Volume Frameworks and Deformation Varieties

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Abstract. A volume framework is a $(d+1)$-uniform hypergraph with real numbers associated to its hyperedges. A realization is given by placing the vertices as points in $\mathbb{R}^d$ in such a way that the volumes of the simplices induced by the hyperdges have the assigned values. A framework realization is rigid if its underlying point set is determined locally up to a volume-preserving transformation, otherwise it is flexible and has a non-trivial deformation space. The study of deformation spaces is a challenging problem requiring techniques from real algebraic geometry. Complementing a previous paper on *Realizations of volume frameworks*, we study several properties of deformation spaces, including singularities, for families of volume frameworks associated to polygons.

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1 Introduction

In this paper we investigate *deformation varieties* (or *configuration spaces*) for volume frameworks associated to polygons.

Volume Frameworks. A bar-and-joint framework in $\mathbb{R}^d$ may be conceived as a configuration of n labeled points respecting a system of distance constraints: certain pairs of points have a prescribed distance. Similarly, a volume framework in $\mathbb{R}^d$ may be conceived as a configuration of n points respecting a system of volume constraints: certain ordered $(d+1)$ subsets of the configuration span simplices of prescribed (signed) volume. In the first case, the system of constraints is given abstractly as a weighted simple graph on n vertices and equivalence of configurations is up to distance-preserving transformations of $\mathbb{R}^d$. In the second case, the system of constraints is given as a weighted $(d+1)$-uniform hypergraph on n vertices and equivalence of configurations is up to affine volume-preserving transformation.

Rigid and Flexible Structures. Classical questions in rigidity theory concern the characterization of *minimally rigid structures* [28], computing their *number of distinct realizations* [6,7], or, for flexible structures, obtaining descriptions of

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their space of *deformations.* These are notoriously challenging problems even for the well studied bar-and-joint case, where a full characterization is known only in dimension two [28,29]. Generalizing finite frameworks to a periodic setting [8–10] has led to many advances in understanding bar-and-joint frameworks. In a similar spirit, we have initiated in [7] the study of rigidity theoretic properties of volume frameworks, where the generalization goes from graphs with fixed edge lengths to hypergraphs with fixed hyperedge volumes.

Deformation Spaces of Polygons. Deformation spaces of bar-and-joint structures have been characterized only in special cases [23,30]. Particularly studied, and in different geometries, is the polygonal case [5,24–27]. In this paper we initiate the study of *deformation varieties* or *configuration spaces* for volume frameworks associated to *polygons.* More precisely, for given $d \geq 2$ and $n \geq d+3$, we examine volume frameworks corresponding to hypergraphs on vertices $1, ..., n$ with n (cyclically marked) hyperedges: $12...(d+1)$, $23...(d+2)$, ..., $n12...d$. Such *cyclic volume frameworks* provide interesting deformation spaces, especially when considered from the *complex projective* point of view.

Related Work. In [7] we considered minimally rigid volume frameworks and obtained bounds on the number of non-equivalent realizations. We also showed that *sparsity* conditions on the hypergraph are *not* sufficient for characterizing minimal rigidity for $d \geq 2$, just as Maxwell's sparsity conditions [29] for bar-and-joint frameworks are *not* sufficient for characterizing minimal rigidity for $d \geq 3$.

Here, we give particular attention to the *planar case* not only for comparison and contrast purposes with other instances of area constraint problems [15,17,18,20], but mostly for the explicit and geometrically revealing character of *singular configurations.*

Our Contribution. In this paper we show that, when comparing deformation varieties of polygonal bar-and-joint and volume frameworks, there are significant similarities, but also noticeable distinctions, already in dimension two. While singular configurations for bar-and-joint frameworks remain singular under *projective* transformations, this *is not the case for singular configurations of area frameworks.* This disproves an expectation formulated by Whiteley in [2].

Nevertheless, singular configurations of cyclic area frameworks are remarkable in their own right and are best understood in relation to classical theorems of Desargues and Brianchon. The more general phenomenon of singular configurations on cyclic volume frameworks is related to points on *rational normal curves.*

The planar family may be compared to bar-and-joint polygon spaces described in [5], since both scenarios lead to complex deformation spaces of a distinctive type: elliptic curves, $K3$ surfaces and higher dimensional varieties related to Calabi-Yau manifolds. Varieties of this type play a key role in Mirror Symmetry [12].

Our investigation advances the theory of volume frameworks in the context of a more general but natural theory of frameworks associated to various *classical groups* [31]. Volume frameworks correspond to special linear groups in the same way that bar-and-joint frameworks correspond to orthogonal groups.

2 Preliminaries

We review some basic definitions and formulations involving Grassmanians. See also [7].

Let $H = (V, \mathcal{E})$ be a $(d+1)$-uniform hypergraph with vertex set V and hyperedge set $\mathcal{E}$ made of certain ordered subsets I of cardinality $(d+1)$. Actually, we may take $V = \{1, 2, ..., n\}$, and refer to hyperedges $I \in \mathcal{E}$ as multi-indices $I = (i_0, ..., i_d)$,, with distinct $i_k \in V$, $k = 0, ..., d < n$.

