Chapter 2
Periodic Replacement Overtime

In this chapter, we suppose that the system is large and complex which consists of many kinds of units, and it operates for a job with random working cycles introduced in Chap. 1. The system undergoes minimal repairs at failures [1, p.96], [2, p.95], [3] and can be quickly resumed after minimal repairs. As preventive replacement policies, the system is planned to be replaced at time T, at working cycle N or at number K of failures.

In Sect. 2.1, we suppose that the unit is replaced at time T or at working cycle N, whichever occurs first. Respective policies are called *periodic replacement* and *random replacement* [4, p.53, 6–8] . The expected cost rates are obtained and optimum T_P^* and N_R^* which minimize them are derived analytically. Furthermore, we compare theoretically periodic replacement with time T and random replacement with cycle N [9, 10]. It is shown that when both replacement costs for time T and cycle N are the same, periodic replacement is better than random replacement.

In Sect. 2.2, we propose *periodic replacement overtime* [4, p.66, 11] in which the unit is replaced at the first completion of working cycles over time T discussed in Sect. 1.2. Optimum replacement time T_O^* which minimizes the expected cost rate is derived analytically. In Sect. 2.3, to compare random replacement with replacement overtime, we take up *replacement overtime first* in which the unit is replaced at cycle N or over time T, whichever occurs first. When both replacement costs for cycle N and overtime T are the same, it is also shown that replacement overtime is better than random replacement.

In Sect. 2.4, we propose *replacement overtime last* in which the unit is replaced at cycle N or over time T, whichever occurs last, and compare replacement overtime first and last. It is of interest that if replacement number N is less than some number N_O, then overtime last is better than overtime first, and vice versa.

As one of modified replacement policies in Sect. 2.5, we consider another overtime replacement in which the unit is replaced at failure K or at the first failure over time T in order to operate continuously. Two overtime replacement first and last policies are considered, and optimum policies which minimize the expected cost rates are discussed [8]. Finally, we take up *preventive maintenance overtime* in which the unit undergoes imperfect preventive maintenance [4, p.171, 6] when it finishes each

T. Nakagawa and X. Zhao, *Maintenance Overtime Policies in Reliability Theory*,
Lecture Notes in Production Engineering, DOI 10.1007/978-3-319-20813-8_2

work and is replaced at the first completion of working cycles over time T. The expected cost rates for two kinds of imperfect preventive maintenances are obtained.

Throughout this chapter, it is assumed that working cycles Y_j are independent and have an identical distribution $G(t) \equiv \Pr\{Y_j \le t\}$ with finite mean $1/\theta \equiv \int_0^\infty \overline{G}(t)\mathrm{d}t$, $G^{(j)}(t)$ $(j = 1, 2, \ldots)$ denotes the j−fold convolution of $G(t)$ with itself, $G^{(0)}(t) \equiv 1$ for $t \ge 0$, and $M(t) \equiv \sum_{j=1}^{\infty} G^{(j)}(t)$. In addition, the unit has a failure distribution $F(t)$ with finite mean μ, a density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$, the failure rate $h(t) \equiv f(t)/\overline{F}(t)$, and the cumulative hazard rate $H(t) \equiv \int_0^t h(u)\mathrm{d}u$, which represents the expected number of failures in $[0, t]$. It is assumed that the failure rate $h(t)$ increases from $h(0) = 0$ to $h(\infty) \equiv \lim_{t\to\infty} h(t)$.

2.1 Periodic and Random Replacements

A new unit begins to operate at time 0 and undergoes minimal repairs at failures, where the time for minimal repair is negligible. Suppose that the unit has to operate for a job with random working cycles Y_j $(j = 1, 2, \ldots)$ defined in Sect. 1.1. As preventive replacement, the unit is planned to be replaced at time T $(0 < T \le \infty)$ or at working cycle N $(N = 1, 2, \ldots)$, whichever occurs first. This is called *periodic replacement first*. Then, the probability that the unit is replaced at cycle N is $G^{(N)}(T)$, and the probability that it is replaced at time T is $1 - G^{(N)}(T)$. Thus, the mean time to replacement is

$$T[1 - G^{(N)}(T)] + \int_0^T t\mathrm{d}G^{(N)}(t) = \int_0^T [1 - G^{(N)}(t)]\mathrm{d}t,$$

and the total expected number of failures until replacement is

$$H(T)[1 - G^{(N)}(T)] + \int_0^T H(t)\mathrm{d}G^{(N)}(t) = \int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t.$$

Therefore, the expected cost rate is

$$C_F(T, N) = \frac{c_T + (c_N - c_T)G^{(N)}(T) + c_M \int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t}{\int_0^T [1 - G^{(N)}(t)]\mathrm{d}t}, \qquad (2.1)$$

where c_T = replacement cost at time T, c_N = replacement cost at cycle N, and c_M = minimal repair cost at each failure.

In particular, when the unit is replaced only at time T, which is called *standard periodic replacement*,

$$C_P(T) \equiv C_F(T, \infty) = \lim_{N\to\infty} C_F(T, N) = \frac{c_T + c_M H(T)}{T}. \qquad (2.2)$$

An optimum policy which minimizes $C_P(T)$ is [1, p.102], [2, p.101]:

(i) If $\int_0^\infty t\mathrm{d}h(t) > c_T/c_M$, then there exists a finite and unique T_P^* $(0 < T_P^* < \infty)$ which satisfies

$$\int_0^T [h(T) - h(t)]\mathrm{d}t = \frac{c_T}{c_M} \quad \text{or} \quad \int_0^T t\mathrm{d}h(t) = \frac{c_T}{c_M}, \tag{2.3}$$

and the resulting cost rate is

$$C_P(T_P^*) = c_M h(T_P^*). \tag{2.4}$$

(ii) If $\int_0^\infty t\mathrm{d}h(t) \le c_T/c_M$, then $T_P^* = \infty$, i.e., the unit always undergoes minimal repair at each failure, and the expected cost rate is

$$C_P(\infty) \equiv \lim_{T\to\infty} C_P(T) = c_M h(\infty). \tag{2.5}$$

When the unit is replaced only at cycle N, which is called *periodic random replacement* [4, p.75],

$$C_R(N) \equiv \lim_{T\to\infty} C_F(T, N) = \frac{c_N + c_M \int_0^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t}{N/\theta} \quad (N = 1, 2, \ldots). \tag{2.6}$$

We find optimum N_R^* which minimizes $C_R(N)$. From the inequality $C_R(N+1) - C_R(N) \ge 0$,

$$\frac{N}{\theta} H_1(N) - \int_0^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t \ge \frac{c_N}{c_M},$$

or

$$\int_0^\infty [1 - G^{(N)}(t)][H_1(N) - h(t)]\mathrm{d}t \ge \frac{c_N}{c_M}, \tag{2.7}$$

where

$$H_1(T, N) \equiv \frac{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]h(t)\mathrm{d}t}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]\mathrm{d}t} \le h(T),$$

$$H_1(N) \equiv \lim_{T\to\infty} H_1(T; N) = \theta \int_0^\infty [G^{(N)}(t) - G^{(N+1)}(t)]h(t)\mathrm{d}t.$$

Thus, if $H_1(N)$ increases strictly to $H_1(\infty)$, then the left-hand side of (2.7) increases strictly with N. Therefore, an optimum policy which minimizes $C_R(N)$ is:

(i) If $H_1(N)$ increases strictly to $H_1(\infty)$ and $\int_0^\infty[H_1(\infty)-h(t)]\mathrm{d}t > c_N/c_M$, then there exists a finite and unique minimum N_R^* $(1 \le N_R^* < \infty)$ which satisfies (2.7), and the resulting cost rate is

$$c_M H_1(N_R^*-1) < C_R(N_R^*) \le c_M H_1(N_R^*). \tag{2.8}$$

(ii) If $\int_0^\infty[H_1(\infty)-h(t)]\mathrm{d}t \le c_N/c_M$, then $N_R^* = \infty$, and the expected cost rate is given in (2.5).

When $G(t) = 1 - \mathrm{e}^{-\theta t}$, i.e., $G^{(N)}(t) = \sum_{j=N}^\infty[(\theta t)^j/j!]\mathrm{e}^{-\theta t}$ $(N = 0, 1, 2, \ldots)$, from Appendix 3.1,

$$H_1(N) = \int_0^\infty \frac{\theta(\theta t)^N}{N!}\mathrm{e}^{-\theta t}h(t)\mathrm{d}t = \sum_{j=0}^N \int_0^\infty \frac{(\theta t)^j}{j!}\mathrm{e}^{-\theta t}\mathrm{d}h(t)$$

increases strictly with N to $h(\infty)$. In this case, if

$$\int_0^\infty t\mathrm{d}h(t) > \frac{c_N}{c_M},$$

then a finite and unique minimum N_R^* $(1 \le N_R^* < \infty)$ exists.

When $F(t) = 1 - \mathrm{e}^{-(t/10)^2}$ and $G(t) = 1 - \mathrm{e}^{-t}$, Table 2.1 presents optimum T_P^* and N_R^*, and their expected cost rates for c_i/c_M $(i = T, N)$. This indicates that optimum T_P^* and N_R^* increase with c_i/c_M $(i = T, N)$, and $T_P^* \ge N_R^*/\theta$, however, they are almost the same, and when $c_T = c_N$, their cost rates are $C_P(T_P^*) < C_R(N_R^*)$. When $c_N < c_T$, e.g., when $c_T = 0.2$ and $c_N = 0.1$, $C_P(T_P^*) > C_R(N_R^*)$.

