Chapter 2
Managing a Portfolio of Risks

2.1 Introduction

Basic ideas concerning risk pooling and risk transfer, presented in Chap. 1, are progressed further in the present chapter, mainly with the following purposes:

1. to discuss key features of premium calculation when non-homogeneous portfolios are concerned, namely portfolios consisting of risks with various claim probabilities;
2. to analyze, more deeply, the riskiness of a portfolio and the tools which can be used to face potential losses, in particular introducing the role of the shareholders' capital;
3. to illustrate the possibility, for an insurance company, to transfer, in its turn, risk of losses to another insurer, namely the possibility to resort to reinsurance;
4. to address dynamic aspects of the management of insurance portfolios in a multi-year context.

As we will see, the actions undertaken by an insurer in order to deal with potential losses (see points 1 and 3 above) constitute important examples of risk management actions, in the specific framework of *insurance risk management*.

The "basic" insurance cover, namely the cover related to Case 2 (Possible loss with fixed amount) widely used in Chap. 1, will still be addressed while dealing with the issues mentioned above, in order to keep the presentation at an acceptable level of complexity.

2.2 Rating: The Basics

2.2.1 Some Preliminary Ideas

We refer to a portfolio of "basic" insurance covers, as defined in Chap. 1 (see, in particular Case 2 in Sects. 1.2.3, 1.4.2, 1.6.1 and 1.7.2), and we focus on the calculation of net premiums (i.e., not including loadings for expenses).

A. Olivieri and E. Pitacco, *Introduction to Insurance Mathematics*,
EAA Series, DOI 10.1007/978-3-319-21377-4_2

We assume that, for each risk, the premium is proportional to the benefit (that we also call the “sum insured”) paid in the case of a claim. Denoting (as in Chap. 1) with x the benefit for the generic risk, the premium is then given by $x\hat{p}$, where the quantity $\hat{p}$ represents the premium for one monetary unit of benefit. In the insurance language, $\hat{p}$ is commonly called the *premium rate*.

The following are natural choices:

a. set $\hat{p}$ equal to the probability of a claim, p, as implied by the equivalence principle (see, for example, Sect. 1.7.4 and formula (1.7.3) in particular), implemented on the realistic basis;
b. set $\hat{p}$ equal to the adjusted probability of a claim, p', so that riskiness is accounted for via an implicit safety loading (see formula (1.7.14) in particular).

Although we now do not deal with implicit safety loadings, the first choice is not the only feasible one, as we will see in the next sections. Anyhow, the premium rate should reflect, at least to some extent, the probability of a claim. As a consequence, a number of premium rates, $\hat{p}_1, \hat{p}_2, \ldots$, should be used for calculating the premiums for risks with various claim probabilities. The set of rules which link the premium rates to the claim probabilities constitutes a *rating system*. The rating system is the basis underlying the construction of an *insurance tariff* (which also includes loading for expenses, possible discounts, and so on).

2.2.2 The Portfolio Structure

We refer to a portfolio P which consists of n basic risks. As usual, let $X^{(j)}$ denote the random loss and hence the benefit for the jth risk:

$$X^{(j)} = \begin{cases} x^{(j)} & \text{in the case of claim} \\ 0 & \text{otherwise} \end{cases} \tag{2.2.1}$$

Let $p^{(j)}$ denote the (realistic) claim probability for the risk j:

$$p^{(j)} = \mathbb{P}[X^{(j)} = x^{(j)}] \tag{2.2.2}$$

The total portfolio payout is given by:

$$X^{[\mathrm{P}]} = \sum_{j=1}^{n} X^{(j)} \tag{2.2.3}$$

and the portfolio result (whatever the premium calculation principle and the premium rates) by:

$$Z^{[\mathrm{P}]} = \text{Total premium income} - X^{[\mathrm{P}]} \tag{2.2.4}$$