A *placement* $p : V \to \mathbb{R}^d$ of the vertices in $\mathbb{R}^d$ allows the interpretation of the hyperedges $I \in \mathcal{E}$ as (possibly degenerated) marked oriented simplices in $\mathbb{R}^d$ and retaining their signed volume provides weights for $\mathcal{E}$. Assuming that at least one of the volumes is non-zero, the pair (H, p) gives a *volume framework* in $\mathbb{R}^d$.

Other placements $\tilde{p} : V \to \mathbb{R}^d$ which induce the same weights as (H, p) will be called *realizations* of the framework (H, p). Two realizations which differ by an affine volume-preserving transformation of $\mathbb{R}^d$ are considered equivalent. Equivalence classes of realizations define the *configuration space* of the framework. The realization space can be envisaged as a real algebraic variety and has thereby a topological structure. The configuration space is considered with the quotient topology. The connected component of the configuration represented by (H, p) is called the *deformation space* of the framework (H, p).

Since placements are assumed to have at least one full dimensional marked simplex, equivalence under affine volume-preserving transformations can be factored out by *pinning* one such simplex, that is, the configuration space can be represented by all realizations maintaining the initial placement for the chosen simplex. This shows that the configuration space of a volume framework can be represented as (the real points) of an affine real algebraic variety.

Another approach, used in [7] and adopted in the present paper, factors out equivalence by resorting to Grassmann manifolds and Plücker embeddings [16]. We denote by $G(k, m)$ the Grassmannian consisting of all k-dimensional vector subspaces in a vector space of dimension m. The ground field will be $\mathbb{R}$ or $\mathbb{C}$, according to context.

Let $p_1, ..., p_n \in \mathbb{R}^d$ denote the placements of the vertices by p. With p_i as column vectors, we consider the $(d+1) \times n$ matrix:

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \tag{1}$$

The $(d+1)$ rows of this matrix are independent and define a $(d+1)$-dimensional vector subspace in $\mathbb{R}^n$ and thereby a point of the Grassmannian $G(d+1, n)$. The Plücker (projective) coordinates of this point are given by all $(d+1) \times (d+1)$ minors:

$$V_I(p) = det \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_{i_0} & p_{i_1} & \dots & p_{i_d} \end{pmatrix} = det \left(p_{i_1} - p_{i_0} \dots p_{i_d} - p_{i_0} \right) \tag{2}$$

for $I = (i_0, ..., i_d)$, $1 \leq i_0 < ... < i_d \leq n$. It folows immediately from (1) and (2) that replacing p_i, $i = 1, ..., n$ by their images under an affine transformation does not change the corresponding point of the Grassmannian.

With obvious adaptation of signs, we may assume the hyperedges given as increasing subsets of $(d+1)$ indices, hence the volume constraints take the ratio form:

$$\frac{V_I(p)}{\delta_I} = \frac{V_J(p)}{\delta_J} \tag{3}$$

that is:

$$\delta_J V_I(p) = \delta_I V_J(p), \quad \text{for } I, J \in \mathcal{E}$$

where δ_I is the prescribed signed volume for hyperedge I. Note that the scaling aspect is resolved through the use of the projective setting. The essential parameters are the prescribed volumes up to proportionality.
Operating as above in $G(d+1, n)$ makes immediate the correspondence between prescribed volumes and marked Plücker coordinates. However, since the matrix (1) contains the row (11...1) the points under consideration can be identified with the points of the Grassmannian $G(d, n-1)$, in the quotient vector space $\mathbb{R}^n/\mathbb{R}(11...1)$.

Thus, for $\mathcal{E}$ of cardinality m and $\delta = (\delta_1 : \delta_2 : ... : \delta_m) \in \mathbb{P}_{m-1}$, an independent system of volume constraints (3) gives a projective linear section of the Grassmannian $G(d, n-1)$ of codimension $m-1$.

Since the dimension of $G(d, n-1)$ is $d(n-d-1)$, a necessary *sparsity* condition for *minimally rigid volume frameworks* is that $m = d(n-d-1)+1$ and on any subset of $n' > d$ vertices there are at most $d(n'-d-1)+1$ hyperedges.

3 Area Frameworks on Five Vertices

We start with some elementary examples and identify their singular configurations.

For three or four vertices there is, up to relabeling, a unique minimally rigid area framework hypergraph. With simplified notation, $\mathcal{E}$ is made of 123, for three vertices and 123, 234, 341 for four vertices. The latter is obtained from the former by an elementary operation of *'vertex addition'*. This can be defined in arbitrary dimension d and consists in adding a vertex and marking d new simplices which have the new point as a vertex. Clearly, vertex addition takes minimally rigid graphs to minimally rigid graphs.

