

Chapter 2

Waves in a Channel on the Mid-latitude β -Plane

In this chapter, Trapped wave solutions of system (1.2) subject to the boundary conditions $v(y = \pm L/2) = 0$ will be derived. The amplitudes of these waves decay with distance from the channel wall located closer to the equator (i.e., $y = -L/2$ in the Northern Hemisphere and $y = +L/2$ in the Southern Hemisphere) which will be referred to as the equatorward wall. These Trapped wave solutions supplement the Harmonic waves of Sect. 1.1.1 whose amplitudes spread over the entire channel. Trapped waves dominate the solution when the channel width L is much larger than the radius of deformation $R_d = (gH)^{1/2}/(2\Omega \sin \phi_0)$.

Before attempting to solve the set (1.2), it is instructive to non-dimensionalize it in order to reduce the number of parameters that determine the solution. The scaling procedure is not unique and one simple way of non-dimensionalizing (1.2) is to scale time, t , on $(2\Omega)^{-1}$; x and y on a , earth's radius; u and v on $2\Omega a$; and η on H , the mean layer's thickness. With this scaling $\sin \phi_0$ and $\cos \phi_0$ designate f_0 and β , respectively, while the non-dimensional y is the latitude angle measured relative to ϕ_0 (since the latitude at a dimensional distance y^{dim} is $\phi_0 + y^{\text{nondim}} = \phi_0 + y^{\text{dim}}/a$). These scales transform the set (1.2) to the non-dimensional form (where variables are designated by the same symbols as their dimensional counterparts):

$$\begin{aligned}\frac{\partial u}{\partial t} - (\sin \phi_0 + y \cos \phi_0)v &= -\alpha \frac{\partial \eta}{\partial x}, \\ \frac{\partial v}{\partial t} + (\sin \phi_0 + y \cos \phi_0)u &= -\alpha \frac{\partial \eta}{\partial y}, \\ \frac{\partial \eta}{\partial t} &= -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),\end{aligned}\tag{2.1}$$

where $\alpha = gH/(2\Omega a)^2$ is the only parameter of this non-dimensional differential system that plays the role of g in the dimensional system (1.2). This number is the square of non-dimensional speed of Gravity waves and its inverse is known as Lamb number or Lamb parameter. On earth, $2\Omega a = 931$ m/s and since the speed of gravity waves, $(gH)^{1/2}$, varies from 2 to 3 m/s in a baroclinic ocean to about 200 m/s in a barotropic ocean/atmosphere, the value of α varies between 5×10^{-6}

and 5×10^{-2} . The boundary conditions imposed on the solutions of (2.1) are $v(y = \pm \delta\phi/2) = 0$ where $\delta\phi = L/a$ is the non-dimensional channel width expressed as a meridional angle (i.e. in radians).

Since there are no x - or t -dependent coefficients in system (2.1) Fourier theorem guarantees that its solutions can be written as the sum (more generally as an integral) of zonally propagating waves that vary as $A(y)e^{ik(x-Ct)}$ where k is the wave number, C is the phase speed (so its frequency is $\omega = kC$) and $A(y)$ is latitude-dependent amplitude. The amplitude structure function $A(y)$ and the dispersion relation $C(k)$ have yet to be determined. For such zonally propagating wave solutions, system (2.1) becomes:

$$\begin{bmatrix} 0 & f(y) & \alpha \\ \frac{f(y)}{k^2} & 0 & \frac{\alpha}{k^2} \frac{\partial}{\partial y} \\ 1 & -\frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{V} \\ \tilde{\eta} \end{bmatrix} = C \begin{bmatrix} \tilde{u} \\ \tilde{V} \\ \tilde{\eta} \end{bmatrix} \quad (2.2)$$

where $\tilde{V} = \frac{iv}{k}$; $f(y) = \sin \phi_0 + y \cos \phi_0$; and the $(\tilde{u}(y), \tilde{V}(y), \tilde{\eta}(y))$ column vector is the y -dependent amplitudes of u , v and η (the tilde designation of the amplitudes will be omitted from this point onwards since we will refer only to the amplitudes when writing u , V and η).

An examination of the first equation in (2.1) or (2.2), i.e., the x -momentum equation, shows that for traveling wave solutions, it yields the following algebraic relation between u , v and η :

$$u = \frac{f(y)}{C} V + \frac{\alpha}{C} \eta = \frac{\sin \phi_0 + y \cos \phi_0}{C} V + \frac{\alpha}{C} \eta. \quad (2.3)$$

This algebraic (i.e., not differential) relation is a reflection of the fact that the linearized x -momentum equation does not contain y -derivatives.

Substituting the expression for u , (2.3), in the other two equations in (2.2) yields the second-order differential system:

$$\begin{aligned} \frac{dV}{dy} &= \frac{\sin \phi_0 + y \cos \phi_0}{C} V + \left(\frac{\alpha}{C} - C \right) \eta, \\ \frac{d\eta}{dy} &= \frac{k^2 C^2 - (\sin \phi_0 + y \cos \phi_0)^2}{\alpha C} V - \frac{\sin \phi_0 + y \cos \phi_0}{C} \eta. \end{aligned} \quad (2.4)$$

This system can be transformed to a single, second-order, equation for either V or η by eliminating the other variable. However, in the special case where the constant coefficient of η in the dV/dy equation, $(\alpha - C^2)/C$, vanishes, this elimination cannot be carried out since the equation for V decouples from the η -equation. This special case yields Kelvin waves.

