_1^{|x|^2/t} \sigma^{-n/2}\mathrm{e}^{\sigma/4}\,\mathrm{d}\sigma + C\Big).$$

Since $E(t,x)$ is defined and regular for $t = 1$ and $x = 0$, we conclude that the constant D must vanish, i.e.,

$$E_n = C_n Y(t) t^{-n/2}\mathrm{e}^{-|x|^2/(4t)} \in L^1_{\mathrm{loc}}(\mathbf{R}^{n+1}), \qquad C_n \in \mathbf{R},$$

where we indicate the dependence on the space dimension n now by an additional index.

Let us yet determine the value of C_n by Hadamard's method of descent, cf. Delache and Leray [56], p. 317. The distribution

$$W = E_n * (\delta_{(t,x')} \otimes 1_{x_n}) - E_{n-1} \otimes 1_{x_n} \in \mathcal{S}'(\mathbf{R}^{n+1}), \qquad x' = (x_1, \ldots, x_{n-1}),$$

is well-defined and satisfies $(\partial_t - \Delta_n)W = 0$ and $W = 0$ for $t < 0$. By Proposition 2.4.13, we have $W = 0$, i.e., $E_n * (\delta_{(t,x')} \otimes 1_{x_n}) = E_{n-1} \otimes 1_{x_n}$. For

$t = 1, x' = 0$, this implies

$$C_{n-1} = E_{n-1}(1,0) = \int_{\mathbf{R}} E_n(1,0,x_n)\,\mathrm{d}x_n = C_n \int_{-\infty}^{\infty} \mathrm{e}^{-x_n^2/4}\,\mathrm{d}x_n = 2\sqrt{\pi}\,C_n.$$

Because of $E_0 = Y(t)$, i.e., $C_0 = 1$, we finally obtain

$$E_n = \frac{Y(t)}{(4\pi t)^{n/2}}\,\mathrm{e}^{-|x|^2/(4t)}$$

in accordance with (1.3.14) in Example 1.3.14 (d), see also Examples 1.6.16 and 2.3.8 (b). □

Example 2.6.4 We shall apply Proposition 2.6.1 also in the case of the *Sobolev operator* $P(\partial) = \partial_t - \partial_1\partial_2\partial_3$ in $\mathbf{R}^4$, cf. Example 2.3.8 (c). This operator is quasi-hyperbolic with respect to t, and hence there exists one and only one fundamental solution E in the subspace

$$\mathcal{H} = \{T \in \mathcal{S}'(\mathbf{R}^4);\ T = 0 \text{ for } t < 0\}.$$

The relations $P \circ A = cP$ for $A = \begin{pmatrix} c & 0 \\ 0 & B \end{pmatrix} \in \mathrm{Gl}_4(\mathbf{R}),\ B = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$, $c_j \in \mathbf{R}$ with $c = c_1c_2c_3 > 0$ then yield $E(ct, c_1x_1, c_2x_2, c_3x_3) = c^{-1}E$. This implies that E can be represented by composition with the function $h(t,x) = x_1x_2x_3/t$. More precisely, if

$$U = \{(t,x) \in \mathbf{R}^4;\ t \neq 0, (x_1x_2, x_1x_3, x_2x_3) \neq 0\}$$
$$\text{and } h : U \longrightarrow \mathbf{R} : (t,x) \longmapsto \frac{x_1x_2x_3}{t},$$

then h is $\mathcal{C}^\infty$ and submersive, and

$$E|_U = \frac{Y(t)}{t}\,h^*(\Lambda) = \frac{Y(t)}{t}\,\Lambda\Big(\frac{x_1x_2x_3}{t}\Big) \in \mathcal{D}'(U)$$

holds for some distribution in one variable $\Lambda \in \mathcal{D}'(\mathbf{R})$.