Proposition 1. *Up to relabeling, a minimally rigid area framework on five vertices 1,...,5 is one of the following:*
(1) 512, 523, 534, 541, 524
(2) 541, 542, 543, 531, 421
(3) 345, 145, 245, 125, 123
(4) 512, 523, 534, 541, 123
(5) 123, 234, 345, 451, 512

Proof: It is more convenient to mark a triangle by the pair of vertices (edge) it does not contain. Sparsity permits at most three edges from any vertex. Since five edges on five vertices must form at least one cycle, cases (1), (2) and (3) result from a triangular cycle, (4) from a quadrangular cycle and (5) from a full pentagonal cycle. □

Remark: (1) to (4) can be obtained from a triangle by vertex additions and have, generically, unique realizations. Thus (5), which can be obtained from a minimally rigid graph on four vertices by a vertex splitting operation [7] is the more interesting case.

We determine here the *infinitesimally flexible* configurations for area frameworks on five vertices indexed 1,...,5, with marked triangles 123, 234,..., 512. There is no loss of generality if we 'pin' the triangle 123 with vertex 2 at the origin and vertices 1 and 3 at $(1, 0)$, respectively $(0, 1)$. Then 4 and 5 have coordinates at (β, y), respectively (x, α), with α and β fixed by the areas of 512 and 234.

The two remaining areas 345 and 451 impose conditions of the form:

$$det \begin{pmatrix} 1 & 1 & 1 \\ 0 & \beta & x \\ 1 & y & \alpha \end{pmatrix} = \alpha\beta - \beta - xy + x = b$$

$$det \begin{pmatrix} 1 & 1 & 1 \\ \beta & x & 1 \\ y & \alpha & 0 \end{pmatrix} = \alpha\beta - \alpha - xy + y = a$$

This gives:

$$y = x + c \quad \text{ with } c = (a + \alpha) - (b + \beta)$$

$$x^2 + (c - 1)x + (b + \beta - \alpha\beta) = 0$$

Infinitesimal flexibility means a double root in the last equation, which results in:

$$x = \frac{1 - c}{2}, \quad y = \frac{1 + c}{2} \quad i.e. \quad x + y = 1$$

Geometrically, this means that the parallel to edge $(1, 2)$ through vertex 4 meets the parallel to edge (23) through vertex 5 on the line (13). We'll mark this point as 2'.

Since infinitesimal flexibility is not affected by cyclic permutations of the indices, the above considerations yield the following:

Condition ($\circledast_5$): For all indices $i \in Z_5$, taken mod 5, the parallel to edge $(i, i+1)$ through vertex $i+3$ meets the parallel to edge $(i+1, i+2)$ through vertex $i+4$ on the line $(i, i+2)$. We denote this point as $(i+1)'$.

Infinitesimally Flexible Configurations: The pentagon $1'2'3'4'5'$ has edges parallel to the corresponding edges of the pentagon 12345 and is mutually inscribed with the pentagon 13524. Together, the vertices and edges of the pentagons $1'2'3'4'5'$ and 13524 determine a configuration of type 10_3, with each point on three lines and each line passing through three points [13,21,22]. In fact, as shown in Fig. 1, it is a *Desargues configuration*. Two triangles in perspective are $41'5'$ and $3'54'$, with center of perspective at 2 and pairs of corresponding edges meeting at the three collinear points 2', 3 and 1.

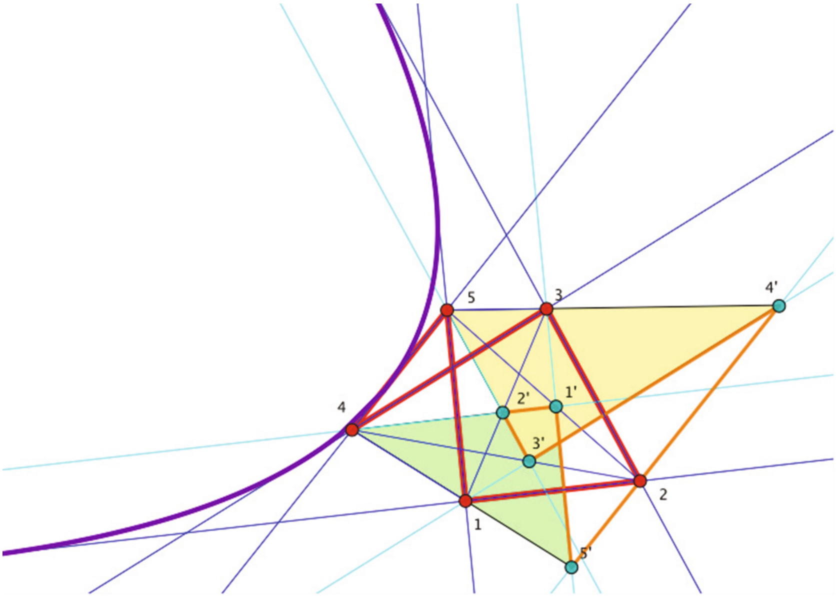

Fig. 1. Infinitesimal flexibility of a pentagon leads to a Desargues configuration. The lines supporting the pentagon edges are tangent to a parabola.

