Chapter 2
Description of the Most Important Elements of Leibniz's Planetary Theory

This chapter is divided into four parts according to an ideal division of the *Tentamen*. In the first part Leibniz dealt with harmonic circulation and introduced paracentric motion; in the second one he analysed the properties of paracentric motion; in the third one he dealt with the inverse square law and the elliptic movements of the planets; in the fourth one Leibniz provided a summary of his model. Every paragraph is divided into two subparagraphs: 1. Leibniz's assertions; 2. commentaries.

2.1 Physical Presuppositions, the *Circulatio harmonica* and the *Motus paracentricus*

2.1.1 Leibniz's Assertions

In the published version of the *Tentamen*, Leibniz, after a general historical introduction concerning the development of astronomy and vortex theory, clarified the physical assumption on which his planetary theory is based: the planets are moved by a rotating fluid in which they are situated, because: a) planets' orbits are curved lines; b) each body moving in a curved line is subject to a *conatus* to recede along the tangent, that is a centrifugal force; c) the planets do not recede along the tangent; d) hence it is necessary that something exists allowing the planets to continue their curved paths; e) thus, the only physical possibility to explain this motion is the hypothesis of a moving fluid vortex, which surrounds every planet and transports the planet by means of its motion. The planets are afloat in the vortex which communicates them its movement.

After these physical considerations, Leibniz introduced the definition of *Circulatio Harmonica* (harmonic circulation) like this:

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> I call a *circulation* a *harmonic* one if the velocities of circulation in some body are inversely proportional to the radii or distances from the centre of circulation, or (what is the same) if the velocities of circulation round the centre decrease proportionally as the distances from the centre increase, or most briefly, if the velocities of circulation increase proportionally to the closeness.[1]

According to Leibniz, the harmonic circulation can characterize the arcs of every curve, not only the arcs of a circle.

The next step consists in two different possible decompositions of the curvilinear motion (see Fig. 2.1). Let a body move along a curve $M_1M_2M_3$ describing the elementary arcs M_1M_2 and M_2M_3 in equal time, then its motion can be decomposed into: a) a circular motion around the centre Θ (M_2T_1 and M_3T_2 are, in this case, infinitesimal circular arcs) plus a rectilinear motion as T_1M_1 and T_2M_2; b) the motion of a rigid ruler around the centre Θ plus the rectilinear motion of the body M along the rotating ruler. The motion of M along the ruler was called by Leibniz *motus paracentricus* (paracentric motion). Leibniz adopted this second decomposition of the curvilinear motion. Then, without considering for the moment the paracentric motion, a circulation is harmonic if the infinitesimal circulations M_2T_1 and M_3T_2, completed in equal elements of time, are inversely as the radii ΘM_2 and ΘM_3. Leibniz wrote:

> For since these arcs of elementary circulations are as the times and the speeds combined, and the elements of time are taken to be equal, the circulations will be as the velocities, and consequently the velocities inversely as the radii, and therefore the circulation will be called harmonic.[2]

Leibniz could now prove that the area law is valid for bodies which move according to a harmonic circulation. Actually, rather than a demonstration, the area law is a definitory property of the harmonic circulation, once specified the proportionality between elementary circulations and speeds.

In the sixth paragraph of the *Tentamen* Leibniz claimed that, since the planets move according to the area law and given the logical equivalence between area law and harmonic circulation, the planets move with a harmonic circulation.

The seventh paragraph deals briefly with a problem which is important in order to understand Leibniz's way of reasoning, which runs as follows: a) as already seen a body which is posed in a fluid does not move spontaneously in a curved line, this means that the aether itself is not at rest; b) it is reasonable to think (*rationis est*

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[1] Translation drawn from Bertoloni Meli (1993, pp. 129–130). Original latin text: "Circulationem voco Harmonicam, si velocitates circulandi, quae sunt in aliquo corpore, sint radiis seu distantiis a centro circulationis reciproce proportionales, vel (quod idem) si ea proportione decrescant velocitates circulandi circa centrum, in qua crescunt distantiae a centro, vel brevissime, si crescant velocitates circulandi proportione viciniarum." (Leibniz 1689, 1860, 1962, VI, pp. 149–150).

[2] Translation drawn from Bertoloni Meli (1993, p. 130). Original latin text: "Cum enim arcus isti elementarium circolationum sunt in ratione composita temporum et velocitatum, tempora autem elementaria assumantur equalia, erunt circulationes ut velocitates, itaque et velocitates reciproce ut radii erunt, adeoque circulatio dicetur harmonica." (Leibniz 1689, 1860, 1962, VI, p. 150).

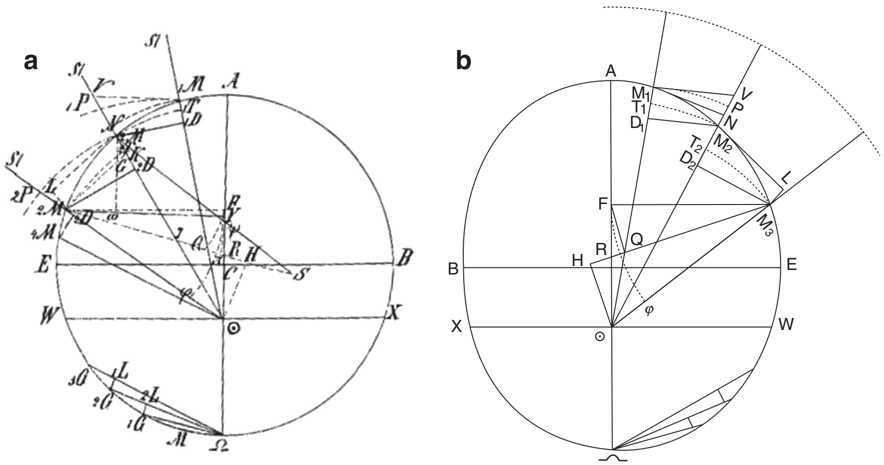

Fig. 2.1 Leibniz's planetary theory model. (**a**) This is Leibniz's original figure posed by Gerhardt at the end of Leibniz 1860, 1962. The diagram is unclear. There are many letters and this makes it difficult to clearly read the diagram. There is a typo because the $_2M$ written immediately over $_4M$ is a mistake. The right form is $_3M$. Furthermore there is the habit to write the index of a letter before the letter, while nowadays we write after the letter. Because of all these reasons—if I do not specify otherwise—I will refer to (**b**), which is written in a more modern form but does not betray Leibniz's thought, at all. This diagram is drawn from Aiton (1960, p. 69)

credere, *ivi*, p. 151) that the movement of the aethereal fluid has the same features as planet's movement, hence, it follows: c) the motion of the fluid itself is harmonic.

Leibniz imagined the situation like this: the planet moves in an ellipsis (he dealt with the properties of the elliptic motion in the next paragraphs of the *Tentamen*) of harmonic circulation. Let us consider the part of aether, which constitutes a ring, whose centre is in the sun, whose major radius is the distance sun-aphelion and whose minor radius is the distance sun-perihelion. This ring can be thought as divided into concentric circumferences of small thickness (*exiguae crassitudinis*, *ivi*, p. 152), centred in the sun with the property that the fluid composing every circumference moves harmonically. Therefore, the planet moves harmonically on an ellipsis, every aethereal fluid's circular section of infinitesimal thickness moves harmonically, this means that the whole aethereal fluid moves harmonically according to a circular motion. Therefore (par. 8), the motion of a planet can be considered as decomposed in the harmonic motion of the fluid plus the paracentric motion along the ruler. When a planet, at the time t, moves in the circumference C of the aethereal fluid, the planet itself does not retain the impetus of circulation (*impetus circulandi*, *ivi*, p. 152) it had got while moving along a different circumference at the time $t_i < t$; rather it assumes immediately the harmonic movement of the circumference in which it is at the time t.

This assertion in paragraph 8 concludes ideally the first part of the *Tentamen*, in which the essential properties of the planetary harmonic circulation are explained. The second part will face the paracentric motion.

2.1.2 Commentaries

In these commentaries I will remain strictly adherent to Leibniz's text, while dealing with more general questions in Chap. 3.