It was shown numerically that when both replacement costs are the same, periodic replacement is better than random replacement. Next, we discuss theoretically which policy is better to replace the unit at time T or at cycle N. For this purpose, we find optimum T_F^* and N_F^* which minimize the expected cost rate $C_F(T, N)$ in (2.1). Differentiating $C_F(T, N)$ with respect to T and setting it equal to zero,

Table 2.1 Optimum T_P^* and N_R^*, and their expected cost rates when $F(t) = 1 - \mathrm{e}^{-(t/10)^2}$ and $G(t) = 1 - \mathrm{e}^{-t}$

c_i/c_M	T_P^*	$C_P(T_P^*)/c_M$	N_R^*	$C_R(N_R^*)/c_M$
0.1	3.162	0.063	3	0.073
0.2	4.472	0.089	4	0.100
0.5	7.071	0.141	7	0.151
1.0	10.000	0.200	10	0.210
2.0	14.142	0.283	14	0.293
5.0	22.361	0.447	22	0.457

$$c_M \int_0^T [1 - G^{(N)}(t)][h(T) - h(t)]\mathrm{d}t$$
$$- (c_T - c_N) \int_0^T [1 - G^{(N)}(t)][r_N(T) - r_N(t)]\mathrm{d}t = c_T, \tag{2.9}$$

where $r_N(t)$ is given in (1.13). From the inequality $C_F(T, N+1) - C_F(T, N) \geq 0$,

$$c_M \int_0^T [1 - G^{(N)}(t)][H_1(T, N) - h(t)]\mathrm{d}t$$
$$+ (c_T - c_N) \int_0^T [1 - G^{(N)}(t)] \left\{ \frac{G^{(N)}(T) - G^{(N+1)}(T)}{\int_0^T [G^{(N)}(u) - G^{(N+1)}(u)]\mathrm{d}u} + r_N(t) \right\} \mathrm{d}t \geq c_T. \tag{2.10}$$

Substituting (2.9) for (2.10),

$$c_M[H_1(T, N) - h(T)] + (c_T - c_N) \left\{ \frac{G^{(N)}(T) - G^{(N+1)}(T)}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]\mathrm{d}t} + r_N(T) \right\} \geq 0. \tag{2.11}$$

Thus, when $c_T \leq c_N$, there does not exist any finite optimum N_F^* for $T > 0$ because $H_1(T, N) \leq h(T)$, i.e., $N_F^* = \infty$. In this case, the unit should be replaced only at time T.

In particular, when $G(t) = 1 - \mathrm{e}^{-\theta t}$ and $c_T > c_N$,

$$r_N(T) = \frac{\theta(\theta T)^{N-1}/(N-1)!}{\sum_{j=0}^{N-1}[(\theta T)^j/j!]}$$

decreases strictly with N from θ to 0 and increases strictly with T from 0 to θ for $N \geq 2$, $r_1(T) = \theta$ for $T \geq 0$ from Appendix 1.1, and

$$\frac{G^{(N)}(T) - G^{(N+1)}(T)}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)]\mathrm{d}t} = \frac{\theta(\theta T)^N/(N)!}{\sum_{j=N+1}^{\infty}[(\theta T)^j/j!]}$$

increases strictly with N to ∞ and decreases strictly with T from ∞ to 0 from Appendix 1.1. Thus, because $\lim_{N\to\infty} H_1(T, N) = h(T)$, there exists a finite N_F^* $(1 \leq N_F^* < \infty)$ which satisfies (2.11) for $T > 0$. Furthermore, the left-hand side of (2.9) goes to

$$c_M \int_0^\infty [1 - G^{(N)}(t)][h(\infty) - h(t)]\mathrm{d}t - (c_T - c_N)(N-1)$$

as $T \to \infty$. Therefore, if $\int_0^\infty [1-G^{(N)}(t)][h(\infty)-h(t)]\mathrm{d}t > [c_T N - c_N(N-1)]/c_M$, then there exists a finite T_F^* $(0 < T_F^* < \infty)$ which satisfies (2.9). It can be clearly seen that if $h(\infty) = \infty$, both finite T_F^* and N_F^* exist in case of $c_T > c_N$.

On the other hand, suppose that the unit is replaced at time T or at cycle N, whichever occurs last. This is called *periodic replacement last*. Because the probability that the unit is replaced at cycle N is $1 - G^{(N)}(T)$, and the probability that it is replaced at time T is $G^{(N)}(T)$, the mean time to replacement is

$$TG^{(N)}(T) + \int_T^\infty t\mathrm{d}G^{(N)}(t) = T + \int_T^\infty [1 - G^{(N)}(t)]\mathrm{d}t, \tag{2.12}$$

and the expected number of failures until replacement is

$$H(T)G^{(N)}(T) + \int_T^\infty H(t)\mathrm{d}G^{(N)}(t) = H(T) + \int_T^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t. \tag{2.13}$$

Then, by the similar method of obtaining (2.1), the expected cost rate is [4, p.79]

$$C_L(T,N) = \frac{c_N + (c_T - c_N)G^{(N)}(T) + c_M\{H(T) + \int_T^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t\}}{T + \int_T^\infty [1 - G^{(N)}(t)]\mathrm{d}t}. \tag{2.14}$$

Clearly,

$$C_L(0,N) \equiv \lim_{T\to 0} C_L(T,N) = C_F(\infty,N) = C_R(N)$$

in (2.6), and

$$C_L(T,0) \equiv \lim_{N\to 0} C_L(T,N) = C_F(T,\infty) = C_P(T)$$

in (2.2). We could make similar discussions of deriving optimum policies to minimize the expected cost rate $C_L(T,N)$ in (2.14).

2.2 Replacement Overtime

Suppose that the unit is replaced at the first completion of working cycles Y_j $(j = 1, 2, \ldots)$ over time T $(0 \le T \le \infty)$, which has been introduced in Sect. 1.2. Then, the mean time to replacement is

$$\sum_{j=0}^{\infty}\int_0^T\left[\int_{T-t}^{\infty}(t+u)\mathrm{d}G(u)\right]\mathrm{d}G^{(j)}(t)$$
$$=T+\int_T^{\infty}\overline{G}(t)\mathrm{d}t+\int_0^T\left[\int_T^{\infty}\overline{G}(u-t)\mathrm{d}u\right]\mathrm{d}M(t),\tag{2.15}$$

and the expected number of failures until replacement is

$$\sum_{j=0}^{\infty}\int_0^T\left[\int_{T-t}^{\infty}H(t+u)\mathrm{d}G(u)\right]\mathrm{d}G^{(j)}(t)$$
$$=H(T)+\int_T^{\infty}\overline{G}(t)h(t)\mathrm{d}t+\int_0^T\left[\int_T^{\infty}\overline{G}(u-t)h(u)\mathrm{d}u\right]\mathrm{d}M(t),\tag{2.16}$$

which agrees with (2.15) when $H(t)=t$, i.e., $h(t)=1$.

Therefore, the expected cost rate is

$$C_O(T)=\frac{c_O+c_M\{H(T)+\int_T^{\infty}\overline{G}(t)h(t)\mathrm{d}t+\int_0^T[\int_T^{\infty}\overline{G}(u-t)h(u)\mathrm{d}u]\mathrm{d}M(t)\}}{T+\int_T^{\infty}\overline{G}(t)\mathrm{d}t+\int_0^T[\int_T^{\infty}\overline{G}(u-t)\mathrm{d}u]\mathrm{d}M(t)},\tag{2.17}$$

where $c_O=$ replacement cost over time T and c_M is given in (2.1). In particular,

$$C_O(0)\equiv\lim_{T\to 0}C_O(T)=C_R(1)$$

in (2.6) when $c_O=c_N$, and

$$C_O(\infty)\equiv\lim_{T\to\infty}C_O(T)=C_P(\infty)$$

in (2.5). Differentiating $C_O(T)$ with respect to T and setting it equal to zero,

$$\int_0^{\infty}\theta\overline{G}(t)\Bigg(Th(T+t)-H(T)+\int_T^{\infty}\overline{G}(u)[h(T+t)-h(u)]\mathrm{d}u$$
$$+\int_0^T\left\{\int_T^{\infty}\overline{G}(u-x)[h(T+t)-h(u)]\mathrm{d}u\right\}\mathrm{d}M(x)\Bigg)\mathrm{d}t=\frac{c_O}{c_M},\tag{2.18}$$

whose left-hand side increases strictly from 0 to $\int_0^{\infty}t\mathrm{d}h(t)$. Therefore, if $\int_0^{\infty}t\mathrm{d}h(t)>c_O/c_M$, then there exists a finite and unique T_O^* $(0<T_O^*<\infty)$ which satisfies (2.18), and the resulting cost rate is

$$C_O(T_O^*)=c_M\int_0^{\infty}\theta\overline{G}(t)h(t+T_O^*)\mathrm{d}t.\tag{2.19}$$

When $G(t) = 1 - e^{-\theta t}$,

$$C_O(T) = \frac{c_O + c_M[H(T) + \int_0^\infty e^{-\theta t} h(t+T)dt]}{T + 1/\theta}. \tag{2.20}$$

From (2.18), optimum T_O^* satisfies

$$T \int_0^\infty \theta e^{-\theta t} h(t+T)dt - H(T) = \frac{c_O}{c_M}, \tag{2.21}$$

whose left-hand side increases strictly with T from 0 to $\int_0^\infty t dh(t)$, and decreases strictly with θ to $Th(T) - H(T)$.