Remark: Since arbitrary projective transformations won't preserve this special class of affine polygons 12345, we see that the property of *infinitesimal flexibility* of volume frameworks is not invariant under projective transformations, in contrast to the bar-and-joint case. This aspect is emphasized in the following equivalent formulation:

Proposition 2. *A cyclic area framework on five vertices, with marked triangles* $(i-1, i, i+1)$*,* $i \in Z_5$*, is infinitesimally flexible if and only if all edges* $(i, i+1)$ *are tangent to a parabola.*

Proof: Infinitesimal flexibility amounts to condition $(\circledast_5)$ identified above, namely: the parallel to edge $(i, i+1)$ through vertex $i+3$ meets the parallel to edge $(i+1, i+2)$ through vertex $i+4$ on the line $(i, i+2)$, for all $i \in Z_5$. We have to prove its equivalence with the fact that the pentagon has edges tangent to a parabola. For one implication, we may observe that an affine parabola may be treated as a projective conic with a distinguished tangent ℓ_∞ (that tangent being the line at infinity). Suppose our pentagon has edges tangent to the conic

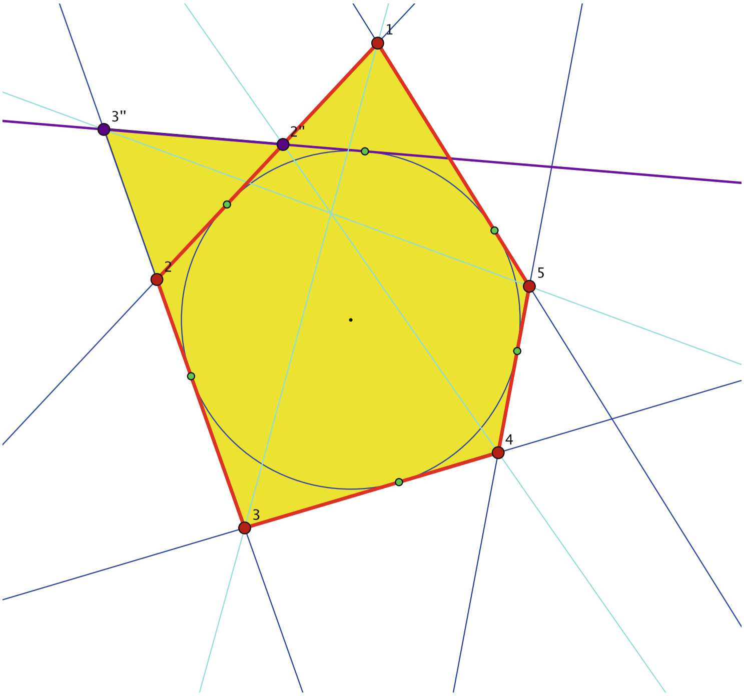

Fig. 2. Singular configuration of a pentagon. The line at infinity ℓ_∞ passes through points $2''$ and $3''$.

as illustrated in Fig. 2 and we want to establish condition $(\circledast_5)$ for $i = 1$. We mark the following intersections of lines:

$$(1,2) \cap \ell_\infty = 2'', \quad (2,3) \cap \ell_\infty = 3''$$

Then, $15433''2''$ is a hexagon with edges tangent to the conic. Thus, by Brianchon's theorem [21], the diagonals $(1,3),(5,3''),(4,2'')$ are concurrent. Affinely, this is condition $(\circledast_5)$.

The other implication amounts to using Brianchon's converse. $\square$

4 Cyclic volume frameworks

In this section we outline a relation between configuration spaces for certain volume frameworks and varieties expected to allow natural desingularizations to Calabi-Yau manifolds. It is analogous to the relation established in [5] between polygon spaces and Darboux varieties (which have natural resolutions to Calabi-Yau manifolds).

Definition. A $(d+1)$-uniform hypergraph will be called *cyclic* when defined on $n \geq d+3$ vertices 1,...,n by marking (cyclically) the n hyperedges: $12...(d+1)$, $23...(d+2)$, ..., $n12...d$.