1. The role of harmonic circulation of the aethereal fluid is twofold:
   a) from a kinematical point of view, it has to provide the mean motion of the planet. The deviation from the uniform circular motion is given by the paracentric motion.
   b) from a physical-structural point of view, the aethereal vortex is a real existing entity, according to Leibniz. As we will see, he proposed, at least, two hypotheses on the features of the vortices when he needed to better specify some dynamical properties of gravity or to explain the movements of the comets inside his system, but Leibniz never doubted the physical existence of the vortices and of their harmonic circulation. In this regard, the correspondence with Huygens is significant: it is well known that both Leibniz and Huygens did not accept the idea of action at a distance, both of them sustained vortex theory, but Huygens never accepted the role ascribed by Leibniz to the harmonic motion of the aethereal vortex. He saw harmonic circulation as a useless additional hypothesis, because the area law was given for granted in this hypothesis and, as to gravity, the harmonic circulation—not the vortices in themselves—seemed to play no role. Therefore Huygens was not able to understand the meaning of harmonical vortices.

   In a brief but dense passage of a letter to Leibniz on 11 July 1992, Huygens wrote:

   > It is sure that the gravities (*pesanteurs*) of the planets are in inverse double reason as their distances from the sun, which, together with the centrifugal virtue (*vertu*), provides Kepler's eccentrical ellipses. But I was never able to understand, relying upon your explanation given in the *Acta* of Leipzig [the published version of the *Tentamen*], how you deduce the same ellipses, replacing your harmonic circulation and maintaining the same proportions of gravities. I do not see how you find the place for a kind of Descartes' deferent-vortex, which you want to maintain, since the mentioned proportion of gravity, joined with the centrifugal force, produces—by itself—Keplerian ellipses, according to the proof given by Mr. Newton. For a long time, you promised me to clarify this difficulty.[3]

[3] LSB, III, 5, p. 337. Original French text: "Il est certain que les pesanteurs des Planetes estant posees en raison double reciproque de leurs distances du soleil, cela, avec la vertu Centrifuge, donne les Eccentriques Elliptiques de Kepler. Mais comment en substituant vostre Circulation Harmonique, et retenant la mesme proportion des pesanteurs, vous en deduisez les mesmes Ellipses, c'est ce que je n'ay jamais pu comprendre par vostre explication qui est aux *Acta* de Leipsich; ne voiant pas comment vous trouvez place à quelque espece de Tourbillon deferant de des Cartes, que vous voulez conserver; puisque la dite proportion de pesanteur, avec la force Centrifuge produisent elles seules les Ellipses Keplerienes selon la demonstration de Mr Newton. Vous m'aviez promis il y a longtemps d'eclaireir cette difficulté".

Aiton claims:

> Since the harmonic vortex played no part in the motion of a planet in its orbit, this vortex may be left out of account in the analysis of Leibniz's theory.[4]

And again:

> What he [Leibniz] still failed to see clearly was that the harmonic circulation of the planet followed from the attraction, so that his resolution of the orbital motion into transverse and radial components, which gave a correct mathematical representation, had a sufficient physical foundation in the attraction without the addition of the harmonic vortex.[5]

As a matter of fact, Aiton's observation is similar to Huygens': the harmonic hypothesis is useless for the theory,[6] which is certainly true if the aim is a mere mathematical analysis of the paracentric motion. However, from a conceptual point of view the harmonic motion has an important role because it allowed Leibniz to prove the area law without resorting to the immediate action at a distance of a centripetal force. On the other hand, to admit a harmonic circulation means, essentially, to postulate, not to prove, the area law. The situation looks like this: Leibniz was going to provide a theory which described the real structure and functioning of the solar system, not only a kinematical and dynamical model, but a very physical-structural theory. The harmonic vortex has a fundamental role because it describes something really existing, not exclusively a model. Leibniz preferred to sacrifice the empirical content of his theory—because he almost postulated the area law—rather than to admit a Newtonian force, for which no mechanical support had been given. It is necessary to add that a further problem exists: Leibniz condemned the action at a distance and every action which should be immediately transmitted without respecting the principle of continuity. But, if one reflects on the way Leibniz imagined the harmonic motion in the planetary ellipses, one discovers a problem similar to the immediate action (even though not at a distance): we have seen that every circumference of infinitesimal thickness of the aethereal vortex included between aphelion and perihelion moves harmonically and that the planet, while moving from a circumference C to another D assumes *immediately* the motion of D without retaining the one of C. But this is exactly an action which is immediate, though by contact. The principle of continuity is not respected because the motion should instantaneously lose its previous properties. Not only: this immediate action should take place in every instant because the planet changes its distance from the sun in every instant and hence, in a finite time, there should be $2^{\aleph_0}$—to use a Cantorian language—immediate adaptions of the planet to its new condition of harmonic motion. Every point of the space-

[4] Aiton (1964, p. 112).

[5] Aiton (1972, p. 136).

[6] Huygens' and Aiton's aims are, however, different, which is obvious: Huygens seems to invite Leibniz to abandon the harmonic circulation, while Aiton has the intention to prove that the mathematical treatment of the paracentric motion is independent of harmonic circulation.

temporal *continuum* would represent a point of discontinuity in the motion of the planet. Leibniz was against an immediate action in physics, also considering the action with contact: his well known ideas on the collisions—which, according to him, can never be considered as if they took place among perfectly hard bodies—and his oppositions to the existence of the atoms are, in great part, based exactly on the refusal of an immediate action, which changes the condition of the bodies-motion. Whereas the elliptic harmonical circulation of the planets needed more than a denumerable infinity of these immediate changes in a finite time. It seems difficult to conceive a physical mechanism which allows a body to completely cancel its preceding motion-state, at least as far as the transversal direction is concerned and Leibniz was absolutely clear that this is a property of the harmonic circulation shared with no other kind of motion. For, he wrote to Huygens in 1690:

> And the body itself is moved in the aether, as if it tranquilly navigated, without either impetuosity or residue of the preceding impressions. The body only obeys to the aether, which surrounds it. [...] But in each other circulation, excluded that harmonic, the bodies maintain the preceding impression.[7]

Therefore, from a logical point of view the fact that a body does not retain any data of its preceding physical state seems to be in conflict with Leibniz refusal of an immediate action and with his principle of continuity; from a physical standpoint, the one described is a mechanism which is difficult to conceive. Anyway, the harmonic vortices aimed at: a) supplying the real structure of the solar system; b) offering an alternative to Newton's model; c) avoiding the action at a distance.

2. The kind of velocity, of which Leibniz was speaking about while referring to the *velocitas circulandi*.

   There is no doubt after Aiton's contributions: he was considering a reference frame in polar coordinates, whose pole is in the sun and the model applied is that of the rotating-ruler plus paracentric motion. Leibniz considers the situation from the perspective of the rotating planet and analyses, in every moment, the planetary movement in terms of the physical quantities experienced by the planet. The *velocitas circulandi* is the transverse velocity. Under the condition that such a velocity is harmonic, it is trivial to prove that (in modern terms) the angular moment—even though this is not a concept, to which Leibniz explicitly referred—is conserved and that, which is equivalent, the areolar velocity is a constant of the motion, that is the area law.

[7] Leibniz (1690a, 1860, 1962, VI, pp. 189–190). This is a letter written in October 1690 and edited by Gerhardt in *Ivi*, pp. 187–193. This letter was never sent to Huygens. On this see Aiton (1964, p. 114, note 16). Original French text: "El le même corps aussi est mû dans l'ether comme s'il y nageoit tranquillement sans avoir aucune impetuosité propre, *ny aucun reste des impressions precedentes*, et ne faisoit qu'obeïr absolument à l'ether qui l'environne [...] Mais quelque autre circulation qu'on suppose hors l'harmonique, le corps gardant l'impression precedente [...]".

A confirmation that the *velocitas circulandi* is the transverse velocity is given by the above mentioned letter to Huygens, where Leibniz wrote that, if we compare velocities' modules of the different planets in their orbits, then they are as square root of the distance (as Newton had proved in *Principia*, I, prop. IV, cor. 6), but if we consider a single planet in its orbit, then in the different points of the orbit, the *velocitas circulandi* is as the inverse distance from the sun, which supplies the area law. Thence there is no contradiction between the two assertions because they are referred to different kinds of velocity. Leibniz is clear, for he wrote:

> Perhaps, Mister, you will immediately say that the hypothesis of the squares of the velocities equal to the reciprocal of the distaces is not in agreement with the harmonic circulation. But I answer that the harmonic circulation is valid for each singular body, if ones compares its different distances [from the sun], but the harmonic circulation *in potentia* (where the squares of velocities are reciprocal to the distances) is valid when one compares the different bodies, both in the cases in which they describe a circular line, or when one considers their mean movement [. . .] for the circular orbit they describe.[8]

2.2 The *Motus Paracentricus* and Its Properties

2.2.1 Leibniz's Assertions

The *circulatio harmonica* provides the mean motion of the planets, while the *motus paracentricus* is the motion of approaching and moving away of the planet from centre of gravity along the radius-vector. It is the radial motion. The paracentric motion is due to two opposite tendencies: 1) the *impressio excussoria circulationis*; 2) the *attractio solaris* (*ivi*, par. 9, p. 152).