2.1 Kelvin Waves

When $C^2 = \alpha$, the coefficient of η in the dV/dy equation vanishes so V satisfies a first-order equation and the only way for this solution to vanish at the two channel walls, $y = \pm\delta\phi/2$, is for $V(y)$ to vanish everywhere. In this case, η can be solved by setting $V = 0$ in the second equation whose solutions are the non-dimensional counterpart of Kelvin waves in a channel, (1.4), that is,

$$\begin{aligned} V &= 0, \\ u &= \frac{\alpha}{C} \eta_0 e^{\frac{-(\sin \phi_0 + \cos \phi_0 y)^2}{2C \cos \phi_0}} e^{ik(x-Ct)}, \\ \eta &= \eta_0 e^{\frac{-(\sin \phi_0 + \cos \phi_0 y)^2}{2C \cos \phi_0}} e^{ik(x-Ct)}, \end{aligned} \quad (2.5)$$

where $C = \pm\alpha^{1/2}$. As in the dimensional case, the amplitude of the eastward-propagating wave ($C = +\alpha^{1/2}$) decays with the increase in y while that of the westward-propagating wave ($C = -\alpha^{1/2}$) decays with the decrease in y .

2.2 Inertia-Gravity and Planetary Waves

In the general case, when $C^2 \neq \alpha$, one can eliminate either V or η from (2.4) to obtain a single second-order equation. However, since the boundary conditions involve only V it is natural to eliminate η and retain V as the only dependent variable of the sought equation. To eliminate η one differentiates the V -equation with respect to y and employs the original equations in (2.4) to eliminate the terms proportional to $d\eta/dy$ and η from the resulting equation. It turns out that the elimination yields an equation that has no dV/dy term and the resulting second-order equation and its accompanying boundary conditions are:

$$\frac{d^2 V}{dy^2} + \left(\frac{\omega^2}{\alpha} - \frac{\cos \phi_0}{C} - k^2 - \frac{(\sin \phi_0 + y \cos \phi_0)^2}{\alpha} \right) V = 0; \quad V\left(y = \pm \frac{\delta\phi}{2}\right) = 0. \quad (2.6)$$

This is a classical eigenvalue problem (equation and boundary conditions) whose solutions are determined by imposing the boundary conditions on the general solutions of the differential equation. The general solution of the equation in (2.6) is a linear combination of parabolic cylinder functions (Abramowitz and Stegun 1972) but there is no explicit way of applying the boundary conditions to this form of the solution. To advance, we first note that $\delta\phi$ appears in the boundary conditions of (2.6) so its role in determining the eigensolutions is more cumbersome to identify compared with the parameters of the differential equation itself.

The change of variables $z = 2y/\delta\phi$ leads to a slightly modified problem in which the boundary conditions are applied at $z = \pm 1$ regardless of the channel width, while the parameter $\delta\phi$ appears in the equation itself instead of in the boundary conditions. In terms of z , (2.6) is written as:

$$\varepsilon^2 \frac{d^2 V}{dz^2} + \left(E - (\sin \phi_0 + zb)^2 \right) V = 0; \quad V(z = \pm 1) = 0, \quad (2.7)$$

where

$$\varepsilon = \frac{2\sqrt{\alpha}}{\delta\phi}, \quad E = \omega^2 - \frac{\alpha \cos \phi_0}{C} - \alpha k^2 \quad \text{and} \quad b = \frac{\delta\phi}{2} \cos \phi_0. \quad (2.8)$$

The eigenvalue problem (2.7) has the form of a time-independent Schrödinger eigenvalue problem in one dimension in which E is the energy (or eigenvalue to use a more mathematical notion), $f^2 = (\sin \phi_0 + bz)^2$ is the potential and ε is the counterpart of $\hbar(2m)^{-1/2}$ where m is the mass of the particle and $\hbar$ is Planck constant, h , divided by 2π . It has an infinite number of discrete energy values (or levels), E_n , where $n = 0, 1, 2, \dots$ and each level has an associated eigenfunction, $V_n(z)$, that has exactly n zero-crossings in the $z = (-1, 1)$ interval (i.e., in the inner points excluding the boundaries at the endpoints $z = -1$ and $z = 1$).

In the special case where $\phi_0 = 0$ Eq. (2.7) reduces to the equatorial wave problem so in this Schrödinger eigenvalue problem formulation the equatorial problem is a regular limit of the mid-latitude one, which is the focus of Chap. 3. For general $\phi_0 \neq 0$ Eq. (2.7) can be divided through by $\sin^2 \phi_0$ which results in the mid-latitude eigenvalue problem:

$$\varepsilon^2 \frac{d^2 V}{dz^2} + \left(E - (1 + zb)^2 \right) V = 0; \quad V(z = \pm 1) = 0, \quad (2.9)$$

where the parameters of this differential equation are

$$\varepsilon = \frac{2\sqrt{\alpha}}{\delta\phi \sin \phi_0}, \quad E = \frac{1}{\sin^2 \phi_0} \left(\omega^2 - \frac{\alpha \cos \phi_0}{C} - \alpha k^2 \right) \quad \text{and} \quad b = \frac{\delta\phi \cos \phi_0}{2 \sin \phi_0}. \quad (2.10)$$

The classical wave theory outlined in the Introduction constitutes the particular $b = 0$ case of (2.9) which is a legitimate mathematical limit but its consistency with the inclusion of $\alpha \cos \phi_0 / C$ in the expression for E in (2.10) is not obvious since the latter term originates from the differentiation of $f(y) = \sin \phi_0 + y \cos \phi_0$ so its presence in E implies that the potential, f^2 , is not constant. However, despite its possible inconsistency the $b = 0$ case provides a check on the numerical calculations below and establishes a connection with the classical/harmonic theory. With $b = 0$

(2.9) becomes a constant coefficient equation, so the solution is $V(z) = V_0 \sin\left(\frac{(n+1)\pi}{2}(z+1)\right)$ where V_0 is an arbitrary normalization constant. This special form of the general solution satisfies the boundary conditions at $z = \pm 1$ provided $E_n = 1 + (\varepsilon\pi(n+1)/2)^2$. This particular form of $V(z)$ is precisely the non-dimensional counterpart of the solution for $v(y)$ in (1.6).

