

Chapter 2

Weak Solutions

In this chapter we prove Theorem 1.4.1 on the global existence of weak solutions of problem (VKH) in the space $\mathcal{Y}_{m,0}(T)$, for $m \geq 2$ and given $T > 0$. To the best of our knowledge, uniqueness of weak solutions to problem (VKH) is open, and presumably not to be expected; in contrast, uniqueness does hold in the physically relevant case of the von Karman equations (3) and (4) in $\mathbb{R}^2$, that is, when $m = 1$; we briefly comment of this result, due to Favini et al., [16], in Chap. 6. In addition, it turns out that the cases $m > 2$ and $m = 2$ require a slightly different regularity assumption on the source term φ , as described by the fact that, as per (1.137),

$$S_{m,0}(T) = \begin{cases} C([0, T]; H^{m+2}) & \text{if } m > 2, \\ C([0, T]; H^5) & \text{if } m = 2. \end{cases} \quad (2.1)$$

As remarked just before the statement of Lemma 1.2.4 of the previous chapter, this seems to be due to the restrictions imposed by the limit case of the Sobolev imbeddings, as we can see in (2.29) and (2.31) below; we do not know if the additional regularity of φ required when $m = 2$ is actually necessary. On the other hand, we point out that weak solutions of problem (VKH) are global in time; that is, they are defined on the whole time interval $[0, T]$ on which the source term φ is given. In particular, when $T = +\infty$ and $\varphi(t) \rightarrow 0$ in an appropriate norm as $t \rightarrow +\infty$, one could study the asymptotic stability properties of the corresponding weak solutions of problem (VKH), as done for example by Chuesov and Lasiecka, [9], for various types of initial-boundary value problems for the von Karman equations in $\mathbb{R}^2$.

2.1 Existence of Weak Solutions

In accord with Theorem 1.4.1, we first prove

Theorem 2.1.1 *Let $m \geq 2$, $T > 0$, and assume that $u_0 \in H^m$, $u_1 \in L^2$, and that $\varphi \in S_{m,0}(T)$ [see (2.1)]. There exists $u \in \mathcal{Y}_{m,0}(T)$, which is a weak solution of problem (VKH).*

Proof

1) We construct a weak solution u of problem (VKH) by means of a Galerkin approximation algorithm. Following Lions, [21, Chap. 1, Sect. 4], we consider a total basis $\mathcal{W} = (w_j)_{j \geq 1}$ of H^m , orthonormal with respect to the scalar product induced by the norm (1.16), that is

$$\langle u, v \rangle_m := \langle u, v \rangle + \langle \nabla^m u, \nabla^m v \rangle. \quad (2.2)$$

(For the existence of such a basis, see, e.g., Cherrier and Milani, [8, Chap. 1, Sect. 6].) For each $n \geq 1$ we set $\mathcal{W}_n := \text{span}\{w_1, \dots, w_n\}$, and

$$u_0^n := \sum_{j=1}^n \langle u_0, w_j \rangle_m w_j; \quad (2.3)$$

thus,

$$u_0^n \rightarrow u_0 \quad \text{in } H^m. \quad (2.4)$$

Note that u_0^n is the orthogonal projection, in the sense of (2.2), of u_0 onto $\mathcal{W}_n$. Since H^m is dense in L^2 , the span of $\mathcal{W}$ is dense in L^2 ; thus, there is a strictly increasing sequence $(a_k)_{k \geq 1} \subseteq \mathbb{N}$, as well as a sequence $(\tilde{u}_1^k)_{k \geq 1} \subseteq H^m$, such that, for each $k \geq 1$,

$$\tilde{u}_1^k \in \mathcal{W}_{a_k} \quad \text{and} \quad \|\tilde{u}_1^k - u_1\|_0 \leq \frac{1}{k}. \quad (2.5)$$

For $n \geq 1$ we define

$$u_1^n := \begin{cases} \tilde{u}_1^k & \text{if } a_k \leq n < a_{k+1}, \\ 0 & \text{if } 1 \leq n < a_1. \end{cases} \quad (2.6)$$

Then, $u_1^n \in \mathcal{W}_n$ for each n , and

$$u_1^n \rightarrow u_1 \quad \text{in } L^2. \quad (2.7)$$

We denote by P_n the orthogonal projection, with respect to the scalar product of L^2 , of L^2 onto $\mathcal{W}_n$; that is, for $u \in L^2$, $v = P_n(u) \in \mathcal{W}_n$ is defined as the (unique) solution of the $n \times n$ algebraic system

$$\langle v, w_j \rangle = \langle u, w_j \rangle, \quad j = 1, \dots, n. \quad (2.8)$$

Note that if $\mathcal{H} = (h^n)_{n \geq 1}$ is a total orthonormal basis of L^2 derived from $\mathcal{W}$ by the Gram-Schmidt procedure, then

$$P_n(u) = \sum_{j=1}^n \langle u, h_j \rangle h_j \quad (2.9)$$

for all $u \in L^2$. We can then project Eq. (13) onto $\mathcal{W}_n$; that is, we look for a solution of the form

$$u^n = u^n(t, x) = \sum_{j=1}^n \alpha_{nj}(t) w_j(x) \quad (2.10)$$

to the equation

$$\begin{aligned} u_{tt}^n + \Delta^m u^n &= P_n N(f^n, (u^n)^{(m-1)}) + P_n N(\varphi^{(m-1)}, u^n) \\ &=: P_n (A_n + B_n), \end{aligned} \quad (2.11)$$

where $f^n := f(u^n)$ is defined in analogy to (12), that is by the equation

$$\Delta^m f^n = -M(u^n); \quad (2.12)$$

we remark explicitly that, in general, $f^n(t) \notin \mathcal{W}_n$. We attach to (2.11) the initial conditions

$$u^n(0) = u_0^n, \quad u_t^n(0) = u_1^n, \quad (2.13)$$

with u_0^n and u_1^n defined, respectively, in (2.3) and (2.6). Equation (2.11) is equivalent to the system

$$\begin{cases} \langle u_{tt}^n + \Delta^m u^n, w_j \rangle = \langle A_n + B_n, w_j \rangle \\ j = 1, \dots, n, \end{cases} \quad (2.14)$$

which is in fact a system of second order ODEs in the coefficients $\alpha_n = (\alpha_{n1}, \dots, \alpha_{nn})$ of u^n in its expansion (2.10). We clarify this point by considering

the case $m = 2$ and $\varphi \equiv 0$ for simplicity. By (2.12),

$$\begin{aligned} f^n &= -\Delta^{-2}N\left(\sum_{h=1}^m \alpha_{nh} w_h, \sum_{k=1}^m \alpha_{nk} w_k\right) \\ &= -\sum_{h,k=1}^n \alpha_{nh} \alpha_{nk} \Delta^{-2}N(w_h, w_k); \end{aligned} \quad (2.15)$$

thus, recalling (2.11),

$$\begin{aligned} A_n &= N(f^n, (u^n)^{(m-1)}) \\ &= -\sum_{h,k=1}^n \alpha_{nh} \alpha_{nk} N\left(\Delta^{-2}N(w_h, w_k), \sum_{\ell=1}^n \alpha_{n\ell} w_\ell\right) \\ &= -\sum_{h,k,\ell=1}^n \alpha_{nh} \alpha_{nk} \alpha_{n\ell} \underbrace{N(\Delta^{-2}N(w_h, w_k), w_\ell)}_{=: \Psi_{hkl}}. \end{aligned} \quad (2.16)$$

From this, it follows that (2.14) reads

$$\begin{aligned} &\sum_{k=1}^n (\alpha''_{nk} \langle w_k, w_j \rangle + \alpha_{nk} \langle \Delta w_k, \Delta w_j \rangle) \\ &= -\sum_{h,k,\ell=1}^n \alpha_{nh} \alpha_{nk} \alpha_{n\ell} \langle \Psi_{hkl}, w_j \rangle. \end{aligned} \quad (2.17)$$