The *impressio excussoria circulationis* (translated by Bertoloni Meli as "outward impression of the circulation", p. 132) is the centrifugal force due to the harmonic circulation. The centrifugal force tends outwards. Leibniz's problem is to find a geometrical representation and an analytical expression for the instantaneous centrifugal acceleration that he called *conatus centrifugus* or *conatus excussorius circulationis*. In paragraph 10 and 11 of the *Tentamen* Leibniz solved the problem to find a geometrical representation of the *conatus centrifugus*. For, he wrote:

[8] *Ivi*, p. 192. See also Aiton (1964, pp. 113–115). Original French text: "Vous dirés peutestre d'abord, Monsieur que l'hypothese de quarrés des vistesses reciproques aux distances ne s'accorde pas avec la circulation harmonique. Mais la réponse ast aisée: la circulation harmonique se rencontre dans châque corps a part, comparant les distances differentes qu'il a, mais la circulation harmonique en puissance (où le quarrés des velocités sont reciproques aux distances) se rencontre en comparant des differens corps, soit qu'ils décrivent une ligne circulaire, ou qu'on prenne leur moyen movement [. . .] pour l'orbe circulaire qu'ils décrivent".

> This conatus can be measured by the perpendicular from the following point to the tangent at the inassignably distant preceding point.[9]

This means (*Tentamen*, par. 11) that the *conatus excussorius* can be represented by PN (see Fig. 2.1b), namely the versed sine of the angle of circulation $M_1\Theta N$. For, the versed sine—Leibniz continues—"is equal to the perpendicular drawn from one end-point of the arc of a circle to the tangent from the other end-point".[10] The versed sine can be identified with D_1T_1, the inassignable difference between two infinitely near radii-vector. This means that, in general, the *conatus escussorius* can be represented by segments of the type D_iT_i, for every position of the radius vector. It is then easy to prove that the *conatus centrifugus* is equal to PV.[11]

Leibniz is here imagining the trajectory as composed of an infinite number of infinitesimal circular arcs whose radii have infinitesimal differences and are all centrated in the sun. Given this situation, the infinitesimal arcs of circumference can be considered as sides of a polygon. In the commentaries we will see that the consideration of the trajectory as composed of infinitesimal arcs or of infinitesimal sides of a polygon implies a problem as to the concept of tangent, with the consequence that Leibniz wrongly added a factor 2. This mistake did not have remarkable effects on the coherence of Leibniz's theory.

With regard to the analytical expression of the *conatus centrifugus*, if the motion is circular and uniform, than the *conatus* is as V^2, where V is the transverse velocity, since the versed sine is proportional to the square of the chord and the transverse velocity is proportional to the chord. If two or more circles are considered in which the movement is uniform, then the *conatus* are as V^2/R, where R is the radius. From this expression for the centrifugal force, Leibniz deduced another expression which is fundamental in his reasoning: if a body moves with a harmonic circulation, the *conatus centrifugus* is inversely proportional to the radius vector. This happens because of the inverse proportion between transverse velocity and radius vector in the *circulatio harmonica* and because of the relation $c = V^2/R$, where c is the centrifugal conate. From here another expression is possible: Leibniz considered a fixed elementary area, completed by the radius-vector in an infinitesimal time dt (the area law is valid), which he indicated by ϑa and assumed it equal to the double of the elementary triangle $M_2M_3\Theta$, namely equal to $D_2M_3 \cdot \Theta M_2$. The expressions ΘM_n can be indicated by $r =$ radius, because the difference between ΘM_i and ΘM_{i-1} is an infinitesimal, which can be neglected in this calculation. Therefore $D_2M_3 = \vartheta a/r$ and the centrifugal conate $D_2T_2 = (D_2M_3)^2/2\Theta M_3$. Thus, in conclusion

[9] Translation drawn from Bertoloni Meli (1993, p. 132). Original Latin text: "Hunc conatum metiri licebit perpendiculari ex puncto seguenti in tangentem puncti praecedentis inassignabiliter distantis." (Leibniz 1689, 1860, 1962, VI, p. 152).

[10] Translation drawn from Bertoloni Meli (1993, p. 133). Original latin text: " [...] aequatur perpendiculari ex uno extremo arcus circuli puncto in tangentem alterius ductae [...]." (Leibniz 1689, 1860, 1962, VI, p. 153).

[11] See Leibniz (1689, 1860, 1962, VI, paragraph 11, p. 153).

$$D_2T_2 = \frac{\vartheta^2 a^2}{2r^{3.}}$$

That is: the centrifugal conate is as the inverse of the radius-cube.

This means that Leibniz is considering a non-inertial reference frame in polar coordinates, whose origin is posed in the rotating planet. From the point of view of the planet, the acceleration along the radius is given by two components: one outwards, which is the *conatus centrifugus* due to the harmonic circulation; the other one is due to gravity or levity. Leibniz thought that this second component can be either inwards (gravity), which is the normal experienced case, or outwards (levity), which is a theoretical case. The acceleration along the radius is the algebraic sum of the two components, which is an arithmetical difference in case of gravity and an arithmetical sum in case of levity. Considering the case of gravity, if the *conatus centrifugus* prevails,[12] the radial acceleration is directed outwards. While, if the *solicitatio gravitatis* prevails, the radial acceleration is directed inwards.

We have seen how Leibniz represented the *conatus centrifugus*. As to the *solicitatio gravitatis*, Leibniz claimed:

> Paracentric solicitation, whether of gravity or levity is expressed by the straight line M_3L drawn from the point M_3 of the curve to the tangent M_2L (produced to L), of the preceding inassignably distant point M_2 parallel to the preceding radius ΘM_2 (drawn from the centre to the preceding point M_2).[13]

Leibniz imagined hence that, given an infinitesimal arc M_1M_2, which can be approximeted by its chord, the inertial motion of a body moving in such an arc can be approximated by the prolongation of the chord (the *tangent* in the Leibnizian sense) rather than by the Euclidean *tangent* (on this, see the following commentaries) without a detectable mistake. This kind of representation, as well as the idea that the trajectory can be considered a polygon with infinitesimal sides, is evidently the same as the one used by Newton in the proposition I of the first book of his *Principia*.

The section of the *Tentamen*, which concludes the part concerning the general properties of the paracentric motion is the 15th paragraph, where Leibniz determined geometrically the element of the *impetus paracentricus*, that is the instantaneous acceleration along the radius. He claimed that in every harmonic circulation the element of *impetus paracentricus* is the difference or the sum of the paracentric

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[12] Leibniz wrote "[...] differentia vel summa solicitationis paracentricae [...] et *dupli* conatus centrifugi [...]" (my italics, Leibniz 1689, 1860, 1962, VI, p. 154), referring to the double centrifugal conate and not to the simple centrifugal conate. This is a mistake highlighted by Varignon. For an explanation see next Sect. 2.2.2. *Commentaries*.

[13] Translation drawn from Bertoloni Meli (1993, p. 134). Original Latin: "Solicitatio paracentrica, seu gravitatis vel levitatis exprimitur recta M_3L ex puncto curvae M_3 in puncti praecedentis inassignabiliter distantis M_2 tangentem M_2L (productam in L) acta, radio praecedenti ΘM_2 (ex centro Θ in punctum precedens M_2 ducto) parallela". (Leibniz 1689, 1860, 1962, VI, p. 154).

solicitation and of the double centrifugal conatus. We refer to Leibniz reasoning because:

A) it is an example of what one could call *infinitesimal geometry applied to physics*, that is both the finite and the infinitesimal quantities are represented by means of geometrical constructions and, at least in this paragraph, there is not a transcription into analytical terms;
B) it is an example which clearly shows the use of differentials of different degree in a geometrical context (for more details see the next commentaries).