Another particular case in which an analytical solution of (2.9) exists is $b = 1$. In this case the potential $(1 + bz)^2 = (1 + z)^2$ vanishes at the equatorward wall, $z = -1$, i.e., the channel extends from the equator to latitude $\delta\phi$. When the $z = (-1, +1)$ domain is doubled to $z = (-3, +1)$ the potential $(1 + z)^2$ becomes a symmetric parabola whose minimum is located at $z = -1$. In this case, (2.9) turns into the well-known eigenvalue equation of Harmonic Oscillator of Quantum Mechanics. The symmetry of solutions of the Harmonic Oscillator is determined by the mode number n , i.e., symmetric solutions have even n and antisymmetric solutions have odd n . Since eigenfunctions of the Harmonic Oscillator are solutions of (2.9) only if they vanish at $z = -1$ the eigenfunctions relevant to (2.9) must be antisymmetric (so they vanish at the mid-point $z = -1$) i.e., those with odd n . The conclusion based on this comparison with the Harmonic Oscillator, where the eigenvalues are $E_m = (2m + 1)\varepsilon$, is that for $b = 1$, the eigenvalues of (2.9) are the eigenvalues of Harmonic Oscillator with odd m , i.e., $m = 2n + 1$ so $E_n = (4n + 3)\varepsilon$.

Numerical solutions of either (2.9) or (2.7) can be obtained straightforwardly using a variety of methods, e.g., shooting methods and finite-difference or collocation methods (the latter methods turn the differential eigenvalue problem into a matrix eigenvalue problem). These numerical methods yield solutions of E_n and V_n for any pair of ε and b values.

An example of such numerical solutions of (2.9) that yield $E_0(\varepsilon, b)$ is shown in Fig. 2.1 (see Paldor et al. 2007). It clearly demonstrates that for large ε the contours are nearly horizontal so E_0 at $b > 0$ is well approximated by its value at $b = 0$. In contrast, at small ε the value of E_0 is very sensitive to the value of b . This finding and the definition of ε in (2.10) suggest that the harmonic theory that applies formally at $b = 0$ yields accurate estimates of the eigensolutions for $b > 0$ only in sufficiently narrow channels where the width, $\delta\phi$, is of the order of the radius of deformation $\alpha^{1/2} \sin \phi_0$.

The definition of E in (2.10) implies that each E_n value is associated with 3 values of C_n given by the roots of the cubic polynomial:

$$k^2 C^2 - (E_n \sin^2 \phi_0 + \alpha k^2) - \frac{\alpha \cos \phi_0}{C} = 0. \quad (2.11)$$

These three roots can be approximated by noting that the absolute value of two of them (one positive and one negative) is large while that of the third root is small. For the small root, we neglect the C^2 term and obtain the approximate expression for the phase speed of the slowly propagating Planetary (Rossby) wave:

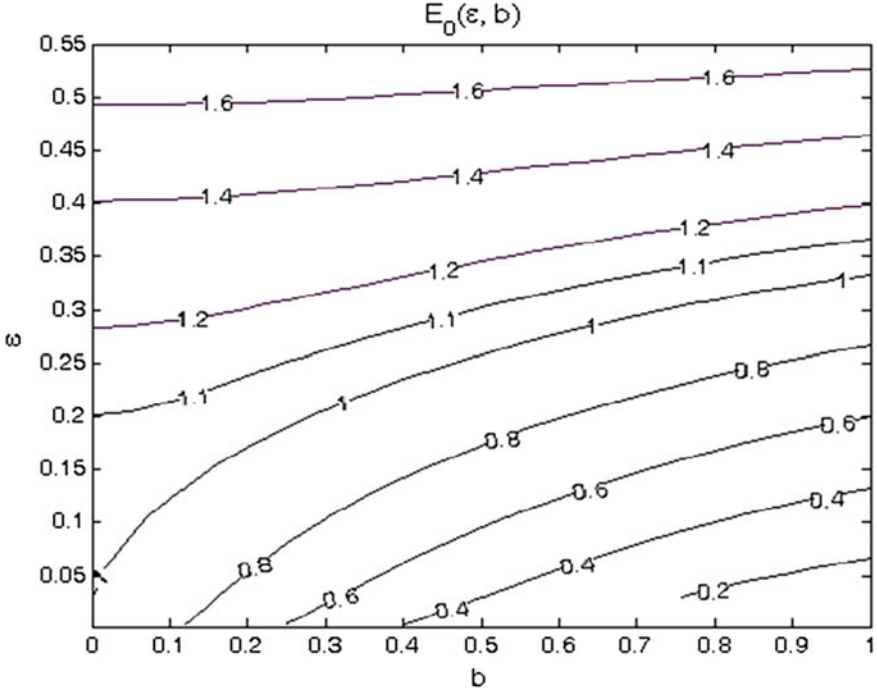


Fig. 2.1 Numerically calculated contours of $E_0(\varepsilon, b)$, the first ($n = 0$) eigenvalue of (2.9). The analytical solutions $E_n = 1 + (\pi\varepsilon(n+1)/2)^2 \geq 1$ along $b = 0$ and $E_n = (4n+3)\varepsilon$ along $b = 1$ are fully confirmed by these numerical results. For $0 \ll b < 1$ and for sufficiently small ε the values of E_0 are appreciably smaller than 1. For large ε (e.g., $\varepsilon > 0.5$), E_0 varies only slightly with b . Permission from American Meteorological Society: J. Phys. Oceanogr. doi:[10.1175/JPO2986.1](https://doi.org/10.1175/JPO2986.1)

$$C_n^{\text{Rossby}} \approx \frac{-\cos \phi_0}{k^2 + \frac{\sin^2 \phi_0}{\alpha} E_n}. \quad (2.12)$$

The approximate expression for the two fast Inertia-Gravity waves is obtained by neglecting $\alpha \cos \phi_0 / C$ which yields the approximate expression:

$$(C_n^{\text{Poincare}})^2 \approx \alpha + E_n \frac{\sin^2 \phi_0}{k^2}. \quad (2.13)$$

When the explicit approximate dispersion relation of Planetary waves, (2.12), is combined with the effect that b has on the contours of E_0 in Fig. 2.1, it becomes evident that estimates of E_0 based on setting $b = 0$ (where $E_n > 1$ for all ε) are

associated with lower phase speeds than those of $b > 0$ (where E_n can be much less than 1). The opposite conclusion holds for Inertia-Gravity waves and (2.13).

