

Chapter 2

The Irregularity Strength of a Graph

Throughout Chaps. 2–7, we will be concerned with connected graphs G of order $n \geq 3$ and size m and an unrestricted edge coloring of G , that is, no condition is placed on the manner in which colors are assigned to the edges of G .

The unrestricted edge colorings inducing vertex colorings that have attracted the most attention are those where the vertex colorings are either vertex-distinguishing or neighbor-distinguishing. In this chapter, we consider a particular example of the first of these.

A nontrivial graph has been called *irregular* if its vertices have distinct degrees. It is well known that there is no such graph; that is, no graph is irregular. This observation led to a concept introduced by Gary Chartrand at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne in 1986.

For a connected graph G , a *weighting* w of G is an assignment of numbers (usually positive integers) to the edges of G , where $w(e)$ denotes the weight of an edge e of G . This then converts G into a weighted graph in which the (*weighted*) *degree* of a vertex v is defined as the sum of the weights of the edges incident with v . A weighted graph G is then *irregular* if the vertices of G have distinct degrees. Later this concept was viewed in another setting.

2.1 Sum-Defined Vertex Colorings: Irregularity Strength

Rather than consider connected graphs G of order at least 3 whose edges are assigned weights, resulting in irregular weighted graphs, we can view this as *vertex-distinguishing* edge colorings of G where the induced vertex coloring is sum-defined and where then the vertices of G have distinct colors. Such vertex colorings are also referred to as *rainbow vertex colorings*.

We now formally define such vertex-distinguishing edge colorings. Let \mathbb{N} denote the set of positive integers and let E_v denote the set of edges incident with a vertex v in a graph G . An unrestricted edge coloring $c : E(G) \rightarrow \mathbb{N}$ induces a vertex coloring $c' : V(G) \rightarrow \mathbb{N}$, defined by

$$c'(v) = \sum_{e \in E_v} c(e) \text{ for each vertex } v \text{ of } G. \quad (2.1)$$

Proposition 2.1. *Let G be a nontrivial connected graph and let $c : E(G) \rightarrow \mathbb{N}$ be an edge coloring of G , where $c' : V(G) \rightarrow \mathbb{N}$ is the induced vertex coloring defined in (2.1). Then there exists an even number of vertices of odd color.*

Proof. Let $E(G) = \{e_1, e_2, \dots, e_m\}$. Since

$$\sum_{v \in V(G)} c'(v) = 2 \sum_{i=1}^m c(e_i)$$

is even, there exists an even number of vertices of odd color. \square

While no edge coloring of the graph K_2 can induce a rainbow vertex coloring defined in this manner, there is a vertex-distinguishing edge coloring for every connected graph G of order 3 or more. To see this, let $E(G) = \{e_1, e_2, \dots, e_m\}$ where then $m \geq 2$ and let c be the edge coloring of G defined by $c(e_i) = 2^{i-1}$ for $1 \leq i \leq m$. Since no two vertices are incident with the same set of edges, c induces a rainbow vertex coloring. This edge coloring shows that there is always a vertex-distinguishing edge coloring of a connected graph of size $m \geq 2$ where the largest color used is 2^{m-1} . In general, there exist vertex-distinguishing edge colorings of a graph of size m whose largest color is considerably less than 2^{m-1} .

For a connected graph G of size $m \geq 2$, the minimum of the largest colors used among the vertex-distinguishing edge colorings of G is called the *irregularity strength* of G and is denoted by $s(G)$. (The *strength* of a multigraph M is the maximum number of parallel edges joining two vertices of M .) Therefore, for a connected graph G of order at least 3, there exists an edge coloring $c : E(G) \rightarrow [k] = \{1, 2, \dots, k\}$ for every integer k with $k \geq s(G)$ such that the induced (sum-defined) vertex coloring c' is vertex-distinguishing but there is no such edge coloring $c : E(G) \rightarrow [k]$ with this property for any integer k with $1 \leq k < s(G)$.

Since no nontrivial graph is irregular, it follows that every connected graph of order at least 3 must have irregularity strength at least 2. It is well known that there is exactly one connected graph G_n of order n for each $n \geq 2$ containing exactly two vertices having the same degree. All of these graphs have irregularity strength 2.

Proposition 2.2. *If G_n is the unique connected graph of order $n \geq 3$ containing exactly two vertices of equal degree, then $s(G_n) = 2$.*

Proof. As mentioned above, $s(G_n) \geq 2$ for every integer $n \geq 2$. Each such graph G_n can be described as having vertex set $V(G_n) = \{v_1, v_2, \dots, v_n\}$ where $v_i v_j \in E(G_n)$ if and only if $i + j \leq n + 1$. Consequently,

$$\deg v_i = \begin{cases} n - i & \text{if } 1 \leq i \leq \lceil n/2 \rceil \\ n + 1 - i & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n. \end{cases} \quad (2.2)$$

So $\deg v_{\lceil \frac{n}{2} \rceil} = \deg v_{\lceil \frac{n}{2} \rceil + 1} = \lfloor n/2 \rfloor$. Let c be the edge coloring of G_n in which each edge is assigned the color 2 except for $v_1 v_{\lceil \frac{n}{2} \rceil + 1}$, which is colored 1. Then

$$c'(v_i) = \begin{cases} 2 \deg v_i & \text{if } i \neq 1, \lceil n/2 \rceil + 1 \\ 2 \deg v_i - 1 & \text{if } i = 1, \lceil n/2 \rceil + 1. \end{cases}$$

Since c' is vertex-distinguishing, $s(G) \leq 2$ and so $s(G) = 2$. \square

To show that every complete graph of order $n \geq 3$ has irregularity strength 3, we first make an observation concerning the irregularity strength of every regular graph.

Proposition 2.3. *The irregularity strength of every regular graph of order 3 or more is at least 3.*

Proof. Suppose that there exists an edge coloring of a regular graph G of order at least 3 with the colors 1 and 2 and that H is the spanning subgraph of G whose edges are color 1. Then H has two vertices u and v of equal degree. Since u and v have the same induced color in G , it follows that $s(G) \geq 3$. \square

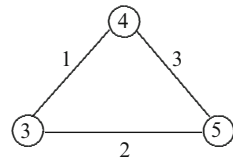
Theorem 2.4 ([24]). *For each integer $n \geq 3$, $s(K_n) = 3$.*

Proof. By Proposition 2.3, it follows that $s(K_n) \geq 3$. To establish the inequality $s(K_n) \leq 3$, we show that there is a vertex-distinguishing edge coloring of K_n with the colors 1, 2 and 3. Since the edge coloring of K_3 given in Fig. 2.1 has this property, we may assume that $n \geq 4$.

Let G_n be the unique connected graph of order $n \geq 4$ having exactly two vertices of equal degree that is described in the proof of Theorem 2.2. Thus $V(G_n) = \{v_1, v_2, \dots, v_n\}$ whose degrees are given in (2.2). As noted there, these equal degrees are $\lfloor n/2 \rfloor$. Assign the color 2 to the edges of G_n and the color 1 to the edges of its complement $\overline{G_n}$. The induced vertex colors $c^*(v_i)$ for this edge coloring of K_n are then

$$c^*(v_i) = 2 \deg_{G_n} v_i + (n - 1 - \deg_{G_n} v_i) = n - 1 + \deg_{G_n} v_i \quad (2.3)$$

Fig. 2.1 Showing $s(K_3) = 3$



for $1 \leq i \leq n$. Next, increase the color of each of the edges $v_1 v_2, v_1 v_3, \dots, v_1 v_{\lceil \frac{n}{2} \rceil}$ by 1, resulting in an edge coloring c using the colors 1, 2, 3. By (2.3), the induced vertex coloring c' of K_n satisfies

$$c'(v_i) = \begin{cases} (2n-2) + (\lceil n/2 \rceil - 1) & \text{if } i = 1 \\ n + \deg_{G_n} v_i & \text{if } 2 \leq i \leq \lceil n/2 \rceil \\ (n-1) + \deg_{G_n} v_i & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n. \end{cases}$$

It then follows by (2.2) that

$$c'(v_i) = \begin{cases} 2n + \lceil n/2 \rceil - 3 & \text{if } i = 1 \\ 2n - i & \text{if } 2 \leq i \leq n. \end{cases}$$

Since the revised edge coloring c of K_n is vertex-distinguishing, $s(K_n) \leq 3$ and so $s(K_n) = 3$. \square

2.2 On the Irregularity Strength of Regular Graphs

We saw in Proposition 2.3 that the irregularity strength of every regular graph of order 3 or more is at least 3 and in Theorem 2.4 that the irregularity strength of the complete graph K_n , $n \geq 3$, an $(n-1)$ -regular graph, is 3. We now investigate the irregularity strength of regular graphs in more detail. First, we present a lower bound for the irregularity strength of a graph G in terms of the number of vertices of a specific degree in G .