Leibniz reasoned like this:

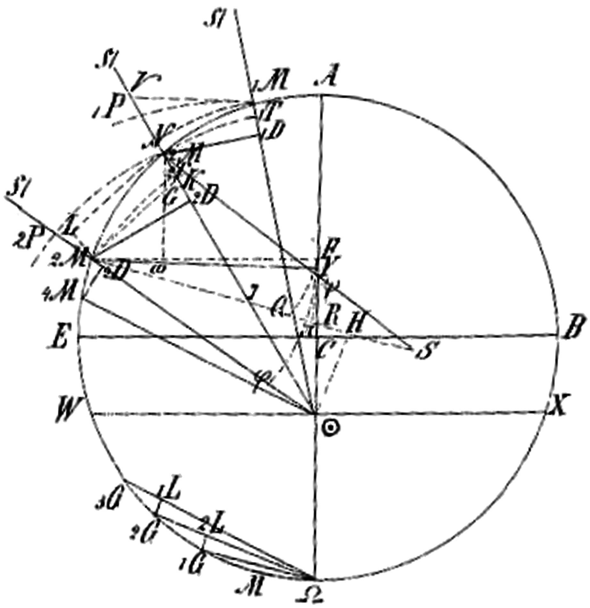

Fig. 2.2 Enlarged imagine of Leibniz's planetary theory-figure. I offer here the reader an enlarged imagine of Leibniz's planetary theory. The imagine is the same as Fig. 2.1a. I present this imagine because in Aiton's the point G is not represented, while it is quite important in the context I am dealing with. I hope this imagine can help the reader to follow the mathematical reasoning developed in the running text. Let us remind the reader that the symbol $_2M$ near $_4M$ has to be replaced with $_3M$ (in my running text M_3)

1. let M_1N and M_3D_2 be the perpendiculars from M_1 and M_3 to ΘM_2.
2. The circulation is harmonic, hence the triangles $M_1M_2\Theta$ and $M_2M_3\Theta$ are congruent. Therefore their altitudes M_1N and M_3D_2 are congruent.
3. Let M_2G be congruent to LM_3 and M_3G parallel to M_2L.
4. The triangles M_1NM_2 and M_3D_2G are congruent.[14] Therefore it is $M_1M_2 = GM_3$ and $NM_2 = GD_2$.
5. Let us assume $\Theta P = \Theta M_1$ and $\Theta T_2 = \Theta M_3$, so.

[14] I remind the reader that the two triangles are congruent because: a) $M_3D_2 = M_1N$; b) they are right triangles; c) For the angles the following identities are valid: $M_1M_2N = D_2M_2L$ and $D_2M_2L = D_2GM_3$, because of the parallels M_3G and M_2L. Thus, $M_1M_2N = D_2GM_3$. Hence, the thesis follows.

6. $PM_2 = \Theta M_1 - \Theta M_2$ and $T_2M_2 = \Theta M_2 - \Theta M_3$.
7. $PM_2 (= NM_2) = GD_2 + NP$ and $T_2M_3 = M_2G + GD_2 - D_2T_2$. Hence.
8. $PM_2 - T_2M_2 = NP + D_2T_2 - M_2G$. But.
9. $NP = D_2T_2$ because they are the versed sines of two angles and radii whose differences are inassignable. Hence.
10. $PM_2 - T_2M_2 = 2D_2T_2 - M_2G$.
11. The difference of the radii expresses the paracentric velocity; the difference of the differences expresses the element of the paracentric velocity (that is the paracentric acceleration). But D_2T_2 or NP is the centrifugal conatus of circulation and M_2G or M_3G is the paracentric solicitation. This proves the theorem.

In this demonstration: the segments, one extremum of which is the centre of gravity Θ are finite; all other elements used in the proof are infinitesimal. The quantities P_2M_2 and T_2M_2 are first differences and represent the instantaneous radial velocity; their difference $PM_2 - T_2M_2$ is a second difference and represents the radial instantaneous acceleration.

With this demonstration, Leibniz completed the description and the explanation of the basic elements of his theory. He then applied these elements to the case of the elliptical orbits, the ones which are relevant for the planetary motions. In particular: at the moment Leibniz has been able to determine both a geometrical and an algebraic-analytical form with regard to the *conatus centrifugus*, while, for the *solicitatio paracentrica*, he has only given the geometrical form. His next step is to prove that such a solicitation is as the inverse of the square distance.

2.2.2 Commentaries

1. Relation between harmonic circulation and paracentric motion.

   Let us summarize the results obtained by Leibniz till the paragraph 17 of the *Tentamen*: Leibniz considered the situation from the point of view of an observer posed in the rotating planet, which is subject to three actions:

   1) the action due to the *circulatio harmonica*, which determines the transverse velocity of the planet;
   2) the centrifugal force due to the rotating vortex. In this case it is necessary to underline that the physical cause of the transverse velocity and of the centrifugal force is the same, that is the harmonic vortex, but, while the area law depends on the fact that the circulation of the vortex is harmonic so that the areal velocity is constant, the centrifugal force depends on the rotation, not on the fact that the rotation is harmonic;
   3) the solicitation of gravity or of levity. In the case of the solar system, the solicitation of gravity due to the sun. Centrifugal force plus solicitation of gravity provide the *paracentric motion*.

A brief physical explanation is maybe useful: from the point of view of an inertial reference frame, the so called centrifugal force is not a really existing force. However, from the point of view of the rotating observer the situation is different: for him the centrifugal force is a real force and depends on the rotation originated—according to Newton–by two physical quantities and situations:

A) *The centripetal force;*
B) *The initial conditions of the motion; basically the initial inertial velocity.*[15]

The conditions A) and B) determine the rotation of the planet and hence the intensity and the direction of the physical quantities in the rotating system, in particular, of the centrifugal force. The physicists call it *fictitious centrifugal force* and we can call *Leibnizian centrifugal force*. This force simply depends on the fact that a system is rotating, independently of the dynamical causes of the rotation, because the rotating observer experiences a centrifugal force in the case of planetary motion (and this, in Newtonian terms, depends on centripetal force plus initial velocity), but also, for example, in a roundabout, where no centripetal force exists. When the intensity of the centripetal force is equal to that of Leibnizian centrifugal force, then the motion is circular and uniform, otherwise it is not.

An explanation in modern terms can be useful for a complete understanding of Leibniz's reasoning. In a rotating reference frame the forces equation can be written, using polar coordinates like this:

$$F(r) = m\left(\left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathrm{r}}\right) + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}) \tag{2.1}$$

where r is the variable radius vector, θ is the angular distance from an angular position of the radius vector assumed equal to 0, $\hat{r}$ is the radial versor and $\hat{\theta}$ is the versor in the direction perpendicular to $\hat{r}$. Since we are in a field of central forces, the transverse component of the acceleration $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ is zero, the whole acceleration is radial and is expressed by the term $\ddot{r} - r\dot{\theta}^2$. Therefore if one wonders how the acceleration along the radius vector varies, one gets the equation

$$m\ddot{r} = F(r) + r\dot{\theta}^2 . \tag{2.2}$$

Since in a central field the angular moment $L = m\dot{\theta}r^2$ is conserved, Eq. (2.2) gets the form

[15] The explanation of the centrifugal force in terms of A) and B) could be called an inertial interpretation of a non-inertial reference frame. Historically, Leibniz did not resort to it. However, this explanation is useful to catch the situation from a physical point of view and to better understand the correct reasoning of Leibniz as to the centrifugal force.

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}. \tag{2.3}$$

The term $\frac{L^2}{mr^3}$ is called *centrifugal force*. We have seen that the centrifugal conate is expressed by Leibniz as $D_2T_2 = \frac{\vartheta^2 a^2}{2r^3}$. Since ϑa represents an infinitesimal area, it can be indicated by dA. In modern terms the relation between the infinitesimal area swept by the radius vector and the angular moment is $L = 2m\frac{dA}{dt}$. If one does not take into account the constant factor m and considers (so to say)—as Leibniz did—a unitary infinitesimal time, then the relation becomes $L = 2dA$. Therefore, if we exclude a constant factor 8, Leibniz's result is perfectly correct.[16] This is an important and new result in history of physics. Let us add that, if in Eq. (2.3), we consider *F(r)* acting as gravity acceleration, namely as $-\frac{1}{r^2}$, one gets exactly the situation taken into account by Leibniz.