Now, recalling the definition (2.2) of the scalar product in H^2 , and that $\mathcal{W}$ is orthonormal in H^2 ,

$$\langle \Delta w_k, \Delta w_j \rangle = \langle w_k, w_j \rangle_2 - \langle w_k, w_j \rangle = \delta_{kj} - \langle w_k, w_j \rangle, \quad (2.18)$$

where δ_{kj} is the Kronecker delta. From this, it follows that (2.17) reads

$$\begin{aligned} &\sum_{k=1}^n (\alpha''_{nk} - \alpha_{nk}) \langle w_k, w_j \rangle + \alpha_{nj} \\ &+ \sum_{h,k,\ell=1}^n \alpha_{nh} \alpha_{nk} \alpha_{n\ell} \langle \Psi_{hkl}, w_j \rangle = 0. \end{aligned} \quad (2.19)$$

Since $\mathcal{W}$ is a linearly independent system, the Gram matrix $\mathcal{G} = [\langle w_j, w_k \rangle]_{j,k=1}^n$ is invertible, and (2.19) has the form

$$\frac{d^2}{dt^2} \alpha_n^* - \alpha_n^* + \mathcal{G}^{-1} ((\alpha_n + B(\alpha_n))^*) = 0, \quad (2.20)$$

where the apex $*$ means transposition, and $B(\alpha_n)$ is the vector whose components are

$$B(\alpha_n)_j := \sum_{h,k,\ell=1}^n \alpha_{nh} \alpha_{nk} \alpha_{n\ell} \langle \Psi_{hk\ell}, w_j \rangle, \quad 1 \leq j \leq n. \quad (2.21)$$

Equation (2.20) is the explicit form of the second order system of ODEs (2.14) when $m = 2$ and $\varphi = 0$. In accord with (2.13), the initial conditions on α_n attached to (2.20) are

$$\alpha_{nj}(0) = \langle u_0^n, w_j \rangle, \quad \alpha'_{nj}(0) = \langle u_1^n, w_j \rangle. \quad (2.22)$$

We now return to the general system (2.14), which can be translated into a system analogous to (2.20) in a similar way. By Carathéodory's theorem, this system admits a local solution $u^n \in C([0, t_n]; \mathcal{W}_n)$, with $u_t^n \in AC([0, t_n]; \mathcal{W}_n)$, for some $t_n \in [0, T]$.

2) We establish an a priori estimate on u^n which allows us to extend each u^n to all of $[0, T]$.

Proposition 2.1.1 *There exists $R_0 \geq 1$, independent of n and t_n , such that for all $t \in [0, t_n]$,*

$$\|u_t^n(t)\|_0^2 + \|u^n(t)\|_m^2 + \frac{1}{m} \|f^n(t)\|_m^2 \leq R_0^2. \quad (2.23)$$

Proof Multiplying (2.14) by α'_{nj} and then summing the resulting identities for $1 \leq j \leq n$, we obtain

$$\frac{d}{dt} (|u_t^n|_2^2 + |\nabla^m u^n|_2^2) = 2\langle A_n + B_n, u_t^n \rangle. \quad (2.24)$$

Recalling (2.12), we compute that

$$\begin{aligned} 2\langle A_n, u_t^n \rangle &= 2\langle N(f^n, (u^n)^{(m-1)}), u_t^n \rangle \\ &= 2\langle N((u^n)^{(m-1)}, u_t^n), f^n \rangle \\ &= \frac{2}{m} \langle \partial_t (M(u^n)), f^n \rangle = \frac{2}{m} \langle -\Delta^m f_t^n, f^n \rangle \\ &= -\frac{2}{m} \langle \nabla^m f^n, \nabla^m f_t^n \rangle = -\frac{1}{m} \frac{d}{dt} |\nabla^m f^n|_2^2. \end{aligned} \quad (2.25)$$

Replacing (2.25) into (2.24) and adding the identity

$$\frac{d}{dt} |u^n|_2^2 = 2\langle u^n, u_t^n \rangle, \quad (2.26)$$

we obtain that

$$\frac{d}{dt} \underbrace{(\|u_t^n\|_0^2 + \|u^n\|_m^2 + \frac{1}{m} \|\nabla^m f^n\|_0^2)}_{=: \Psi(u^n)} = 2 \langle B_n + u^n, u_t^n \rangle. \quad (2.27)$$

To estimate the term with B_n , let first $m \geq 3$. Then, $\nabla^2 \varphi \in H^m \hookrightarrow L^p$ for all $p \in [2, +\infty[$; hence, choosing p such that

$$\frac{m-1}{p} + \frac{1}{m} = \frac{1}{2} \quad (2.28)$$

and recalling that $H^{m-2} \hookrightarrow L^m$, we can proceed with

$$\begin{aligned} |\langle B_n, u_t^n \rangle| &\leq C |\nabla^2 \varphi|_p^{m-1} |\nabla^2 u^n|_m |u_t^n|_2 \\ &\leq C \|\varphi\|_{m+2}^{m-1} \|u^n\|_m |u_t^n|_2 \\ &\leq C_\varphi (\|u_t^n\|_0^2 + \|u^n\|_m^2), \end{aligned} \quad (2.29)$$

where

$$C_\varphi := C \max\{1, \|\varphi\|_{S_{m,0}(T)}^{m-1}\}. \quad (2.30)$$

If instead $m = 2$, $\nabla^2 \varphi \in H^3 \hookrightarrow L^\infty$, so that, again,

$$|\langle B_n, u_t^n \rangle| \leq C |\nabla^2 \varphi|_\infty |\nabla^2 u^n|_2 |u_t^n|_2 \leq C_\varphi (\|u_t^n\|_0^2 + \|u^n\|_2^2). \quad (2.31)$$

Replacing (2.29) or (2.31) into (2.27) yields

$$\frac{d}{dt} \Psi(u^n) \leq 2 C_\varphi \Psi(u^n), \quad (2.32)$$

from which we deduce, via Gronwall's inequality, that for all $t \in [0, t_n]$,

$$\Psi(u^n(t)) \leq \Psi(u^n(0)) e^{2 C_\varphi t}. \quad (2.33)$$

By (2.12) and (1.117) at $t = 0$,

$$\|\nabla^m f^n(0)\|_0 \leq C \|u^n(0)\|_m^m = C \|u_0^n\|_m^m; \quad (2.34)$$

thus, keeping in mind that, by (2.4) and (2.7), the sequences $(u_0^n)_{n \geq 1}$ and $(u_1^n)_{n \geq 1}$ are bounded in, respectively, H^m and L^2 , it follows that there is $D_0 \geq 1$, independent of n and t_n , such that $\Psi(u^n(0)) \leq D_0^2$. Consequently, we deduce from (2.33) that for

all $t \in [0, t_n]$,

$$\Psi(u^n(t)) \leq D_0^2 e^{2C_\varphi t}, \quad (2.35)$$

from which (2.23) follows, with $R_0 = D_0 e^{C_\varphi T}$. $\square$

3) Since R_0 is independent of t_n , the function u^n can be extended to all of $[0, T]$, with estimate (2.23) valid for all $t \in [0, T]$. Since R_0 is also independent of n , the sequences $(u^n)_{n \geq 1}$, $(u_t^n)_{n \geq 1}$, and $(f^n)_{n \geq 1}$ are bounded, respectively, in $C([0, T]; H^m)$, $C([0, T]; L^2)$, and $C([0, T]; \bar{H}^m)$. Consequently, there are functions $u \in L^\infty(0, T; H^m)$ and $f \in L^\infty(0, T; \bar{H}^m)$, with $u_t \in L^\infty(0, T; L^2)$, such that, up to subsequences,¹

$$u^n \rightarrow u \quad \text{in } L^\infty(0, T; H^m) \text{ weak}^*, \quad (2.36)$$

$$u_t^n \rightarrow u_t \quad \text{in } L^\infty(0, T; L^2) \text{ weak}^*, \quad (2.37)$$

$$f^n \rightarrow f \quad \text{in } L^\infty(0, T; \bar{H}^m) \text{ weak}^*. \quad (2.38)$$