Proposition 2.5 ([24]). *Let G be a connected graph of order $n \geq 3$ with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ containing n_i vertices of degree i for each integer i with $\delta(G) \leq i \leq \Delta(G)$. Then*

$$s(G) \geq \max \left\{ \frac{n_i - 1}{i} + 1 : \delta(G) \leq i \leq \Delta(G) \right\}.$$

Proof. Suppose that $s(G) = s$. Let there be given a vertex-distinguishing edge coloring of G with the colors $1, 2, \dots, s$ and let $v \in V(G)$ where $\deg v = i$. Then the induced vertex color $c'(v)$ satisfies $i \leq c'(v) \leq si$. Hence each vertex of degree i has one of the $si - i + 1 = i(s-1) + 1$ induced colors in the set $\{i, i+1, \dots, si\}$ and so $n_i \leq i(s-1) + 1$. Therefore,

$$s(G) = s \geq \frac{n_i - 1}{i} + 1$$

for each i with $\delta(G) \leq i \leq \Delta(G)$. \square

If G is a regular graph, then Proposition 2.5 has the following corollary.

Corollary 2.6 ([24]). *If G is a connected r -regular graph, $r \geq 2$, of order $n \geq 3$, then*

$$s(G) \geq \frac{n-1}{r} + 1.$$

When $n \equiv 2 \pmod{4}$ or $n \equiv 3 \pmod{4}$, Corollary 2.6 can be improved a bit.

Corollary 2.7 ([24]). *If G is a connected r -regular graph of order $n \geq 3$ where $n \equiv 2 \pmod{4}$ or $n \equiv 3 \pmod{4}$, then*

$$s(G) > \frac{n-1}{r} + 1.$$

Proof. Suppose that $n \equiv 2 \pmod{4}$ and assume, to the contrary, that $s(G) = s = \frac{n-1}{r} + 1$. Then there is a vertex-distinguishing edge coloring of G with the colors $1, 2, \dots, s$. Hence each induced vertex color is one of the $sr - r + 1$ colors $r, r + 1, \dots, sr$. By assumption, $n = sr - r + 1$ and so the induced vertex colors are precisely the n colors $r, r + 1, \dots, sr$. However, $n/2$ of these colors are odd, that is, G has an odd number of vertices of odd color, contradicting Proposition 2.1.

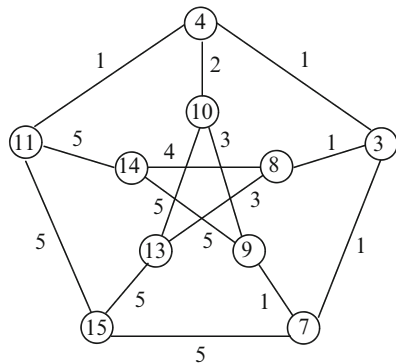
The argument when $n \equiv 3 \pmod{4}$ is similar. \square

By Corollary 2.7, the irregularity strength of the Petersen graph P satisfies $s(P) > \frac{10-1}{3} + 1 = 4$, that is, $s(P) \geq 5$. Since the edge coloring of the Petersen graph with the colors $1, 2, \dots, 5$ shown in Fig. 2.2 is vertex-distinguishing, $s(P) \leq 5$ and so $s(P) = 5$.

Since, by Theorem 2.4, $s(K_n) = 3$ for every integer $n \geq 3$, it follows that the complete n -partite graph in which every partite set consists of a single vertex has irregularity strength 3. We now see that this is also true when each partite set consists of exactly two vertices. For each integer $r \geq 2$, we write $K_{r(2)}$ for the $(2r-2)$ -regular complete r -partite graph where each partite set consists of two vertices.

Theorem 2.8 ([52]). *For each integer $r \geq 2$, $s(K_{r(2)}) = 3$.*

Fig. 2.2 An edge coloring of the Petersen graph



Proof. Since it is easy to see that $s(C_4) = 3$ and $K_{2(2)} = C_4$, we may assume that $r \geq 3$. Let $G = K_{r(2)}$. Since G is a $(2r-2)$ -regular graph of order $2r$, it follows by Corollary 2.6 that $s(G) \geq 3$. We show that $s(G) \leq 3$ by describing a vertex-distinguishing edge coloring $c : E(G) \rightarrow \{1, 2, 3\}$.

Denote the partite sets of G by V_1, V_2, \dots, V_r , where $V_i = \{x_i, y_i\}$ for $1 \leq i \leq r$. We now relabel the vertices of G by u_1, u_2, \dots, u_n , where $n = 2r$, such that $u_i = x_i$ for $1 \leq i \leq r$ and $u_{n+1-i} = y_i$ for $1 \leq i \leq r$. Let H be the spanning subgraph of G where $u_i u_j \in E(H)$ if $1 \leq i < j \leq n$ and $i + j \leq n$. Thus

$$\deg_H u_i = \begin{cases} 2r-1-i & \text{if } 1 \leq i \leq r \\ 2r-i & \text{if } r+1 \leq i \leq n. \end{cases} \quad (2.4)$$

Thus $\deg_H u_1 \geq \deg_H u_2 \geq \dots \geq \deg_H u_n$ and $\deg_H u_i = \deg_H u_{i+1}$ only when $i = r$. Next, we define an edge coloring $\bar{c} : E(G) \rightarrow \{1, 2, 3\}$ of G by assigning the color 1 to each edge of H and the color 3 to the remaining edges of G . The induced vertex coloring \bar{c}' is then defined by

$$\bar{c}'(u_i) = \deg_H u_i + 3(2r-2 - \deg_H u_i) = 6r-6 - 2 \deg_H u_i$$

for $1 \leq i \leq n$. Hence $\bar{c}'(u_1) \leq \bar{c}'(u_2) \leq \dots \leq \bar{c}'(u_n)$ with equality only for $\bar{c}'(u_r)$ and $\bar{c}'(u_{r+1})$. In particular, $\bar{c}'(u_r) = \bar{c}'(u_{r+1}) = 4r-4$.

We now revise the edge coloring \bar{c} by replacing the color 1 of $u_1 u_r$ by 2, producing a new edge coloring c of G . The induced vertex coloring c' then satisfies the following

$$c'(u_i) = \begin{cases} 2r-1 & \text{if } i = 1 \\ 6r-6 - 2 \deg_H u_i & \text{if } 2 \leq i \leq r-1 \text{ or } r+1 \leq i \leq n \\ 4r-3 & \text{if } i = r. \end{cases}$$

This is illustrated for $K_{4(2)}$ in the following table.

u_1, u_2, \dots, u_8	x_1	x_2	x_3	x_4	y_4	y_3	y_2	y_1
$\deg_H u_i$	6	5	4	3	3	2	1	0
$\bar{c}'(u_i)$	6	8	10	12	12	14	16	18
$c'(u_i)$	7	8	10	13	12	14	16	18

Since c is a vertex-distinguishing edge coloring, it follows that $s(G) \leq 3$ and so $s(G) = 3$. \square

Even though each complete multipartite graph in which every partite set consists of exactly one vertex or every partite set consists of exactly two vertices has irregularity strength 3, this is not the case if every partite set consists of exactly three vertices, as we now illustrate with the graph $K_{3,3}$. By Corollary 2.7, $s(K_{3,3}) \geq 3$. Assume to the contrary that $s(K_{3,3}) = 3$. Then there is a vertex-distinguishing edge coloring c of $G = K_{3,3}$ with induced vertex coloring c' . Therefore, $\{c'(v) : v \in V(G)\} \subseteq S = \{3, 4, \dots, 9\}$. Since the order of G is 6 and $|S| = 7$, every integer in

S is a vertex color of G except for one color in S . Because S consists of four odd integers and three even integers and every graph has an even number of vertices of odd color (by Proposition 2.1), each of the integers 3, 5, 7, 9 is the color of exactly one vertex of G . Suppose that $c'(x) = 3$ and $c'(y) = 9$. Then the three edges incident with x are colored 1 and the three edges incident with y are colored 3. This implies that x and y belong to the same partite set U of G . Thus each vertex belonging to the other partite set W of G is incident with at least one edge colored 1 and at least one edge colored 3. Thus, the colors of the three vertices in W are 5, 6 and 7. Since the sum of the colors of the three vertices of W is 18, the the sum of the colors of the three vertices of U is also 18, which implies that the colors of the three vertices in U are 3, 6 and 9. This is impossible, however, since there is a vertex of W colored 6. Therefore, $s(K_{3,3}) \geq 4$. We now show that not only $s(K_{3,3}) = 4$ but provide information about the value of $s(K_{r,r})$ for every integer $r \geq 2$.