Therefore, we have the optimum policy:

(i) If $\int_0^\infty t dh(t) > c_O/c_M$, then there exists a finite and unique T_O^* $(0 < T_O^* < \infty)$ which satisfies (2.21), and the resulting cost rate is

$$C_O(T_O^*) = c_M \int_0^\infty \theta e^{-\theta t} h(t + T_O^*)dt = \frac{c_O + c_M H(T_O^*)}{T_O^*}. \tag{2.22}$$

(ii) If $\int_0^\infty t dh(t) \le c_O/c_M$, then $T_O^* = \infty$, and the expected cost rate is given in (2.5).

Note that T_O^* decreases with $1/\theta$ from T_P^* given in (2.3).

When $F(t) = 1 - e^{-(t/10)^2}$ and $G(t) = 1 - e^{-\theta t}$, Table 2.2 presents optimum T_O^* and its expected cost rate. This indicates that optimum T_O^* increases with c_O/c_M and decreases with $1/\theta$ from T_P^*. Compared to Table 2.1 when $1/\theta = 1$, $T_O^* < T_P^* < T_O^* + 1/\theta$ and $C_R(N_R^*) > C_O(T_O^*) > C_P(T_P^*)$, however, their differences are very small as c_O/c_M becomes large. So that, if $c_N < c_O < c_T$ then random replacement might be better than replacement overtime, and replacement overtime might be better than periodic replacement, respectively.

Table 2.2 Optimum T_O^* and its expected cost rate when $F(t) = 1 - e^{-(t/10)^2}$ and $G(t) = 1 - e^{-\theta t}$

c_O/c_M	$1/\theta = 1$		$1/\theta = 2$		$1/\theta = 5$	
	T_O^*	$C_O(T_O^*)/c_M$	T_O^*	$C_O(T_O^*)/c_M$	T_O^*	$C_O(T_O^*)/c_M$
0.1	2.317	0.066	1.742	0.075	0.916	0.118
0.2	3.583	0.091	2.899	0.098	1.709	0.134
0.5	6.141	0.143	5.348	0.147	3.661	0.173
1.0	9.050	0.201	8.198	0.204	6.182	0.224
2.0	13.177	0.284	12.283	0.286	10.002	0.300
5.0	21.383	0.448	20.450	0.449	17.915	0.458

2.3 Comparisons of Periodic and Random Replacements

Compare theoretically replacement overtime to periodic and random replacements with time T $(0 < T \leq \infty)$ and cycle N $(N = 1, 2, \ldots)$ when $c_T = c_N = c_R = c_O$ and $h(\infty) = \infty$. In this case, finite T_P^*, N_R^* and T_O^* always exist. In addition, because T_P^* is an optimum solution of minimizing $C_P(T)$ in (2.2), $C_O(T_O^*)$ is greater than $C_P(T_P^*)$ from (2.22), i.e., periodic replacement is better than replacement overtime. If $c_O < c_T$ then replacement overtime might be rather than periodic replacement. In this case, we could compute numerically $C_P(T_P^*)$ in (2.4) and $C_O(T_O^*)$ in (2.22), and compare them.

We have already compared numerically random replacement and replacement overtime in Tables 2.1 and 2.2. Next, we compare theoretically random replacement and replacement overtime. For this purpose, we propose the following extended replacement with time T and cycle N, which is called *replacement overtime first*: The unit is replaced at cycle N $(N = 1, 2, \ldots)$ or over time T, whichever occurs first. Then, the probability that the unit is replaced at cycle N is $G^{(N)}(T)$, and the probability that it is replaced over time T is $1 - G^{(N)}(T)$, where it is counted as replacement done over time T when the Nth working cycle occurs over time T. Then, the mean time to replacement is

$$\int_0^T t\mathrm{d}G^{(N)}(t) + \sum_{j=0}^{N-1} \int_0^T \left[\int_{T-t}^{\infty} (t+u)\mathrm{d}G(u)\right] \mathrm{d}G^{(j)}(t)$$
$$= \int_0^T [1 - G^{(N)}(t)]\mathrm{d}t + \sum_{j=0}^{N-1} \int_0^T \left[\int_T^{\infty} \overline{G}(u-t)\mathrm{d}u\right] \mathrm{d}G^{(j)}(t), \tag{2.23}$$

and the expected number of failures until replacement is

$$\int_0^T H(t)\mathrm{d}G^{(N)}(t) + \sum_{j=0}^{N-1} \int_0^T \left[\int_{T-t}^{\infty} H(t+u)\mathrm{d}G(u)\right] \mathrm{d}G^{(j)}(t)$$
$$= \int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t + \sum_{j=0}^{N-1} \int_0^T \left[\int_T^{\infty} \overline{G}(u-t)h(u)\mathrm{d}u\right] \mathrm{d}G^{(j)}(t), \tag{2.24}$$

which agrees with (2.23) when $h(t) = 1$.

Therefore, the expected cost rate is

$$C_{OF}(T, N) = \frac{\begin{array}{l} c_O + (c_N - c_O)G^{(N)}(T) + c_M\{\int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t \\ \quad + \sum_{j=0}^{N-1} \int_0^T [\int_T^{\infty} \overline{G}(u-t)h(u)\mathrm{d}u]\mathrm{d}G^{(j)}(t)\} \end{array}}{\int_0^T [1 - G^{(N)}(t)]\mathrm{d}t + \sum_{j=0}^{N-1} \int_0^T [\int_T^{\infty} \overline{G}(u-t)\mathrm{d}u]\mathrm{d}G^{(j)}(t)}. \tag{2.25}$$

It can be clearly seen that $C_{OF}(\infty, N) = C_R(N)$ in (2.6) and $C_{OF}(T, \infty) = C_O(T)$ in (2.17).

When $G(t) = 1 - \mathrm{e}^{-\theta t}$ $(0 < \theta < \infty)$, i.e., $G^{(N)}(t) = \sum_{j=N}^{\infty}[(\theta t)^j/j!]\mathrm{e}^{-\theta t}$, and $c_O = c_N$, (2.25) is rewritten as

$$C_{OF}(T, N) = \frac{\begin{array}{c} c_O + c_M\{\int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t \\ +[1 - G^{(N)}(T)]\int_T^\infty \mathrm{e}^{-\theta(t-T)}h(t)\mathrm{d}t\} \end{array}}{\int_0^T [1 - G^{(N)}(t)]\mathrm{d}t + (1/\theta)[1 - G^{(N)}(T)]}. \tag{2.26}$$

We discuss optimum T_{OF}^* and N_{OF}^* which minimize $C_{OF}(T, N)$. In particular, when $N = 1$, $C_{OF}(T, 1) = C_R(1)$, and hence, $T_{OF}^* = \infty$. For $N \geq 2$, differentiating $C_{OF}(T, N)$ with respect to T and setting it equal to zero,

$$\int_T^\infty \theta \mathrm{e}^{-\theta(t-T)}h(t)\mathrm{d}t \int_0^T [1 - G^{(N)}(t)]\mathrm{d}t - \int_0^T [1 - G^{(N)}(t)]h(t)\mathrm{d}t = \frac{c_O}{c_M},$$

i.e.,

$$\int_0^T [1 - G^{(N)}(t)] \left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u + T) - h(t)]\mathrm{d}u\right\} \mathrm{d}t = \frac{c_O}{c_M}, \tag{2.27}$$

whose left-hand side increases strictly with T. Therefore, if

$$\int_0^\infty [1 - G^{(N)}(t)][h(\infty) - h(t)]\mathrm{d}t > \frac{c_O}{c_M},$$

then there exists a finite and unique T_{OF}^* $(0 < T_{OF}^* < \infty)$ which satisfies (2.27), and the resulting cost rate is

$$C_{OF}(T_{OF}^*, N) = c_M \int_0^\infty \theta \mathrm{e}^{-\theta t}h(t + T_{OF}^*)\mathrm{d}t. \tag{2.28}$$

In addition, because the left-hand side of (2.27) increases with N, T_{OF}^* decreases with N to T_O^* given in (2.21). So that, from (2.28), optimum T_{OF}^* and N_{OF}^* which minimize $C_{OF}(T, N)$ for $N \geq 2$ is $T_{OF}^* = T_O^*$ and $N_{OF}^* = \infty$.

From the above discussions, and from (2.6) and (2.22), if

$$C_O(T_O^*) < C_R(1),$$

i.e.,

$$c_M \int_0^\infty \mathrm{e}^{-\theta t}h(t + T_O^*)\mathrm{d}t < c_O + c_M \int_0^\infty \mathrm{e}^{-\theta t}h(t)\mathrm{d}t,$$

or

$$\int_0^\infty e^{-\theta t}[h(t+T_O^*)-h(t)]dt < \frac{c_O}{c_M},$$

then replacement overtime is better than random replacement.

For example, when $H(t)=(\lambda t)^2$, from (2.21), T_O^* satisfies

$$(\lambda T_O^*)^2 + \frac{2\lambda^2}{\theta}T_O^* = \frac{c_O}{c_M}.$$

Then, from (2.22),

$$\begin{aligned}
&c_O + c_M\int_0^\infty e^{-\theta t}h(t)dt - c_M\int_0^\infty e^{-\theta t}h(t+T_O^*)dt \\
&= c_O + c_M\frac{2\lambda^2}{\theta^2} - c_M\frac{2\lambda^2}{\theta}\left(T_O^*+\frac{1}{\theta}\right) \\
&= c_M\left[(\lambda T_O^*)^2 + \frac{2\lambda T_O^*}{\theta} + \frac{2\lambda^2}{\theta^2} - \frac{2\lambda^2}{\theta}\left(T_O^*+\frac{1}{\theta}\right)\right] > 0,
\end{aligned}$$

which shows that replacement overtime is better than random replacement.