If we express $(\partial_t - \partial_1\partial_2\partial_3)E$ by the function $\Lambda(s)$ of the variable $s = \frac{x_1x_2x_3}{t}$, we obtain that

$$\begin{aligned}(\partial_t - \partial_1\partial_2\partial_3)E &= -\frac{Y(t)}{t^2}\big(s^2\Lambda''' + 3s\Lambda'' + (s+1)\Lambda' + \Lambda\big) \circ h \\ &= -\frac{Y(t)}{t^2}\Big[\Big(s\frac{\mathrm{d}}{\mathrm{d}s} + 1\Big)\Big(s\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{\mathrm{d}}{\mathrm{d}s} + 1\Big)\Lambda\Big] \circ h\end{aligned}$$

holds in $\mathcal{D}'(U)$. From this we conclude that $(s\frac{\mathrm{d}}{\mathrm{d}s}+1)(s\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\mathrm{d}}{\mathrm{d}s}+1)\Lambda = 0$ in $\mathcal{D}'(\mathbf{R})$.

The solutions of $s \cdot T' + T = 0$ in $\mathcal{D}'(\mathbf{R}^1_s)$ are given by $T = A\delta + B\,\mathrm{vp}(\frac{1}{s}),\ A, B \in \mathbf{C}$. Hence Λ fulfills in $\mathcal{D}'(\mathbf{R})$ the ordinary differential equation

$$s\Lambda'' + \Lambda' + \Lambda = A\delta + B\,\mathrm{vp}(\tfrac{1}{s}).$$

For $s > 0$, we substitute $u = 2\sqrt{s}$ and write $M(u) = \Lambda(s) = \Lambda(u^2/4)$. Then $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{2}{u}\frac{\mathrm{d}}{\mathrm{d}u}$ and thus M fulfills $M'' + \frac{1}{u}M' + M = 4Bu^{-2}$. From Gradshteyn and Ryzhik [113], Eqs. 8.577, 8.571, we then obtain

$$\begin{aligned}\Lambda(s) &= C_1 J_0(2\sqrt{s}) + C_2 N_0(2\sqrt{s}) \\ &\quad + C_3\Big[J_0(2\sqrt{s})\int_{2\sqrt{s}}^{\infty} N_0(u)\,\frac{\mathrm{d}u}{u} - N_0(2\sqrt{s})\int_{2\sqrt{s}}^{\infty} J_0(u)\,\frac{\mathrm{d}u}{u}\Big], \qquad s > 0.\end{aligned}$$

Similarly, for $s < 0$, we substitute $u = 2\sqrt{-s}$ and obtain

$$\begin{aligned}\Lambda(s) &= C_4 I_0(2\sqrt{-s}) + C_5 K_0(2\sqrt{-s}) \\ &\quad + C_6\Big[I_0(2\sqrt{-s})\int_{1}^{2\sqrt{-s}} K_0(u)\,\frac{\mathrm{d}u}{u} - K_0(2\sqrt{-s})\int_{1}^{2\sqrt{-s}} I_0(u)\,\frac{\mathrm{d}u}{u}\Big], \qquad s < 0,\end{aligned}$$

cf. Sobolev [255], Eq. (7), p. 1247.

Let us consider now $(\partial_t - \partial_1\partial_2\partial_3)Y(t)t^{-1}\Lambda(x_1x_2x_3/t)$ in $\mathcal{D}'(\mathbf{R}^4)$. If we specify Λ by

$$\Lambda(s) = Y(s)N_0(2\sqrt{s}) - \tfrac{2}{\pi}Y(-s)K_0(2\sqrt{-s})$$

then the asymptotic expansions of N_0 and K_0 at 0 (see Gradshteyn and Ryzhik [113], Eqs. 8.444.1, 8.447.3) imply that

$$\Lambda(s) = \frac{1}{\pi}(2\gamma + \log|s|) + O(s^2\log|s|), \qquad s \to 0, \tag{2.6.2}$$