It should be noted that while the phase speed of Inertia-Gravity waves in (2.13) differs from that of Planetary waves in (2.12) the eigenfunction $V(y)$ in (2.9) of the two waves is identical because the two waves belong to the same eigensolution of the eigenvalue equation. Since the expressions that relate $u(y)$ and $\eta(y)$ to $V(y)$ and dV/dy involve the phase speed, C (i.e., (2.3) for u and the first equation in (2.4) for η), these functions are not identical in the two waves.

Before turning to an analysis of the eigenfunctions V_n associated with the eigenvalues E_n we turn our attention to Inertial waves that do not exist in the harmonic theory.

2.3 Inertial Waves

The difference between the values of E_0 for $b = 0$, where $E_0 > 1$ and $b > 0$, where E_0 can be smaller than 1 brings about a fundamental difference between these two cases when it comes to Inertial waves. As explained in the Introduction, these waves should be thought of as a limiting case of Inertia-Gravity waves when no pressure gradient forces act on the fluid, i.e., when the total wave number, $k^2 + n^2$, vanishes in (1.5). In the harmonic, $b = 0$, theory these waves cannot satisfy the two boundary conditions at the channel walls (see the discussion at the end of Sect. 3.9 in Pedlosky 1987) and they are therefore regarded as “spurious solutions” of the differential equations.

As implied by their names the frequency of Inertial waves is the Coriolis frequency, f_0 , so in the present non-dimensional formulation becomes $\omega^2 = \sin^2 \phi_0$ and in this case (2.10) implies that $\sin^2 \phi_0 (1 - E) = \alpha k (k + \cos \phi_0 / \omega)$. Therefore, for $k = 0$ these waves can exist only when $E = 1$ while in the case where $k + \cos \phi_0 / \omega > 0$ (i.e., either $\omega = +\sin \phi_0$ or $\omega = -\sin \phi_0$ and $k > \cot \phi_0$) E should be smaller than 1 so Inertial waves cannot exist for $E > 1$. In the $b = 0$ case, $E_n = 1 + (\pi \varepsilon (n + 1))^2 / 4 > 1$ and therefore no Inertial waves can exist in this case and the only solution of eigenvalue problem (2.9) is the trivial $V = 0$ which is the foundation of the conclusion reached in Pedlosky (1987). In contrast, Fig. 2.1 clearly shows that contours with $E_0 \leq 1$ do exist provided $b > 0$. Thus, in terms of the eigenvalue problem (2.9) Inertial waves with either $k = 0$ or $k > 0$ are “spurious solutions” only for $b = 0$, i.e., in the harmonic theory but these waves are legitimate solutions of the Trapped wave theory where $E_0 \leq 1$ eigenvalues exist for $b > 0$.

2.4 Analytic Solutions of the Eigenvalue Problem and Eigenfunctions

In order to obtain analytic insight into the structure of the eigenfunctions and in order to derive explicit expression for E_n when $b > 0$ we note that in the β -plane approximation second-order terms in $f(y)$ (i.e., second-order terms in the expansion of $\sin \phi$ near ϕ_0) are neglected. Therefore, terms of order y^2 should be neglected in the potential of (2.6), i.e., the $b^2 z^2$ term should be neglected in the potentials of (2.7) and (2.9) for these equations to be consistent with the neglect of second-order terms in the expansion of $f(y)$ on the β -plane. The neglect of $b^2 z^2$ is consistent with the restriction on values of b ($= \beta L / f_0 = L \cot \phi_0 / 2a$) which has to be smaller than 1 but larger than 0. Since eigensolutions of two Schrödinger eigenvalue problems with close potentials are always close to one another analytic insight can be gained by approximating the potential in (2.6) by a linear function in y (and that of (2.9) by a linear function of z).

When $b^2 z^2$ is neglected in (2.9) this equation becomes:

$$\varepsilon^2 \frac{d^2 V}{dz^2} + (E - (1 + 2zb))V = 0; \quad V(z = \pm 1) = 0, \quad (2.14)$$

and the parameters in this equation maintain their original definitions, (2.10). The elimination of $b^2 z^2$ yields an equation that can be transformed (by dividing Eq. (2.15) through by ε^2 and defining a new independent variable) to have only two parameters, $(E - 1)/\varepsilon^2$ and $2b/\varepsilon^2$, instead of three (ε , E and b) in (2.9). This reduction in the number of parameters that determine the solution of the equation greatly simplifies the analysis of these solutions.

The linear potential in (2.14) suggests that the differential equation can be transformed to an Airy equation (see Sect. 10.4 in Abramowitz and Stegun 1972). As was shown in Paldor and Sigalov (2008) transforming the independent variable z to new independent variable Z defined by

$$Z = (4b^2 \varepsilon^2)^{-1/3} (2bz - E + 1) \quad (2.15)$$

transforms the differential Eq. (2.14) to Airy equation

$$\frac{d^2 V}{dZ^2} - ZV = 0. \quad (2.16)$$

As a solution of a second-order differential equation, $V(Z)$ in (2.16) can be expressed as a linear combination of two independent solutions: $\text{Ai}(Z)$ and $\text{Bi}(Z)$. One can get a rough idea on the qualitative behavior of Airy functions by replacing the coefficient Z in front of V in (2.16) by a constant in which case the behavior of the solutions of the modified, constant coefficient, equation is well known. When the “constant” Z is positive the solutions decay/grow while when it is negative the