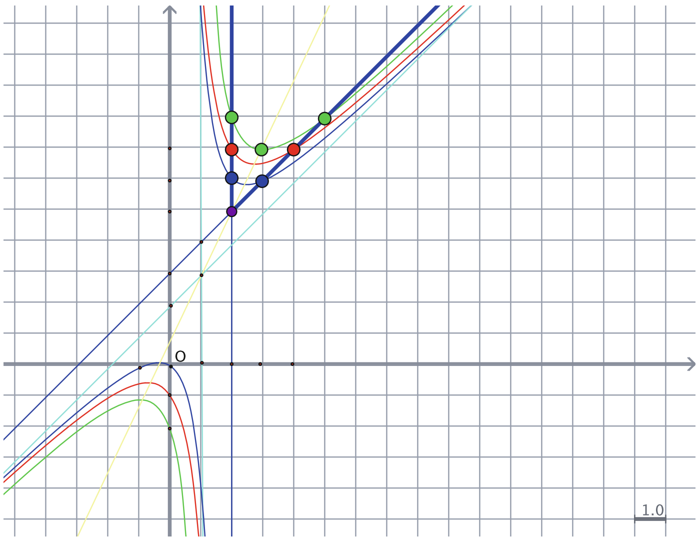

Fig. 3. The (d, n)-plane with hyperbolas $d(n-d-1)-(n-1) = D$ depicted for $D = 1, 2, 3$. There is an affine involution $(d, n) \mapsto (n-d-1, n)$ preserving all hyperbolas in this pencil.

The presentation will be streamlined if we adopt directly a *projective set-up*. Since the scale factor is determined by any non-zero volume, we may consider the equivalence of point configurations up to affine transformations and impose the *volume constraints as ratios of two volumes*. Thus, as seen above, point configurations modulo affine equivalence correspond to $G(d, n-1) \subset \mathbb{P}_{\binom{n-1}{d}-1}$ and constraints become hyperplane sections.

For cyclic frameworks we have $n-1$ (ratio) prescriptions resulting in a codimension $(n-1)$ linear section of the Grassmannian. While generic linear sections of this codimension do yield, over the complex field, smooth submanifolds with trivial canonical class i.e. *Calabi-Yau manifolds*, we have yet to determine particular properties of the sections dictated by framework constraints.

For n vertices in dimension d the space of parameters (the chosen volume ratios) is $\mathbb{P}_{n-1}$ and the complex configuration spaces have dimension: $D(d, n) = d(n-d-1)-(n-1) = (d-1)(n-1)-d^2$.

Recalling that $d \geq 2$ and $n \geq d+3$, one may notice in Fig. 3 the symmetry of this array of dimensions under the involution $(d, n) \mapsto (n-d-1, n)$ which exchanges the two border lines $d = 2$ and $n = d+3$:

$$D(d, n) = d(n-d-1)-(n-1) = D(n-d-1, n) \tag{4}$$

4.1 Association

Formula (4) is an indication that *association* is implicated in this pairing. The notion of association goes back to Castelnuovo and Coble [11]. It expresses the fact that the projective invariants of configurations of $(n+1)$ ordered points in $\mathbb{P}_d$ can be identified with the projective invariants of associated configurations of $(n+1)$ points in $\mathbb{P}_{n-d-1}$. See also [4,14].

We note first that a cyclic volume configuration of type (d,n) is completely described by the affine hyperplanes $[12...d], [23...(d+1)], ..., [n1...(d-1)]$. This gives a *projective* configuration of $(n+1)$ hyperplanes when we retain the hyperplane at infinity as the last element of this ordered list. Thus, we have $(n+1)$ points in the dual projective space $\mathbb{P}_d^*$. For general configurations, association provides, up to projective equivalence, corresponding configurations of $(n+1)$ points in $\mathbb{P}_{n-d-1}^*$, which give $(n+1)$ hyperplanes in $\mathbb{P}_{n-d-1}$. With the last hyperplane interpreted as the hyperplane at infinity, we have a configuration of n affine hyperplanes in C^{n-d-1}, up to affine equivalence i.e. a cyclic volume configuration of type $(n-d-1,n)$.

4.2 Cyclic Area Frameworks

Here the emphasis will be on $d=2$ i.e. *cyclic area frameworks*. We are going to use the following abbreviations:

$$\Delta_{ijk} = det \begin{pmatrix} 1 & 1 & 1 \\ p_i & p_j & p_k \end{pmatrix} \quad , \quad \Delta_{(i-1)i(i+1)} = \Delta_i$$

With indices considered *mod* n, the equations defining the (projective) configuration space of a cyclic area framework on n vertices are:

$$\frac{\Delta_1}{\delta_1} = \frac{\Delta_2}{\delta_2} = ... = \frac{\Delta_n}{\delta_n} \tag{5}$$

that is:

$$\delta_i \Delta_j = \delta_j \Delta_i, \quad 1 \le i < j \le n, \quad \delta = (\delta_1 : ... : \delta_n) \in \mathbb{P}_{n-1}$$

Solutions with $\Delta_1 = ... = \Delta_n = 0$ will be called *degenerate solutions*. For any $\delta_i \neq 0$, the hyperplane section $\Delta_i = 0$ consists of precisely these 'degenerate' solutions. As a divisor, or divisor class, it will be called the *degeneracy divisor* or the *divisor at infinity*.

If we look back at the source of our set-up, we have

$$G(2,n-1) \subset G(3,n) \subset \mathbb{P}_{(\binom{n}{3})-1} \cdot\cdot \rightarrow \mathbb{P}_{n-1}$$

where the last map is rational, with indeterminacy locus $\Delta_1 = ... = \Delta_n = 0$. Thus, our degeneracy locus is where this indeterminacy locus meets the configuration space.