In particular, (2.36) and (2.37) imply that $u \in L^2(0, T; H^m)$ and $u_t \in L^2(0, T; L^2)$; thus, by the trace theorem [(1.136) of Proposition 1.4.1], $u \in C([0, T]; L^2)$. But then, by (1.135) of the same proposition it follows that $u \in C_{\text{bw}}([0, T]; H^m)$, and the map $t \mapsto \|u(t)\|_{H^m}$ is bounded. In fact, by (2.23), for all $t \in [0, T]$,

$$\|u(t)\|_m \leq \liminf \|u^n(t)\|_m \leq R_0. \quad (2.39)$$

In addition, using the interpolation inequality

$$\|h\|_{m-\delta} \leq C \|h\|_m^{1-\delta/m} \|h\|_0^{\delta/m}, \quad \delta \in]0, m], \quad (2.40)$$

for $h = u(t) - u(t_0)$, $0 \leq t, t_0 \leq T$, the bound of (2.39), and the fact that $u \in C([0, T]; L^2)$, we deduce that

$$u \in C([0, T]; H^{m-\delta}). \quad (2.41)$$

We proceed then to show that the function u defined in (2.36) is a solution of problem (VKH).

¹Here and in the sequel, by this expression we understand that, for example, there is in fact a subsequence $(u^{n_k})_{k \geq 1}$ of $(u^n)_{n \geq 1}$, such that (2.36) and (2.37) hold with u^n replaced u^{n_k} . When there is no danger of ambiguity, we adopt this convention in order to avoid, later on, to keep a cumbersome track of subsequences of subsequences. Furthermore, we will often not repeat the statement “up to subsequences”.

4) Our first step is to prove that the functions u and f introduced in (2.36) and (2.38) are such that $f = f(u)$; that is, that f solves (12). To this end, we first recall that if $v \in \bar{H}^m$, then, by Lemma 1.2.2, $M(v) \in L^1 \cap \bar{H}^{-m}$, and that, if $w \in \bar{H}^m$, Lemma 1.2.2 yields the estimate

$$|\langle M(v), w \rangle_{\bar{H}^{-m} \times \bar{H}^m}| \leq C \|v\|_{\bar{m}}^m \|w\|_{\bar{m}}. \quad (2.42)$$

If in addition $w \in L^\infty$, so that the function $M(v)w$ is integrable, then

$$\langle M(v), w \rangle_{\bar{H}^{-m} \times \bar{H}^m} = \int M(v) w \, dx = I(v, \dots, v, w). \quad (2.43)$$

With abuse of notation, we shall abbreviate

$$\langle M(v), w \rangle_{\bar{H}^{-m} \times \bar{H}^m} =: \langle M(v), w \rangle, \quad (2.44)$$

even though neither of the terms $M(v)$ and w is in L^2 . In the sequel, for $s \geq 0$ we set, again with some abuse of notation, $H_{\text{loc}}^{-s} := (H_{\text{loc}}^s)'$; more precisely, H_{loc}^{-s} is the dual of the Fréchet space H_{loc}^s , and is not to be confused with the space $(H^{-s})_{\text{loc}}$ of the localized distributions in H^{-s} .

We claim:

Proposition 2.1.2 *Let u^n and u be as in (2.36). Then, up to subsequences,*

$$M(u^n) \rightarrow M(u) \quad \text{in } L^\infty(0, T; \bar{H}^{-m}) \text{ weak}^*. \quad (2.45)$$

Proof From (1.72) of Lemma 1.2.2, with $k = 0$, it follows that the sequence $(M(u^n))_{n \geq 1}$ is bounded in $L^\infty(0, T; \bar{H}^{-m})$, and, by (2.23),

$$\|M(u^n(t))\|_{\bar{H}^{-m}} \leq C \|u^n(t)\|_m^m \leq C R_0^m. \quad (2.46)$$

Thus, up to subsequences, there is $\mu \in L^\infty(0, T; \bar{H}^{-m})$ such that

$$M(u^n) \rightarrow \mu \quad \text{in } L^2(0, T; \bar{H}^{-m}) \text{ weak}^*. \quad (2.47)$$

We now show that

$$M(u^n) \rightarrow M(u) \quad \text{in } L^2(0, T; H_{\text{loc}}^{-m-2}); \quad (2.48)$$

then, comparing (2.48) to (2.47) yields (2.45). To show (2.48), let $\Omega \subset \mathbb{R}^{2m}$ be an arbitrary bounded domain, and $\zeta \in L^2(0, T; H^{m+2})$, with $\text{supp}(\zeta(t, \cdot)) \subset \Omega$ for a.a. $t \in [0, T]$. Let $R_\Omega : u \mapsto u|_\Omega$ denote the corresponding restriction operator. Since R_Ω is linear and continuous from $H^m = H^m(\mathbb{R}^{2m})$ to $H^m(\Omega)$ and from $L^2 = L^2(\mathbb{R}^{2m})$ to $L^2(\Omega)$, and since the inclusion $H^m(\Omega) \hookrightarrow L^2(\Omega)$ is compact, (2.36) and (2.37)

imply, by part (4) of Proposition 1.4.1, that, again up to subsequences,

$$R_{\Omega} u^n \rightarrow R_{\Omega} u \quad \text{in } L^2(0, T; H^{m-\delta}(\Omega)), \quad \delta \in]0, m]. \quad (2.49)$$

As in (1.104) we decompose (omitting the reference to the variable t , as well as to R_{Ω})

$$M(u^n) - M(u) = \sum_{j=1}^m \underbrace{N((u^n)^{(m-j)}, u^{(j-1)}, u^n - u)}_{=: N_j(u^n, u)}. \quad (2.50)$$

Thus, by (1.62), (2.23) and (2.39),

$$\begin{aligned} & \int_0^T |\langle N_j(u^n, u), \zeta \rangle| \, dt \\ & \leq C \int_0^T \|\nabla^2 u^n\|_{L^m(\Omega)}^{m-j} \|\nabla^2 u\|_{L^m(\Omega)}^{j-1} \|\nabla(u^n - u)\|_{L^m(\Omega)} \|\nabla \zeta\|_{\infty} \, dt \\ & \leq C \int_0^T \|u^n\|_m^{m-j} \|u\|_m^{j-1} \|u^n - u\|_{H^{m-1}(\Omega)} \|\nabla \zeta\|_{m+1} \, dt \\ & \leq C R_0^{m-1} \int_0^T \|u^n - u\|_{H^{m-1}(\Omega)} \|\zeta\|_{m+2} \, dt. \end{aligned} \quad (2.51)$$

Hence, by (2.49) with $\delta = 1$,

$$\int_0^T \langle M(u^n) - M(u), \zeta \rangle \, dt \rightarrow 0. \quad (2.52)$$

This allows us to deduce (2.48). $\square$

For future reference, we note that, by (2.37) and (2.23), we also have that for a.a. $t \in [0, T]$,

$$\|R_{\Omega} u_t(t)\|_{L^2(\Omega)} \leq \|u_t(t)\|_0 \leq \liminf \|u_t^n(t)\|_0 \leq R_0, \quad (2.53)$$

as well as, of course,

$$\|R_{\Omega} u_t^n(t)\|_{L^2(\Omega)} \leq \|u_t^n(t)\|_0 \leq R_0; \quad (2.54)$$

thus, the already cited trace theorem allows us to deduce from (2.49) and (2.53) that

$$R_{\Omega} u^n \rightarrow R_{\Omega} u \quad \text{in } C([0, T]; L^2). \quad (2.55)$$

Using the interpolation inequality (2.40) with $\delta = 1$ for $h = R_\Omega(u^n(t) - u(t))$, as well as the bound

$$\|R_\Omega(u^n(t) - u(t))\|_{H^m(\Omega)} \leq \|u^n(t) - u(t)\|_m \leq 2R_0, \quad (2.56)$$

which follows from (2.23) and (2.39), we deduce from (2.40) and (2.55) that

$$R_\Omega u^n \rightarrow R_\Omega u \quad \text{in } C([0, T]; H^{m-1}(\Omega)). \quad (2.57)$$