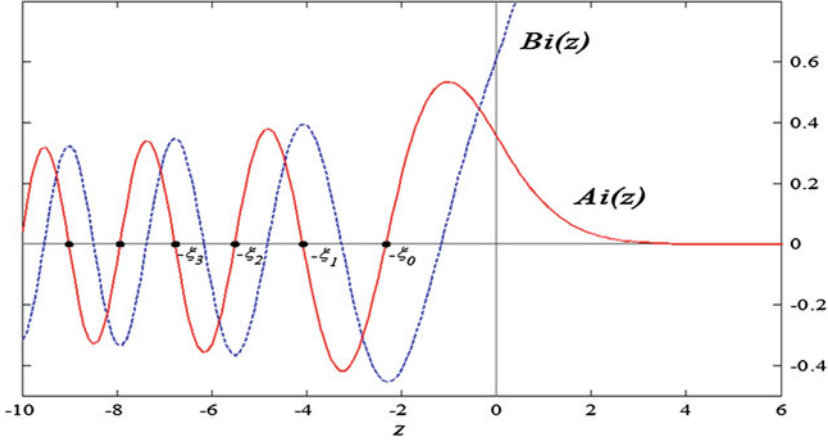


Fig. 2.2 The two independent solutions of Airy equation, (2.16), $Ai(Z)$ and $Bi(Z)$. While $Bi(Z)$ grows faster than exponential in the $Z > 0$ half-plane $Ai(Z)$ decays to zero there faster than exponential. Both functions oscillate a-periodically in the $Z < 0$ half-plane

solutions oscillate periodically. Similarly, at $Z > 0$ $Ai(Z)$ decays while $Bi(Z)$ grows both at a rate faster than exponential and at $Z < 0$, both solutions oscillate but are not periodic. The zeros of Ai and Bi can be easily found either in tables such as that given in Abramowitz and Stegun (1972) or by resorting to readily available mathematical packages such as MATLAB or Mathematica. A sketch of these two solutions is shown in Fig. 2.2.

The final step that completes the transformation of the differential equation is the application of the boundary conditions. To satisfy the boundary condition $V(z = -1) = 0$, the point $Z(z = -1)$ has to be set to the n th zero of Ai , $-\xi_n$, which ensures that $V(Z(z = -1; E = E_n)) = Ai(-\xi_n) = 0$. The requirement $Z(z = -1; E = E_n) = -\xi_n$ in the transformation: $Z = (4b^2\varepsilon^2)^{-1/3}(2bz - E + 1)$ yields $-\xi_n = (4b^2\varepsilon^2)^{-1/3}(-2b - E_n + 1)$ which can be inverted to the following explicit expression for the n th eigenvalue:

$$E_n = 1 - 2b + (2b\varepsilon)^{2/3}\xi_n = 1 - \delta\phi \cot \phi_0 + \left(2 \frac{\cos \phi_0 \sqrt{\alpha}}{\sin^2 \phi_0}\right)^{2/3} \xi_n \quad (2.17)$$

where ξ_n is the absolute value of the n th zero of Ai . The first five ξ_n 's are $\xi_0 = 2.3381$, $\xi_1 = 4.08795$, $\xi_2 = 5.5205$, $\xi_3 = 6.7867$ and $\xi_4 = 7.9441$ and while the value of ξ_n increases with n the difference between two successive zeros of Ai $\xi_n - \xi_{n+1}$ decreases with n .

To satisfy the boundary condition at the poleward wall, $V(z = +1) = 0$, the value of Z at $z = +1$ has to be sufficiently large (and positive) such that $Ai(Z(z = 1))$ is small enough to be regarded as 0. This boundary condition limits the applicability of the present theory to wide channels only. Since $Ai(2) \approx 0.035$ and since

$\text{Ai}(2)/\text{Bi}(2) \leq 0.01$, the choice $Z(z = 1) \geq 2$ guarantees both that V is negligibly tiny at the poleward wall and that the contribution of Bi , the exponentially growing solution of Airy equation, is uniformly negligible throughout the entire channel (the negligible contribution of Bi is an essential element in the determination of E_n based on the location of the zeros of Ai only). Substituting $Z \geq 2$ in the general $Z(z)$ transformation given in Eq. (2.15) $Z = (4b^2\varepsilon^2)^{-1/3}(2bz - E + 1)$, with $z = 1$ and $E = E_n$ (where E_n is given by (2.17)) yields:

$$\xi_n + 2 \leq 4b(2b\varepsilon)^{-2/3} = 4^{2/3}b^{1/3}\varepsilon^{-2/3}. \quad (2.18)$$

Substituting ε and b from (2.10) into (2.18) yields, $\xi_n + 2 \leq \delta\phi \left(\frac{2 \sin \phi_0 \cos \phi_0}{\alpha} \right)^{1/3}$ which can be inverted to yield the following lower bound on the value of the channel width, $\delta\phi$, above which the Trapped wave theory applies:

$$\delta\phi \geq (\xi_n + 2) \left(\frac{\alpha}{\sin 2\phi_0} \right)^{1/3}. \quad (2.19)$$

According to the values of the first five ξ_n 's given above, the coefficient $(\xi_n + 2)$ on the RHS of (2.19) varies between 4.3 for $n = 0$ and 9.9 for $n = 4$. At $n = 20$, $\xi_{20} + 2$ is 22 and at $n \geq 20$ $\xi_{n+1} - \xi_n$, the difference between successive zeros of Ai , is less than 0.5 (and it decreases further with n). Thus, with an error of no more than a factor of 2 and for not-too-large mode number, n , one can set $(\xi_n + 2)$ to 10. For ϕ_0 in the mid-latitudes $\sin^{1/3} 2\phi_0$ is very close to 1 as $\sin^{1/3} 2\phi_0$ varies between 0.95 and 1 when ϕ_0 varies between 30° and 75° . Consequently, the present theory is valid for channel widths satisfying $\delta\phi \geq 10\alpha^{1/3}$ so for a baroclinic ocean where $\alpha \approx 5 \times 10^{-6}$ the present theory is valid provided $\delta\phi > 0.17$ rad, i.e., less than 10° , or 1000 km, of latitude. This is close to the latitudinal range over which the β -plane approximation is valid—at larger ranges, the linearity of $f(y)$ cannot be justified while at lower ranges, the f -plane approximation (where $f(y)$ is assumed constant) is valid. In contrast, in a barotropic ocean (and atmosphere) where $\alpha \approx 5 \times 10^{-2}$ the channel width has to satisfy $\delta\phi > 1$ rad even for $n = 0$, which is much larger than the range of validity of the β -plane approximation. The conclusion from this discussion is that the present Trapped wave theory is applicable to a baroclinic ocean on the β -plane.