and hence $f(t,x) = Y(t)t^{-1}\Lambda(x_1x_2x_3/t)$ fulfills $(\partial_t - \partial_1\partial_2\partial_3)f = 0$ for $t \neq 0$. Therefore, by the jump formula,

$$(\partial_t - \partial_1\partial_2\partial_3)f = \lim_{\epsilon\searrow 0}(\partial_t - \partial_1\partial_2\partial_3)Y(t-\epsilon)f = T\otimes\delta(t), \quad T = \lim_{\epsilon\searrow 0} f(\epsilon, x) \in \mathcal{D}'(\mathbf{R}^3).$$

Because of (2.6.2), the equation $(s\Lambda')' + \Lambda = 0$ holds in $\mathcal{D}'(\mathbf{R})$. The functions $g_\epsilon(s) = \Lambda(s/\epsilon)$ then satisfy $sg_\epsilon'' + g_\epsilon' + \epsilon^{-1}g_\epsilon = 0$, and $\lim_{\epsilon\searrow 0} g_\epsilon = 0$ in $\mathcal{D}'(\mathbf{R})$ inductively yields that $\lim_{\epsilon\searrow 0}\epsilon^\alpha g_\epsilon = 0$ holds in $\mathcal{D}'(\mathbf{R})$ for each $\alpha \in \mathbf{R}$.

In particular, for $\alpha = -1$, we conclude by composition with $h_1(x) = x_1x_2x_3$ that

$$T = \lim_{\epsilon \searrow 0} f(\epsilon, x) = \lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\Lambda\Big(\frac{x_1x_2x_3}{\epsilon}\Big) = h_1^*\Big(\lim_{\epsilon \searrow 0} \frac{1}{\epsilon}\Lambda\Big(\frac{s}{\epsilon}\Big)\Big) = 0$$

holds in $U_1 = \{x \in \mathbf{R}^3;\ (x_1x_2, x_1x_3, x_2x_3) \neq 0\}$, i.e., outside the three coordinate axes.

If $c_1, c_2, c_3 \in \mathbf{R}$ with $c = c_1c_2c_3 > 0$, then $f(ct, c_1x_1, c_2x_2, c_3x_3) = c^{-1}f$ and hence $T(c_1x_1, c_2x_2, c_3x_3) = c^{-1}T$. Since $T|_{U_1} = 0$, this implies that T is a multiple of δ. Thus $(\partial_t - \partial_1\partial_2\partial_3)f = C\delta$, and $C \neq 0$ due to $f \in \mathcal{S}'(\mathbf{R}^4)$ and the uniqueness statement in Proposition 2.4.13. As a consequence,

$$E = C_1\frac{Y(t)}{t}\Big[Y(x_1x_2x_3)N_0\big(2\sqrt{x_1x_2x_3/t}\big) - \frac{2}{\pi}Y(-x_1x_2x_3)K_0\big(2\sqrt{-x_1x_2x_3/t}\big)\Big]$$

is, for appropriate C_1, the unique fundamental solution of $\partial_t - \partial_1\partial_2\partial_3$ satisfying $E \in \mathcal{S}'(\mathbf{R}^4)$ and $E = 0$ for $t < 0$. In fact, (2.3.21) shows that we must choose $C_1 = -\frac{1}{2\pi}$. □

Example 2.6.5

(a) In a very similar way, one could consider the operators $\partial_t - \mathrm{i}\prod_{j=1}^{2m}\partial_j$ and $\partial_t - \prod_{j=1}^{2m+1}\partial_j$, which are quasihyperbolic with respect to t and hence have one and only one temperate fundamental solution E with support in the half-space where $t \geq 0$.