The expected portfolio result is then expressed by:

$$\begin{aligned}\mathbb{E}[Z^{[P]}] &= \text{Total premium income} - \mathbb{E}[X^{[P]}] \\ &= \text{Total premium income} - \sum_{j=1}^{n} \mathbb{E}[X^{(j)}]\end{aligned} \tag{2.2.5}$$

2.2.3 Homogeneous Risks

First, we assume that the n risks, which constitute the portfolio, are homogeneous in probability. Hence:

$$p^{(j)} = p \quad \text{for } j = 1, 2, \ldots, n \tag{2.2.6}$$

According to the equivalence principle, implemented on a realistic basis, the net premium for the jth risk, $P^{(j)}$, is then given by:

$$P^{(j)} = \mathbb{E}[X^{(j)}] = x^{(j)}\, p \tag{2.2.7}$$

Thus, the premium rate is equal to the (realistic) probability p, so that no safety loading is included in the premium.

At the portfolio level, the premiums expressed by (2.2.7) lead to the so-called *technical equilibrium* (clearly, in terms of expected value). Indeed, we have

$$\mathbb{E}[Z^{[P]}] = \sum_{j=1}^{n} P^{(j)} - \sum_{j=1}^{n} \mathbb{E}[X^{(j)}] = 0 \tag{2.2.8}$$

Thus:

$$\text{Total premium income} = \text{Total expected outgo} \tag{2.2.9}$$

Equation (2.2.9) expresses the *equivalence principle at the portfolio level*.

2.2.4 Non-homogeneous Risks

We now shift to non-homogeneous portfolios, namely portfolios consisting of risks with various claim probabilities. For simplicity, we refer to a portfolio P which consists of n_1 risks with claim probability p_1, and n_2 risks with claim probability p_2. Let $n = n_1 + n_2$. Without loss of generality, we assume $p_1 < p_2$.

The portfolio P can be split into two homogeneous *sub-portfolios*, P1 and P2, whose total payments are respectively given by:

$$X^{[\mathrm{P1}]} = \sum_{j=1}^{n_1} X^{(j)} \tag{2.2.10a}$$

$$X^{[\mathrm{P2}]} = \sum_{j=n_1+1}^{n} X^{(j)} \tag{2.2.10b}$$

Further, the related random results are given by:

$$Z^{[\mathrm{P1}]} = \text{Total premium income in P1} - X^{[\mathrm{P1}]} \tag{2.2.11a}$$

$$Z^{[\mathrm{P2}]} = \text{Total premium income in P2} - X^{[\mathrm{P2}]} \tag{2.2.11b}$$

The obvious choice for premium calculation consists in charging each risk with a premium calculated according to the related claim probability. This means that we set:

- in the sub-portfolio P1, i.e., for $j = 1, 2, \dots, n_1$:

$$P^{(j)} = \mathbb{E}[X^{(j)}] = x^{(j)}\, p_1 \tag{2.2.12}$$

- in the sub-portfolio P2, i.e., for $j = n_1 + 1, n_1 + 2, \dots, n$:

$$P^{(j)} = \mathbb{E}[X^{(j)}] = x^{(j)}\, p_2 \tag{2.2.13}$$

We have:

$$\mathbb{E}[Z^{[\mathrm{P1}]}] = \sum_{j=1}^{n_1} P^{(j)} - \sum_{j=1}^{n_1} \mathbb{E}[X^{(j)}] = 0 \tag{2.2.14a}$$

$$\mathbb{E}[Z^{[\mathrm{P2}]}] = \sum_{j=n_1+1}^{n} P^{(j)} - \sum_{j=n_1+1}^{n} \mathbb{E}[X^{(j)}] = 0 \tag{2.2.14b}$$

The random result for the portfolio P is given by:

$$Z^{[\mathrm{P}]} = Z^{[\mathrm{P1}]} + Z^{[\mathrm{P2}]} \tag{2.2.15}$$

We then find:

$$\mathbb{E}[Z^{[\mathrm{P}]}] = \mathbb{E}[Z^{[\mathrm{P1}]}] + \mathbb{E}[Z^{[\mathrm{P2}]}] = 0 \tag{2.2.16}$$

Hence, the premiums defined by (2.2.12) and (2.2.13) ensure the technical equilibrium, as expressed by (2.2.9), in both the sub-portfolios P1 and P2, and then, of course, in the whole portfolio P.