Similar arguments can be made for delineating the regime of channel width where Harmonic waves can be expected to prevail. An upper bound on the channel width below which Harmonic wave theory is valid can be estimated by noting that if the poleward wall, $z = 1$, is located at $Z = 0$ then, as is evident from Fig. 2.2, the two Airy solutions oscillate throughout the entire channel and in such an oscillatory regime the purely oscillatory Harmonic waves provide an accurate solution. For the $n = 0$ mode and at 45° , (2.19) yields $\delta\phi \geq 4.3\alpha^{1/3}$ as the lower bound on the channel width above which Trapped waves can be expected to prevail while for $\delta\phi \leq 2.3\alpha^{1/3}$ Harmonic waves should prevail. Thus, in a barotropic ocean where

$\alpha = 0.05$, the first Harmonic mode prevails at all physically acceptable widths (i.e., smaller than $0.85 \text{ rad} \approx 50^\circ$ so the channel extends from above 20° to below 70°), while in a baroclinic ocean where $\alpha = 5 \times 10^{-6}$ the first Trapped mode prevails at all channels with widths exceeding 0.0735 rad , i.e., wider than about 4° , which is hardly the range that justifies the application of the β -plane approximation (the f -plane suffices).

For values of α and $\delta\phi$ satisfying (2.19) the dispersion relations of Planetary and Inertia-Gravity waves are obtained by substituting (2.17) in (2.12) and (2.13), respectively:

$$\omega_n^{\text{Rossby}} \approx \frac{-\cos \phi_0 k}{k^2 + \frac{\sin^2 \phi_0}{\alpha} E_n} = \frac{-\cos \phi_0 k}{k^2 + \frac{\sin^2 \phi_0}{\alpha} - \frac{\delta\phi \sin 2\phi_0}{2} + \alpha^{-2/3} \xi_n (\sin 2\phi_0)^{2/3}} \quad (2.20)$$

and

$$(\omega_n^{\text{Poincare}})^2 \approx \alpha k^2 + \sin^2 \phi_0 E_n = \sin^2 \phi_0 + \alpha k^2 - \frac{\delta\phi}{2} \sin 2\phi_0 + \alpha^{1/3} \xi_n (\sin 2\phi_0)^{2/3}. \quad (2.21)$$

These dispersion relations should be compared with their classical dimensional counterparts, (1.5) for Planetary waves and (1.7) for Inertia-Gravity waves. The comparison clarifies that in both waves the arbitrary (i.e., independent of the radius of deformation) meridional wave number of the harmonic theory, $n = (2N + 1) \cdot \pi/L$ (where L is the channel width) is replaced in the Trapped wave theory by two terms: The first term is proportional to the channel width, $\delta\phi$, and the second term (proportional to $\alpha^{-2/3}$ in the case of planetary waves) has the coefficient ξ_n which determines the meridional mode number, n , as the number of zeros that $V(y)$ crosses inside the channel, i.e., the mode number and channel width appear in different terms of the dispersion relation instead of $nL = \text{constant}$ in the Harmonic theory.

Numerical solutions of the eigenvalue problem associated with zonally propagating waves in a mid-latitude channel (including Kelvin, Planetary, and Inertia-Gravity waves) can be obtained by solving system (2.2) numerically. A powerful method of solution of such differential-matrix eigenvalue system is the Chebyshev spectral collocation method on a grid (see e.g., Trefethen (2000), Poulin and Flierl (2003) and De-Leon and Paldor (2009) for more details) in which the (u, V, η) system, (2.2), is solved as a matrix eigenvalue problem of linear algebra by transforming the differential $\partial/\partial y$ operators to an algebraic form (as in the case when derivatives are written as differences). The eigenvalues of this matrix eigenvalue problem are the phase speeds, C , while the corresponding eigenvectors consist of the three amplitudes $u(y)$, $V(y)$, and $\eta(y)$.

A numerical solution of the eigenvectors in a channel with $b = 0.15$ and $\varepsilon = 0.055$ (so the channel width, $\delta\phi$, is 36 deformation radii, $\alpha^{1/2}/\sin \phi_0$) where the present theory applies is presented in Fig. 2.3, reproduced from Paldor et al. (2007). Consistent with the analytical conclusions reached above, the $V(y)$ eigenfunction is