For $P(\partial) = \partial_t - \mathrm{i}\partial_1\partial_2$, formula (1.4.14) yields

$$E = \frac{Y(t)}{2\pi t}\,\mathrm{e}^{\mathrm{i}x_1x_2/t} \in \mathcal{C}\big([0, \infty), \mathcal{S}'(\mathbf{R}^2)\big).$$

On the other hand, the invariance method as in Example 2.6.4 leads to the representation

$$E = \frac{Y(t)}{t}\,\Lambda\Big(\frac{x_1x_2}{t}\Big) = \frac{Y(t)}{t}\,\Lambda(s), \qquad s = \frac{x_1x_2}{t},$$

and to the ordinary differential equation $(\frac{\mathrm{d}}{\mathrm{d}s}s)(1 + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s})\Lambda = 0$, which has the correct solution $C\mathrm{e}^{\mathrm{i}s},\ c \in \mathbf{C}$.

(b) By partial Fourier transformation as in Example 2.3.8 (c), the fundamental solutions of $\partial_t - \mathrm{i}\partial_1\partial_2\partial_3\partial_4$ and of $\partial_t - \partial_1\partial_2\partial_3\partial_4\partial_5$, respectively, can be expressed as simple definite integrals involving Bessel functions. E.g., for $P(\partial) = \partial_t - \partial_1\partial_2\partial_3\partial_4\partial_5$, we obtain the following representation of the fundamental solution

E by a simple definite integral:

$$E = \frac{2Y(t)}{\pi^4 t} \int_0^\infty \Big\{ K_0(u) \Big[Y(X) K_0\Big(\frac{4}{u}\sqrt{\frac{X}{t}}\Big) - \frac{\pi}{2} Y(-X) N_0\Big(\frac{4}{u}\sqrt{-\frac{X}{t}}\Big)\Big] \\ - \frac{\pi}{2} N_0(u) \Big[Y(-X) K_0\Big(\frac{4}{u}\sqrt{-\frac{X}{t}}\Big) - \frac{\pi}{2} Y(X) N_0\Big(\frac{4}{u}\sqrt{\frac{X}{t}}\Big)\Big]\Big\} \frac{\mathrm{d}u}{u}$$

where $X = \prod_{j=1}^5 x_j$. In contrast, the invariance method leads to the fourth-order ordinary differential equation $\big[1 + \frac{\mathrm{d}}{\mathrm{d}s}(s\frac{\mathrm{d}}{\mathrm{d}s})^3\big]\Lambda(s) = 0$. □

Example 2.6.6 Let us finally apply the invariance method to the *Klein–Gordon operator* $P(\partial) = \partial_t^2 - \Delta_n + c^2$, $c \in \mathbf{C} \setminus (-\infty, 0]$, which is invariant under Lorentz transformations. For the method in general, see Szmydt and Ziemian [267, 268].

Since the operator $P(\partial)$ is hyperbolic, there exists one and only one fundamental solution E in

$$\mathcal{H} = \{T \in \mathcal{D}'(\mathbf{R}^{n+1});\ T = 0 \text{ for } t < 0\},$$

see Proposition 2.4.11. If we define the Lorentz product by

$$\left[\begin{pmatrix} t \\ x \end{pmatrix}, \begin{pmatrix} t \\ x \end{pmatrix}\right] = t^2 - |x|^2, \qquad \begin{pmatrix} t \\ x \end{pmatrix} \in \mathbf{R}^{n+1},$$

then $A = (a_{ij})_{0 \le i,j \le n} \in \mathrm{Gl}_{n+1}(\mathbf{R})$ is called a *proper Lorentz transformation* if $a_{00} > 0$ and $\big[A\binom{t}{x}, A\binom{t}{x}\big] = t^2 - |x|^2$ for each $\binom{t}{x} \in \mathbf{R}^{n+1}$. Since $P(\partial)$ and $\mathcal{H}$ are invariant under proper Lorentz transformations A, Proposition 2.6.1 yields $E = E \circ A$.