2.4 Replacement Overtime Last

We have already obtained the expected cost rate of replacement overtime first in which the unit is replaced at cycle N before time T in Sect. 2.3. Next, we propose *replacement overtime last* in which the unit is replaced at cycle N or over time T, whichever occurs last. The probability that the unit is replaced at cycle N is $1-G^{(N)}(T)$, and the probability that it is replaced over time T is $G^{(N)}(T)$. Then, the mean time to replacement is

$$\begin{aligned}
&\int_T^\infty t\,dG^{(N)}(t) + \sum_{j=N}^\infty\int_0^T\left[\int_{T-t}^\infty (t+u)dG(u)\right]dG^{(j)}(t) \\
&= T + \int_T^\infty [1-G^{(N)}(t)]dt + \sum_{j=N}^\infty\int_0^T\left[\int_T^\infty \overline{G}(u-t)du\right]dG^{(j)}(t), \qquad (2.29)
\end{aligned}$$

and the expected number of failures until replacement is

$$\int_T^\infty H(t)\mathrm{d}G^{(N)}(t) + \sum_{j=N}^\infty \int_0^T \left[\int_{T-t}^\infty H(t+u)\mathrm{d}G(u)\right]\mathrm{d}G^{(j)}(t)$$
$$= H(T) + \int_T^\infty [1-G^{(N)}(t)]h(t)\mathrm{d}t + \sum_{j=N}^\infty \int_0^T \left[\int_T^\infty \overline{G}(u-t)h(u)\mathrm{d}u\right]\mathrm{d}G^{(j)}(t). \tag{2.30}$$

Therefore, the expected cost rate is

$$C_{OL}(T,N) = \frac{\begin{array}{l} c_O + (c_N - c_O)[1-G^{(N)}(T)] \\ +c_M\{H(T) + \int_T^\infty [1-G^{(N)}(t)]h(t)\mathrm{d}t \\ +\sum_{j=N}^\infty \int_0^T [\int_T^\infty \overline{G}(u-t)h(u)\mathrm{d}u]\mathrm{d}G^{(j)}(t)\}\end{array}}{T + \int_T^\infty [1-G^{(N)}(t)]\mathrm{d}t + \sum_{j=N}^\infty \int_0^T [\int_T^\infty \overline{G}(u-t)\mathrm{d}u]\mathrm{d}G^{(j)}(t)}. \tag{2.31}$$

It can be easily seen that $C_{OL}(0,N) = C_{OF}(\infty,N) = C_R(N)$ in (2.6) and $C_{OL}(T,0) = C_{OF}(T,\infty) = C_O(T)$ in (2.17). Note that when $c_N = c_O$, $C_{OL}(T,0) = C_{OL}(T,1)$, in which the unit is always replaced over time T.

When $G(t) = 1 - \mathrm{e}^{-\theta t}$, i.e., $G^{(N)}(t) = \sum_{j=N}^\infty [(\theta t)^j/j!]\mathrm{e}^{-\theta t}$, and $c_O = c_N$, (2.31) is rewritten as

$$C_{OL}(T,N) = \frac{\begin{array}{l} c_O + c_M\{H(T) + \int_T^\infty [1-G^{(N)}(t)]h(t)\mathrm{d}t \\ +G^{(N)}(T)\int_T^\infty \mathrm{e}^{-\theta(t-T)}h(t)\mathrm{d}t\}\end{array}}{T + \int_T^\infty [1-G^{(N)}(t)]\mathrm{d}t + (1/\theta)G^{(N)}(T)}. \tag{2.32}$$

We discuss optimum T_{OL}^* and N_{OL}^* which minimize $C_{OL}(T,N)$. Differentiating $C_{OL}(T,N)$ with respect to T and setting it equal to zero,

$$\int_0^\infty \theta \mathrm{e}^{-\theta t} h(t+T)\mathrm{d}t \left\{T + \int_T^\infty [1-G^{(N)}(t)]\mathrm{d}t\right\}$$
$$- H(T) - \int_T^\infty [1-G^{(N)}(t)]h(t)\mathrm{d}t = \frac{c_O}{c_M},$$

or

$$\int_0^T \left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u+T) - h(t)]\mathrm{d}u\right\}\mathrm{d}t$$
$$+ \int_T^\infty [1-G^{(N)}(t)]\left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u+T)-h(t)]\mathrm{d}u\right\}\mathrm{d}t = \frac{c_O}{c_M}, \tag{2.33}$$

whose left-hand increases strictly with T to $\int_0^\infty t\mathrm{d}h(t)$. Therefore, if

$$\int_0^\infty t\mathrm{d}h(t) > \frac{c_O}{c_M},$$

then there exists a finite and unique T_{OL}^* $(0 \le T_{OL}^* < \infty)$ which satisfies (2.33), and the resulting cost rate is

$$C_{OL}(T_{OL}^*) = c_M \int_0^\infty \theta \mathrm{e}^{-\theta t} h(t + T_{OL}^*)\mathrm{d}t. \tag{2.34}$$

Note that $C_{OL}(T_{OL}^*)$ agrees with $C_{OF}(T_{OF}^*)$ in (2.28) when $T_{OL}^* = T_{OF}^*$.

Furthermore, we prove that the left-hand side of (2.33) decreases with N as follows:

$$\begin{aligned}
&\int_T^\infty \frac{(\theta t)^N}{N!}\mathrm{e}^{-\theta t}\left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u+T) - h(t)]\mathrm{d}u\right\}\mathrm{d}t \\
&= \int_T^\infty \frac{(\theta t)^N}{N!}\mathrm{e}^{-\theta t}\left\{\int_T^\infty \theta \mathrm{e}^{-\theta(u-T)}[h(u) - h(t)]\mathrm{d}u\right\}\mathrm{d}t \\
&= \int_T^\infty \frac{(\theta t)^N}{N!}\mathrm{e}^{-\theta t}\left\{-\int_T^t \theta \mathrm{e}^{-\theta(u-T)}[h(t) - h(u)]\mathrm{d}u\right. \\
&\quad \left. + \int_t^\infty \theta \mathrm{e}^{-\theta(u-T)}[h(u) - h(t)]\mathrm{d}u\right\}\mathrm{d}t.
\end{aligned} \tag{2.35}$$

Furthermore,

$$\begin{aligned}
&\int_T^\infty \frac{(\theta t)^N}{N!}\mathrm{e}^{-\theta t}\left\{\int_t^\infty \theta \mathrm{e}^{-\theta(u-T)}[h(u) - h(t)]\mathrm{d}u\right\}\mathrm{d}t \\
&= \int_T^\infty \theta \mathrm{e}^{-\theta(t-T)}\left\{\int_T^t \frac{(\theta u)^N}{N!}\mathrm{e}^{-\theta u}[h(t) - h(u)]\mathrm{d}u\right\}\mathrm{d}t.
\end{aligned}$$

Thus, (2.35) becomes

$$\int_T^\infty \left\{\int_T^t [h(t) - h(u)]\theta \mathrm{e}^{-\theta(t+u-T)}\left[\frac{(\theta u)^N}{N!} - \frac{(\theta t)^N}{N!}\right]\mathrm{d}u\right\}\mathrm{d}t \le 0,$$

which follows that the left-hand side of (2.33) decreases with N. This shows that T_{OL}^* increases with N from T_O^* given in (2.21). So that, from (2.34), optimum T_{OL}^* and N_{OL}^* which minimize $C_{OL}(T; N)$ is $T_{OL}^* = T_O^*$ and $N_{OL}^* = 0$ or 1.

Next, we compare the expected cost rates $C_{OF}(T, N)$ in (2.26) and $C_{OL}(T, N)$ in (2.32) for a fixed $N \ge 1$. From the inequality (2.27)–(2.33) ≥ 0,

$$\begin{aligned}
&\int_T^\infty [1 - G^{(N)}(t)]\left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(t) - h(u+T)]\mathrm{d}u\right\}\mathrm{d}t \\
&\ge \int_0^T G^{(N)}(t)\left\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u+T) - h(t)]\mathrm{d}u\right\}\mathrm{d}t.
\end{aligned} \tag{2.36}$$

Noting that from (2.35), its left-hand side increases with N from 0, and conversely, its right-hand side decreases with N to 0. So that, there exists a finite and unique minimum N_O $(1 \le N_O < \infty)$ which satisfies

$$\frac{\int_T^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t + \int_0^T G^{(N)}(t)h(t)\mathrm{d}t}{\int_T^\infty [1 - G^{(N)}(t)]\mathrm{d}t + \int_0^T G^{(N)}(t)\mathrm{d}t} \ge \int_0^\infty \theta \mathrm{e}^{-\theta t} h(t+T)\mathrm{d}t. \quad (2.37)$$

Therefore, if $N \ge N_O$ then the inequality (2.37) holds, and hence, $T_{OL}^* \ge T_{OF}^*$, i.e., replacement overtime first is better than replacement overtime last. Conversely, if $N < N_O$ then $T_{OF}^* > T_{OL}^*$, i.e., replacement overtime last is better than replacement overtime first.

2.5 Replacement Overtime with Number of Failures

The unit is replaced at periodic times in standard periodic replacement as shown in Sect. 2.1. However, some units should be replaced when they have failed rather than a planned time [2, p.104] in order to operate continuously without stopping. This section proposes two policies with the number of failures in which the unit is replaced at a planned number K of failures or over time T, whichever occurs first or last.