For two disjoint subsets A and B of the vertex set of a graph G , let $[A, B]$ denote the set of edges joining a vertex of A and a vertex of B .

Theorem 2.9 ([24, 51]). *For an integer $r \geq 2$,*

$$s(K_{r,r}) = \begin{cases} 3 & \text{if } r \text{ is even} \\ 4 & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Denote the partite sets of $G = K_{r,r}$ by

$$U = \{u_1, u_2, \dots, u_r\} \text{ and } W = \{w_1, w_2, \dots, w_r\}.$$

By Corollary 2.6, $s(G) \geq 3$. Assume first that r is even. Then $r = 2k$ for some integer k . Define an edge coloring $c : E(G) \rightarrow \{1, 2, 3\}$ by

$$c(u_i w_j) = \begin{cases} 1 & \text{if } j > i \text{ or } i = j \geq k + 1 \\ 2 & \text{if } i = j \leq k \\ 3 & \text{if } j < i. \end{cases}$$

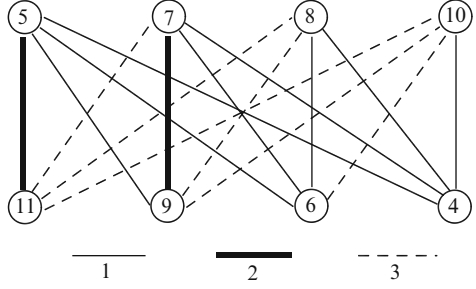
Then the induced vertex coloring c' satisfies the following

$$c'(u_i) = \begin{cases} r + (2i - 1) & \text{if } 1 \leq i \leq k \\ r - 2 + 2i & \text{if } k + 1 \leq i \leq 2k \end{cases}$$

$$c'(w_i) = \begin{cases} 3r + 1 - 2i & \text{if } 1 \leq i \leq k \\ 3r - 2i & \text{if } k + 1 \leq i \leq 2k. \end{cases}$$

Consequently, $c' : V(G) \rightarrow \{r, r + 1, \dots, 3r - 1\}$ is vertex-distinguishing. The colorings c and c' are illustrated for $K_{4,4}$ in Fig. 2.3. Since c is a vertex-distinguishing edge coloring, it follows that $s(G) \leq 3$ and so $s(G) = 3$ if r is even.

Fig. 2.3 An edge coloring of $K_{4,4}$



Next, assume that $r \geq 3$ is odd. First, we show that $s(G) \geq 4$. Assume, to the contrary, that $s(G) = 3$. Then there exists an edge coloring $\bar{c} : E(G) \rightarrow \{1, 2, 3\}$ such that $\bar{c}' : V(G) \rightarrow \{r, r+1, \dots, 3r\} = T$ is vertex-distinguishing. Since $|T| = 2r+1$ and there is an even number of vertices of odd color, there is an even integer $t \in T$ that is not the color of any vertex in G .

If 1 is subtracted from each edge color, then we obtain a vertex-distinguishing edge coloring $c : E(G) \rightarrow \{1, 2\}$ such that $c' : V(G) \rightarrow \{0, 1, \dots, 2r\}$. Hence the odd color $i = t - r$ is not the color of any vertex of G .

Let $V(G) = S \cup L$, where $|S| = |L| = r$, such that S is the set of vertices of G having the smallest r colors and L is the set of vertices of G having the largest r colors. Let

$$\sigma(S, L) = \sum_{e \in [S, L]} c(e),$$

$U_L = U \cap L$, $W_L = W \cap L$, $a = |U_L|$ and $b = |W_L|$. Then $a + b = r$. If $x \in U_L$, then $\sum_{e \in [\{x\}, W_L]} c(e) \leq 2b$; while if $x \in W_L$, then $\sum_{e \in [\{x\}, U_L]} c(e) \leq 2a$. Therefore,

$$\sigma(S, L) \geq \sum_{x \in U_L} [c'(x) - 2b] + \sum_{x \in W_L} [c'(x) - 2a] = \left[\sum_{x \in L} c'(x) \right] - 4ab.$$

Since $a + b = r$ and r is odd, the maximum value of ab is $\frac{1}{4}(r^2 - 1)$. Hence

$$\sigma(S, L) \geq \left[\sum_{x \in L} c'(x) \right] - (r^2 - 1). \quad (2.5)$$

We consider two cases, according to whether $i \leq r$ or $i \geq r+2$.

Case 1. $i \leq r$. Since $\{c'(x) : x \in L\} = \{r+1, r+2, \dots, 2r\}$, it follows by (2.5) that

$$\sigma(S, L) \geq (r+1 + r+2 + \dots + 2r) - (r^2 - 1) = \frac{r^2 + r + 2}{2}. \quad (2.6)$$

On the other hand, $\{c'(x) : x \in S\} = \{0, 1, 2, \dots, r\} - \{i\}$ and the sum of these colors is maximum when $i = 1$. Thus,

$$\sigma(S, L) \leq 0 + 2 + 3 + \dots + r = \frac{r^2 + r - 2}{2},$$

which contradicts (2.6).

Case 2. $i \geq r + 2$. Then $\{c'(x) : x \in L\} = \{r, r + 1, \dots, 2r\} - \{i\}$ and the sum of these colors is minimum when $i = 2r - 1$. It then follows by (2.5) that

$$\sigma(S, L) \geq [r + (r + 1) + \dots + (2r - 2) + 2r] - (r^2 - 1) = \frac{r^2 - r + 4}{2}. \quad (2.7)$$

On the other hand, $\{c'(x) : x \in S\} = \{0, 1, 2, \dots, r - 1\}$. Hence

$$\sigma(S, L) \leq 0 + 1 + 2 + \dots + (r - 1) = \frac{r^2 - r}{2},$$

which contradicts (2.7). Therefore, $s(G) \geq 4$.

It remains to show that there is a vertex-distinguishing edge coloring $c : E(G) \rightarrow \{1, 2, 3, 4\}$. Since $r \geq 3$ is odd, $r = 2k + 1$ for some positive integer k . Define an edge coloring $c : E(G) \rightarrow \{1, 2, 3, 4\}$ by

$$c(u_i w_j) = \begin{cases} 1 & \text{if } j > i \text{ and } (i, j) \neq (k + 1, k + 2) \\ & \text{or } i = j = k + 1 \\ 2 & \text{if } i = j \leq k \text{ or } (i, j) = (k + 1, 2k + 1) \\ 3 & \text{if } i = j \geq k + 2 \\ 4 & \text{if } j < i. \end{cases}$$

Then the induced vertex coloring c' satisfies the following

$$c'(u_i) = \begin{cases} r - 2 + 3i & \text{if } 1 \leq i \leq k + 1 \\ r - 1 + 3i & \text{if } k + 2 \leq i \leq 2k + 1 \end{cases}$$

$$c'(w_i) = \begin{cases} 4r + 1 - 3i & \text{if } 1 \leq i \leq k \\ 4r - 3i & \text{if } i = k + 1 \\ 4r + 2 - 3i & \text{if } k + 2 \leq i \leq 2k \\ r + 3 & \text{if } i = 2k + 1. \end{cases}$$

The vertex coloring $c' : V(G) \rightarrow \{r, r + 1, \dots, 4r\}$ is vertex-distinguishing. This is illustrated in Fig. 2.4 for $K_{5,5}$. Since c is a vertex-distinguishing edge coloring, it follows that $s(G) \leq 4$ and so $s(G) = 4$ if r is odd. \square

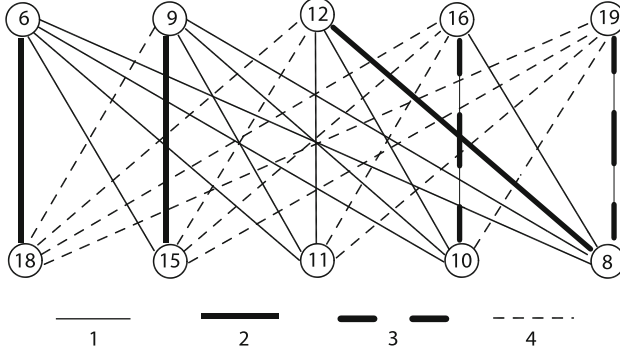


Fig. 2.4 An edge coloring of $K_{5,5}$

In [41], it was shown that if G is a regular complete k -partite graph where $k \geq 3$, then $s(G) = 3$. We now verify this statement by giving a proof along the same lines as the proofs of Proposition 2.2 and Theorems 2.4 and 2.8.