Therefore, outside the origin, E is the pull-back of a one-dimensional distribution T by the submersive function

$$h : \mathbf{R}^{n+1} \setminus \{0\} \longrightarrow \mathbf{R} : \begin{pmatrix} t \\ x \end{pmatrix} \longmapsto t^2 - |x|^2,$$

cf. Proposition 1.2.13 and Gårding and Lions [96], i.e.,

$$E|_{\mathbf{R}^{n+1}\setminus\{0\}} = Y(t) h^*(T), \qquad T \in \mathcal{D}'(\mathbf{R}^1), \qquad \operatorname{supp} T \subset [0, \infty).$$

Note that $\operatorname{supp} h^*(T)$ is contained in the union $\{\binom{t}{x} \in \mathbf{R}^{n+1};\ |t| \ge |x|\}$ of the forward and the backward propagation cones, and that therefore $Y(t)$ and $h^*(T)$ can be multiplied in $\mathbf{R}^{n+1} \setminus \{0\}$ without difficulties.

The equation $(\partial_t^2 - \Delta_n + c^2)E|_{\mathbf{R}^{n+1}\setminus\{0\}} = 0$ then furnishes for $T \in \mathcal{D}'(\mathbf{R}^1_u)$ the ordinary differential equation

$$u \cdot T'' + \frac{n+1}{2}T' + \frac{c^2}{4}T = 0.$$

From Gradshteyn and Ryzhik [113], Eq. 8.491.3 (corrected), p. 971, we infer that

$$T|_{(0,\infty)} = u^{(1-n)/4}\big[C_1 J_{(1-n)/2}(c\sqrt{u}) + C_2 N_{(1-n)/2}(c\sqrt{u})\big] \in \mathcal{C}^\infty((0,\infty)).$$

By the recursion formula

$$\frac{\mathrm{d}}{\mathrm{d}u}\big(u^{\nu/2} Z_\nu(c\sqrt{u})\big) = -\frac{c}{2}u^{(\nu-1)/2} Z_{\nu-1}(c\sqrt{u}),$$

where $u > 0$ and $Z_\nu = J_\nu$ or $Z_\nu = N_\nu$ (cf. Gradshteyn and Ryzhik [113], Eq. 8.472.3, p. 968), we see that T coincides on $\mathbf{R} \setminus \{0\}$ with the distribution

$$S = \frac{\mathrm{d}^k}{\mathrm{d}u^k}\big[Y(u)u^{\nu/2}\big(D_1 J_\nu(c\sqrt{u}) + D_2 N_\nu(c\sqrt{u})\big)\big] \in \mathcal{D}'(\mathbf{R}),$$

if $\nu = \left\{\begin{array}{ll} 0: & n \text{ odd}, \\ \frac{1}{2}: & n \text{ even} \end{array}\right\}$ and $k = \nu + (n-1)/2 = [\frac{n}{2}]$, $D_1, D_2 \in \mathbf{C}$. From Proposition 1.3.15, we conclude that

$$T = S + \sum_{j=0}^{m} a_j \delta^{(j)}, \qquad a_j \in \mathbf{C}.$$

Since, by Leibniz' formula,

$$u \cdot \frac{\mathrm{d}^{k+2}U}{\mathrm{d}u^{k+2}} = \frac{\mathrm{d}^{k+2}}{\mathrm{d}u^{k+2}}(u \cdot U) - (k+2)\frac{\mathrm{d}^{k+1}U}{\mathrm{d}u^{k+1}}$$

holds for $U \in \mathcal{D}'(\mathbf{R}^1_u)$, we obtain, due to $\frac{n+1}{2} - (k+2) = -(\nu+1)$,