5) Recalling the definitions (2.12) of f^n and (12) of $f(u)$, we deduce from (2.45) that

$$\Delta^m f^n \rightarrow \Delta^m f(u) \quad \text{in } L^\infty(0, T; \bar{H}^{-m}) \text{ weak}^*. \quad (2.58)$$

On the other hand, (2.38) implies that

$$\Delta^m f^n \rightarrow \Delta^m f \quad \text{in } L^\infty(0, T; \bar{H}^{-m}) \text{ weak}^*; \quad (2.59)$$

comparing (2.58) and (2.59) yields, via (1.45), that $f = f(u)$, as claimed. This is an identity in $L^\infty(0, T; \bar{H}^m)$; however, since $u \in C_{\text{bw}}([0, T]; H^m)$, the map $t \mapsto f(u(t))$ is well-defined and bounded from $[0, T]$ into $\bar{H}^m$. In fact, by (2.38) and (2.23),

$$\|f(t)\|_{\bar{m}} \leq \liminf \|f^n(t)\|_{\bar{m}} \leq \sqrt{m} R_0 \quad (2.60)$$

for all $t \in [0, T]$.

6) We proceed to show that $f \in C_{\text{bw}}([0, T]; \bar{H}^m)$. To this end, we recall that $\Delta^m f \in L^\infty(0, T; \bar{H}^{-m}) \hookrightarrow L^\infty(0, T; H^{-m}) \hookrightarrow L^\infty(0, T; H^{-m-2})$. We first show that, in fact, $\Delta^m f \in C([0, T]; H^{-m-2})$. Similarly to (2.50), we decompose

$$\begin{aligned} M(u(t)) - M(u(t_0)) &= \sum_{j=1}^m \langle N((u(t))^{(m-j)}, (u(t_0))^{(j-1)}, u(t) - u(t_0)) \rangle \\ &=: \sum_{j=1}^m \tilde{N}_j(t, t_0). \end{aligned} \quad (2.61)$$

Fix $\psi \in H^{m+2}$ with $\|\psi\|_{m+2} = 1$, and denote by $\langle \langle \cdot, \cdot \rangle \rangle$ the duality pairing between H^{-m-2} and H^{m+2} . Then, for $0 \leq t, t_0 \leq T$,

$$\langle \langle \Delta^m(f(t) - f(t_0)), \psi \rangle \rangle = - \sum_{j=1}^m \langle \langle \tilde{N}_j(t, t_0), \psi \rangle \rangle. \quad (2.62)$$

Since $\nabla\psi \in H^{m+1} \hookrightarrow L^\infty$, we obtain from (2.62) that

$$\begin{aligned}
 & \left| \langle \tilde{N}_j(t, t_0), \psi \rangle \right| \\
 & \leq C |\nabla^2 u(t)|_m^{m-j} |\nabla^2 u(t_0)|_m^{j-1} |\nabla(u(t) - u(t_0))|_m |\nabla\psi|_\infty \\
 & \leq C R_0^{m-1} |\nabla^{m-1}(u(t) - u(t_0))|_2 \|\nabla\psi\|_{m+1} \\
 & \leq C R_0^{m-1} \|u(t) - u(t_0)\|_{m-1}
 \end{aligned} \tag{2.63}$$

By (2.41), the right side of (2.63) vanishes as $t \rightarrow t_0$; thus, (2.63) implies that $\Delta^m f \in C([0, T]; H^{-m-2})$, as claimed. Since also $\Delta^m f \in L^\infty(0, T; H^{-m})$, by (1.135) it follows that $\Delta^m f \in C_{\text{bw}}([0, T]; H^{-m})$; thus, for each $h \in H^m$, the map

$$t \mapsto \langle \Delta^m f(t), h \rangle_{H^{-m} \times H^m} = \langle \nabla^m f(t), \nabla^m h \rangle_0 = \langle f(t), h \rangle_{\bar{m}} \tag{2.64}$$

is continuous. By the density of H^m into $\bar{H}^m$, it follows that the map $t \mapsto \langle f(t), h \rangle_{\bar{m}}$ is also continuous for each $h \in \bar{H}^m$. To see this, given $h \in \bar{H}^m$, let $(h_n)_{n \geq 1} \subset H^m$ be such that $h_n \rightarrow h$ in $\bar{H}^m$. Then, for $0 \leq t, t_0 \leq T$,

$$\begin{aligned}
 \langle f(t) - f(t_0), h \rangle_{\bar{m}} &= \langle f(t) - f(t_0), h - h_n \rangle_{\bar{m}} \\
 &+ \langle f(t) - f(t_0), h_n \rangle_{\bar{m}} =: A_n(t, t_0) + B_n(t, t_0).
 \end{aligned} \tag{2.65}$$

Let $\varepsilon > 0$. By (2.60), there is $n_0 \geq 1$ such that

$$|A_n(t, t_0)| \leq 2\sqrt{m}R_0 \|h - h_n\|_{\bar{m}} \leq \varepsilon. \tag{2.66}$$

Fix $n = n_0$. By (2.64), there is $\delta > 0$ such that

$$|B_{n_0}(t, t_0)| \leq \varepsilon \tag{2.67}$$

if $|t - t_0| \leq \delta$. Replacing (2.66) and (2.67) into (2.65) shows the asserted continuity of the map $t \mapsto \langle f(t), h \rangle_{\bar{m}}$; hence, $f \in C_{\text{bw}}([0, T]; \bar{H}^m)$, as claimed.

7) We now set

$$F(u) := N(f(u), u^{(m-1)}), \tag{2.68}$$

and prove

Proposition 2.1.3 *Let u and f be as in (2.36) and (2.38). Then, up to subsequences,*

$$F(u^n) \rightarrow F(u) \quad \text{in } L^\infty(0, T; \bar{H}^{-m}) \text{ weak}^*. \tag{2.69}$$

Proof As in (2.46), the sequence $(N(f^n, (u^n)^{(m-1)}))_{n \geq 1}$ is bounded in $L^\infty(0, T; \bar{H}^{-m})$, with

$$\|N(f^n, (u^n)^{(m-1)})\|_{\overline{m}} \leq C \|f^n\|_{\overline{m}} \|u^n\|_m^{m-1} \leq C R_0^m, \quad (2.70)$$

as follows from (2.60) and (2.39). Thus, up to subsequences, there is $v \in L^\infty(0, T; \bar{H}^{-m})$ such that

$$N(f^n, (u^n)^{(m-1)}) \rightarrow v \quad \text{in } L^\infty(0, T; \bar{H}^{-m}) \text{ weak}^*. \quad (2.71)$$

On the other hand, we also have that

$$F(u^n) \rightarrow F(u) \quad \text{in } L^2(0, T; \bar{H}_{\text{loc}}^{-m-2}). \quad (2.72)$$

Indeed, with the same Ω and ζ as in the proof of Proposition 2.1.2, we can decompose, as in (2.50),

$$\begin{aligned} & \int_0^T |\langle F(u) - F(u^n), \zeta \rangle| dt \\ & \leq \int_0^T |\langle N(f - f^n, u^{(m-1)}), \zeta \rangle| dt \\ & \quad + \sum_{j=2}^m \int_0^T |\langle N(f^n, u^{(m-j)}, (u^n)^{(j-2)}, u - u^n), \zeta \rangle| dt \\ & =: W_1^n + \sum_{j=2}^m W_j^n. \end{aligned} \quad (2.73)$$

At first,

$$W_1^n = \int_0^T |\langle N(\zeta, u^{(m-1)}), f - f^n \rangle| dt \rightarrow 0 \quad (2.74)$$

by (2.38). As for the other terms W_j^n , acting as in (2.51) of the proof of Proposition 2.1.2 and recalling (2.60) and (2.39), we estimate

$$\begin{aligned} |W_j^n| & \leq C \int_0^T \|f^n\|_{\overline{m}} \|u\|_m^{m-j} \|u^n\|_m^{j-2} \|u - u^n\|_{H^{m-1}(\Omega)} |\nabla \zeta|_\infty dt \\ & \leq C R_0^{m-1} \int_0^T \|u - u^n\|_{H^{m-1}(\Omega)} \|\zeta\|_{m+2} dt. \end{aligned} \quad (2.75)$$