It is assumed that the unit undergoes minimal repair between replacements. Then, failures occur at a nonhomogeneous Poisson process with mean value function $H(t)$, i.e., the probability that j failures occur exactly in $[0, t]$ is $p_j(t) \equiv [H(t)^j/j!]\mathrm{e}^{-H(t)}$ $(j = 0, 1, 2, \ldots)$ [5, p.27], and the probability that more than j failures occur in $[0, t]$ is $\sum_{i=j}^\infty p_i(t) = P_j(t)$ and $\overline{P}_j(t) \equiv 1 - P_j(t) = \sum_{i=0}^{j-1} p_i(t)$. Note that $p_0(t) = \mathrm{e}^{-H(t)} = \overline{F}(t) = \overline{P}_1(t)$, $P_j(0) = 0$, $\overline{P}_j(0) = 1$, $P_0(t) = 1$, and $\overline{P}_0(t) = 0$.

2.5.1 Replacement Overtime First with Number of Failures

Suppose that the unit is replaced at failure K $(K = 1, 2, \ldots)$ or at the first failure over time T $(0 \le T \le \infty)$, whichever occurs first, i.e., it is replaced either at failure K before time T or over time T before failure K.

The probability that the unit is replaced at failure K is $\sum_{j=K}^\infty p_j(T) = P_K(T)$, and the probability that it is replaced over time T is $\sum_{j=0}^{K-1} p_j(T) = \overline{P}_K(T)$. Thus, the expected number of failure until replacement is

$$\begin{aligned} K P_K(T) + \sum_{j=0}^{K-1}(j+1)p_j(T) &= K - \sum_{j=0}^{K-1}(K-1-j)p_j(T) \\ &= \int_0^T \overline{P}_K(t)h(t)\mathrm{d}t + \overline{P}_K(T), \end{aligned} \quad (2.38)$$

where note that any failure at replacement is always counted. Because the probability that some failure occurs in $(u, u + \mathrm{d}u]$ for $u > t$, given that a failure have occurred at time t is $f(u)\mathrm{d}u/\overline{F}(t)$ [2, p.96], the mean time to replacement is

$$\begin{aligned}
&\int_0^T t\mathrm{d}P_K(t) + \int_0^T \left[\frac{1}{\overline{F}(t)}\int_T^\infty u\mathrm{d}F(u)\right]\mathrm{d}\overline{P}_K(t) \\
&= \int_0^T \overline{P}_K(t)\mathrm{d}t + \overline{P}_K(T)\int_T^\infty \mathrm{e}^{-[H(t)-H(T)]}\mathrm{d}t,
\end{aligned} \tag{2.39}$$

which agrees with (2.38) when $H(t) = t$, i.e., $h(t) = 1$.

Therefore, the expected cost rate is

$$C_{OF}(T, K) = \frac{c_O + (c_K - c_O)P_K(T) + c_M[\int_0^T \overline{P}_K(t)h(t)\mathrm{d}t + \overline{P}_K(T)]}{\int_0^T \overline{P}_K(t)\mathrm{d}t + \overline{P}_K(T)\int_T^\infty \mathrm{e}^{-[H(t)-H(T)]}\mathrm{d}t}, \tag{2.40}$$

where c_O = replacement cost over time T, c_K = replacement cost at failure K, and c_M = minimal repair cost at each failure. In particular, when $T = \infty$, i.e., the unit is replaced only at failure K, the expected cost rate is, from [2, p.106],

$$C(K) \equiv \lim_{T\to\infty} C_{OF}(T, K) = \frac{c_K + c_M K}{\int_0^\infty \overline{P}_K(t)\mathrm{d}t} \quad (K = 1, 2, \ldots). \tag{2.41}$$

If $h(\infty) > c_K/c_M$ then there exists a finite and unique minimum K^* $(1 \le K^* < \infty)$ which satisfies

$$\frac{1}{\int_0^\infty p_K(t)\mathrm{d}t}\int_0^\infty \overline{P}_K(t)\mathrm{d}t - K \ge \frac{c_K}{c_M}, \tag{2.42}$$

and the resulting cost rate is

$$\frac{c_M}{\int_0^\infty p_{K^*-1}(t)\mathrm{d}t} < C(K^*) \le \frac{c_M}{\int_0^\infty p_{K^*}(t)\mathrm{d}t}. \tag{2.43}$$

On the other hand, when $K = \infty$, i.e., the unit is replaced only at the first failure over time T, the expected cost rate is

$$C_{OF}(T) \equiv \lim_{K\to\infty} C_{OF}(T, K) = \frac{c_O + c_M[H(T) + 1]}{T + 1/Q(T)}, \tag{2.44}$$

where

$$Q(T) \equiv \frac{1}{\int_T^\infty \mathrm{e}^{-[H(t)-H(T)]}\mathrm{d}t} = \frac{\overline{F}(T)}{\int_T^\infty \overline{F}(t)\mathrm{d}t} \ge h(T),$$

which increases strictly with T from $1/\mu$ to $h(\infty)$ [2, p.9] from Appendix 1.2. Clearly, when $T = 0$, i.e., when the unit is always replaced at the first failure,

$$C_{OF}(0) \equiv \lim_{T \to 0} C_{OF}(T) = \frac{c_O + c_M}{\mu} = C(1) \tag{2.45}$$

for $c_O = c_K$, and when $T = \infty$, i.e., there is no replacement to be made,

$$C_{OF}(\infty) \equiv \lim_{T \to \infty} C_{OF}(T) = c_M h(\infty) = C(\infty). \tag{2.46}$$

We find optimum T_O^* which minimizes $C_{OF}(T)$ in (2.44). Differentiating $C_{OF}(T)$ with respect to T and setting it equal to zero,

$$TQ(T) - H(T) = \frac{c_O}{c_M}, \tag{2.47}$$

whose left-hand side increases strictly with T from 0 to

$$\int_0^\infty [h(\infty) - h(t)]\mathrm{d}t = \int_0^\infty t\mathrm{d}h(t).$$

Therefore, we have the following optimum policy:

(i) If $\int_0^\infty t\mathrm{d}h(t) > c_O/c_M$, then there exists a finite and unique T_O^* $(0 < T_O^* < \infty)$ which satisfies (2.47), and the resulting cost rate is

$$C_{OF}(T_O^*) = c_M Q(T_O^*). \tag{2.48}$$

(ii) If $\int_0^\infty t\mathrm{d}h(t) \le c_O/c_M$, then $T_O^* = \infty$, and the expected cost rate is given in (2.46).

It can be easily seen that when $h(\infty) = \infty$, a finite T_O^* $(0 < T_O^* < \infty)$ always exits.

When $F(t) = 1 - \mathrm{e}^{-(t/10)^2}$, Table 2.3 presents optimum K^*, T_O^* and their expected cost rates. This indicates that optimum K^* and T_O^* increase with c_i/c_M $(i = K, O)$ and $C_{OF}(T_O^*) < C(K^*)$. In this case, the mean time to replacement to the K^*th failure is $\mu_{K^*} = 1/\int_0^\infty p_{K^*}(t)\mathrm{d}t = 10\Gamma(K^* + 0.5)/\Gamma(K^*)$, and $\mu_{K^*} > T_O^* > \mu_{K^*-1}$, however, their differences are small as c_i/c_M is large.

Next, we derive optimum T_F^* and K_F^* which minimize $C_{OF}(T, K)$ in (2.40) when $c_O = c_K$ and $h(\infty) = \infty$. Differentiating $C_{OF}(T, K)$ with respect to T and setting it equal to zero,

$$\int_0^T \overline{P}_K(t)[Q(T) - h(t)]\mathrm{d}t = \frac{c_O}{c_M}, \tag{2.49}$$

Table 2.3 Optimum T_O^*, K^* and their expected cost rates for c_i/c_M $(i = K, O)$ when $F(t) = 1 - \mathrm{e}^{-(t/10)^2}$

c_i/c_M	T_O^*	$C_{OF}(T_O^*)/c_M$	K^*	$C(K^*)/c_M$	μ_K^*
1.0	6.936	0.214	1	0.226	8.862
2.0	11.476	0.289	2	0.301	13.293
3.0	14.959	0.350	3	0.361	16.616
4.0	17.862	0.403	4	0.414	19.386
5.0	20.394	0.449	5	0.461	21.809
6.0	22.665	0.491	6	0.507	23.990
7.0	24.738	0.530	7	0.552	25.990
8.0	26.657	0.567	8	0.598	27.846
9.0	28.447	0.601	9	0.648	29.586
10.0	30.123	0.633	10	0.700	31.230

whose left-hand side increases strictly with T from 0 to ∞. Thus, there exists a finite and unique T_F^* $(0 < T_F^* < \infty)$ which satisfies (2.49), and the resulting cost rate is

$$C_{OF}(T_F^*, K) = c_M Q(T_F^*). \tag{2.50}$$

Furthermore, noting that T_F^* decreases with K to T_O^*, optimum policy which minimizes $C_{OF}(T, K)$ is $T_F^* = T_O^*$ given in (2.47) and $K_F^* = \infty$, i.e., the unit should be replaced only over time T_O^*.

