Preface

This book provides an introduction to and an overview of the most important concepts and structures of modern mathematics. Therefore, I try to motivate and explain those concepts and structures and illustrate them with examples, but I often do not provide the complete proofs of the basic results. Therefore, this book has a broader scope and is less complete and detailed than standard mathematical textbooks. Its main intention is to describe and develop the conceptual, structural, and abstract thinking of mathematics. Specific mathematical structures then illustrate that conceptual approach, and their mutual relationships and their abstract common features should also provide deeper insight into each of them.

The book thus could be used

1. simply as an overview of the panorama of mathematical structures and the relations between them, to be supplemented by more detailed texts whenever you want to acquire a working knowledge of some structure,
2. by itself as a first introduction to abstract mathematics,
3. together with existing textbooks, to put their results into a more general perspective,
4. after having studied such textbooks, in order to gain a new and hopefully deeper perspective.

Putting it differently, this means that the readers should use this book in the manner that best suits their individual needs and aims. It is not my intention to suggest or prescribe a particular way of reading this book. Of course, you could, in principle, try to read the book from the beginning to the end. You could also, however, browse through it and see what strikes you as especially interesting or useful. Or you could simply search for particular topics that you want to learn or understand better. Whichever way you approach it, I hope that this book will prove useful for you.

In any case, however, I should emphasize that mathematics can only be properly learned by going through and understanding the proofs of the main results in detail and by carrying out exercises and computing examples. This book does not systematically provide such proofs and exercises, so it needs to be supplemented by suitable other texts. (Some such textbooks are listed in the bibliography and are cited in the individual chapters of this book. Of course, there exist many more good textbooks than listed, and my selection may appear somewhat random to some experts in the respective fields).

After having dutifully expressed this warning, let me also issue a warning in an opposite direction. Beware of those self-appointed guardians of certain narrow subfields of mathematics who claim that you cannot understand what their subfield and work is about unless you study it with the insiders over many years. Often, such people are just afraid of competition from outsiders. On the contrary, I maintain that every important conceptual mathematical idea can be understood with some effort. Perhaps it requires somewhat more effort than the present book might offer to you, and in any case you will also need some ability in flexible and abstract thinking. But do not let this deter you. This book is intended to offer you some help and guidance on your path toward a deeper understanding of mathematics. This book wants to convey a positive, encouraging message, and not a negative, discouraging one.

Let me now describe the contents and topics. The first chapter gives an informal overview. Chapter 2 introduces some basic structures, like graphs, monoids, and groups, rings and fields, lattices and Boolean and Heyting algebras, as well as the basic notions of category theory. In Chap. 3, we first treat relations in an abstract manner and then discuss graphs as the mathematical structures encoding binary relations. As a more concrete example of mathematical reasoning, we discuss the representation theory of finite groups and apply this to illustrate the space of all graphs on the same set of vertices. In Chap. 4, we introduce topological spaces, as well as the more general class of pretopological spaces. A topological structure on a set distinguishes a subclass of the Boolean algebra of its subsets. The members of this subclass are called the open sets of the topology, and they constitute a Heyting algebra. (Since in general, the complement of an open set need not be open itself, they no longer form a Boolean algebra.) On a topological space, we can define sheaves and cohomology groups and thereby obtain algebraic invariants. Also, we introduce measures and with their help supplement the algebraic invariants by geometric ones. In the next Chap. 5, we analyze the concept of space from the points of view of topology, differential geometry, and algebraic geometry. In differential geometry, we identify the basic notion of curvature, whereas the algebro-geometric approach is based on the concept of a scheme. In the next Chap. 6, we turn to algebraic topology in more detail and discuss general homology theory. We illustrate this for simplicial complexes, and this also allows us to develop a dynamical picture of topology. This can be seen as a discrete analog of Conley theory, the extension of Morse theory to dynamical systems. I then insert Chap. 7, where the generation of structures through specified operations, perhaps obeying certain constraints, is discussed. In Chap. 8, we return to categories and provide an introduction to the basic results of category theory, like the Yoneda lemma and its applications. In Chap. 9, devoted to topoi, we combine algebraic structures such as Boolean and Heyting algebras, the geometric perspective of categorial concepts like presheafs, with the abstract approach of mathematical logic. The last chapter is something of an anticlimax. It reviews the various structures that can be imposed upon or induced by the simplest examples, the sets with no, one, or two elements. While much of this is, of course, rather trivial, it should give the reader the opportunity to review the basic concepts. Of course, to study these examples, one does not have to wait until the end, but can also utilize them while reading about a certain type of structure in another chapter. In the other chapters, a box in the page margin indicates that one of the recurring standard examples is discussed in the text. Occasionally, I use the abbreviation “iff” for “if and only if,” which has become customary in mathematical texts.

Also, at certain places, I point out—possible or already existing—connections with theoretical biology. A systematic conceptual framework for theoretical biology still awaits to be developed, but I believe that some of the concepts presented in this book could yield important ingredients.

While some aspects of the present book may be new, like the discrete Conley theory or some items in the discussion of the concept of space, most of this book simply condenses and illustrates what can be found in existing textbooks. The motivation, as already mentioned in the beginning, is to provide a comprehensive overview and an orientation among the many important structures of modern mathematics. Of course, there are many omissions in this book. In particular, the most fundamental concepts of analysis like compactness are not treated, nor are such important structures as Banach spaces. Also, no mention is made of number theory beyond the elementary concept of a prime number.

Since this book covers a wide range of mathematical topics, some conflicts with established notation in certain fields are inevitable, because the different notational conventions are not always compatible. One point that the reader should be alerted to is that I use the symbol 1 for the truth value *true*, although some other texts use 0 instead.

Clearly, in many of the special fields discussed here, I am not an expert in the technical sense. I wanted to share the understanding that I do possess, however, with my students and collaborators, and to guide them into the powerful realm of abstract modern mathematical concepts, and therefore, I lectured on these topics in graduate courses at Leipzig. I hope that the present book can likewise serve to share that understanding with its readers.

Also, the style of the book leads to inconsistencies, in the sense that certain mathematical concepts are presupposed, but not explained, at various places. Foremost among them is the core concept of analysis, that of differentiation. We need this because some of the most important concepts and examples depend on differential calculus. This is clearly inconsistent, because I shall explain more basic principles, like that of continuity, carefully. The reason is that, to understand the essential thrust of this book and its examples, we need not go into the conceptual foundations of calculus, but can hopefully get away with what the reader knows from her basic calculus course. In any case, all the required material from calculus can be found in my textbook [59]. Also, at certain places, I use constructions and results from linear algebra without further explanation.

Moreover, the style of this book is not uniform. Some passages are rather elementary, with many details, whereas others are much denser and technically more difficult. You don't have to read this book linearly. It might be most efficient to first select those parts that you find easiest to understand and only subsequently proceed to the more technical sections.

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