Theorem 2.10. *If G is a regular complete k -partite graph where $k \geq 3$, then*

$$s(G) = 3.$$

Proof. Let $G = K_{k(r)}$ where $k \geq 3$. Thus G is a $(k-1)r$ -regular graph of order kr . By Proposition 2.3, $s(G) \geq 3$. Thus, it remains to show that G has a vertex-distinguishing edge coloring using the colors 1, 2, 3. Let V_1, V_2, \dots, V_k denote the k partite sets of G where

$$V_i = \{v_1^{(i)}, v_2^{(i)}, \dots, v_r^{(i)}\} \text{ for } 1 \leq i \leq k.$$

First, suppose that r is even, say $r = 2\ell$ for some positive integer ℓ . We now construct an ordered list L of the n vertices of G , separated into r blocks B_1, B_2, \dots, B_r of k vertices each. The first block is $B_1 : v_1^{(1)}, v_1^{(2)}, \dots, v_1^{(k)}$. In general, for $1 \leq j \leq \ell$, the block B_j is

$$B_j : v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(k)}. \quad (2.8)$$

For $\ell + 1 \leq j \leq r$, the block B_j is

$$B_j : v_j^{(k)}, v_j^{(k-1)}, \dots, v_j^{(2)}, v_j^{(1)}. \quad (2.9)$$

Consequently, the list L is

$$L : B_1, B_2, \dots, B_\ell, B_{\ell+1}, B_{\ell+2}, \dots, B_r. \quad (2.10)$$

We relabel the vertices of L as u_1, u_2, \dots, u_n . Next, we construct a spanning subgraph H of G as follows. For integers i and j with $1 \leq i < j \leq n$, the vertex u_i is adjacent to u_j in H if $i + j \leq n + 1$ and u_i and u_j do not belong to the same partite set of G . Thus $\deg_H u_1 \geq \deg_H u_2 \geq \dots \geq \deg_H u_n$ and $\deg_H u_i = \deg_H u_{i+1}$

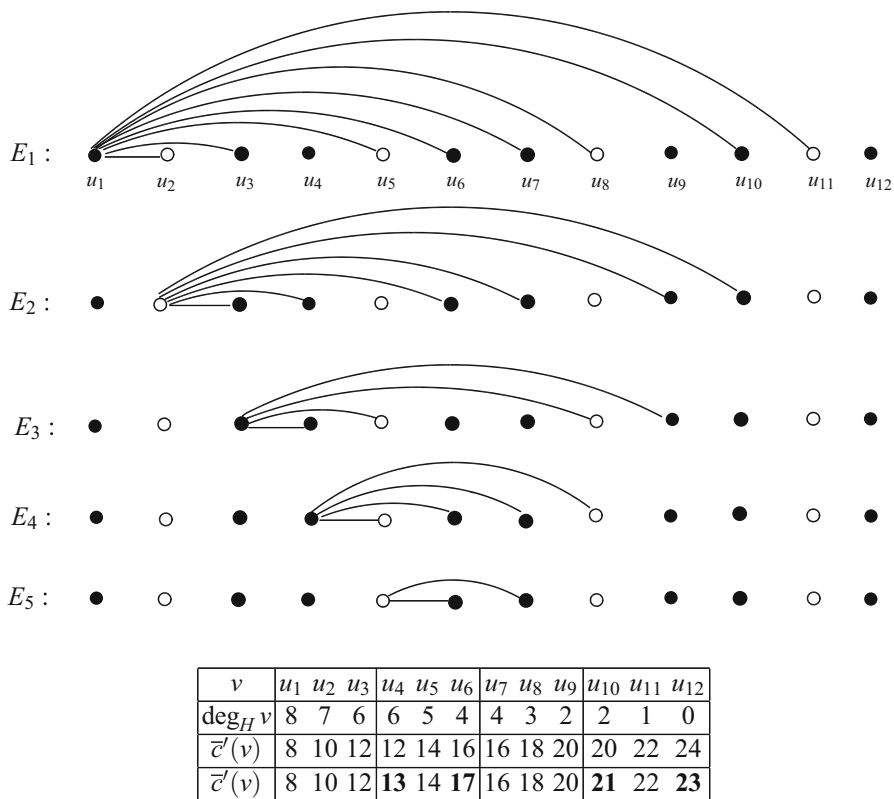


Fig. 2.5 Constructing the graph H in $K_{3(4)}$

only when $i < n$ and $i \equiv 0 \pmod{k}$. For $G = K_{3(4)}$, the edge set $\cup_{i=1}^5 E_5$ of the graph H is shown in Fig. 2.5 where $E_i = \{u_i u_j \in E(G) : i + j \leq 13\}$ for $1 \leq i \leq 5$.

First, we define an edge coloring $\bar{c} : E(G) \rightarrow \{1, 3\}$ of G by assigning the color 1 to each edge of H and the color 3 to each edge in $G - E(H)$. The induced vertex coloring $\bar{c}' : V(G) \rightarrow \mathbb{N}$ satisfies the following:

- (1) $\bar{c}'(u_i)$ is even for all i ($1 \leq i \leq n$),
- (2) $\bar{c}'(u_1) \leq \bar{c}'(u_2) \leq \dots \leq \bar{c}'(u_n)$ and
- (3) $\bar{c}'(u_i) = \bar{c}'(u_{i+1})$ only when $i < n$ and $i \equiv 0 \pmod{k}$.

We now revise the edge coloring $\bar{c} : E(G) \rightarrow \{1, 3\}$ by constructing a new edge coloring $c : E(G) \rightarrow \{1, 2, 3\}$ as follows:

$$c(e) = \begin{cases} \bar{c}(e) + 1 & \text{if } e = u_{(j-1)k+1} u_{jk}, j \text{ even}, 2 \leq j \leq \ell, \\ \bar{c}(e) - 1 & \text{if } e = u_{(j-1)k+1} u_{jk}, j \text{ even}, \ell + 1 \leq j \leq 2\ell \\ \bar{c}(e) & \text{otherwise.} \end{cases}$$

Then the induced vertex coloring $c' : V(G) \rightarrow \mathbb{N}$ satisfies

$$c'(v) = \begin{cases} \bar{c}'(v) + 1 & \text{if } v = u_{(j-1)k+1}, u_{jk}, j \text{ even}, 2 \leq j \leq \ell \\ \bar{c}'(v) - 1 & \text{if } v = u_{(j-1)k+1}, u_{jk}, j \text{ even}, \ell + 1 \leq j \leq 2\ell \\ \bar{c}'(v) & \text{otherwise.} \end{cases}$$

It then follows by properties (1)–(3) of the vertex coloring \bar{c}' that c' is vertex-distinguishing. This is also illustrated for $K_{3(4)}$ in Fig. 2.5.

Next, suppose that $r \geq 3$ is odd, say $r = 2\ell + 1$ for some positive integer ℓ . We now construct an ordered list L of the n vertices of G , separated into r blocks B_1, B_2, \dots, B_r of k vertices each. For $1 \leq j \leq \ell + 1$, the block B_j is the one in (2.8). For $\ell + 2 \leq j \leq r$, the block B_j is the one in (2.9). Consequently, the list L is as described in (2.10). Then relabel the vertices of L as u_1, u_2, \dots, u_n . We now construct a spanning subgraph H of G as in the case when r is even. That is, for integers i and j with $1 \leq i < j \leq n$, the vertex u_i is adjacent to u_j in H if $i + j \leq n + 1$ and u_i and u_j do not belong to the same partite set of G . Thus $\deg_H u_1 \geq \deg_H u_2 \geq \dots \geq \deg_H u_n$ and $\deg_H u_i = \deg_H u_{i+1}$ only when either

$$(1) i \equiv 0 \pmod{k} \text{ and } i \neq n, (\ell + 1)k \text{ or } (2) i = \left\lceil \frac{n}{2} \right\rceil.$$

First, we define an edge coloring $\bar{c} : E(G) \rightarrow \{1, 3\}$ of G by assigning the color 1 to each edge of H and the color 3 to each edge in $G - E(H)$. The induced vertex coloring $\bar{c}' : V(G) \rightarrow \mathbb{N}$ satisfies the following:

- (1) $\bar{c}'(u_i)$ is odd for all i ($1 \leq i \leq n$) if $k - 1$ is odd and $\bar{c}'(u_i)$ is even ($1 \leq i \leq n$) if $k - 1$ is even,
- (2) $\bar{c}'(u_1) \leq \bar{c}'(u_2) \leq \dots \leq \bar{c}'(u_n)$ and
- (3) $\bar{c}'(u_i) = \bar{c}'(u_{i+1})$ only when either $i \equiv 0 \pmod{k}$ and $i \neq n, (\ell + 1)k$ or $i = \left\lceil \frac{n}{2} \right\rceil$.