Thus, by (2.49), also $W_j^n \rightarrow 0$, and (2.72) follows. Comparison of (2.71) and (2.72) yields (2.69). $\square$

8) We now consider the equation of (2.14) for fixed j and $n \geq j$, multiply it by an arbitrary $\psi \in C_0^1([0, T])$, and integrate by parts, to obtain

$$\begin{aligned} & \int_0^T (\langle -u_t^n, \psi' w_j \rangle + \langle \nabla^m u^n, \psi \nabla^m w_j \rangle) dt \\ &= \int_0^T \langle A_n + B_n, \psi w_j \rangle dt. \end{aligned} \quad (2.76)$$

Letting then $n \rightarrow \infty$ (along the last of the sub-subsequences determined in all the previous steps), by (2.37), (2.36) and (2.69) we deduce that

$$\begin{aligned} & \int_0^T (\langle -u_t, \psi' w_j \rangle + \langle \nabla^m u, \psi \nabla^m w_j \rangle) dt \\ &= \int_0^T \langle N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u), \psi w_j \rangle dt. \end{aligned} \quad (2.77)$$

Since $\mathcal{W}$ is a total basis in H^m , we can replace w_j in (2.77) by an arbitrary $w \in H^m$. Recalling that $N(f, u^{(m-1)})$ and $N(\varphi^{(m-1)}, u) \in H^{-m} \hookrightarrow H^{-m}$, by Fubini's theorem we can rewrite the resulting identities as

$$\langle B, w \rangle_{H^{-m} \times H^m} = 0, \quad (2.78)$$

where $B \in H^{-m}$ is defined as the Bochner integral

$$B := \int_0^T (-\psi' u_t + \psi(\Delta^m u - N(f, u^{(m-1)}) - N(\varphi^{(m-1)}, u))) dt. \quad (2.79)$$

The arbitrariness of $w \in H^m$ in (2.78) implies that $B = 0$ in H^{-m} ; in turn, this means that the identity

$$u_{tt} = -\Delta^m u + N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u) =: \Lambda \quad (2.80)$$

holds in $\mathcal{D}'([0, T]; H^{-m}) = \mathcal{L}(\mathcal{D}([0, T]); H^{-m})$. Now, (2.36) and (2.38) imply that $\Lambda \in L^\infty(0, T; H^{-m})$; in fact, since $u \in C_{\text{bw}}([0, T]; H^m)$ and $f \in C_{\text{bw}}([0, T]; \tilde{H}^m)$, (1.75) implies that the map $t \mapsto \Lambda(t) \in H^{-m}$ is well-defined and bounded on $[0, T]$. Thus, Eq. (13) holds in H^{-m} for all $t \in [0, T]$, as desired. In addition, by the trace theorem, $u_t \in C([0, T]; H^{-m})$. Arguing then as we did for u , we conclude that $u_t \in C_{\text{bw}}([0, T]; L^2)$, and that the map $t \mapsto \|u_t(t)\|_0$ is bounded. In fact, as in (2.53),

$$\|u_t(t)\|_0 \leq \liminf \|u_t^n(t)\|_0 \leq R_0, \quad (2.81)$$

and this bound is now valid for all $t \in [0, T]$.

9) To conclude the proof of Theorem 2.1.1, we still need to show that u takes on the correct initial values (5). By (2.57), $u^n(0) \rightarrow u(0)$ in H_{loc}^{m-1} . On the other hand, (2.4) implies that $u^n(0) = u_0^n \rightarrow u_0$ in H^m ; thus, $u(0) = u_0$. Next, we proceed as in part (8) of this proof, but now take $\psi \in \mathcal{D}(\cdot - T, T]$ with $\psi(0) = 1$, so that, by (2.7), the identity corresponding to (2.78) reads

$$\langle B, w \rangle_{H^{-m} \times H^m} = \langle u_1, w \rangle. \quad (2.82)$$

On the other hand, multiplying Eq. (13) by ψw and integrating by parts we obtain that

$$\langle B, w \rangle_{H^{-m} \times H^m} = \langle u_t(0), w \rangle. \quad (2.83)$$

Comparing this with (2.82) we conclude that $u_t(0) = u_1$, as desired. This ends the proof of Theorem 2.1.1. $\square$

2.2 Continuity at $t = 0$

We now prove the second claim of Theorem 1.4.1; that is,

Theorem 2.2.1 *Let $m \geq 2$, $T > 0$, and $u \in \mathcal{Y}_{m,0}(T)$ be one of the weak solution of problem (VKH), corresponding to data $u_0 \in H^m$, $u_1 \in L^2$, and $\varphi \in S_{m,0}(T)$, obtained by means of Theorem 2.1.1. Then, u , u_t and f are continuous at $t = 0$, in the sense that*

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_m = 0, \quad \lim_{t \rightarrow 0} \|u_t(t) - u_1\|_0 = 0, \quad (2.84)$$

and

$$\lim_{t \rightarrow 0} \|\nabla^m(f(t) - f(0))\|_0 = 0. \quad (2.85)$$

Proof

1) We recall from (2.41) that $u \in C([0, T]; L^2)$; thus, we can replace claim (2.84) by

$$\lim_{t \rightarrow 0} \|\nabla^m(u(t) - u_0)\|_0 = 0, \quad \lim_{t \rightarrow 0} \|u_t(t) - u_1\|_0 = 0. \quad (2.86)$$

Next, we note that since u , u_t and f are weakly continuous from $[0, T]$ into H^m , L^2 and $\tilde{H}^m$ respectively, in order to prove (2.86) and (2.85) it is sufficient to show that the function

$$t \mapsto \Phi(u(t)) := \|u_t(t)\|_0^2 + \|\nabla^m u(t)\|_0^2 + \frac{1}{m} \|\nabla^m f(t)\|_0^2 \quad (2.87)$$

satisfies the inequality

$$\Phi(u(t)) \leq \Phi(u(0)) + 2 \int_0^t \langle N(\varphi^{(m-1)}, u), u_t \rangle d\theta =: G(t) \quad (2.88)$$

for all $t \in [0, T]$. Indeed, the weak continuity of u , u_t and f with respect to t implies that

$$\Phi(u(0)) \leq \liminf_{t \rightarrow 0^+} \Phi(u(t)) . \quad (2.89)$$

On the other hand, from (2.88) it would follow that

$$\limsup_{t \rightarrow 0^+} \Phi(u(t)) \leq \limsup_{t \rightarrow 0^+} G(t) = \lim_{t \rightarrow 0^+} G(t) = \Phi(u(0)) , \quad (2.90)$$

which, together with (2.89), implies that

$$\lim_{t \rightarrow 0^+} \Phi(u(t)) = \Phi(u(0)) , \quad (2.91)$$

and (2.86), (2.85) would follow.