If we denote by $\mathbb{P}_{\binom{n-1}{2}-n}(\delta)$ the codimension $(n-1)$ projective space defined by equations (5) in the linear span of the Grassmannian $G(2,n-1)$, the projective configuration space of a cyclic area framework on n vertices is the linear section:

$$X_{n-5} = X_{n-5}(\delta) = G(2,n-1) \cap P_{\binom{n-1}{2}-n}(\delta) \subset \mathbb{P}_{\binom{n-1}{2}-n} \tag{6}$$

4.3 Other Birational Models

The projective configuration space (6) obtained above involves no particular choice, but one may consider birationally equivalent realizations by 'pinning' a certain triangle. Suppose (up to rescaling and relabeling if necessary) that $\delta_1 = 1/2$ and pin triangle $(n12)$ with vertex 1 at the origin, and the two edges from it corresponding to the standard basis. This eliminates the action of affine transformations and the remaining vertices from 3 to $n-1$ can be parametrized by $(\mathbb{P}_2)^{n-3}$. If we label the hyperplane classes in each factor by h_i, according to the corresponding vertex, the remaining $n-1$ area prescriptions yield divisors of the following type (pull-back classes being represented by the same symbols):

$$h_3,\ h_3+h_4,\ h_3+h_4+h_5,\ h_4+h_5+h_6,\ ...,\ h_{n-3}+h_{n-2}+h_{n-1},\ h_{n-2}+h_{n-1},\ h_{n-1}$$

Note that the sum of these divisors on $(\mathbb{P}_2)^{n-3}$ is precisely the canonical divisor $3\sum_3^{n-1} h_i$, endorsing the expectation of Calabi-Yau birational models for our configuration spaces.

Remark: Triangles formed at the vertices of a polygon have been considered in some other contexts. We mention [17] and [18]. We briefly review here another realization of the configuration space, based on a simple extension of the approach used in [18] for equal areas.

Let $s_i = p_{i+1} - p_i,\ i \in Z_n$ be the edge vectors. Equations (5) can be written as:

$$\frac{1}{\delta_i} s_{i-1} \times s_i = \frac{1}{\delta_{i+1}} s_i \times s_{i+1}$$

which require the existence of scalars x_i such that:

$$s_{i-1} + \frac{\delta_i}{\delta_{i+1}} s_{i+1} + x_i s_i = 0, \quad i \in Z_n \tag{7}$$

With $\mu_i = \delta_i/\delta_{i+1}$, we define the $(n+1) \times n$ matrix:

$$X_\mu = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ x_1 & \mu_1 & 0 & ... & 0 & 1 \\ 1 & x_2 & \mu_2 & ... & 0 & 0 \\ 0 & 1 & x_3 & ... & 0 & 0 \\ . & . & . & .... & . & . \\ 0 & 0 & 0 & ... & x_{n-1} & \mu_{n-1} \\ \mu_n & 0 & 0 & ... & 1 & x_n \end{pmatrix}$$

The first row with all entries 1 will take account of the fact that the edge vectors of the polygon have zero sum. If we denote by S the $n \times 2$ matrix with rows s_i, equations (7) take the matrix form:

$$X_\mu S = 0 \tag{8}$$

Thus, a non-degenerate configuration gives a rank two S and therefore a rank $\leq (n-2)$ matrix X_μ. Conversely, when X_μ has rank $\leq (n-2)$, the choice of two independent vectors in its kernel provide a solution for (7). Generically, with rank $(n-2)$, all solutions are affinely equivalent, hence the affine algebraic variety defined in the n-space with coordinates x_i by the condition:

$$rank(X_\mu) \leq n-2 \tag{9}$$

gives another birational model of the configuration space of the area framework. There are some advantages from the point of view of elimination, but the projective completion offered by this realization seems entangled.

4.4 Hexagons and Elliptic Curves

We have seen above that, for generic area prescriptions, the cyclic area framework on five vertices i.e. the case of a generic pentagon, has a configuration space consisting of two points, corresponding to the degree two of the Grassmann-Plücker quadric $G(2,4) \subset \mathbb{P}_5$.

For a cyclic area framework on a hexagon, we shall determine first the divisor at infinity. In all cases, the degenerate configurations have only three distinct vertices and are completely described by the corresponding coalescence of vertices. They are:

$$(12,34,56),\ (23,45,61),(1,4,2356),\ (2,5,3461),\ (3,6,4512)$$

Note that they must be simple points on our curve since $deg(G(2,5)) = 5$. For the affine part $\Delta_1 \neq 0$ and we may consider the triangle 612 as 'pinned'. Then 3 runs on a parallel (3) to line (12) at a distance prescribed by the ratio δ_2/δ_1. Then we put $m = m(3) = (61) \cap (23)$ and with vertices $1m456$ we are in the pentagonal case. Thus, our curve is presented as a ramified double covering of $\mathbb{P}_1$ represented by the completed affine line (3).