We now revise the edge coloring $\bar{c} : E(G) \rightarrow \{1, 3\}$ by constructing a new edge coloring $c : E(G) \rightarrow \{1, 2, 3\}$ as follows:

$$c(e) = \begin{cases} \bar{c}(e) + 1 & \text{if } e = u_{\ell k+1} u_{\left\lceil \frac{n}{2} \right\rceil} \text{ or} \\ & e = u_{(j-1)k+1} u_{jk}, j = \ell - i, i \text{ odd}, 1 \leq i \leq \ell - \\ \bar{c}(e) - 1 & \text{if } e = u_{(j-1)k+1} u_{jk}, j = \ell + i, i \text{ odd}, 3 \leq i \leq \ell + 1 \\ \bar{c}(e) & \text{otherwise.} \end{cases}$$

Then the induced vertex coloring $c' : V(G) \rightarrow \mathbb{N}$ satisfies

$$c'(v) = \begin{cases} \bar{c}'(v) + 1 & \text{if } v = u_{\ell k+1}, v = u_{\left\lceil \frac{n}{2} \right\rceil} \text{ or} \\ & v = u_{(j-1)k+1}, u_{jk}, j = \ell - i, i \text{ odd}, 1 \leq i \leq \ell - 1 \\ \bar{c}'(v) - 1 & \text{if } v = u_{(j-1)k+1}, u_{jk}, j = \ell + i, i \text{ odd}, 3 \leq i \leq \ell + 1 \\ \bar{c}'(v) & \text{otherwise.} \end{cases}$$

It then follows by properties (1)–(3) of the vertex coloring \bar{c}' that c' is vertex-distinguishing. This is illustrated for $K_{4(3)}$, $K_{5(3)}$ and $K_{4(5)}$ in the following three tables. \square

$V(K_{4(3)})$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
$\deg_H v$	9	8	7	6	6	5	5	4	3	2	1	0
$\bar{c}'(v)$	9	11	13	15	15	17	17	19	21	23	25	27
$c'(v)$	9	11	13	15	16	18	17	19	21	23	25	27

$V(K_{5(3)})$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
$\deg_H v$	12	11	10	9	8	8	7	6	6	5	4	3	2	1	0
$\bar{c}'(v)$	12	14	16	18	20	20	22	24	24	26	28	30	32	34	36
$c'(v)$	12	14	16	18	20	21	22	25	24	26	28	30	32	34	36

$V(K_{4(5)})$	u_1, u_2, u_3, u_4	u_5, u_6, u_7, u_8	$u_9, u_{10}, u_{11}, u_{12}$	$u_{13}, u_{14}, u_{15}, u_{16}$	$u_{17}, u_{18}, u_{19}, u_{20}$
$\deg_H v$	15, 14, 13, 12	12, 11, 10, 9	9, 8, 8, 7	6, 5, 4, 3	3, 2, 1, 0
$\bar{c}'(v)$	15, 17, 19, 21	21, 23, 25, 27	27, 29, 29, 31	33, 35, 37, 39	39, 41, 43, 45
$c'(v)$	16 , 17, 19, 22	21, 23, 25, 27	28 , 30 , 29, 31	33, 35, 37, 39	38 , 41, 43, 44

The following corollary then summarizes all results on the irregularity strength of regular complete multipartite graphs.

Corollary 2.11. *If G is a regular complete multipartite graph of order at least 3, then*

$$s(G) = \begin{cases} 4 & \text{if } G = K_{r,r} \text{ where } r \geq 3 \text{ is odd} \\ 3 & \text{otherwise.} \end{cases}$$

2.3 The Irregularity Strength of Paths and Cycles

We now turn our attention to two other well-known classes of graphs, namely paths and cycles. The next theorem gives the irregularity strength of all paths.

Theorem 2.12 ([24]). *For an integer $n \geq 3$,*

$$s(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \dots, v_n)$ and $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$. First, we establish a lower bound for $s(P_n)$. If $c : E(P_n) \rightarrow \mathbb{N}$ is a vertex-distinguishing edge

coloring with induced vertex coloring c' , then $c'(v_j) \geq n$ for some vertex v_j . If v_j is an end-vertex, say $v_j = v_1$, then $c(e_1) \geq n$; while if $\deg v_j = 2$, then either $c(e_{j-1}) \geq n/2$ or $c(e_j) \geq n/2$. Thus $s(P_n) \geq n/2$ when n is even and $s(P_n) \geq (n+1)/2$ when n is odd. If $n \equiv 2 \pmod{4}$ and $s(P_n) = n/2$, then $\{c'(v_i) : 1 \leq i \leq n\} = [n]$ and so $\sum_{i=1}^n c'(v_i)$ is odd, contradicting Proposition 2.1. Hence $s(P_n) \geq (n+2)/2$ when $n \equiv 2 \pmod{4}$.

Next, we show that each of these lower bounds for $s(P_n)$ is also an upper bound. If $n \equiv 0 \pmod{4}$, then $n = 4k$ for some positive integer k . Define the edge coloring $c : E(P_n) \rightarrow \mathbb{N}$ by

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2k \\ n-2 \lfloor \frac{i}{2} \rfloor & \text{if } 2k+1 \leq i \leq n-1. \end{cases}$$

For the induced vertex coloring c' , we then have

$$c'(v_i) = \begin{cases} 2i-1 & \text{if } 1 \leq i \leq 2k \\ 2n-2i+2 & \text{if } 2k+1 \leq i \leq n. \end{cases}$$

This is illustrated in Fig. 2.6 for $n = 8$. Since c is a vertex-distinguishing edge coloring whose largest color is $c(e_{2k}) = 2k = n/2$, it follows that $s(P_n) \leq n/2$ and so $s(P_n) = n/2$ if $n \equiv 0 \pmod{4}$.

Assume next that n is odd. Then $n = 2k+1$ for some positive integer k . If $n \equiv 3 \pmod{4}$, then define the edge coloring $c : E(P_n) \rightarrow \mathbb{N}$ by

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq k \\ n+1-2 \lceil \frac{i}{2} \rceil & \text{if } k+1 \leq i \leq n-1. \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} 2i-1 & \text{if } 1 \leq i \leq k+1 \\ 2n-2i+2 & \text{if } k+2 \leq i \leq n. \end{cases}$$

If $n \equiv 1 \pmod{4}$, then define the edge coloring $c : E(P_n) \rightarrow \mathbb{N}$ by

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq k-1 \text{ or } i = k+1 \\ k+1 & \text{if } i = k \\ n+1-2 \lceil \frac{i}{2} \rceil & \text{if } k+2 \leq i \leq n-1. \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} 2i-1 & \text{if } 1 \leq i \leq k-1 \\ 2i & \text{if } i = k, k+1 \\ 2i-3 & \text{if } i = k+2 \\ 2n-2i+2 & \text{if } k+3 \leq i \leq n. \end{cases}$$

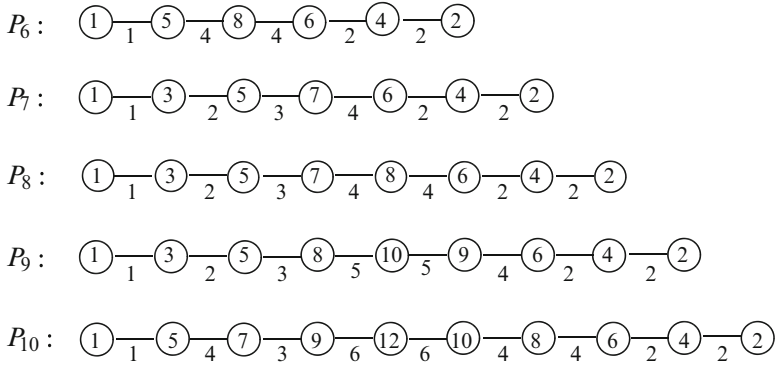


Fig. 2.6 Edge colorings of P_n in the proof of Theorem 2.12 for $6 \leq n \leq 10$

This is illustrated in Fig. 2.6 for $n = 7, 9$. In each case, c is a vertex-distinguishing edge coloring whose largest color is $c(e_{k+1}) = (n+1)/2$, it follows that $s(P_n) \leq (n+1)/2$ and so $s(P_n) = (n+1)/2$ when n is odd.