2) Our first step towards establishing (2.88) is to integrate (2.27), which yields that for all $t \in [0, T]$,

$$\Phi(u^n(t)) = \Phi(u^n(0)) + 2 \int_0^t \langle N(\varphi^{(m-1)}, u^n), u_t^n \rangle d\theta ; \quad (2.92)$$

that is, (2.88) is satisfied, as an equality, by each of the Galerkin approximants of u . Next, we note that (2.4) implies that

$$\|\nabla^m(f^n(0) - f(0))\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty : \quad (2.93)$$

indeed, this is a consequence of the fact that

$$\|\Delta^m(f^n(0) - f(0))\|_{\tilde{H}^{-m}} = \|M(u_0^n) - M(u_0)\|_{\tilde{H}^{-m}} \rightarrow 0 , \quad (2.94)$$

which in turn follows from the estimate (compare to (2.50) of Proposition 2.1.2)

$$\begin{aligned} & \|M(u_0^n) - M(u_0)\|_{\tilde{H}^{-m}} \\ & \leq C \sum_{j=1}^m \|N((u_0^n)^{(m-j)}, u_0^{(j-1)}, u_0^n - u_0)\|_{\tilde{H}^{-m}} \\ & \leq C \sum_{j=1}^m |\nabla^m u_0^n|_2^{m-j} |\nabla^m u_0|_2^{j-1} |\nabla^m(u_0^n - u_0)|_2 , \end{aligned} \quad (2.95)$$

via (2.4). Since (2.4), (2.7) and (2.93) imply that

$$\Phi(u^n(0)) \rightarrow \Phi(u(0)) \quad \text{as } n \rightarrow \infty, \quad (2.96)$$

and since for each $t \in [0, T]$,

$$\Phi(u(t)) \leq \liminf_{n \rightarrow \infty} \Phi(u^n(t)), \quad (2.97)$$

in order to prove (2.88) it is sufficient to show that, for all $t \in [0, T]$,

$$\lim_{n \rightarrow \infty} J_n(t) = 0, \quad (2.98)$$

where, for $n \geq 1$,

$$J_n(t) := \int_0^t (\langle N(\varphi^{(m-1)}, u^n), u_t^n \rangle - \langle N(\varphi^{(m-1)}, u), u_t \rangle) d\theta. \quad (2.99)$$

We shall prove (2.98) separately for $m \geq 3$ and $m = 2$.

3) We consider first the case $m \geq 3$. We recall the following density result, a proof of which is reported for convenience at the end of this section.

Lemma 2.2.1 *Let $T > 0$ and $r \in \mathbb{R}_{\geq 0}$. The space $\mathcal{D}([-T, 2T[\times \mathbb{R}^N)$ is dense in $C_0([-T, 2T[; H^r)$.*

Assuming this, we extend the source term φ to a function, still denoted φ , such that $\varphi \in C_0([-T, 2T[; H^{m+2})$. Given any $\eta > 0$, by Lemma 2.2.1 we determine $\tilde{\varphi} \in \mathcal{D}([-T, 2T[\times \mathbb{R}^{2m})$ such that

$$\max_{-T \leq t \leq 2T} \|\varphi(t) - \tilde{\varphi}(t)\|_{m+2} \leq \eta, \quad (2.100)$$

and rewrite

$$\begin{aligned} J_n(t) &= \int_0^t \langle N(\varphi^{(m-2)}, \varphi - \tilde{\varphi}, u^n), u_t^n \rangle d\theta \\ &\quad + \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u^n), u_t^n \rangle d\theta \\ &\quad - \int_0^t \langle N(\varphi^{(m-2)}, \varphi - \tilde{\varphi}, u), u_t \rangle d\theta \\ &\quad - \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u), u_t \rangle d\theta \\ &=: \Gamma_n^1(t) + \Gamma_n^2(t) - \Gamma^1(t) - \Gamma^2(t). \end{aligned} \quad (2.101)$$

Acting as in (2.29), with $p = \frac{2m(m-1)}{m-2}$ as in (2.28), and using (2.100), we estimate

$$\begin{aligned}
 |\Gamma_n^1(t)| &\leq C \int_0^t |\nabla^2 \varphi|_p^{m-2} |\nabla^2(\varphi - \tilde{\varphi})|_p |\nabla^2 u^n|_m |u_t^n|_2 d\theta \\
 &\leq C \int_0^t \|\varphi\|_{m+2}^{m-2} \|\varphi - \tilde{\varphi}\|_{m+2} \|u^n\|_m \|u_t^n\|_0 d\theta \\
 &\leq C_\varphi \eta R_0^2 T =: C_1 \eta .
 \end{aligned} \tag{2.102}$$

The same exact estimate holds for $\Gamma^1(t)$; thus, from (2.101) we obtain that

$$|J_n(t)| \leq 2 C_1 \eta + |\Gamma_n^2(t) - \Gamma^2(t)| . \tag{2.103}$$

We further decompose

$$\begin{aligned}
 \Gamma_n^2(t) - \Gamma^2(t) &= \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u^n - u), u_t^n \rangle d\theta \\
 &\quad + \int_0^t \langle N(\varphi^{(m-2)}, \tilde{\varphi}, u), u_t^n - u_t \rangle d\theta \\
 &=: \Gamma_n^3(t) + \Gamma_n^4(t) .
 \end{aligned} \tag{2.104}$$

Next, we note that $N(\varphi^{(m-2)}, \tilde{\varphi}, u) \in L^2(0, t; L^2)$ for each $t \in]0, T]$; thus, (2.37) implies that

$$\Gamma_n^4(t) \rightarrow 0 . \tag{2.105}$$

Finally, let $\Omega \subset \mathbb{R}^{2m}$ be a domain such that $\text{supp}(\tilde{\varphi}) \subset]-T, 2T[\times \Omega$. Then, identifying functions with their restriction on Ω ,

$$\Gamma_n^3(t) = \int_0^t \int_\Omega N(\varphi^{(m-2)}, \tilde{\varphi}, u^n - u) u_t^n dx dt . \tag{2.106}$$

By (2.49), $u^n \rightarrow u$ in $L^2(0, T; H^{m-\delta}(\Omega))$; in addition, $H^{m-\delta-2}(\Omega) \hookrightarrow L^q(\Omega)$ for $\delta \in]0, 1[$ and $q = \frac{2m}{\delta+2}$. Thus, taking r such that

$$\frac{m-2}{r} + \frac{\delta+2}{2m} = \frac{1}{2} \tag{2.107}$$

(compare to (2.28); note that $r > 2$), we estimate

$$\begin{aligned}
 |\Gamma_n^3(t)| &\leq C \int_0^t |\nabla^2 \varphi|_r^{m-2} |\nabla^2 \tilde{\varphi}|_\infty |\nabla^2(u^n - u)|_{L^q(\Omega)} |u_t^n|_2 \, d\theta \\
 &\leq C \int_0^t \|\varphi\|_{m+2}^{m-2} \|\tilde{\varphi}\|_{m+3} \|u^n - u\|_{H^{m-\delta}(\Omega)} \|u_t^n\|_0 \, d\theta \\
 &\leq C_\varphi C_{\tilde{\varphi}} \left(\int_0^T \|u^n - u\|_{H^{m-\delta}(\Omega)}^2 \, dt \right)^{1/2} \left(\int_0^T \|u_t^n\|_0^2 \, dt \right)^{1/2} \\
 &\leq C_\varphi C_{\tilde{\varphi}} \|u^n - u\|_{L^2(0,T;H^{m-\delta}(\Omega))} R_0 \sqrt{T} .
 \end{aligned} \tag{2.108}$$

From this it follows that

$$\Gamma_n^3(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty . \tag{2.109}$$

Inserting (2.105) and (2.109) into (2.104), we conclude that

$$\Gamma_n^2(t) - \Gamma^2(t) \rightarrow 0 . \tag{2.110}$$

Together with (2.103), this implies (2.98), when $m \geq 3$ (under the stipulation that Lemma 2.2.1 holds).