Expecting generically a smooth curve, it must be elliptic, since with trivial canonical class. Thus, we should have four ramification points.

Smoothness at affine points can be verified as follows: assume $\delta_1 = 1$ and pin triangle 612 with $6 = e_2, 1 = 0, 2 = e_1$. Then 3 and 5 must run on lines $(3) = \delta_2 e_2 + a e_1$, respectively $(5) = \delta_6 + b e_2$. The locus of $4 = x$ is then obtained by eliminating a and b from:

$$\Delta_k = \delta_k, \quad k = 3,4,5$$

This process gives:

$$a = 1 + \frac{1}{x_2}(\delta_3 - \delta_2 + \delta_2 x_1), \quad b = 1 + \frac{1}{x_1}(\delta_5 - \delta_6 + \delta_6 x_2)$$

and results in the affine cubic:

$$x_1^2 x_2 + x_1 x_2^2 - \delta_2 x_1^2 - \delta_6 x_2^2 - (1 + \delta_2 + \delta_6 - \delta_3 - \delta_4 - \delta_5) x_1 x_2 +$$

$$+[\delta_2(\delta_6 - \delta_5) + (\delta_2 - \delta_3)]x_1 + [\delta_6(\delta_2 - \delta_3) + (\delta_6 - \delta_5)]x_2 - (\delta_2 - \delta_3)(\delta_6 - \delta_5) = 0$$

The non-singular example $\delta_2 = \delta_6 = 0$, $\delta_3 = \delta_5 = 1$, $\delta_4 = -1$ shows that the cubic is smooth in the generic case. On the other hand, an example of singularity at $(0,0)$ is obtained for $\delta_2 - \delta_3 = \delta_6\delta_5 = 0$. For $\delta_2 = ... = \delta_6 = 0$ the cubic is made of the three supporting lines of the triangle 612.

Remark: An examination of the pencil of projective cubics corresponding to $\delta_2 = \delta_6 = 0$, $\delta_3 = \delta_5 = 1$, illustrates the possibility of one or two loops for the curve of real points, when smooth. The pencil has the transposition symmetry $(x_1\ x_2)$ and takes the form:

$$x_1x_2(x_1 + x_2) - x_0^2(x_1 + x_2) - x_0^3 + \lambda x_0x_1x_2 = 0$$

For $\lambda = 0$ the cubic is smooth and the real points form a single loop. For $\lambda = 1$ the cubic decomposes into a line and a conic:

$$(x_0 + x_1 + x_2)(x_1x_2 - x_0^2) = 0$$

The real points avoid the singularities and give two loops. The actual topological transition happens when passing through the nodal case corresponding to the real solution (between 0 and 1) of the equation:

$$3\lambda^3 + 4\lambda^2 + 54\lambda - 33 = 0$$

Returning to the map defined above between the configuration curve $X = X(\delta) \subset G(2,5)$ and the projective completion of the affine cubic $\mathcal{C}_1 = \mathcal{C}_1(\delta) \subset \mathbb{P}_2$, one finds that the points 'at infinity' on $\mathcal{C}_1$ correspond with the degenerations: $(12,34,56)$, $(23,45,61)$ and $(1,4,2356)$. More precisely, with $(x_0 : x_1 : x_2)$ as homogeneous coordinates on $\mathbb{P}_2$, the correspondence is:

$$(0:1:0) = (12,34,56), \quad (0:0:1) = (23,45,61), \quad (0:1:-1) = (1,4,2356)$$

Furthermore, by letting 3 go to infinity on line (3), that is $a \to \infty$, and similarly for 6, we find that:

$$(1 : \delta_6 - \delta_5 : 0) = (2,5,3461), \quad (1 : 0 : \delta_2 - \delta_3) = (3,6,4512)$$

This explains the singularity observed above for $\delta_2 - \delta_3 = \delta_6 - \delta_5 = 0$ at $(1:0:0)$ as a self-intersection produced by the map $X \to \mathcal{C}_1$.

As point 3 moves on line (3), the parallels to line (23) (on which point 4 must lie) run through another intersection of the cubic with $x_2 = 0$, namely $p_{12} = (\delta_2 : \delta_2 - \delta_3 : 0)$. This implies that the involution of X defined by projecting on (3) can be interpreted on the cubic as the exchange of the two points of residual intersection of the cubic with a line running through $(\delta_2 : \delta_2 - \delta_3 : 0)$.

One can be more precise about *singular configurations* and observe that a singularity of the configuration curve must be a ramification point for all maps $X \to X/\sigma = \mathbb{P}_1$ associated to double coverings (and corresponding involutions σ) as described above for the case where $\mathbb{P}_1$ was the completed line (3). In fact, being a ramification point for two such maps will be enough. Looking back at the case of the pentagon, the condition for singularity amounts to the following geometric feature:

Condition ($\circledast_6$): the parallel to edge $(i+1, i+2)$ through vertex $i+5$ of the hexagon must meet the parallel to edge $(i+3, i+4)$ through vertex $i+6$ on the diagonal $(i+1, i+4)$, $i \in Z_6$.