Finally, assume that $n \equiv 2 \pmod{4}$. Then $n = 4k+2$ for some positive integer k . Define the edge coloring $c : E(P_n) \rightarrow \mathbb{N}$ by

$$c(e_i) = \begin{cases} i & \text{if } i = 1, 3, \dots, 2k-1 \\ i+2 & \text{if } i = 2, 4, \dots, 2k \\ n-2 \lfloor \frac{i}{2} \rfloor & \text{if } 2k+1 \leq i \leq n-1. \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} 1 & \text{if } i = 1 \\ 2i+1 & \text{if } 2 \leq i \leq 2k \\ 2n-2i+2 & \text{if } 2k+1 \leq i \leq n. \end{cases}$$

This is illustrated in Fig. 2.6 for $n = 6, 10$. Since c is a vertex-distinguishing edge coloring having the largest color $c(e_{2k+1}) = (n+2)/2$, it follows that $s(P_n) \leq (n+2)/2$ and so $s(P_n) = (n+2)/2$ when $n \equiv 2 \pmod{4}$. \square

The next theorem gives the irregularity strength of cycles (see [41]).

Theorem 2.13. For an integer $n \geq 3$,

$$s(C_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \equiv 3 \pmod{4}. \end{cases} \quad (2.11)$$

Proof. By Corollaries 2.6 and 2.7, each of the expressions in (2.11) is a lower bound for $s(C_n)$. Thus, it remains to verify that each of these expressions is also an upper bound. Let $C_n = (v_1, v_2, \dots, v_n, v_{n+1} = v_1)$ where $n \geq 3$.

We first consider the case when $n \equiv 1 \pmod{4}$. Then $n = 4q + 1$ for some positive integer q and so $\frac{n+1}{2} = 2q + 1$. Define an edge coloring $c : E(C_{4q+1}) \rightarrow [2q + 1]$ by

$$c(v_i v_{i+1}) = \begin{cases} 2q + 1 - 2 \lfloor \frac{i}{2} \rfloor & \text{for } 1 \leq i \leq 2q + 1 \\ i - 2q & \text{for } 2q + 2 \leq i \leq 4q + 1. \end{cases}$$

Then the induced vertex coloring c' satisfies the following

$$c'(v_i) = \begin{cases} 4q + 4 - 2i & \text{if } 1 \leq i \leq 2q + 1 \\ 2i - 4q - 1 & \text{if } 2q + 2 \leq i \leq 4q + 1. \end{cases}$$

This is illustrated in Fig. 2.7 for C_9 and C_{13} . Since c is a vertex-distinguishing edge coloring whose largest color is $2q + 1$, it follows that $s(C_{4q+1}) \leq 2q + 1$ and so $s(C_{4q+1}) = 2q + 1$.

Next, we show that if n is even, the lower bound $\frac{n+2}{2}$ for $s(C_n)$ is also an upper bound. Then $n = 2k$ for some integer $k \geq 2$ and so $\frac{n+2}{2} = k + 1$. Define an edge coloring $c : E(C_{2k}) \rightarrow [k + 1]$ by considering two cases, according to whether k is odd or k is even. If k is odd, then let

$$c(v_i v_{i+1}) = \begin{cases} k + 1 - 2 \lfloor \frac{i}{2} \rfloor & \text{for } 1 \leq i \leq k \\ k + 2 - 2 \lfloor \frac{i}{2} \rfloor = 1 & \text{for } i = k + 1, k + 2 \\ i + 1 - k & \text{for } k + 3 \leq i \leq 2k \end{cases}$$

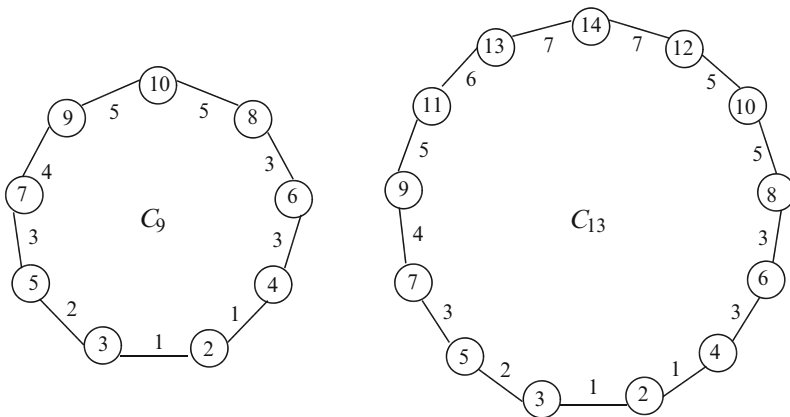


Fig. 2.7 Edge colorings of C_9 and C_{13}

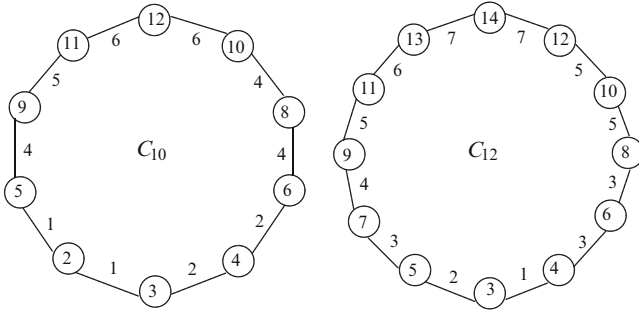


Fig. 2.8 Edge colorings of C_{10} and C_{12}

Then the induced vertex coloring c' satisfies the following

$$c'(v_i) = \begin{cases} 2k + 4 - 2i & \text{if } 1 \leq i \leq k \\ 3 & \text{if } i = k + 1 \\ 2 & \text{if } i = k + 2 \\ 5 & \text{if } i = k + 3 \\ 2i - 2k + 1 & \text{if } k + 4 \leq i \leq 2k. \end{cases}$$

This is illustrated in Fig. 2.8 for C_{10} . If k is even, then let

$$c(v_i v_{i+1}) = \begin{cases} k + 1 - 2 \lfloor \frac{i}{2} \rfloor & \text{for } 1 \leq i \leq k \\ i + 1 - k & \text{for } k + 1 \leq i \leq 2k \end{cases}$$

Then the induced vertex coloring c' satisfies the following

$$c'(v_i) = \begin{cases} 2k + 4 - 2i & \text{if } 1 \leq i \leq k \\ 2i - 2k + 1 & \text{if } k + 1 \leq i \leq 2k. \end{cases}$$

This is illustrated in Fig. 2.8 for C_{12} . Since c is a vertex-distinguishing edge coloring whose largest color is $k + 1$, it follows that $s(C_{2k}) \leq k + 1$ and so $s(C_{2k}) = k + 1$.

Finally, we consider the case where n is odd and $n \equiv 3 \pmod{4}$. In this case, $n = 4q + 3$ for some positive integer q . Define an edge coloring $c : E(C_{4q+3}) \rightarrow [2q + 3]$ by

$$c(v_i v_{i+1}) = \begin{cases} 2q + 3 - 2 \lfloor \frac{i}{2} \rfloor & \text{for } 1 \leq i \leq 2q + 3 \\ i - (2q + 2) & \text{for } 2q + 4 \leq i \leq 4q + 2 \\ 2q + 3 & \text{for } i = 4q + 3. \end{cases}$$

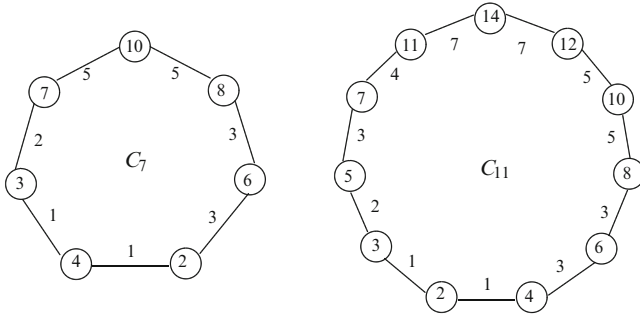


Fig. 2.9 Edge colorings of C_7 and C_{11}

Then the induced vertex coloring c' satisfies the following

$$c'(v_i) = \begin{cases} 4q + 8 - 2i & \text{if } 1 \leq i \leq 2q + 3 \\ 2i - 4q - 5 & \text{if } 2q + 4 \leq i \leq 4q + 2 \\ 4q + 3 & \text{if } i = 4q + 3. \end{cases}$$

This is illustrated in Fig. 2.9 for C_7 and C_{11} . Since c is a vertex-distinguishing edge coloring whose largest color is $2q + 3$, it follows that $s(C_{4q+3}) \leq 2q + 3$ and so $s(C_{4q+3}) = 2q + 3$. \square

2.4 Additional Bounds for the Irregularity Strength of a Graph

A graph G is said to be *factorable* into the factors (spanning subgraphs of G) F_1, F_2, \dots, F_t if these factors are (pairwise) edge-disjoint and $\cup_{i=1}^t E(F_i) = E(G)$. If G is factored into F_1, F_2, \dots, F_t , then $\{F_1, F_2, \dots, F_t\}$ is called a *factorization* of G . If a graph G has a factorization into two factors, one of which is regular, then the irregularity strength of the other factor provides an upper bound for the irregularity strength of G .