On the other hand, suppose that T $(0 \le T < \infty)$ is fixed. From the inequality $C_{OF}(T, K+1) - C_{OF}(T, K) \ge 0$,

$$\begin{aligned} &Q_2(T, K-1)\left[\int_0^T \overline{P}_K(t)\mathrm{d}t + \overline{P}_K(T)\frac{1}{Q(T)}\right] \\ &- \int_0^T \overline{P}_K(t)h(t)\mathrm{d}t - \overline{P}_K(T) \ge \frac{c_O}{c_M}, \end{aligned} \tag{2.51}$$

where

$$Q_2(T, K-1) \equiv \frac{\int_0^T p_{K-1}(t)h(t)\mathrm{d}t}{\int_0^T p_{K-1}(t)[h(t)/Q(t)]\mathrm{d}t} = \frac{\int_0^T p_K(t)h(t)\mathrm{d}t + p_K(T)}{\int_0^T p_K(t)\mathrm{d}t + p_K(T)/Q(T)},$$

which increases strictly with K to $Q(T)$ from Appendix 3.2. Thus, the left-hand of (2.51) increases strictly with K to $TQ(T) - H(T)$, which agrees with that of (2.47). Therefore, if $T > T_O^*$, then there exists a finite and unique minimum K_F^* $(1 \le K_F^* < \infty)$ which satisfies (2.51), and conversely, if $T \le T_O^*$, then $K_F^* = \infty$.

2.5.2 Replacement Overtime Last with Number of Failures

Suppose that the unit is replaced at failure K $(K = 0, 1, 2, \ldots)$ or at the first failure over time T $(0 \le T \le \infty)$, whichever occurs last, i.e., it is replaced either at failure K after time T or over time T after failure K.

The probability that the unit is replaced at failure K after time T is $\sum_{j=0}^{K-1} p_j(T) = \overline{P}_K(T)$, and the probability that it is replaced over time T after failure K is $\sum_{j=K}^{\infty} p_j(T) = P_K(T)$. Thus, the expected number of failures until replacement is

$$\begin{aligned} K\overline{P}_K(T) + \sum_{j=K}^{\infty}(j+1)p_j(T) &= K + \sum_{j=K}^{\infty}(j-K+1)p_j(T) \\ &= H(T) + \int_T^{\infty} \overline{P}_K(t)h(t)\mathrm{d}t + P_K(T) \\ &= H(T) + \int_T^{\infty} \overline{P}_{K-1}(t)h(t)\mathrm{d}t, \end{aligned} \tag{2.52}$$

and the mean time to replacement is

$$\begin{aligned} &\int_T^{\infty} t\mathrm{d}P_K(t) + \sum_{j=K}^{\infty}\int_0^T \left[\frac{1}{\overline{F}(t)}\int_T^{\infty} u\mathrm{d}F(u)\right]\mathrm{d}P_j(t) \\ &= \int_T^{\infty} t\mathrm{d}P_K(t) + \int_T^{\infty} u\mathrm{d}F(u)\sum_{j=K-1}^{\infty}\int_0^T \frac{H(t)^j}{j!}h(t)\mathrm{d}t \\ &= T + \int_T^{\infty}\overline{P}_K(t)\mathrm{d}t + P_K(T)\int_T^{\infty} \mathrm{e}^{-[H(t)-H(T)]}\mathrm{d}t, \end{aligned} \tag{2.53}$$

which agrees with (2.52) when $H(t) = t$.

Therefore, the expected cost rate is

$$C_{OL}(T,K) = \frac{c_O + (c_K - c_O)\overline{P}_K(T) + c_M[H(T) + \int_T^{\infty}\overline{P}_K(t)h(t)\mathrm{d}t + P_K(T)]}{T + \int_T^{\infty}\overline{P}_K(t)\mathrm{d}t + P_K(T)\int_T^{\infty}\mathrm{e}^{-[H(t)-H(T)]}\mathrm{d}t}. \tag{2.54}$$

Clearly, $C_{OL}(T,0) = C_{OL}(T,1)$, $C_{OL}(0,K) = C_{OF}(\infty,K) = C(K)$ in (2.41) for $c_O = c_K$, and $C_{OL}(T,0) = C_{OF}(T,\infty) = C_{OF}(T)$ in (2.44).

We find optimum T_L^* and K_L^* which minimize $C_{OL}(T,K)$ in (2.54), when $c_O = c_K$ and $h(\infty) = \infty$. Differentiating $C_{OL}(T,K)$ with respect to T and setting it equal to zero,

$$Q(T)\left[T + \int_T^{\infty}\overline{P}_K(t)\mathrm{d}t\right] - \left[H(T) + \int_T^{\infty}\overline{P}_K(t)h(t)\mathrm{d}t\right] = \frac{c_O}{c_M}, \tag{2.55}$$

whose left-hand side increases strictly with T from

$$\frac{1}{\mu}\sum_{j=0}^{K-1}\left[\int_0^\infty p_j(t)\mathrm{d}t-\mu\right]<0$$

to ∞. Thus, there exists a finite and unique minimum T_L^* $(0<T_L^*<\infty)$ which satisfies (2.55), and the resulting cost rate is

$$C_{OL}(T_L^*,K)=c_M Q(T_L^*). \tag{2.56}$$

Furthermore, letting $L_1(K,T)$ be the left-hand side of (2.55),

$$L_1(K,T)-L_1(K+1,T)=\int_T^\infty p_K(t)\mathrm{d}t\left[\frac{\int_T^\infty p_K(t)h(t)\mathrm{d}t}{\int_T^\infty p_K(t)\mathrm{d}t}-Q(T)\right]>0,$$

because $\int_T^\infty H(t)^K\mathrm{d}F(t)/\int_T^\infty H(t)^K\overline{F}(t)\mathrm{d}t$ increases strictly with K from $Q(T)$ by similar proof in Appendix 3.3, i.e., $L_1(K,T)$ decrease with K from that of (2.47). Thus, T_L^* increases with K from T_O^*, and optimum policy which minimizes $C_{OL}(T,K)$ is $T_L^*=T_O^*$ given in (2.47) and $K_L^*=0$.

On the other hand, suppose that T $(0\le T<\infty)$ is fixed. From the inequality $C_{OL}(T,K+1)-C_{OL}(T,K)\ge 0$,

$$\begin{aligned}&\widetilde{Q}_2(T,K-1)\left[T+\int_T^\infty \overline{P}_K(t)\mathrm{d}t+\frac{P_K(T)}{Q(T)}\right]\\&-\left[H(T)+\int_T^\infty \overline{P}_K(t)h(t)\mathrm{d}t+P_K(T)\right]\ge\frac{c_O}{c_M},\end{aligned} \tag{2.57}$$

where

$$\widetilde{Q}_2(T,K-1)\equiv\frac{\int_T^\infty p_{K-1}(t)h(t)\mathrm{d}t}{\int_T^\infty p_{K-1}(t)[h(t)/Q(t)]\mathrm{d}t}.$$

The left-hand side of (2.57) increases strictly with K from

$$\widetilde{Q}_2(T,0)\left[T+\frac{1}{Q(T)}\right]-H(T)-1>TQ(T)-H(T)$$

in (2.47), because $\widetilde{Q}_2(T,0)>Q(T)$ from Appendix 3.3. Letting T_O be a solution of

$$\widetilde{Q}_2(T,0)\left[T+\frac{1}{Q(T)}\right]-H(T)-1=\frac{c_O}{c_M},$$

we have $T_O < T_O^*$. Thus, if $T \geq T_O$, then $K_L^* = 0$ or 1, and conversely, if $T < T_O$, then there exists a finite and unique K_L^* $(1 \leq K_L^* < \infty)$ which satisfies (2.57).

From the above interesting results for a fixed T that if $T < T_O$ then we should adopt replacement overtime last, if $T_O \leq T \leq T_O^*$ then we should adopt replacement overtime, and if $T > T_O^*$ then we should adopt replacement overtime first.

2.6 Replacement Overnumber

In this section, we propose two overnumber policies in which the unit is replaced at the first failure over number of cycle N and at the first working cycle over number of failure K.

2.6.1 Replacement Over Number N

The unit is replaced at the first failure over number N $(N = 0, 1, 2 \ldots)$ of working cycles. Then, the mean time to replacement is, when $F(t) = 1 - \mathrm{e}^{-H(t)}$,

$$\begin{aligned}
&\int_0^\infty \left[\frac{1}{\overline{F}(t)} \int_t^\infty u \mathrm{d}F(u)\right] \mathrm{d}G^{(N)}(t) \\
&= \int_0^\infty [1 - G^{(N)}(t)]\mathrm{d}t + \int_0^\infty \left[\frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(u) du\right] \mathrm{d}G^{(N)}(t) \\
&= \mu + \int_0^\infty [1 - G^{(N)}(t)] \frac{h(t)}{Q(t)} \mathrm{d}t. \qquad (2.58)
\end{aligned}$$

The expected number of failures until replacement is

$$1 + \int_0^\infty H(t)\mathrm{d}G^{(N)}(t) = 1 + \int_0^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t, \qquad (2.59)$$

which agrees with (2.58) when $H(t) = t$, i.e., $h(t) = Q(t) = \mu = 1$.

Therefore, the expected cost rate is

$$C_O(N) = \frac{c_N + c_M\{1 + \int_0^\infty [1 - G^{(N)}(t)]h(t)\mathrm{d}t\}}{\mu + \int_0^\infty [1 - G^{(N)}(t)][h(t)/Q(t)]\mathrm{d}t} \quad (N = 0, 1, 2, \ldots). \qquad (2.60)$$

When $N = 0$, i.e., the unit is replaced at the first failure,

$$C_O(0) = \frac{c_N + c_M}{\mu} = C(1), \qquad (2.61)$$

in (2.45) when $c_N = c_O$, and when $N = \infty$, i.e., it always undergoes only minimal repair,

$$C_O(\infty) \equiv \lim_{N\to\infty} C_O(N) = c_M h(\infty), \tag{2.62}$$

in (2.5).