The structure in terms of forces is now complete, as to its fundamental elements. Leibniz had still to determine the specific expression of the solicitation of gravity. With regard to the physical structure of the world, the harmonic vortex produces the first two actions; as to the mechanical cause of gravity, Leibniz—as we will see—faced the problem in various works, but in the *Tentamen* the question is merely outlined, hence, for the moment, I will not deal with it. The examination of the paracentric motion along the radius vector is basically correct and—from the standpoint of history of physics—is an important contribution. It is however significant that Newton criticized[17] the way in which Leibniz presented the centrifugal force. For Newton wrote, speaking in third person:

> Eleventh proposition of the Tentamen: the centrifugal conate can be expressed by means of circulation angle's versed sine. This proposition is true, when the circulation takes place in a circle, without the paracentric motion. But when the movement takes place in an eccentric orbit, the proposition is not true. The centrifugal conate is always equal to gravity and is directed in the opposite direction, according to the third law of motion of Newton's Principia Mathematica, and the force of gravity cannot be expressed by the versed sine of circulation's angle, but it is reciprocal to the distance square.[18]

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[16] For a slightly different explanation of this result by Leibniz, see Aiton (1960, pp. 61–62; 1964, pp. 117–121).

[17] The documents in which Newton and Keill criticized Leibniz are three: 1) Newton's writing titled "Epistola cujusdam ad amicum", published in Edleston 1850. Edleston claims that, probably this letter was written in 1712; 2) a second document sent by Newton to Keill and titled "Notae in Acta Eruditorum an. 89 p. 84 et sequ", available in the University Library of Cambridge, Add. MS 3985 f. 6; 3) the only published work on this question, that is Keill (1714). Keill's work is almost completely based upon Newton's ideas. For a complete report on these critics, see Aiton (1962).

[18] Newton in Edleston 1850, p. 311. Original latin text: "Undecima Tentaminis Propositio est haec: *Conatus centrifugus exprimi potest per sinum versum anguli circulationis*. Et vera quidem est haec propositio ubi circulatio fit in circulo sine motu paracentrico. Sed ubi fit in Orbe excentrico propositio vera non est. Conatus centrifugus semper equalis est vi gravitatis et in contrarias partes dirigitur per tertiam motus Legem in Principiis Mathematicis Newtoni, et vis gravitatis esprimi

And again:

> Propositions 20th (*sic*) and 25th are false, because they show a centrifugal force which is less than planet's gravity towards the sun. Therefore they are false. The motion of a planet in its orbit does not depend on the excess of gravity upon centrifugal force (as Leibniz believes), but the orbit is incurved only by gravity's action, to which the centrifugal force (as reaction or resistance) is always equal and opposed, as to the direction, according the third law posed by Newton.[19]

The situation is like this: Newton believes that the centrifugal force is a mere reaction to the centripetal force, which is the real force acting on the planets. This is in agreement with the third law. Considering the question under this perspective, one could claim that Newton did not correctly understand Leibniz's way of reasoning, in particular the fact that Leibniz was looking at the situation from the point of view of the rotating planet. This is probably part of the truth. The other part of the truth is that, likely, in Newton's eyes the whole *Tentamen* seemed something odd. We will deal with this general question in the fourth section of this book, while analysing the final version of Leibniz's planetary theory written in 1706, after David Gregory's critics in 1702.[20]

Anyway, according to Leibniz's aims and way of thinking, the correct expression for the movement along the radius vector is an instrument in his hands to present his system of the world. If he had considered such an examination just as a contribution to mathematical-physics, it would have been only a different presentation of results already obtained by Newton—although Newton did not recognize this point—, it would have been something like "some new points of view in Newtonian physics", not certainly a new system of the world alternative to Newton's, whereas Leibniz intended to construct such a system. Because of this it is necessary to follow the way in which Leibniz continued to construct his planetary theory.

2. The concept of tangent and the second order differences.

   In the item 4) of Leibniz's demonstration, the triangles M_1NM_2 and M_3D_2G are congruent, so $NM_2 = GD_2$. Newton criticized this assertion by Leibniz[21]: if M_2L is the Euclidean tangent in the point M_2, the direction is not the same as M_1M_2, therefore GM_3 is not parallel to M_1M_2 and the triangles M_1NM_2 and M_3D_2G are not congruent, hence NM_2 is not equal to GD_2. Aiton provides a

non potest per sinum versum anguli circulationis, sed est reciproce ut quadratum radii". Italics in the text.

[19] Ivi, p. 313. Original latin text: "Propositio vigesima (*sic*) prima et vigesima quinta, minorem exhibent vim centrifugam quam gravitatem Planetae in Solem ideoq: falsae sunt. Motus Planetae in orbe non pendet ab excessu gravitatis supra vim centrifugam (ut credit Leibnitius) sed Orbis incurvatur a gravitatis actione sola, cui vis centrifuga (ut reactio vel resistentia) semper est equalis et contraria per motus Legem tertiam a Newtono positam".

[20] See Gregory (1702, pp. 99–104).

[21] Newton in Edleston 1850, p. 312.

different interpretation[22]: the segment M_2L is not the Euclidean tangent, but the prolongation of the chord M_1M_2, that is the model presented by Leibniz is the “polygonal model“, in which the trajectory is interpreted as composed of a polygon with infinitesimal sides.[23] This interpretation is surely the correct one, taking into account that Leibniz in the *Illustratio Tentaminis* explicitly claimed:

> Furthermore, in general, let us consider (Fig. 31) two sides M_1M_2 and M_2M_3 of the polygon which constitutes the curve, and let us prolong one of them, M_1M_2, till L, so that the straight line M_2L represents the velocity, with which the mobile tends to continue its motion along the same line, after having passed through M_1M_2.[24]

Therefore Aiton’s interpretation is correct and no mistake is present in this mathematical reasoning by Leibniz.

A further question, connected to the preceding one, concerns the calculation of the centrifugal force: Varignon calculated the centrifugal force according to the concept of Leibniz’s tangent and discovered that its value is double that computed by Leibniz. He wrote to Leibniz on 6 December 1704.[25] Leibniz corrected the mistake and expressed his gratitude to Varignon for having discovered and communicated the mistake to him. In paragraph 12 of the *Illustratio Tentaminis*, Leibniz highlighted all the occurrences[26] of the *Tentamen* in which the expression double *conatus centrifugus* has to be replaced with *conatus centrifugus*.

Newton and the Newtonians also criticized Leibniz for the problem of second order differences: Newton and Keill objected that Leibniz’s assumption, according to which NP and D_2T_2 are equal (assumption 9) is not correct because

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[22] Aiton (1962, p. 37; 1964, pp. 119, 120; 1972, pp. 138–142), where the most clear explanation is provided. See also Bertoloni Meli (1993, p. 188).

[23] In his work *Nova Methodus pro Maximis et Minimis, itemque tangentibus [...]* (see Leibniz 1684, 1858, 1962, V, p. 223), Leibniz explicitly claimed that the tangent can be considered as the ordinary Euclidean tangent or as the prolongation of the side of the infinitangular polygon which can be thought as equivalent to the curve, at least as far as some mathematical considerations are concerned. For, Leibniz wrote: “to find the tangent is to draw the straight line which joins two points of a curve, whose distance is infinitely small, or the prolonged side of the infinitangular polygon, which, for us, is equivalent to the curve”. Original Latin text: “[...]*tangentem* invenire esse rectam ducere, quae duo curvae puncta distantiam infinite parvam habentia jungat, seu latus productum polygoni infinitanguli, quod nobis *curvae* equivalet.” (I am grateful to Professor Dr. Eberhard Knobloch for this indication). In the case I am analysing, the two representations of the tangent as ordinary tangent or as prolongation of the infinitangular polygon, are not equivalent as the mathematical consequences are different, according to which representation one uses. However: Leibniz had already spoken of the two representations, as the mentioned passage confirms, hence this makes Aiton’s interpretation quite plausible.

[24] Leibniz (1706, 1860, 1962, VI, p. 261). Original latin text: “Porro generatim concipiendo (fig. 31) duo Latera polygon curvam constituentis M_1M_2 et M_2M_3, et unum ex illis M_1M_2 continuando in L ita, ut recta M_2L celeritatem repraesentet, quo mobile post percursam M_1M_2 in eadem recta pergere tendit[...]”.

[25] Varignon to Leibniz 6 December 1704 in Leibniz (1859, 1962, IV, pp. 113–127).