Proposition 2.14. *If $\{F_1, F_2\}$ is a factorization of a graph G where F_2 is regular, then $s(G) \leq s(F_1)$.*

Proof. Suppose that $s(F_1) = s$ and F_2 is r -regular. Then there is a vertex-distinguishing edge coloring $\bar{c} : E(F_1) \rightarrow \{1, 2, \dots, s\}$. Let $c : E(G) \rightarrow \{1, 2, \dots, s\}$ be the edge coloring where

$$c(e) = \begin{cases} \bar{c}(e) & \text{if } e \in E(F_1) \\ 1 & \text{if } e \in E(F_2). \end{cases}$$

Fig. 2.10 Illustrating that the inequality in Proposition 2.14 can be strict

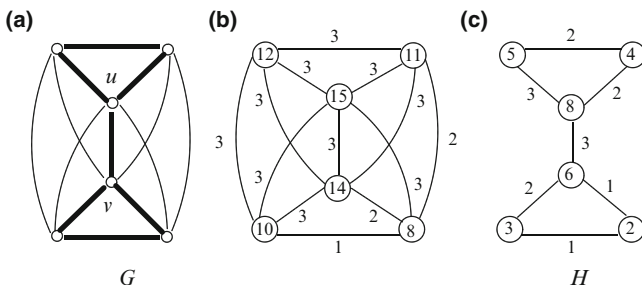
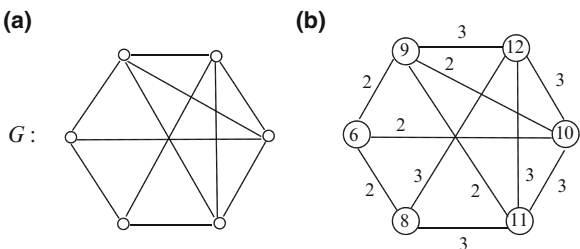


Fig. 2.11 Illustrating the equality in Proposition 2.14

Then $c'(v) = \bar{c}'(v) + r$ for every vertex v of G . Hence c is a vertex-distinguishing edge coloring of G and so $s(G) \leq s(F_1)$. \square

The inequality in Proposition 2.14 can be strict. For example, consider the graph G of Fig. 2.10a. Thus, $\{P_6, C_6\}$ is a factorization of G . By Theorem 2.12, $s(P_6) = \frac{6+2}{2} = 4$. The edge coloring of G in Fig. 2.10b shows that $s(G) \leq 3$. If there was an edge coloring of G with the colors in $\{1, 2\}$, then the vertex colors would belong to the set $\{3, 4, 5, 6, 7, 8\}$. However, there must be an even number of vertices of odd color by Proposition 2.1, a contradiction. Hence $s(G) \geq 3$, implying that $s(G) = 3$.

Equality in Proposition 2.14 is also possible. For example, consider the graph G of Fig. 2.11a with the spanning subgraph H . Thus, $\{H, C_6\}$ is a factorization of G , where the edges of H are drawn by bold lines in Fig. 2.11a. We show that $s(G) = s(H) = 3$. First, we verify that $s(G) = 3$. The edge coloring of G in Fig. 2.11b shows that $s(G) \leq 3$. If there was an edge coloring of G with the colors in $\{1, 2\}$, then the set of vertex colors would be a subset of $S = \{4, 5, 6, 7, 8, 9, 10\}$. So only one element in S is not a vertex color of G , necessarily an odd color. This implies that 10 is the color of some vertex of G . So either u or v has all of its incident edges colored 2, say u is colored 10. Since u is adjacent to all other vertices of G , every vertex is incident with an edge colored 2. Thus, the colors of the vertices of degree 4 are $2 + 1 + 1 + 1 = 5$, $2 + 2 + 1 + 1 = 6$, $2 + 2 + 2 + 1 = 7$ and $2 + 2 + 2 + 2 = 8$. Since 5 and 7 are two vertex colors, 9 is not a vertex color. Hence 4 must be the color of v . Since $\deg v = 5$, this is impossible. Hence $s(G) \geq 3$, implying that $s(G) = 3$.

Next, we show that $s(H) = 3$. If $s(H) = 2$, then for an edge coloring of H with the colors in $\{1, 2\}$, the set of vertex colors of H is a subset of $\{2, 3, 4, 5, 6\}$. This is

impossible since the order of H is 6. Thus, $s(H) \geq 3$. Since the edge coloring of H with colors 1, 2, 3 in Fig. 2.11c is vertex-distinguishing, it follows that $s(H) = 3$.

In 1952, Dirac [27] obtained the first theoretical result dealing with Hamiltonian graphs when he proved that if G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian. The following is a consequence of Dirac's theorem, Theorem 2.13 and Proposition 2.14.

Corollary 2.15 ([41]). *If G is an r -regular graph of order $n \geq 3$ such that $r \geq n/2$, then $s(G) \leq \lceil \frac{n}{2} \rceil + 1$.*

While the lower bounds for the irregularity strength of a graph G that we have presented thus far have been expressed primarily in terms of the order of G and the degrees of the vertices of G , the following lower bound is given in terms of the order and the size of G .

Theorem 2.16. *Let G be a connected graph of order $n \geq 3$ and size m . For each integer k with $2 \leq k \leq \Delta(G)$,*

$$s(G) \geq \frac{kn - 2m}{\binom{k}{2}}.$$

Proof. Let c be a vertex-distinguishing edge coloring of G with irregularity strength $s(G) = s$ where c' is the induced vertex coloring of G . For $i = 1, 2, \dots, \Delta(G)$, let n_i denote the number of vertices of degree i in G . For $2 \leq k \leq \Delta(G)$,

$$1 \cdot n_1 + 2 \cdot n_2 + \dots + (k-1)n_{k-1} + k(n - n_1 - n_2 - \dots - n_{k-1}) = \sum_{i=1}^{\Delta(G)} in_i = 2m.$$

Hence,

$$kn = (k-1)n_1 + (k-2)n_2 + \dots + 1 \cdot n_{k-1} + 2m = \sum_{j=1}^{k-1} \left(\sum_{i=1}^j in_i \right) + 2m.$$

For each integer j ($1 \leq j \leq k-1$), the colors of the $\sum_{i=1}^j in_i$ vertices lie between 1 and js . Therefore,

$$\sum_{i=1}^j in_i \leq js.$$

Thus,

$$kn = \sum_{j=1}^{k-1} \left(\sum_{i=1}^j in_i \right) + 2m \leq \left(\sum_{j=1}^{k-1} (js) \right) = s \binom{k}{2} + 2m$$

and so

$$s = s(G) \geq \frac{kn - 2m}{\binom{k}{2}},$$

as desired. \square

For $k = 2$ in Theorem 2.16, we have $s(G) \geq 2n - 2m$. Since G is connected, $m \geq n - 1$. If $m = n - 1$ (and so G is a tree), then $s(G) \geq 2$, which, of course, we already knew. If $k = 3$ in Theorem 2.16, we have $s(G) \geq \frac{3n-2m}{3}$, while if $k = 4$ in Theorem 2.16, we have $s(G) \geq \frac{4n-2m}{6} = \frac{2n-m}{3}$. When G is a tree, $\frac{3n-2m}{3} = \frac{n+2}{3}$ is a better bound for $s(G)$.

Corollary 2.17 ([24]). *If T is a tree of order $n \geq 3$, then $s(T) \geq (n + 2)/3$.*

Following [24], we now see that the lower bound $(n + 2)/3$ for $s(T)$, where T is a tree of order $n \geq 3$, cannot be improved in general by providing an infinite class of trees T of order $n \geq 3$ for which $s(T) = (n + 2)/3$.

For a positive integer q , let $P_{4q-1} = (u_1, u_2, \dots, u_{4q-1})$ be a path of order $4q - 1$. We attach two paths (u_1, v_1, w_1) and (u_1, v_2, w_2) of length 2 at u_1 . For $2 \leq i \leq 4q - 2$, attach a path (u_i, v_{i+1}, w_{i+1}) of length 2 at u_i . In addition, attach two paths $(u_{4q-1}, v_{4q}, w_{4q})$ and $(u_{4q-1}, v_{4q+1}, w_{4q+1})$ of length 2 at u_{4q-1} . Denote the resulting tree by T_q , which has order $12q + 1$. The tree T_2 is shown in Fig. 2.12. By Corollary 2.17, $s(T_q) \geq (n + 2)/3 = (12q + 3)/3 = 4q + 1$. It remains to show that $s(T_q) \leq 4q + 1$.