Next, the unit is replaced at failure K $(K = 1, 2, \ldots)$ or at the first failure over number N $(N = 0, 1, 2, \ldots)$, whichever occurs first. The probability that the unit is replaced at failure K is

$$\int_0^\infty [1 - G^{(N)}(t)]\mathrm{d}P_K(t) = \int_0^\infty P_K(t)\mathrm{d}G^{(N)}(t), \tag{2.63}$$

and the probability that it is replaced over number N is

$$\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t). \tag{2.64}$$

The mean time to replacement is

$$\begin{aligned}
&\int_0^\infty t[1 - G^{(N)}(t)]\mathrm{d}P_K(t) + \int_0^\infty \overline{P}_K(t)\left[\frac{1}{\overline{F}(t)}\int_t^\infty u\mathrm{d}F(u)\right]\mathrm{d}G^{(N)}(t) \\
&= \int_0^\infty \overline{P}_K(t)[1 - G^{(N)}(t)]\mathrm{d}t + \int_0^\infty \overline{P}_K(t)\frac{1}{Q(t)}\mathrm{d}G^{(N)}(t) \\
&= \mu + \int_0^\infty [1 - G^{(N)}(t)]\overline{P}_{K-1}(t)\frac{h(t)}{Q(t)}\mathrm{d}t,
\end{aligned} \tag{2.65}$$

and the expected number of failures until replacement is

$$\begin{aligned}
&K\int_0^\infty P_K(t)\mathrm{d}G^{(N)}(t) + \sum_{j=0}^{K-1}(j+1)\int_0^\infty p_j(t)\mathrm{d}G^{(N)}(t) \\
&= K - \sum_{j=0}^{K-1}(K-1-j)\int_0^\infty p_j(t)\mathrm{d}G^{(N)}(t) \\
&= \sum_{j=0}^{K-1}\int_0^\infty P_j(t)\mathrm{d}G^{(N)}(t),
\end{aligned} \tag{2.66}$$

which agrees with (2.65) when $H(t) = t$, i.e., $h(t) = Q(t) = \mu = 1$.

Therefore, the expected cost rate is

$$C_{OF}(N,K) = \frac{c_K - (c_K - c_N)\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t) + c_M\sum_{j=0}^{K-1}\int_0^\infty P_j(t)\mathrm{d}G^{(N)}(t)}{\mu + \int_0^\infty [1-G^{(N)}(t)]\overline{P}_{K-1}(t)[h(t)/Q(t)]\mathrm{d}t}. \tag{2.67}$$

Clearly, $C_{OF}(N,\infty) = C_O(N)$ in (2.60) and $C_{OF}(\infty,K) = C(K)$ in (2.41).

2.6.2 Replacement over Number K

The unit is replaced at the first completion of working cycles over number K ($K = 0, 1, 2, \ldots$) of failures. Then, the mean time to replacement is

$$\sum_{j=0}^{\infty}\int_0^\infty \left\{\int_0^t \left[\int_{t-u}^\infty (u+y)\mathrm{d}G(y)\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t)$$
$$= \int_0^\infty \overline{P}_K(t)\mathrm{d}t + \sum_{j=0}^{\infty}\int_0^\infty \left\{\int_0^t \left[\int_t^\infty \overline{G}(y-u)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t). \tag{2.68}$$

The expected number of failures until replacement is

$$\sum_{j=0}^{\infty}\int_0^\infty \left(\int_0^t \left\{\int_{t-u}^\infty [H(u+y)-H(t)+K]\mathrm{d}G(y)\right\}\mathrm{d}G^{(j)}(u)\right)\mathrm{d}P_K(t)$$
$$= K + \sum_{j=0}^{\infty}\int_0^\infty \left\{\int_0^t \left[\int_t^\infty \overline{G}(y-u)h(y)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t), \tag{2.69}$$

which agrees with (2.68) when $H(t) = t$.

Therefore, the expected cost rate is

$$C_O(K) = \frac{c_K + c_M(K + \sum_{j=0}^\infty \int_0^\infty \{\int_0^t [\int_t^\infty \overline{G}(y-u)h(y)\mathrm{d}y]\mathrm{d}G^{(j)}(u)\}\mathrm{d}P_K(t))}{\int_0^\infty \overline{P}_K(t)\mathrm{d}t + \sum_{j=0}^\infty \int_0^\infty \{\int_0^t [\int_t^\infty \overline{G}(y-u)\mathrm{d}y]\mathrm{d}G^{(j)}(u)\}\mathrm{d}P_K(t)} \quad (K = 0, 1, 2, \ldots). \tag{2.70}$$

When $K = 0$, i.e., the unit is replaced at the first working cycle is

$$C_O(0) = \frac{c_K + c_M \int_0^\infty \overline{G}(t)h(t)\mathrm{d}t}{1/\theta} = C_R(1)$$

in (2.6) when $c_K = c_N$, and when $K = \infty$, the expected cost rate is given in (2.62).

When $G(t) = 1 - \mathrm{e}^{-\theta t}$, the expected cost rate in (2.70) is

$$C_O(K) = \frac{c_K + c_M(K + \int_0^\infty \overline{P}_K(t)\{\int_0^\infty \theta \mathrm{e}^{-\theta u}[h(u+t) - h(t)]\mathrm{d}u\}\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\theta t}h(t)\mathrm{d}t)}{\int_0^\infty \overline{P}_K(t)\mathrm{d}t + 1/\theta}. \tag{2.71}$$

Next, the unit is replaced at cycle N $(N = 1, 2, \ldots)$ before failure K $(K = 0, 1, 2, \ldots)$ or at the first working cycle over number K of failures, whichever occurs first. The probability that the unit is replaced at cycle N is

$$\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t) = \int_0^\infty G^{(N)}(t)\mathrm{d}P_K(t), \tag{2.72}$$

and the probability that it is replaced over number K is

$$\int_0^\infty [1 - G^{(N)}(t)]\mathrm{d}P_K(t) = \int_0^\infty P_K(t)\mathrm{d}G^{(N)}(t). \tag{2.73}$$

The mean time to replacement is

$$\begin{aligned}
&\int_0^\infty t\overline{P}_K(t)\mathrm{d}G^{(N)}(t) \\
&+ \sum_{j=0}^{N-1} \int_0^\infty \left\{\int_0^t \left[\int_{t-u}^\infty (u+y)\mathrm{d}G(y)\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t) \\
&= \int_0^\infty [1 - G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t \\
&\quad + \sum_{j=0}^{N-1} \int_0^\infty \left\{\int_0^t \left[\int_t^\infty \overline{G}(y-u)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t).
\end{aligned} \tag{2.74}$$

The expected number of failures until replacement is

$$\begin{aligned}
&\sum_{j=0}^{N-1} \int_0^\infty \left\{\int_0^t \left[\int_{t-u}^\infty [H(u+y) - H(t) + K]\mathrm{d}G(y)\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t) \\
&+ \sum_{j=0}^{K-1} j \int_0^\infty p_j(t)\mathrm{d}G^{(N)}(t)
\end{aligned}$$

$$
\begin{aligned}
&= K - \sum_{j=0}^{K}(K-j)\int_0^\infty p_j(t)\mathrm{d}G^{(N)}(t)\\
&\quad + \sum_{j=0}^{N-1}\int_0^\infty \left\{\int_0^t \left[\int_t^\infty \overline{G}(y-u)h(y)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t)\\
&= \int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t\\
&\quad + \sum_{j=0}^{N-1}\int_0^\infty \left\{\int_0^t \left[\int_t^\infty \overline{G}(y-u)h(y)\mathrm{d}y\right]\mathrm{d}G^{(j)}(u)\right\}\mathrm{d}P_K(t),
\end{aligned}
\tag{2.75}
$$

which agrees with (2.74) when $h(t) = t$.

Therefore, the expected cost rate is

$$
C_{OF}(K,N) = \frac{\begin{array}{l} c_K - (c_K - c_N)\int_0^\infty \overline{P}_K(t)\mathrm{d}G^{(N)}(t)\\ + c_M(\int_0^\infty [1-G^{(N)}(t)]\overline{P}_K(t)h(t)\mathrm{d}t\\ + \sum_{j=0}^{N-1}\int_0^\infty\{\int_0^t[\int_t^\infty \overline{G}(y-u)h(y)\mathrm{d}y]\mathrm{d}G^{(j)}(u)\}\mathrm{d}P_K(t))\end{array}}{\begin{array}{l}\int_0^\infty[1-G^{(N)}(t)]\overline{P}_K(t)\mathrm{d}t\\ \quad + \sum_{j=0}^{N-1}\int_0^\infty\{\int_0^t[\int_t^\infty \overline{G}(y-u)\mathrm{d}y]\mathrm{d}G^{(j)}(u)\}\mathrm{d}P_K(t)\end{array}}.
\tag{2.76}
$$

Clearly, $C_{OF}(K,\infty) = C_O(K)$ in (2.70), $C_{OF}(\infty, N) = C_R(N)$ in (2.6) when $c_K = c_N$. When $G(t) = 1 - \mathrm{e}^{-\theta t}$,

$$
C_O(K,N) = \frac{\begin{array}{l} c_K - (c_K - c_N)\int_0^\infty[\theta(\theta t)^{N-1}/(N-1)!]\mathrm{e}^{-\theta t}\overline{P}_K(t)\mathrm{d}t\\ + c_M\{\sum_{j=0}^{N-1}\int_0^\infty[(\theta t)^j/j!]\mathrm{e}^{-\theta t}\overline{P}_K(t)h(t)\mathrm{d}t\\ + \sum_{j=0}^{N-1}\int_0^\infty[(\theta t)^j/j!][\int_t^\infty \mathrm{e}^{-\theta u}h(u)\mathrm{d}u]\mathrm{d}P_K(t)\}\end{array}}{\begin{array}{l}\sum_{j=0}^{N-1}\int_0^\infty[(\theta t)^j/j!]\mathrm{e}^{-\theta t}\overline{P}_K(t)\mathrm{d}t\\ + (1/\theta)\sum_{j=0}^{N-1}\int_0^\infty[(\theta t)^j/j!]\mathrm{e}^{-\theta t}\mathrm{d}P_K(t)\end{array}}.
\tag{2.77}
$$

In general, it would be be very difficult to discuss analytically optimum policies to minimize $C_O(K)$ in (2.71) and $C_{OF}(K,N)$ in (2.76), which would be interesting problems for further studies.