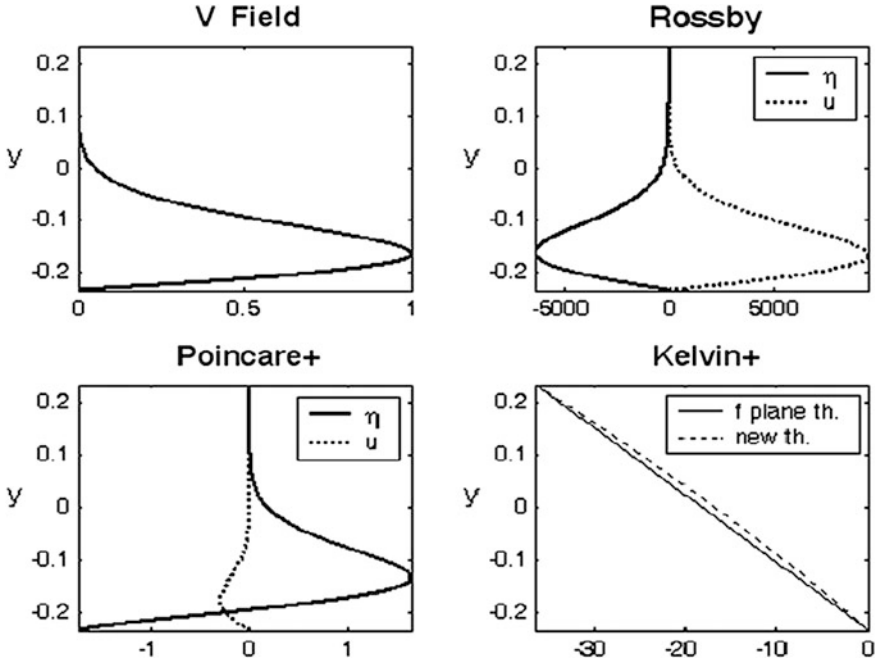


Fig. 2.3 The numerically calculated y -dependent amplitude eigenfunctions of the three wave types in a wide channel ($\varepsilon = 0.055$, i.e., the width, $\delta\phi$, equals 36 deformation radii, $\alpha^{1/2}/\sin\phi_0$). Ordinates are the cross-channel coordinate: $y = z \cdot \delta\phi$ and abscissas are arbitrary amplitudes (normalized in Rossby and Poincaré waves such that $\max\{V\} = 1$). Permission from American Meteorological Society: J. Phys. Oceanogr. doi:[10.1175/JPO2986.1](https://doi.org/10.1175/JPO2986.1)

identical in Planetary and Inertia-Gravity waves (recall that there is a single $V(y)$ eigenfunction to (2.14) or (2.16)), while the $u(y)$ and $\eta(y)$ of the former waves differ from those of the latter because of the different phase speeds of the two wave types i.e., the values of C given in (2.12) and (2.13) affect the relation between η and V and dV/dy in (2.4) and that between u and V and η in (2.3). The trapping of $V(y)$ near the equatorward wall anticipated from the behavior of the regular Airy function is clearly evident in these numerical results which cannot be recovered by the harmonic theory outlined in the Introduction since $E_0 = 0.862$, whereas in the harmonic theory, $E_0 = 1.00187$ in this case.

In contrast, when the channel is sufficiently narrow, the Trapped wave theory does not apply and one can expect the theory outlined in the Introduction to apply. An example for this case is shown in Fig. 2.4 which repeats the calculation shown in Fig. 2.3 but for a channel width that equals the deformation radius. As in Trapped waves, the V eigenfunctions are identical for Planetary and Inertia-Gravity waves, but the u and η functions are not. The main feature of the eigenfunctions in this case is that they spreads all over the channel and are not trapped near the equatorward wall. The theory outlined in the Introduction prevails in this case since $E_0 = 10.871$

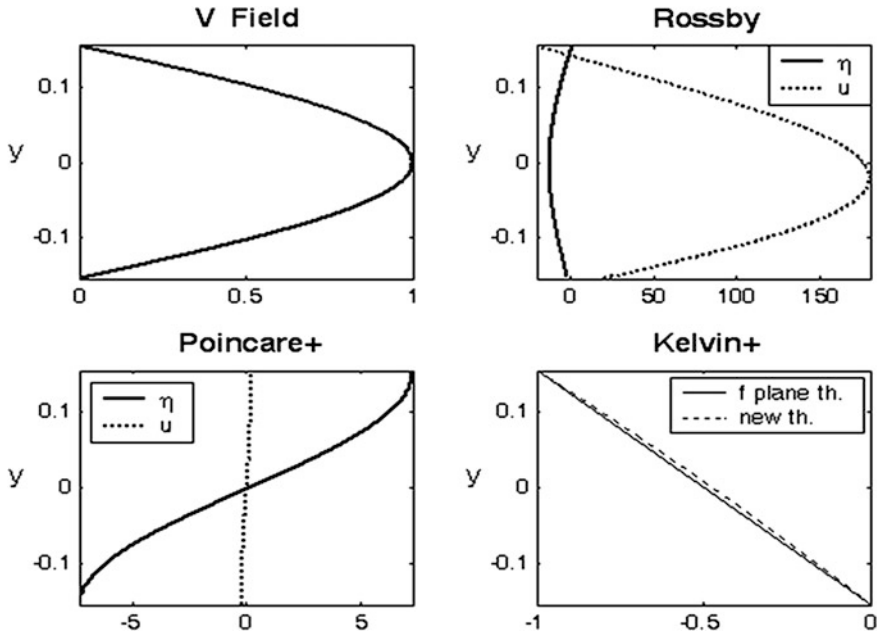


Fig. 2.4 The y -dependent amplitude eigenfunctions in a narrow, $\varepsilon = 2$, channel (i.e., the channel width, $\delta\phi$, equals the deformation radius, $\alpha^{1/2}/\sin\phi_0$). Ordinates, abscissas and normalizations are identical to those in Fig. 2.3. Permission from American Meteorological Society: J. Phys. Oceanogr. doi:[10.1175/JPO2986.1](https://doi.org/10.1175/JPO2986.1)

where the value of b does not affect the solution according to the results shown in Fig. 2.1, so the $b = 0$ results yield accurate estimates for all values of b . These results are relevant to a channel width of $\delta\phi = 0.3$ only for barotropic ocean with $\alpha = 0.05$, while in a baroclinic ocean where $\alpha = 5 \times 10^{-6}$ and at 45° latitude, the channel width where this mode exists is only 0.003 rad or about 0.17° (a dimensional width of less than 20 km!). The numerical solutions underscore the inaccuracy of the theory outlined in the Introduction for Rossby waves since the $u(y)$ eigenfunction of Planetary (Rossby) waves does not vanish at the channel walls in contradiction to the expressions given in (1.8) (note that $V(y)$ in Eq. (2.2) and in Figs. 2.3 and 2.4 is shifted by $\pi/2$ relative to $v(y)$ in (1.8) so $V(y)$ has the same y -dependence as $u(y)$).