4) We now consider the case $m = 2$, in which case (2.98) reads

$$\lim_{n \rightarrow \infty} \int_0^t \langle N(\varphi, u^n), u_t^n \rangle \, d\theta = \int_0^t \langle N(\varphi, u), u_t \rangle \, d\theta . \tag{2.111}$$

We recall that when $m = 2$ we assume that $\varphi \in C([0, T]; H^5)$. Proceeding as we did after the statement of Lemma 2.2.1, given $\eta > 0$ we choose $\tilde{\varphi} \in \mathcal{D}([-T, 2T] \times \mathbb{R}^4)$ such that, as in (2.100),

$$\max_{-T \leq t \leq 2T} \|\varphi(t) - \tilde{\varphi}(t)\|_5 \leq \eta , \tag{2.112}$$

and decompose

$$\begin{aligned}
 &\int_0^t \langle N(\varphi, u^n), u_t^n \rangle \, d\theta \\
 &= \int_0^t \langle N(\varphi - \tilde{\varphi}, u^n), u_t^n \rangle \, d\theta + \int_0^t \langle N(\tilde{\varphi}, u^n), u_t^n \rangle \, d\theta \\
 &=: \Gamma_n^5(t) + \Gamma_n^6(t) .
 \end{aligned} \tag{2.113}$$

As in (2.102),

$$\begin{aligned}
 |\Gamma_n^5(t)| &\leq C \int_0^t |\nabla^2(\varphi - \tilde{\varphi})|_\infty |\nabla^2 u^n|_2 |u_t^n|_2 d\theta \\
 &\leq C \int_0^t \|\varphi - \tilde{\varphi}\|_5 \|u^n\|_2 \|u_t^n\|_0 d\theta \\
 &\leq C \eta R_0^2 T .
 \end{aligned} \tag{2.114}$$

Next, recalling (2.12) and (1.25),

$$\begin{aligned}
 \Gamma_n^6(t) &= \int_0^t \langle N(u^n, u_t^n), \tilde{\varphi} \rangle d\theta = \frac{1}{2} \int_0^t \langle \partial_t M(u^n), \tilde{\varphi} \rangle d\theta \\
 &= -\frac{1}{2} \int_0^t \langle \Delta^2 f_t^n, \tilde{\varphi} \rangle d\theta = -\frac{1}{2} \int_0^t \langle \Delta^2 f_t^n, \tilde{\varphi} \rangle_{(5)} d\theta ,
 \end{aligned} \tag{2.115}$$

where for $r \in \mathbb{N}_{\geq 1}$, we denote by $\langle \cdot, \cdot \rangle_{(r)}$ the duality pairing between H^{-r} and H^r . Assume for the moment the validity of the following

Lemma 2.2.2 *The distributional derivative $\Delta^2 f_t$ is in $L^\infty(0, T; H^{-5})$, with*

$$\int_0^T \langle \Delta^2 f_t, \zeta \rangle_{(5)} dt = -2 \int_0^T \langle N(\zeta, u), u_t \rangle dt \tag{2.116}$$

for all $\zeta \in L^1(0, T; H^5)$; in addition,

$$\Delta^2 f_t^n \rightarrow \Delta^2 f_t \quad \text{in } L^\infty(0, T; H^{-5}) \text{ weak}^* . \tag{2.117}$$

Then, we deduce from (2.115) and (2.116) that, as $n \rightarrow \infty$,

$$\Gamma_n^6(t) \rightarrow -\frac{1}{2} \int_0^t \langle \Delta^2 f_t, \tilde{\varphi} \rangle_{(5)} d\theta = \int_0^t \langle N(\tilde{\varphi}, u), u_t \rangle d\theta . \tag{2.118}$$

Together with (2.114), (2.118) implies (2.111); thus, (2.98) holds also for $m = 2$. As seen in the first part of this proof, this is sufficient to conclude the proof of inequality (2.88).

5) We now prove Lemma 2.2.2. To this end, we recall that $f \in C_{\text{bw}}([0, T]; \bar{H}^2)$; hence, by (1.45) of Proposition 1.1.2, $\Delta^2 f \in L^\infty(0, T; \bar{H}^{-2}) \hookrightarrow L^\infty(0, T; H^{-2})$; hence, we can define $\Delta^2 f_t \in \mathcal{D}'([0, T]; H^{-2}) = \mathcal{L}(\mathcal{D}([0, T]; H^{-2}))$ as usual, by

$$\langle \Delta^2 f_t[\psi], w \rangle_{(2)} := \langle -\Delta^2 f[\psi'], w \rangle_{(2)} , \tag{2.119}$$

where $w \in H^2$ and for $\psi \in \mathcal{D}([0, T])$ we denote by $L[\psi]$ its image in H^{-2} by a distribution $L \in \mathcal{D}'([0, T]; H^{-2})$. Recalling (2.38), we deduce from (2.119), via Fubini's theorem, that, up to subsequences,

$$\begin{aligned}
 \langle \Delta^2 f_t[\psi], w \rangle_{(2)} &= \langle -\Delta^2 f[\psi'], w \rangle_{(2)} \\
 &= \langle -\int_0^T \psi' \Delta^2 f \, dt, w \rangle_{(2)} \\
 &= -\int_0^T \psi' \langle \Delta^2 f, w \rangle_{(2)} \, dt \\
 &= \lim_{n \rightarrow \infty} \left(-\int_0^T \psi' \langle \Delta^2 f^n, w \rangle_{(2)} \, dt \right) \quad (2.120) \\
 &= \lim_{n \rightarrow \infty} \langle -\int_0^T \psi' \Delta^2 f^n \, dt, w \rangle_{(2)} \\
 &= \lim_{n \rightarrow \infty} \langle -\Delta^2 f_t^n[\psi'], w \rangle_{(2)} \\
 &= \lim_{n \rightarrow \infty} \langle \Delta^2 f_t^n[\psi], w \rangle_{(2)} ;
 \end{aligned}$$

that is,

$$\Delta^2 f_t^n \rightarrow \Delta^2 f_t \quad \text{in } \mathcal{D}'([0, T]; H^{-2}). \quad (2.121)$$

On the other hand, the sequence $(\Delta^2 f_t^n)_{n \geq 1}$ is bounded in $L^\infty(0, T; H^{-5})$. Indeed, let $\zeta \in L^1(0, T; H^5)$. Then, as in (2.115),

$$\begin{aligned}
 \int_0^T \langle \Delta^2 f_t^n, \zeta \rangle_{(5)} \, dt &= -2 \int_0^T \langle N(u^n, u_t^n), \zeta \rangle \, dt \\
 &= -2 \int_0^T \langle N(\zeta, u^n), u_t^n \rangle \, dt, \quad (2.122)
 \end{aligned}$$

and since

$$|\langle N(\zeta, u^n), u_t^n \rangle| \leq C |\nabla^2 \zeta|_\infty |\nabla^2 u^n|_2 |u_t^n|_2 \leq C R_0^2 \|\zeta\|_5, \quad (2.123)$$

it follows that

$$\|\Delta^2 f_t^n\|_{L^\infty(0, T; H^{-5})} \leq 2 C R_0^2. \quad (2.124)$$

Thus, again up to subsequences, there is $\lambda \in L^\infty(0, T; H^{-5})$ such that

$$\Delta^2 f_t^n \rightarrow \lambda \quad \text{in } L^\infty(0, T; H^{-5}) \text{ weak}^*. \quad (2.125)$$

Comparing this to (2.121), we conclude that $\Delta^2 f_t = \lambda \in L^\infty(0, T; H^{-5})$, and (2.117) follows from (2.125). This ends the proof of Lemma 2.2.2. $\square$