[26] Leibniz (1706, 1860, 1962, VI, pp. 264–266).

the two segments differ by a second order infinitesimal and, in the context dealt with by Leibniz, where second differences are taken into account, an error of a second order infinitesimal is not acceptable. Aiton has shown that such a mistake does not exist in Leibniz's theory, if one interprets the word *tangent* as *prolongation of the chord* and that the mistake is a third order infinitesimal. We refer to Aiton's works for this problem.[27]

2.3 Elliptical Motion and Inverse Square Law

2.3.1 Leibniz's Assertions

The two paragraphs of the *Tentamen* in which Leibniz faced the motion on an ellipsis, where both the centres of the harmonic circulation and of the gravitational attraction are in the same focus, are the 18th and the 19th. The form in which Leibniz expounded the results is quite different in the published *Tentamen* and in the unpublished *Zweite Bearteitung* because, in this second work, he added the complete demonstrations of his propositions and a series of further mathematical propositions which allowed him to reach interesting astronomical results, whereas in the published version the demonstrations are only outlined and many results are missing. The literature, whose aim has been to provide the general ideas behind Leibniz's planetary theory and the analysis of the problems connected with Huygens', Newton's and Newtonians' critics, has underestimated the importance of the specific contributions expounded in the *Zweite Bearbeitung*.[28] I will face the results and methods of proof explained in this work, because all the results of the *Tentamen* are included here together with further ones.

Leibniz (see, Fig. 2.3) reminded the reader that the velocity of circulation (transverse velocity) can be expressed by the segments T_2M_3 or D_2M_3, since the difference between these two segments is negligible. The paracentric (radial) velocity is expressed by means of D_2M_2 and the velocity of the body in the orbit, which, Leibniz underlined, is composed of the two, by the segment M_2M_3 (*ivi*, par. 18, p. 172).

[27] Aiton (1962, p. 39; 1972, pp. 144–145).

[28] Up to now, the most complete report of the *Zweite Bearbeitung* is in Bertoloni Meli (1993, pp. 155–161).

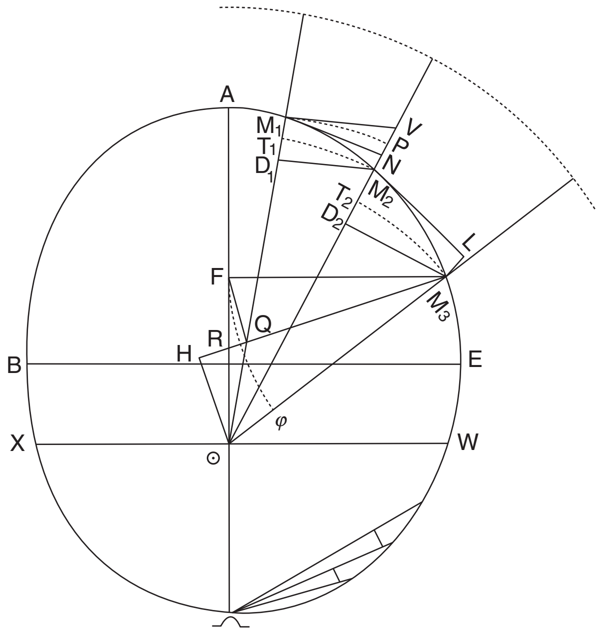

Fig. 2.3 Enlarged view of the Fig. 2.1b. I propose here an enlarged view of the figure Fig. 2.1b, because it can facilitate the reader to follow Leibniz's long reasoning, of which all the steps are explained in the running text.

Leibniz' reasoning (*ivi*, par. 18, pp. 172–174) is developed as follows: for the previous segments, which represent the three velocities, the proportion

$$D_2M_3 : D_2M_2 : M_2M_3 = BE : \sqrt{(F\Theta + \Theta\varphi)(F\Theta - \Theta\varphi)} : 2\sqrt{\Theta M_3 \cdot FM_3} \quad (2.4a)$$

holds, where F is the focus of the ellipsis in which there is not the sun and $FM_3 = \varphi M_3$.

Leibniz proved easily that the following proportion holds:

$$D_2M_3 : D_2M_2 : M_2M_3 = M_3H : H\Theta : \Theta M_3 \quad (2.4b)$$

where M_3H is the perpendicular to the ellipsis in M_3 and FQ and ΘH the perpendiculars from the foci to M_3H.

Therefore he has to prove

1)
$$M_3H : H\Theta : \Theta M_3 = BE : \sqrt{(F\Theta + \Theta\varphi)(F\Theta - \Theta\varphi)} : 2\sqrt{\Theta M_3 \cdot FM_3}$$

Since M_3H is perpendicular to M_2M_3 (ellipsis' arc), that is to its tangent, in M_3, then the angles HM_3F and $HM_3\Theta$ are equal, as follows from the properties of the tangents to the ellipsis, thence

2) the triangles $M_3H\Theta$ and M_3QF are similar and the angle ΘM_3F is bisected by M_3H.

Therefore, if M_3H saws ΘF in R, it holds, from a theorem of elementary geometry

3) $\Theta R : FR = M_3\Theta : M_3F$;
4) the triangles ΘHR and FQR are similar, hence
5) their homologous sides are as $\Theta R : FR$, that is, from 3), as $M_3\Theta : M_3F$ and hence as the homologous sides of the similar triangles $M_3H\Theta$ and M_3QF.
6) $M_3\Theta + M_3F = A\Omega$ because the figure is an ellipsis.
7) Let $M_3\Theta - M_3F = \Theta\varphi$.
8) from the properties of the ellipsis it is: $A\Omega^2 - \Theta F^2 = EB^2 = A\Omega \cdot XW$, where XW is the *latus rectum*.
9) (my addition) given a triangle abc, let l_a be the bisectrix of the angle in A, it is known that its measure is $l_a = \frac{2\sqrt{bc\,p(p-a)}}{b+c}$, where p is the half-perimeter. This expression can also be written as $\frac{\sqrt{bc\left[(b+c)^2-a^2\right]}}{b+c}$, from which the proportion $l_a : \sqrt{(b+c)^2 - a^2} = \sqrt{bc} : (b+c)$ follows. Leibniz applied this proportion to the triangle $FM_3\Theta$, considering the bisectrix M_3R. Therefore he could write:
10) $M_3R : \sqrt{A\Omega^2 - \Theta\varphi^2} = \sqrt{M_3\Omega \cdot M_3F} : A\Omega$. But, because of 8), $\sqrt{A\Omega^2 - \Theta\varphi^2} = BE$ and, elevating to square the relations 6) and 7), and subtracting the results of 7) from that of 6), one gets $M_3\Theta \cdot M_3F = \frac{1}{4}\left(A\Omega^2 - \Theta\varphi^2\right)$, so that Leibniz can obtain the proportion $M_3R : BE = \frac{1}{2}\sqrt{A\Omega^2 - \Theta\varphi^2} : A\Omega$.
11)
$$M_3R(\Theta H + QF) = 2\text{area}\left(\Theta \overset{\Delta}{M}_3 F\right)$$
12) $\frac{1}{2}\sqrt{A\Omega^2 - \Theta F^2} \cdot \sqrt{\Theta F^2 - \Theta\varphi^2} = 2\text{area}\left(\Theta \overset{\Delta}{M}_3 F\right)$, because of the Heron-formula applied at the triangle ΘM_3F, hence:
13) $M_3R(\Theta H + QF) = \frac{1}{2}\sqrt{A\Omega^2 - \Theta F^2} \cdot \sqrt{\Theta F^2 - \Theta\varphi^2}$, which can be written $M_3R : BE = \frac{1}{2}\sqrt{\Theta F^2 - \Theta\varphi^2} : (\Theta H + QF)$.
14) From 10) and 13) one gets $\sqrt{A\Omega^2 - \Theta\varphi^2} : A\Omega = \sqrt{\Theta F^2 - \Theta\varphi^2} : (\Theta H + QF)$, which can, obviously, be written as $(\Theta H + QF) : A\Omega = \sqrt{\Theta F^2 - \Theta\varphi^2} : \sqrt{A\Omega^2 - \Theta\varphi^2}$.
15) Applying 5) one has: $(\Theta H + QF) : (M_3\Theta + M_3F) = \Theta H : M_3\Theta$. But $M_3\Theta + M_3F = A\Omega$, hence from 14) and 15), Leibniz obtained
16) $\Theta H : M_3\Theta = \sqrt{\Theta F^2 - \Theta\varphi^2} : \sqrt{A\Omega^2 - \Theta\varphi^2}$ and elevating to square and subtracting
17) $\left(M_3\Theta^2 - \Theta H^2\right) : M_3\Theta^2 = \left(A\Omega^2 - \Theta F^2\right) : \left(A\Omega^2 - \Theta\varphi^2\right)$, that is $M_3H^2 : M_3\Theta^2 = BE^2 : \left(A\Omega^2 - \Theta\varphi^2\right)$. And finally, from 16) and 17), it follows
18)
$$M_3H : \Theta H : M_3\Theta = BE : \sqrt{\Theta F^2 - \Theta\varphi^2} : \sqrt{A\Omega^2 - \Theta\varphi^2}.$$

At the conclusion of this reasoning, Leibniz can claim:

> If a body is moved in an ellipsis, the velocity of circulation around a focus is at the paracentric velocity, that is the velocity with which the body descends towards the focus, as the minor or transverse axis is at the square root of the difference between the square of the focal distance and the square of the difference of the mobile's distances from the two foci.[29]

From this proposition a series of corollaries follow, which describe important properties of the motion in an elliptical orbit in which the centre of the forces is in one of the foci.