2.7 Preventive Maintenance Overtime

When the unit finishes each work of a job, we do some preventive maintenance (PM) which is imperfect [2, p.171], [6]. It is assumed that the PM is done at the completion of successive working cycles Y_j and let b_j denotes the imperfect PM factor after the

jth PM. The failure rate after the first PM becomes $b_1 h(t)$ when it was $h(t)$ before PM, i.e., the unit has the failure rate $B_j h(t)$ during $(j+1)$th working cycle, where $1 \equiv b_0 < b_1 \le b_2 \le \cdots$, $B_j = \prod_{i=0}^{j} b_i$ $(j = 0, 1, \ldots)$ and $1 = B_0 < B_1 < B_2 \cdots$ [2, p.194].

Suppose that the unit is replaced at the first completion of working cycles Y_j $(j = 1, 2, \ldots)$ over time T $(0 \le T < \infty)$ introduced in Sect. 2.2. The mean time to replacement and the expected number of failures before replacement for perfect PM have been derived in (2.15) and (2.16), respectively. Because the unit has the failure rate $B_j h(t)$ during the $(j+1)$th working cycle, the expected number of failures before replacement in (2.16) is rewritten as

$$\sum_{j=0}^{\infty} B_j \left\{ \int_0^T [G^{(j)}(t) - G^{(j+1)}(t)] h(t) \mathrm{d}t + \int_0^T \left[\int_T^{\infty} \overline{G}(u-t) h(u) \mathrm{d}u \right] \mathrm{d}G^{(j)}(t) \right\}. \tag{2.78}$$

Thus, from (2.15), the expected cost rate is

$$C_{OM}(T) = \frac{\begin{array}{c} c_M \sum_{j=0}^{\infty} B_j \{ \int_0^T [G^{(j)}(t) - G^{(j+1)}(t)] h(t) \mathrm{d}t \\ + \int_0^T [\int_T^{\infty} \overline{G}(u-t) h(u) \mathrm{d}u] \mathrm{d}G^{(j)}(t) \} + c_P M(T) + c_O \end{array}}{T + \sum_{j=0}^{\infty} \int_0^T [\int_T^{\infty} \overline{G}(u-t) \mathrm{d}u] \mathrm{d}G^{(j)}(t)}, \tag{2.79}$$

where $M(T) \equiv \sum_{j=1}^{\infty} G^{(j)}(T)$, c_P = PM cost for the completion of each cycle with $c_P \le c_O$, and c_M and c_O are given in (2.17).

In particular, when $G(t) = 1 - \mathrm{e}^{-\theta t}$,

$$C_{OM}(T) = \frac{\begin{array}{c} c_M \sum_{j=0}^{\infty} B_j \{ \int_0^T [(\theta t)^j / j!] \mathrm{e}^{-\theta t} h(t) \mathrm{d}t \\ + [(\theta T)^j / j!] \int_T^{\infty} \mathrm{e}^{-\theta t} h(t) \mathrm{d}t \} + c_P \theta T + c_O \end{array}}{T + 1/\theta}, \tag{2.80}$$

which agrees with (2.20) when $B_j \equiv 1$ and $c_P \equiv 0$.

We find optimum T_B^* which minimizes $C_{OM}(T)$. Differentiating $C_{OM}(T)$ with respect to T and setting it equal to zero,

$$\begin{aligned} &\left(T + \frac{1}{\theta}\right) \int_0^{\infty} \theta \mathrm{e}^{-\theta t} h(t+T) \mathrm{d}t \sum_{j=0}^{\infty} B_{j+1} \frac{(\theta T)^j}{j!} \\ &- \sum_{j=0}^{\infty} B_j \left[\int_0^T \frac{(\theta t)^j}{j!} \mathrm{e}^{-\theta t} h(t) \mathrm{d}t + \frac{(\theta T)^j}{j!} \int_0^{\infty} \mathrm{e}^{-\theta t} h(t+T) \mathrm{d}t \right] = \frac{c_O - c_P}{c_M}. \end{aligned} \tag{2.81}$$

Letting $L(T)$ be the left-hand side of (2.81), it increases strictly with T from

$$L(0) = (B_1 - B_0) \int_0^{\infty} \mathrm{e}^{-\theta t} h(t) \mathrm{d}t$$

to $L(\infty)$. Thus, if $L(\infty) > (c_O - c_P)/c_M$, then there exists a finite and unique T_B^* $(0 \leq T_B^* < \infty)$ which satisfies (2.81), and the resulting cost rate is

$$C_{OM}(T_B^*) = c_M \sum_{j=0}^{\infty} B_{j+1} \int_0^{\infty} \theta \mathrm{e}^{-\theta t} h(T_B^* + t)\mathrm{d}t. \tag{2.82}$$

Next, it is assumed that when the PM is done at the jth working cycle, the age t is reduced to $a_j t$ $(0 < a_j \leq 1)$ where $a_0 \equiv 1$ [2, p.192], i.e., the age becomes $(1 - a_j)t$ units younger after each PM, where $A_j \equiv \prod_{i=0}^{j} a_i$ $(j = 0, 1, 2, \ldots)$ and $1 = A_0 > A_1 > \cdots$. Then, replacing $B_j H(t)$ in (2.79) with $H(A_j t)$, the expected cost rate is

$$\widetilde{C}_{OM}(T) = \frac{\begin{array}{c} c_M \sum_{j=0}^{\infty}\{\int_0^T [G^{(j)}(t) - G^{(j+1)}(t)]\mathrm{d}H(A_j t) \\ + \int_0^T [\int_T^{\infty} \overline{G}(u-t)\mathrm{d}H(A_j u)]\mathrm{d}G^{(j)}(t)\} + c_P M(T) + c_O \end{array}}{T + \sum_{j=0}^{\infty} \int_0^T [\int_T^{\infty} \overline{G}(u-t)\mathrm{d}u]\mathrm{d}G^{(j)}(t)}. \tag{2.83}$$

In particular, when $G(t) = 1 - \mathrm{e}^{-\theta t}$,

$$\widetilde{C}_{OM}(T) = \frac{\begin{array}{c} c_M \sum_{j=0}^{\infty}\{\int_0^T [(\theta t)^j/j!]\mathrm{e}^{-\theta t}\mathrm{d}H(A_j t) \\ +[(\theta T)^j/j!] \int_T^{\infty} \mathrm{e}^{-\theta t}\mathrm{d}H(A_j t)\} + c_P \theta T + c_O \end{array}}{T + 1/\theta}. \tag{2.84}$$

References

1. Barlow RE, Proschan F (1965) Mathematical theory of reliability. Wiley, New York
2. Nakagawa T (2005) Maintenance theory of reliability. Springer, London
3. Tadj L, Ouali MS, Yacout S, Ait-Kadi S (eds) (2011) Replacement models with minimal repairs. Springer, London
4. Nakagawa T (2014) Random maintenance policies. Springer, London
5. Nakagawa T (2011) Stochastic processes with applications to reliability theory. Springer, London
6. Chen M, Nakamura S, Nakagawa T (2010) Replacement and preventive maintenance models with random working times. IEICE Trans Fundam E 93-A:500–507
7. Zhao X, Nakagawa T (2014) Comparisons of periodic and random replacement policies: Frenkel I et al (eds) Applied reliability engineering and risk analysis, probabilistic models and statistical inference, Wiley, New York, pp 193–204
8. Zhao X, Al-Khalifa KN, Hamouda AMS, Nakagawa T (2015) First and last triggering event approaches for replacement with minimal repairs. IEEE Trans Reliab (To Appear in)
9. Zhao X, Mizutani S, Nakagawa T (2015) Which is better for replacement policies with continuous or discrete schedule time ? Eur J Oper Res 242:477–486
10. Zhao X, Nakagawa T, Zuo M (2014) Optimal replacement last with continuous and discrete policies. IEEE Trans Reliab 63:868–880
11. Zhao X, Qian C, Nakamura S (2014) Optimal age and periodic replacement with overtime policies. J Reliab Qual Saf Eng 21, 1450016 (14 pages)

Springer

http://www.springer.com/978-3-319-20812-1

Maintenance Overtime Policies in Reliability Theory
Models with Random Working Cycles
Nakagawa, T.; Zhao, X.
2015, X, 118 p. 2 illus., Hardcover
ISBN: 978-3-319-20812-1