To summarize this chapter that focuses on waves in a mid-latitude channel on the β -plane it is instructive to compare the phase speed of Trapped Planetary (Rossby) waves, (2.20), with that of Harmonic Planetary waves (1.7). Dividing (1.7) by $2\Omega ak$ yields an expression for the non-dimensional Harmonic phase speed C_{Harm} which can be compared with the phase speed of Trapped waves, C_{Trap} , obtained by dividing (2.20) by k . To enable the comparison, we fix the central latitude, ϕ_0 , to 45° (so $\sin 2\phi_0 = 1$ and $\sin^2 \phi_0 = 1/2$) and assign the radius of deformation, $R_d = (gH)^{1/2}/f_0$, the typical baroclinic value of 25 km so $(gH)^{1/2}/2\Omega = 17.7$ km and $\alpha = (17.7/6400)^2 \approx 7.6 \times 10^{-6}$. The zonal wave number, k , of a wave with

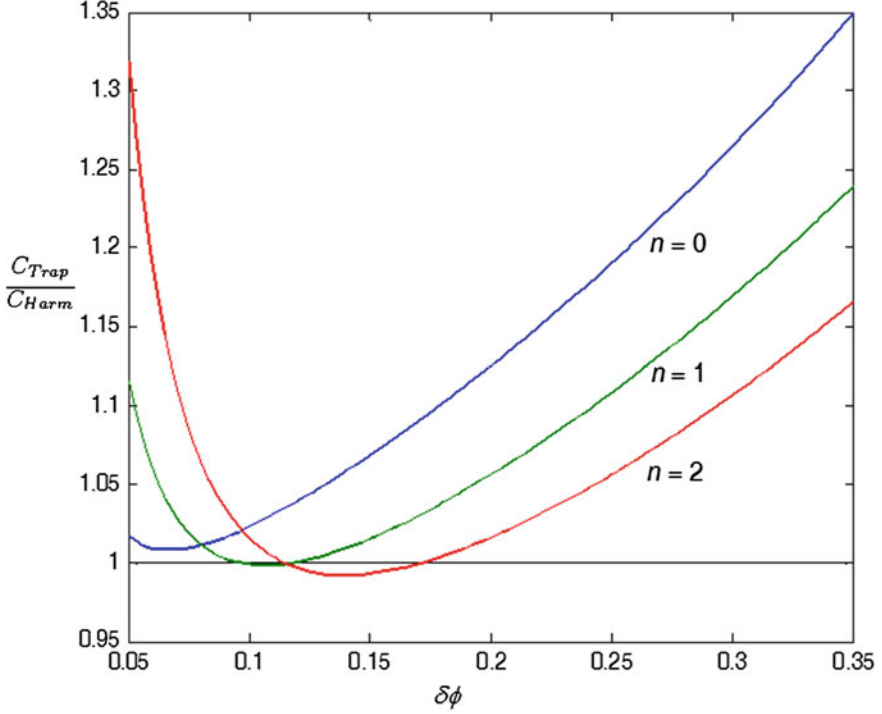


Fig. 2.5 The ratio between the phase speeds of the first three Trapped modes to those of the corresponding Harmonic modes

640 km wavelength, has a non-dimensional value of $k = 2\pi 6400/640 = 20\pi$ so $k^2 \approx 4 \times 10^3 \ll \alpha^{-1} \approx 1.5 \times 10^5$. For these values of ϕ_0 , k^2 and α , the resulting expression for the ratio between Trapped and Harmonic phase speeds of Planetary waves is

$$\begin{aligned} \frac{C_{\text{Trap}}}{C_{\text{Harm}}} &= \frac{k^2 + \frac{\sin^2 \phi_0}{\alpha} + \frac{(n+1)^2 \pi^2}{(\delta\phi)^2}}{k^2 + \frac{\sin^2 \phi_0}{\alpha} + \left(\xi_n \cdot \left(\frac{\sin 2\phi_0}{\alpha} \right)^{2/3} - \frac{\delta\phi \sin 2\phi_0}{2\alpha} \right)}, \\ &\approx \frac{1 + 2\alpha \frac{(n+1)^2 \pi^2}{(\delta\phi)^2}}{1 + 2\xi_n \alpha^{1/3} - \delta\phi} \end{aligned} \quad (2.22)$$

where the index $n = 0, 1, 2, \dots$ is identical in the numerator and denominator (i.e., n was increased by 1 relative to its value in (1.7) so as to match the values of n in (2.17) and (2.20)).

The phase speed ratio, (2.22), is plotted in Fig. 2.5 as a function of the channel width $\delta\phi = L/a$ for $n = 0, 1$, and 2 and it shows that the phase speed of Trapped

waves is larger than that of Harmonic waves in sufficiently wide channels. For $\phi_0 = \pi/4$ and $\alpha = 7.6 \times 10^{-6}$, the right-hand side (RHS) of (2.19) is very close to $0.02 \cdot (\xi_n + 2)$ so for $n = 0, 1$, and 2 , the Trapped wave theory applies to channels of widths exceeding 0.087 , 0.12 , and 0.15 rad, respectively, consistent with the location of the minima of the three curves shown in Fig. 2.5. The monotonic increase of the three curves in Fig. 2.5 at $\delta\phi$ larger than the minimal widths and the fact that in their ascending branches, the ratios are larger than 1 suggest that when wide channels are used to approximate an infinite ocean, the phase propagation of Trapped Planetary waves is appreciably faster than that of Harmonic waves, exceeding a factor 2 in very wide channels.

Having completed the derivation of the Trapped wave theory in a channel on the mid-latitude β -plane it is quite natural to examine the same problem in a channel on the equatorial β -plane as a limiting case of the mid-latitude theory when ϕ_0 is set equal to 0 . This is the subject of the next chapter.

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