6) We conclude the proof of Theorem 2.2.1 by giving a sketch of the proof of Lemma 2.2.1. We recall that we wish to show the density of the space $\mathcal{D}([-T, 2T] \times \mathbb{R}^N)$ in $C_0([-T, 2T]; H^r)$. Thus, let $u \in C_0([-T, 2T]; H^r)$, and extend it to a function $\tilde{u} \in C_0(\mathbb{R}; H^r)$ by setting $\tilde{u}(t) \equiv 0$ if $t \leq -T$ or $t \geq 2T$. We approximate $\tilde{u}$ by mollification and truncation. We first set

$$\tilde{u}^\alpha(t, x) := [\rho^\alpha * \tilde{u}(\cdot, x)](t), \quad (2.126)$$

where ρ^α is the Friedrichs' mollifier with respect to t [see (1.178)]. Then, $\tilde{u}^\alpha \in \mathcal{D}(\mathbb{R}; H^r)$, and $\tilde{u}^\alpha \rightarrow \tilde{u}$ in $C_0([-T, 2T]; H^r)$, which implies that $\mathcal{D}([-T, 2T]; H^r)$ is dense in $C_0([-T, 2T]; H^r)$. Consequently, it is sufficient to show that $\mathcal{D}([-T, 2T] \times \mathbb{R}^N)$ is dense in $\mathcal{D}([-T, 2T]; H^r)$ with respect to the topology of $C_0([-T, 2T]; H^r)$. To this end, given $v \in \mathcal{D}([-T, 2T]; H^r)$, we set [compare to (2.126)]

$$v^\delta(t, x) := \zeta^\delta(x) [\rho^\delta * v(t, \cdot)](x), \quad \delta > 0, \quad (2.127)$$

where now $\zeta^\delta \in C_0^\infty(\mathbb{R}^N)$, with $0 \leq \zeta^\delta(x) \leq 1$ for all $x \in \mathbb{R}^N$, $\zeta^\delta(x) \equiv 1$ for $|x| \leq \frac{1}{\delta}$, and $\zeta^\delta(x) \equiv 0$ for $|x| \geq \frac{2}{\delta}$, and, now, ρ^δ is the Friedrichs' mollifier with respect to x . Then, $v^\delta \in \mathcal{D}([-T, 2T] \times \mathbb{R}^N)$, and $v^\delta \rightarrow v$ in $C([t_0, t_1]; H^r)$ for any compact interval $[t_0, t_1] \subset [-T, 2T]$.² This ends the proof of Lemma 2.2.1; consequently, the proof of Theorem 2.2.1 is now complete. $\square$

Remark We explicitly point out that the proof of Theorem 2.2.1 shows that u and u_t would be continuous at any t_0 such that either

$$\Phi(u(t_0)) = \Phi(u(0)) + 2 \int_0^{t_0} \langle N(\varphi^{(m-1)}, u), u_t \rangle d\theta, \quad (2.128)$$

that is, at any point where (2.88) holds as an equality, or

$$\Phi(u(t_0)) = \lim_{n \rightarrow \infty} \Phi(u^n(t_0)) \quad (2.129)$$

²One way to see this is to argue exactly as in the proof of the claim $u^\delta \rightarrow u$ in $C([0, T]; H^m)$ of Theorem 1.7.1 of Cherrier and Milani, [8, Chap. 1].

[compare to (2.92)]. Indeed, if (2.128) holds, using (2.88) we can repeat the estimates (2.89) and (2.90), with t_0 instead of 0, to deduce that

$$\begin{aligned} \Phi(u(t_0)) &\leq \liminf_{t \rightarrow t_0^+} \Phi(u(t)) \leq \limsup_{t \rightarrow t_0^+} \Phi(u(t)) \\ &\leq \limsup_{t \rightarrow t_0^+} G(t) = G(t_0) = \Phi(u(t_0)), \end{aligned} \quad (2.130)$$

where (2.128) is used for the last step. Hence, the function $t \mapsto \Phi(u(t))$ is continuous at t_0 ; together with the weak continuity of u , u_t and f from $[0, T]$ into H^m , L^2 and $\tilde{H}^m$, respectively, this is enough to deduce the continuity of u and u_t at t_0 . Alternately, (2.129) would be the analogous of (2.96) at t_0 . In particular, both conditions (2.128) and (2.129) hold at $t_0 = 0$. $\diamond$

2.3 Uniqueness Implies Continuity

We conclude by proving the third claim of Theorem 1.4.1; that is,

Theorem 2.3.1 *Let $m \geq 2$ and $T > 0$. Assume that for each choice of data $u_0 \in H^m$, $u_1 \in L^2$ and $\varphi \in S_{m,0}(T)$, there is only one weak solution $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH). Then $u \in \mathcal{X}_{m,0}(T)$.*

Proof We argue as in Majda, [23, Chap. 2, Sect. 1]. Because of Theorem 2.2.1, it is sufficient to prove the continuity of u and u_t [in the sense of (2.86) and (2.85)], at any $t_0 \in]0, T]$.

a) To show the left continuity of u and u_t at t_0 , we note that the function $v(t) := u(t_0 - t)$ solves problem (VKH) on the interval $[0, t_0]$, with initial data $v(0) = u(t_0)$ and $v_t(0) = -u_t(t_0)$ (recall that if $u \in \mathcal{Y}_{m,0}(T)$, then $u(t_0)$ and $u_t(t_0)$ are, for each $t_0 \in [0, T]$, well-defined elements of, respectively, H^m and L^2). The assumed uniqueness of weak solutions in $\mathcal{Y}_{m,0}(T)$ implies that v coincides with the solution provided by Theorem 2.1.1, which by Theorem 2.2.1 is right continuous at $t = 0$. Thus,

$$\lim_{t \rightarrow t_0^-} u(t) = \lim_{\theta \rightarrow 0^+} v(\theta) = v(0) = u(t_0) \quad \text{in } H^m. \quad (2.131)$$

Analogously,

$$\lim_{t \rightarrow t_0^-} u_t(t) = - \lim_{\theta \rightarrow 0^+} v_t(\theta) = -v_t(0) = u_t(t_0) \quad \text{in } L^2. \quad (2.132)$$

This shows that u and u_t are left continuous at $t = t_0$.

b) To show the right continuity of u and u_t at t_0 , with $0 < t_0 < T$, we note that the function $w(t) := u(t_0 + t)$ solves problem (VKH) on the interval $[0, T - t_0]$, with initial data $w(0) = u(t_0)$ and $w_t(0) = u_t(t_0)$. Again, the assumed uniqueness of weak solutions in $\mathcal{Y}_{m,0}(T)$ implies that w coincides with the solution provided by Theorem 2.1.1, which by Theorem 2.2.1 is right continuous at $t = 0$. Thus,

$$\lim_{t \rightarrow t_0^+} u(t) = \lim_{\theta \rightarrow 0^+} w(\theta) = w(0) = u(t_0) \quad \text{in } H^m. \quad (2.133)$$

Analogously,

$$\lim_{t \rightarrow t_0^+} u_t(t) = \lim_{\theta \rightarrow 0^+} w_t(\theta) = w_t(0) = u_t(t_0) \quad \text{in } L^2. \quad (2.134)$$

This shows that u and u_t are right continuous at $t = t_0$. Hence, u and u_t are continuous at $t = t_0$. This concludes the proof of Theorem 2.3.1 (which in fact is really a corollary of Theorem 2.2.1), and Theorem 1.4.1 is now completely proven as well. $\square$

Remark By part (2) of Proposition 1.4.1, the strong continuity of u , u_t and f from $[0, T]$ into, respectively, H^m , L^2 and $\tilde{H}^m$, would follow if u satisfied the same identity (2.27) satisfied by its Galerkin approximants u^n ; that is, if

$$\frac{d}{dt} (\|u_t\|_0^2 + \|u\|_m^2 + \frac{1}{m} \|\nabla^m f\|_0^2) = 2 \langle N(\varphi^{(m-1)}, u) + u, u_t \rangle, \quad (2.135)$$

which yields (2.128). However, (2.135) is formally obtained from Eq. (13) via multiplication by $2u_t$ in L^2 , and none of the individual terms of (13) need be in L^2 if $u \in \mathcal{Y}_{m,0}(T)$ only. In fact, the usual procedure of obtaining (2.135) by means of regularization via Friedrichs' mollifiers fails, precisely because we are not able to determine whether $N(f, u^{(m-1)}) \in L^2$, or not [in general, we can only prove that this nonlinear term is bounded from $[0, T]$ into L^1 , as we see from the estimate

$$\begin{aligned} |N(f, u^{(m-1)})|_1 &\leq C |\nabla^2 f|_m |\nabla^2 u|_m^{m-1} \\ &\leq C |\nabla^m f|_2 \|u\|_m^{m-1} \\ &\leq C \|u\|_m^{2m-1}, \end{aligned} \quad (2.136)$$

which follows from (1.73) and (1.117)]. Thus, we do not know whether (2.135) holds or not, and the problem of the continuity (as well as, of course, that of uniqueness) of weak solutions $u \in \mathcal{Y}_{m,0}(T)$ to problem (VKH) remains open. $\diamond$

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