Define an edge coloring $c : E(T_q) \rightarrow [4q + 1]$ by $c(v_i w_i) = i$ for $1 \leq i \leq 4q + 1$, $c(u_1 v_1) = c(u_{4q-1} v_{4q+1}) = 4q + 1$ and $c(u_i v_{i+1}) = 4q + 1$ for $1 \leq i \leq 4q - 1$. In addition,

$$c(u_i u_{i+1}) = \begin{cases} 2q + \lceil \frac{i}{2} \rceil & \text{for } 1 \leq i \leq 2q + 1 \\ 2q + 1 + \lfloor \frac{i}{2} \rfloor & \text{for } 2q + 2 \leq i \leq 4q - 2. \end{cases}$$

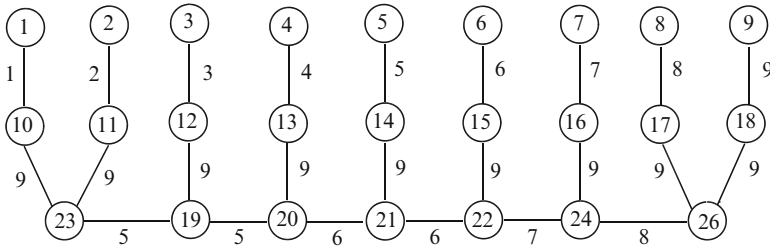


Fig. 2.12 An edge coloring of the tree T_2

This edge coloring c then induces the vertex coloring $c' : V(T_q) \rightarrow \mathbb{N}$ where

$$\begin{aligned} c'(w_i) &= i & \text{for } 1 \leq i \leq 4q + 1 \\ c'(v_i) &= 4q + 1 + i & \text{for } 1 \leq i \leq 4q + 1 \\ c'(u_i) &= \begin{cases} 10q + 3 & \text{if } i = 1 \\ 8q + 1 + i & \text{if } 2 \leq i \leq 2q + 1 \\ 8q + 2 + i & \text{if } 2q + 2 \leq i \leq 4q - 2 \\ 12q + 2 & \text{if } i = 4q - 1. \end{cases} \end{aligned}$$

This is illustrated in Fig. 2.12 for T_2 . Since c is a vertex-distinguishing edge coloring whose largest color is $4q + 1$, it follows that $s(T_q) \leq 4q + 1$ and so $s(T_q) = 4q + 1$.

If G is a unicyclic graph of order n and size m , then $m = n$. Letting $k = 3$ in Theorem 2.16, we have the following corollary.

Corollary 2.18 ([24]). *If G is a unicyclic graph of order $n \geq 3$, then $s(G) \geq n/3$.*

Next, we show that the lower bound $n/3$ for $s(G)$, where G is a unicyclic graph of order $n \geq 3$, is sharp by providing an infinite class of unicyclic graphs G of order $n \geq 3$ for which $s(G) = n/3$.

For a positive integer q , let $C = (u_1, u_2, \dots, u_{4q}, u_{4q+1} = u_1)$ be a cycle of length $4q$. At each vertex u_i ($1 \leq i \leq 4q$), we attach a path (u_i, v_i, w_i) of length 2. The resulting graph is a unicyclic graph G of order $12q$. By Corollary 2.18, $s(G) \geq 12q/3 = 4q$. To show that $s(G) \leq 4q$, define an edge coloring $c : E(G) \rightarrow [4q]$ by

$$\begin{aligned} c(v_i w_i) &= i & \text{for } 1 \leq i \leq 4q \\ c(u_i v_i) &= 4q & \text{for } 1 \leq i \leq 4q \\ c(u_i u_{i+1}) &= \begin{cases} 4q - \lfloor \frac{i-1}{2} \rfloor & \text{for } 1 \leq i \leq 2q + 1 \\ 4q - \lfloor \frac{i}{2} \rfloor & \text{for } 2q + 2 \leq i \leq 4q. \end{cases} \end{aligned}$$

Then the induced vertex coloring c' satisfies the following

$$\begin{aligned} c'(w_i) &= i & \text{for } 1 \leq i \leq 4q \\ c'(v_i) &= 4q + i & \text{for } 1 \leq i \leq 4q \\ c'(u_i) &= \begin{cases} 10q & \text{if } i = 1 \\ 12q + 2 - i & \text{if } 2 \leq i \leq 2q + 1 \\ 12q + 1 - i & \text{if } 2q + 2 \leq i \leq 4q. \end{cases} \end{aligned}$$

For $q = 2$, this graph is shown in Fig. 2.13. Since c is a vertex-distinguishing edge coloring whose largest color is $4q$, it follows that $s(G) \leq 4q$ and so $s(G) = 4q$.

If G is a connected graphs of order n and size $n + 1$, then letting $k = 4$ in Theorem 2.16 provides a lower bound for $s(G)$.

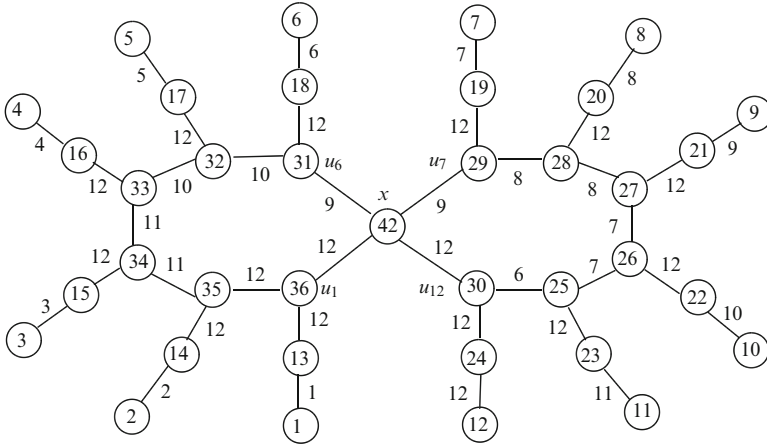


Fig. 2.14 An edge coloring of a connected graph of size $n + 1$

$$c(xu_i) = 3q \quad \text{for } i = 2q, 2q + 1$$

$$c(u_i u_{i+1}) = \begin{cases} 4q - \lfloor \frac{i}{2} \rfloor & \text{for } 1 \leq i \leq 2q - 1 \\ 4q - \lceil \frac{i}{2} \rceil & \text{for } 2q + 1 \leq i \leq 4q - 1. \end{cases}$$

Then the induced vertex coloring c' satisfies the following

$$c'(w_i) = i \quad \text{for } 1 \leq i \leq 4q$$

$$c'(v_i) = 4q + i \quad \text{for } 1 \leq i \leq 4q$$

$$c'(u_i) = \begin{cases} 12q + 1 - i & \text{if } 1 \leq i \leq 2q \\ 12q - i & \text{if } 2q + 1 \leq i \leq 4q - 1. \\ 10q & \text{if } i = 4q \end{cases}$$

$$c'(x) = 14q.$$

This is illustrated in Fig. 2.14 for $q = 3$. Since c is a vertex-distinguishing edge coloring whose largest color is $4q$, it follows that $s(G) \leq 4q$ and so $s(G) = 4q$.

While the results presented on irregularity strength have either dealt with formulas for the irregularity strength of certain classes of graphs or lower bounds, we now present a number of upper bounds. Since the proofs for these results are lengthy and do not provide additional insight into this topic, such results will be stated without proofs.

Theorem 2.20 ([3]). *If G is a connected graph of order $n \geq 4$, then $s(G) \leq n - 1$.*

Since a connected graph G of order $n \geq 3$ and size m has irregularity strength m if and only if G is a star and $m = n - 1$ in this case, the upper bound in Theorem 2.20 is sharp. Because the star of order n is the only tree whose irregularity strength is $n - 1$, there is an improved upper bound for other trees.

Theorem 2.21 ([3]). *If T is a tree of order $n \geq 4$ that is not a star, then $s(T) \leq n - 2$.*

Over the years, many research papers have dealt with the irregularity strength of special classes of graphs. For example, the papers [38, 41, 42] deal with the irregularity strength of regular graphs and [6, 13] concern trees. The papers [25, 40] discuss the irregularity strength of dense graphs (those graphs of order n and size m for which m/n is large). The irregularity strength of circulants and grids has been studied in [11, 26], respectively. Graphs with irregularity strength 2 were studied in [39].



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