The first corollary, which Leibniz deduces easily from the explained reasoning, is: in an ellipsis, given a point P, the ratio between the paracentric (radial) velocity and the velocity of circulation (transverse velocity) is proportional to the ordinate PH, that is: the ratio between the velocity with which the planet approaches to or recedes from the sun is to the velocity of circulation as the distance of the planet from the apses-line.[30]

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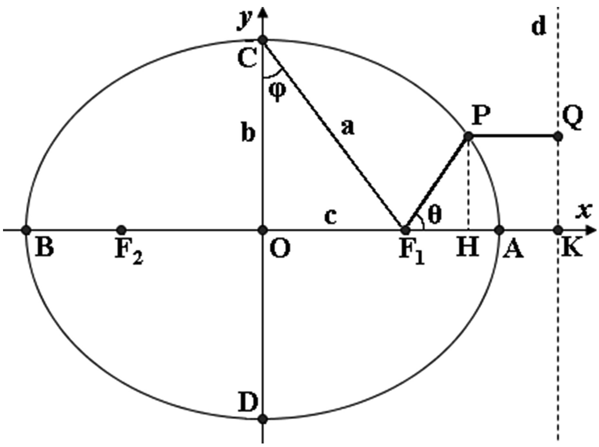
[29] Leibniz (1790?, 1860, 1962, VI, p. 174). Original latin text: "Si quid moveatur in Ellipsi, velocitas circulandi circa focum est ad velocitatem paracentricam, nempe descendendi ad focum vel a foco recedendi, ut axis minor seu transversus est ad latus differentiae inter potestatem distantiae focorum inter se et potestatem differentiae distantiarum mobilis a focis". At the end of the quotation, Leibniz used the Euclidean language to indicate the segments. I have provided a modern translation of "[...] ad latus differentiae inter potestatem distantiae focorum inter se et potestatem differentiae distantiarum mobilis a foci". It is, obviously, possible to give a translation, which is more faithful to Euclid's tradition: "[...] at the side of the difference between the power of foci's distance and the power of the difference of mobile's distances from the foci".

[30] Leibniz (1790?, 1860, 1962, VI, p. 175). This is an important relation between the radial and transverse velocity, which, in modern terms, can be proved like this:



the radial velocity is $v_r = dr/dt$ and the transverse velocity is $v_\theta = r \cdot d\theta/dt$, therefore $v_r/v_\theta = dr/r \cdot d\theta$. Since the orbit is an ellipsis, its polar equation is $r = \frac{ed}{1+e\cos\theta}$, where e is the eccentricity and d is the distance F_1K of the focus F_1 from the directrix d. Differentiating

Two other corollaries proved by Leibniz are:

1) in an ellipsis, given the mobile point P, the ratio between the velocity in the orbit and the velocity of circulation is as the mean proportional between the distances of P from the foci. (This corollary is a direct consequence of 18).
2) The velocities, with which a point M_3 changes its distance from the minor axis BE, are as the velocities with which it changes its distance from the focus Θ.

All these corollaries are missing in the published version of the *Tentamen*. These sets of results show that Leibniz's knowledge of the kinematical aspects of the planetary motions were profound and that he was an original thinker, as to this subject.

Let us now consider how Leibniz approached the problem of determining gravity attraction. In this case, too, the difference between the published and the unpublished version of the *Tentamen* is conspicuous. In both contributions the following reasoning exists:

Positions (referring to Fig. 2.3):

a) $A\Omega = q$; b) $\Theta F = e$ (eccentricity); c) $BE = b$ (minor axis); d) $\Theta M_2 = r$ (radius vector); e) $\Theta\varphi = OM_2 - FM_3 = 2r - q = p$; f) $WX = a = b^2/q$ (latus rectum); g) double area element$= 2M_1M_2\Theta = \vartheta a$, where ϑ is a constant element of time; h) D_2M_2 is the difference between two radii $= dr$; i) $ddr = d^2r$ second difference.

Reasoning:

1) D_2M_3 (=circulation)$= \vartheta a/r$ (for what was proved in paragraph 12);
2) $dr(= D_2M_2) : \vartheta a/r(= D_2M_3) = \sqrt{e^2 - p^2} : b$, for the proved theorem we have seen in details. Therefore
3) $br \cdot dr = \vartheta a\sqrt{e^2 - p^2}$. By differentiating, one gets the second order differences equation
4) $b \cdot dr^2 + br \cdot d^2r = -2pa\vartheta \cdot dr : \sqrt{e^2 - p^2}$. That is, replacing dr with its value deduced from 3), Leibniz got:
5) $d^2r = \left(b^2a^2\vartheta^2 - 2a^2qr\vartheta^2\right)/b^2r^3$.

But: d^2r is the element of paracentric velocity and the first expression in the right-hand member of the equation 5); furthermore $a^2\vartheta^2/r^3$ is the double *conatus centrifugus*. This means that the other expression represents the solicitation of gravity. Since $a = b^2/q$, such expression gets the form $2a\vartheta^2/r^2$. Leibniz multiplies this expression by the constant value $a/2$ and obtains $a^2\vartheta^2/r^2$, that is the square of the circulation. This means that the solicitation of gravity is as the square of the circulation, namely is as the inverse of the radius-square.

This concludes Leibniz's proof, which is explained both in the published and in the unpublished version of the *Tentamen*. However in the unpublished version Leibniz added a series of interesting considerations which do not exist in the

this expression one gets $\frac{dr}{d\theta} = \frac{r^2 \sin\theta}{d}$; therefore $\frac{v_r}{v_\theta} = \frac{r\text{sen}\theta}{d}$; but d is a constant and $r\text{sen}\theta = PH$; this is the corollary of Leibniz (diagram drawn from www.fmboschetto.it/tde2/gravit4.htm).

published one. First of all he developed some remarks and clarifications as to the differential equation in 3). Actually, what is far more interesting from a physical point of view, is the following long observation, which includes almost three pages in the edition by Gerhardt (pp. 178–180): up to this moment, Leibniz provided a representation of the planetary motions using the concept of *conatus centrifugus*. However the *conatus centrifugus* is referred to the harmonic motion of the vortex, that is to a circular harmonic motion. In fact, the orbit is an ellipsis and the movement in the ellipsis is harmonic, too, as Leibniz underlined. This means that another *conatus centrifugus* exists which depends only *indirectly* on the harmonic circulation of the vortex responsible for the mean motion of the planet and *directly* form the elliptical harmonic circulation, that is, from the true orbit of the planet. To indicate this *conatus* Leibniz used the generic expression *conatus excussorius* (used, in the published version of the *Tentamen*, as a synonymous of *conatus centrifugus*, as we have seen), maintaining the expression *conatus centrifugus* only in the case in which the motion is circular. Since the *conatus excussorius* is not, in general, referred to a circular motion, but to every curvilinear motion, Leibniz was in the need to exploit the concept of osculating circle to get a representation of its, which is useful for a mathematical treatment. The aim of Leibniz is rather interesting: he wanted to prove that, even in the case one adopts the representation through the *conatus excussorius*, one obtains the inverse square law, though by different steps than those used while exploiting the concept of *conatus centrifugus*. In the commentaries, I will deal with the possible reasons which induced Leibniz to deal with two different approaches. Leibniz represented the *conatus excussorius* like this (see Fig. 2.2): he considered in the ellipsis two infinitely near points M_2 and M_3, he drew the perpendiculars to the curve in these two points and indicated by S their intersection. This is the centre of the osculating circle. He drew the straight line M_3G, parallel to the line which is the tangent at the ellipsis in M_2. This line saws perpendicularly M_2S in K. Considering M_2M_3 as an infinitesimal arc of the osculating circle and adopting the same representation for gravity and the *conatus excussorius-centrifugus* as that used up to now, one has that M_2G represents the solicitation of gravity and M_2K the *conatus excussorius*. During the proof, Leibniz demonstrated two interesting theorems as to the kinematics of the elliptical motion considering the osculating circle.[31]

[31] The two theorems which Leibniz proved and used to prove the inverse square law by means of the *conatus excussorius* are: 1) in every straight line the solicitation of gravity M_2G is at the excussorius conate M_2K as M_3G (that is M_2M_3, which is the element of the curve or the orbital velocity) is at the velocity of circulation M_3D_2 (Leibniz 1690?, 1860, 1962, VI, pp. 178–179); 2) in every line of motion, it is $M_2K = \frac{M_3K^2}{SM}$, namely, to tell *à la Leibniz*: the *conatus excussori* are as the duplicate ratio of the orbital velocities directly and the simple ratio of the radii of the osculating circle inversely (*Ivi*, p. 179).