$$\begin{aligned} u \cdot S'' + \frac{n+1}{2}S' + \frac{c^2}{4}S = D_1 \frac{\mathrm{d}^k}{\mathrm{d}u^k}\Big[& u\big(Y(u)u^{\nu/2}J_\nu(c\sqrt{u})\big)'' \\ & + (1-\nu)\big(Y(u)u^{\nu/2}J_\nu(c\sqrt{u})\big)' + \frac{c^2}{4}Y(u)u^{\nu/2}J_\nu(c\sqrt{u})\Big] \\ D_2 \frac{\mathrm{d}^k}{\mathrm{d}u^k}\Big[& u\big(Y(u)u^{\nu/2}N_\nu(c\sqrt{u})\big)'' \\ & + (1-\nu)\big(Y(u)u^{\nu/2}N_\nu(c\sqrt{u})\big)' + \frac{c^2}{4}Y(u)u^{\nu/2}N_\nu(c\sqrt{u})\Big]. \end{aligned} \tag{2.6.3}$$

The expressions in the brackets on the right-hand side of (2.6.3) must vanish for $u \neq 0$ since they are classical solutions of $uU'' + (1-\nu)U' + \frac{c^2}{4}U = 0$, $u > 0$. Moreover, this equation holds also in $\mathcal{D}'(\mathbf{R}^1_u)$ for $U = Y(u)u^{\nu/2}J_\nu(c\sqrt{u})$. On the other hand, for $U = Y(u)N_0(c\sqrt{u})$, the asymptotic expansion

$$N_0(z) = \frac{2}{\pi}\Big[\log\frac{z}{2} + \gamma\Big] + O(z^2\log z), \qquad z \searrow 0,$$

yields

$$uU'' + U' + \frac{c^2}{4}U = (uU')' + \frac{c^2}{4}U = \frac{1}{\pi}\,\delta.$$

Similarly, for $\nu = \frac{1}{2}$, we set

$$U = Y(u)u^{1/4}N_{1/2}(c\sqrt{u}) = -\sqrt{\frac{2}{\pi c}}\,Y(u)\cos(c\sqrt{u})$$

and obtain

$$uU'' + \frac{1}{2}U' + \frac{c^2}{4}U = (uU')' - \frac{1}{2}U' + \frac{c^2}{4}U = \frac{1}{\sqrt{2\pi c}}\,\delta.$$

Therefore, eventually,

$$u\cdot S'' + \frac{n+1}{2}S' + \frac{c^2}{4}S = CD_2\delta^{(k)}, \qquad C = \begin{cases} \frac{1}{\pi}: & n \text{ odd},\\ \frac{1}{\sqrt{2\pi c}}: & n \text{ even}.\end{cases}$$

Since a non-trivial linear combination of derivatives of δ, i.e., $U = \sum_{j=0}^m a_j\delta^{(j)} \in \mathcal{D}'(\mathbf{R})\setminus\{0\}$, cannot fulfill an equation of the type

$$uU'' + \frac{n+1}{2}U' + \frac{c^2}{4}U = \alpha\delta^{([n/2])}, \qquad \alpha \in \mathbf{C},$$

we conclude that $D_2 = 0$ and $E|_{\mathbf{R}^{n+1}\setminus\{0\}} = Y(t)h^*(T)$ where T is a multiple of $\dfrac{\mathrm{d}^k}{\mathrm{d}u^k}\big[Y(u)u^{\nu/2}J_\nu(c\sqrt{u})\big] \in \mathcal{D}'(\mathbf{R})$, $\nu = \left\{\begin{matrix} 0: & n \text{ odd},\\ \frac{1}{2}: & n \text{ even}\end{matrix}\right\}$ and $k = [\frac{n}{2}]$.