In agreement with Proposition 2 this condition is equivalent with the following characterization:

Proposition 3. *An affine planar configuration representing $p \in X(\delta)$ is a singular point of the curve if and only if all six lines $(i, i+1)$ defined by the edges of the hexagon are tangent to a parabola.*

Remark: Again, the strictly affine character of this condition for singularity is in contrast with the projective invariance of singular configurations for bar-and-joint frameworks. Even so, our discussion will be cast in projective language.

Proof: An affine parabola is a projective conic with a distinguished tangent ℓ_∞ (that tangent being the line at infinity). Suppose our hexagon has edges tangent to the conic, as illustrated in Fig. 4. We want to establish condition $(\divideontimes_6)$ for $i = 0$. We mark the following intersections of lines:

$$(1,2) \cap \ell_\infty = 2'', \quad (3,4) \cap \ell_\infty = 3''$$

Then, $12''3''456$ is also a hexagon with edges tangent to the conic. Thus, by Brianchon's theorem, the diagonals $(1,4), (2'',5), (3'',6)$ are concurrent. Affinely, this is condition $(\divideontimes_6)$. Brianchon's converse yields the other implication. □

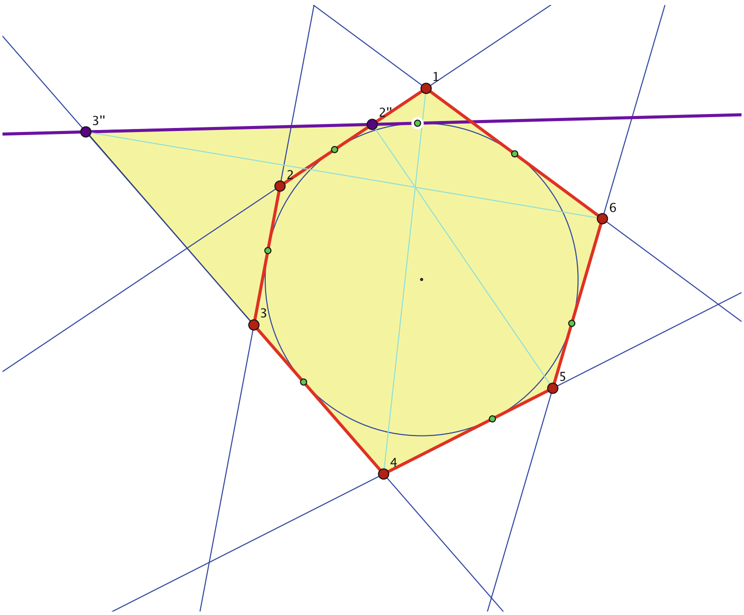

Fig. 4. Singular configuration of a hexagon. The line at infinity ℓ_∞ passes through points $2''$ and $3''$.

4.5 Heptagons and K3 Surfaces

The configuration space for a cyclic area framework on seven vertices will be a codimension six linear section of the Grassmannian $G(2,6) \subset \mathbb{P}_{14}$, hence a surface $Y = Y(\delta) \subset G(2,6)$. The parameter space for δ is $\mathbb{P}_6$.

The divisor at infinity can be described first by tabulating the degenerations of the heptagon which make all triples of vertices $(i-1, i, i+1)$, $i \in Z_7$, collinear. Degenerations must be triangles and we have two types:

$$a_i \equiv [i, (i+1, i+2), (i+5, i+6)](i+3, i+4)$$

$$b_i \equiv i[(i+1, i+2, i+5, i+6), i+3, i+4]$$

where vertices between parentheses (...) coincide, and vertices between brackets [...] are collinear.

Thus, there are fourteen curves of degenerations and they must be all lines since the degree of the Grassmannian $G(2,6)$ is precisely fourteen. We shall retain the notation a_i, b_i, $i \in Z_7$ for the lines themselves, or for the algebraic cycles they define on the surface Y.

Their pattern of intersection may be summarized as follows:

$$a_i b_i = 1, \quad a_i a_{i-2} = a_i a_{i+2} = 1, \quad b_i b_{i-3} = b_i b_{i+3} = 1$$

and all remaining pairs of lines are disjoint.

5 Conclusion

In this paper we considered deformation varieties for volume frameworks in $\mathbb{R}^d$ associated to cyclic hypergraphs on n labeled vertices. Natural parametrizations were obtained as linear sections of Grassmannians. The resulting (d, n) indexed family is invariant under association. Particular attention was given to the planar case $d = 2$, with geometric characterizations of singular configurations. The affine nature of these singularities contrasts with the projective invariance of infinitesimal rigidity and infinitesimal flexes for bar-and-joint frameworks.

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