2.3.2 Commentaries: Two Different Models for Planetary Theory

In the *Zweite Berbeitung* of the *Tentamen*, Leibniz proposes, as a matter of fact, two different models to prove the inverse square law:

a) the model already used in the published version, in which the orbit is imagined as a polygon composed of triangles, with one infinitesimal side (that, whose extrema are the points of the trajectory). The infinitesimal sides of all the triangles compose the polygon.
b) the model in which the osculating circle is used and where, so to say, the main point of the reasoning becomes the variable centre S of the osculating circle.

Both models are referred to rotating reference frames. Bertoloni Meli underlines that:

> The additions to paragraph 19 consist in an attempt of reformulating the demonstration of the equation of paracentric motion without the differential calculus.[32]

This is true. Anyway some further specifications seem to me necessary: the description of the model a) has two conceptual cores:

i) Leibniz provided the geometrical expressions of his physical—both finite and infinitesimal quantities—one could say *à la Newton*.[33]—;
ii) Calculus is used to differentiate the expression of *dr*, so to get *ddr* as a function of centrifugal force and gravitational attraction.

As Bertoloni Meli rightly highlights, in model b) calculus is not used and Leibniz underlined the difference between the methods a) and b), as he writes:

> "[...] exactly as previously, in this same article we had found our result by means of a different way, that is by resorting to our differential calculus and by the theorem proposed in the article 15.[34]

I think the reasons why Leibniz provided a different proof are three:

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[32] Bertoloni Meli (1993, p. 159).

[33] In Newton's *Principia*, one could speak of "infinitesimal geometry" because Newton needs the instantaneous physical quantities, but his resort to calculus is—at least *explicitly*—limited enough in his masterpiece. He provides geometrical demonstrations in which the infinitesimal segments and areas are described as part of a figure. Since in many cases these segments represent potentially infinite quantities, it is possible to speak of *infinitesimal geometry*. The literature on this subject is conspicuous. I provide here only five references in which the problem is faced and explained: Bussotti and Pisano (2014a), in particular pp. 35–37; Bussotti and Pisano (2014b), in particular p. 435; De Gandt (1995), Guicciardini (1998, 1999, 2009). Leibniz uses here a similar technique.

[34] Leibniz (1790?, 1860, 1962, VI, p. 180). Original latin text: "[...] prorsus ut antea in hoc ipso praesente articulo per viam diversam, nempe ope calculi nostri differentialis et theorematis articulo 15 propositi inveneramus".

1) the one indicated by Bertoloni Meli;
2) every great mathematician is pleased to offer different demonstrations of the same proposition. Strictly connected to our context, let us think of Newton's *Principia*, in which numerous propositions are proved in different manners;
3) this is maybe the most important reason: we have to remember that Leibniz had the intention to provide the real physical-structural system of the world, not just a dynamical model. The planet, in its orbit, as a matter of fact, feels the *conatus excussorius*, not the *conatus centrifugus* because its orbit is not a circumference. This means that the model expressed in terms of *conatus excussorius* is more adherent to the forces really experienced by the planet, although the two models are equivalent from a dynamical point of view. This is the reason why Leibniz felt the need to add these considerations on the *conatus excussorius*. This does not mean that the model of the infinitangular polygon cannot be applied to an eccentric path, too.

2.4 The Final Description of the Solar System in the *Tentamen*

2.4.1 Leibniz's Assertions

Leibniz explained the mean motion of a planet in its orbit as due to the constant transverse velocity of the harmonic aethereal vortex in which the planet is afloat and the deviations from the mean motion in terms of two opposite tendencies: the *conatus excussorius/centrifugus*; the *solicitation of gravity*. In the paragraph 27, he supplied a unified vision of his planetary system, also based on two corollaries expounded in the paragraphs 21 and 24. In the former Leibniz proved that the ratio between gravity and centrifugal conate (really the half of the centrifugal conate) are as the distance of the planet from the sun; in the latter that the greatest speed of approaching to or of receding from the sun occurs when the distance of the planet from the sun is equal to ½ *latus rectum* of the ellipsis. This speed is equal to 0 at aphelion and perihelion.

Leibniz summarized his results in this manner: at the aphelion A, gravity is stronger than double centrifugal conate (really centrifugal conate, not double) because of the corollary in paragraph 21, hence the planet approaches the Sun. The speed with which the planet approaches the sun gets a maximum in W (corollary in 24), here the double centrifugal conate (really the simple centrifugal conate) begins to prevail on gravity and the approaching speed diminishes till the perihelion Ω (see Fig. 2.3) where its value is 0 and after Ω, this value becomes negative, this means that the planets begins to recede from the sun till the point X, where the receding velocity has a maximum and where gravity begins to prevail on the double centrifugal conate (really the simple centrifugal conate); the planet continues to recede until the aphelion A, where the receding velocity is null and the cycle begins once again. This is the general mechanism through which the planets rotate around the sun.

Leibniz concluded (paragraph 30) that if the centrifugal conate (really ½ centrifugal conate) is equal to gravity, the trajectory is a parabola; if it is stronger, the trajectory is a hyperbola whose focus is between the sun and the focus of the parabola, if the attraction is an attraction of levity and not of gravity, then the planet is repelled from the sun along the opposite hyperbola.

2.4.2 Commentaries

The description of the planetary motions given by Leibniz in the two versions of the *Tentamen* has its conclusion in the described picture, in which the motion of approaching to or receding from the sun is described as due to the difference between the solar attraction and the centrifugal force, while the deviation from the rectilinear path is due to the harmonic vortex. From a physical point of view, the most interesting aspect is the use made by Leibniz of the initial radial velocities for a given time t. Leibniz is aware that for a time $t_1 > t$ the motion is given by the radial velocity at time t and by the forces acting on the body. For—as we have seen—he underlines that—starting from the aphelion—the approaching velocity of a planet has a maximum when the solar attraction is equal to the *conatus centrifugus*. However, in the moment in which the *conatus* begins to prevail, the velocity of approaching begins to diminish, but this does not mean that the planet begins to recede. This happens only when, at the time t_2, the prevailing *conatus* has produced an effect which is superior to the combined effect of the gravity and of the velocity, which is direct inwards until t_2. This is the case in the perihelion. Therefore Leibniz considered the velocity as an *initial instantaneous datum* for every instant t. This datum changes in every instant. Thence a constant datum as the initial velocity when the elliptic motion is described in terms of centripetal forces *à la Newton* does not exist in Leibniz's description. For every instant the initial velocity changes, but, in that instant, it has to be considered as an initial constant of the motion. It is necessary to highlight that the description of the curvilinear motion using a rotating reference frame is not in contradiction with Newton's work, even if Newton himself thought otherwise, as we have seen. It is a description which uses a different point of view, but there is no contradiction among the two.

However, if the description in kinematical and dynamical terms provided by Leibniz is coherent with Newton's, the situation completely changes when one analyses the physical-structural point of view. In particular: why did Leibniz feel the need to provide such a description of planetary motion? Which are Leibniz's physical convictions and how did they influence his planetary theory? What is the real value of such a theory and in which sense can it represent a real alternative to Newton's conception? Who are the authors who can be considered Leibniz's reference points? The answers to these questions are the subjects of the next chapters.

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