By the recursion formula in Gradshteyn and Ryzhik [113], Eq. 8.472.3, p. 968, the analytic distribution-valued function

$$V: \{\nu \in \mathbf{C};\ \operatorname{Re}\nu > -1\} \longrightarrow \mathcal{D}'(\mathbf{R}^1_u): \nu \longmapsto Y(u)u^{\nu/2}J_\nu(c\sqrt{u})$$

satisfies $\frac{d}{du}V_\nu = -\frac{c}{2}V_{\nu-1}$, and hence it can be extended to an entire function. Therefore, if E_n denotes the fundamental solution of $\partial_t^2 - \Delta_n + c^2$, then

$$E_n|_{\mathbf{R}^{n+1}\setminus\{0\}} = C_n Y(t) h^*(V_{(1-n)/2}) = C_n Y(t) V_{(1-n)/2}(t^2 - |x|^2)$$

for some $C_n \in \mathbf{R}$.

In order to determine the values of the constants C_n, we employ—as in Example 2.6.3—Hadamard's method of descent in the form used in Delache and Leray [56], p. 317. Since

$$W = E_n * (\delta_{(t,x')} \otimes 1_{x_n}) - E_{n-1} \otimes 1_{x_n} \in \mathcal{D}'(\mathbf{R}^{n+1}), \qquad x' = (x_1, \dots, x_{n-1}),$$

is well-defined and satisfies $(\partial_t^2 - \Delta_n + c^2)W = 0$ and $W = 0$ for $t < 0$, we conclude that W vanishes, i.e., $E_n * (\delta_{(t,x')} \otimes 1_{x_n}) = E_{n-1} \otimes 1_{x_n}$. This yields the equation

$$\begin{aligned} 2C_n Y(t - |x'|) \int_0^{\sqrt{t^2-|x'|^2}} (t^2 - |x'|^2 - x_n^2)^{\nu/2} J_\nu\big(c\sqrt{t^2 - |x'|^2 - x_n^2}\big)\, dx_n\Big|_{\nu=(1-n)/2} \\ = C_{n-1} Y(t) V_{(2-n)/2}(t^2 - |x'|^2). \end{aligned}$$

Upon inserting $t = 1$ and $x' = 0$, this implies

$$\begin{aligned} C_{n-1} J_{(n-2)/2}(c) &= 2C_n \int_0^1 (1 - x_n^2)^{\nu/2} J_\nu\big(c\sqrt{1 - x_n^2}\big)\, dx_n\Big|_{\nu=(1-n)/2} \\ &= 2C_n \int_0^1 u^{\nu+1} J_\nu(cu) \frac{du}{\sqrt{1-u^2}}\Big|_{\nu=(1-n)/2} = \sqrt{\frac{2\pi}{c}}\, C_n J_{(2-n)/2}(c), \end{aligned}$$

see Oberhettinger [194], Eq. 4.38, p. 39. Hence $C_n = \sqrt{\frac{c}{2\pi}}\, C_{n-1}$.

Because of

$$E_0 = C_0 Y(t) V_{1/2}(t^2) = C_0 Y(t) \sqrt{t}\, J_{1/2}(ct) = C_0 Y(t) \sqrt{\frac{2}{\pi c}}\, \sin(ct)$$

and $E_0 = Y(t)c^{-1}\sin(ct)$, see Example 1.3.8(a), we have $C_0 = \sqrt{\frac{\pi}{2c}}$ and, consequently,

$$C_n = \Big(\frac{c}{2\pi}\Big)^{n/2} C_0 = \frac{c^{(n-1)/2}}{2^{(n+1)/2}\pi^{(n-1)/2}}.$$

The final formula

$$E_n = \frac{c^{(n-1)/2} Y(t - |x|)}{2^{(n+1)/2} \pi^{(n-1)/2}} \; (t^2 - |x|^2)^{\nu/2} J_\nu(c\sqrt{t^2 - |x|^2})\Big|_{\nu=(1-n)/2}$$

agrees with the result in (2.3.14). For the explicit result for E_n, see also Schwartz [246], (VI, 5; 30), p. 179; Linés [166], 12.9, 12.11, pp. 49, 50; Courant and Hilbert [52], Ch. VI, § 12.6, pp. 693–695; Léonard [162], p. 36; Ortner and Wagner [207], Ex. 5, p. 457. □

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