The technical equilibrium within each sub-portfolio is the natural consequence of adopting the equivalence principle, and implementing this principle with the appropriate claim probabilities. Conversely, the target of achieving the technical equilibrium within each sub-portfolio can be interpreted as a constraint in the premium calculation, and, as such, can be "relaxed," or replaced by weaker constraints.

In particular, we can assume that our aim is charging all the risks with the same premium rate, $\bar{p}$. This premium rate cannot ensure the equilibrium in each sub-portfolio; hence, the target is now the equilibrium within the whole portfolio P. Clearly, we will find $p_1 < \bar{p} < p_2$.

Possible aims of such a rating system are the following ones:

- simplify the insurance tariff;
- charge "reasonable" premiums to risks with a high claim probability, transferring part of the cost to risks with a low claim probability.

We also note that such a system may be mandatory, i.e., imposed by the insurance regulation, for some specific lines of business.

The premium for the jth risk, $j = 1, 2, \ldots, n$, is then given by

$$P^{(j)} = x^{(j)}\,\bar{p} \tag{2.2.17}$$

We have:

$$Z^{[\mathrm{P}]} = \sum_{j=1}^{n} P^{(j)} - \sum_{j=1}^{n} X^{(j)} = \bar{p}\sum_{j=1}^{n} x^{(j)} - \sum_{j=1}^{n} X^{(j)} \tag{2.2.18}$$

and:

$$\mathbb{E}[Z^{[\mathrm{P}]}] = \bar{p}\sum_{j=1}^{n} x^{(j)} - \left[p_1 \sum_{j=1}^{n_1} x^{(j)} + p_2 \sum_{j=n_1+1}^{n} x^{(j)}\right] \tag{2.2.19}$$

Our target is the technical equilibrium in the portfolio P:

$$\mathbb{E}[Z^{[\mathrm{P}]}] = 0 \tag{2.2.20}$$

We then find:

$$\bar{p}\sum_{j=1}^{n} x^{(j)} - \left[p_1 \sum_{j=1}^{n_1} x^{(j)} + p_2 \sum_{j=n_1+1}^{n} x^{(j)}\right] = 0 \tag{2.2.21}$$

and finally:

$$\bar{p} = \frac{p_1 \sum_{j=1}^{n_1} x^{(j)} + p_2 \sum_{j=n_1+1}^{n} x^{(j)}}{\sum_{j=1}^{n} x^{(j)}} \tag{2.2.22}$$

Hence, the premium rate $\bar{p}$ is the arithmetic weighted average of the probabilities p_1 and p_2, and the weights are given by the total amount of sums insured in the sub-portfolios P1 and P2 respectively.

It is interesting to note that, if all the sums insured are equal to x, formula (2.2.22) reduces to

$$\bar{p} = p_1 \frac{n_1}{n} + p_2 \frac{n_2}{n} \tag{2.2.23}$$

Thus, the premium rate $\bar{p}$ is the arithmetic weighted average of the probabilities p_1 and p_2, weighted by the sub-portfolio sizes.

2.2.5 A More General Rating System

Rating systems defined by formulae (2.2.12), (2.2.13) and, respectively, (2.2.17) constitute particular cases of a more general structure.

In order to define a rather general rating system, let $\bar{p}_1, \bar{p}_2$ denote two premium rates, charged to risks with claim probability p_1, p_2 respectively. Premiums are then given by the following formulae:

- in the sub-portfolio P1, i.e., for $j = 1, 2, \ldots, n_1$:

$$P^{(j)} = x^{(j)} \bar{p}_1 \tag{2.2.24}$$

- in the sub-portfolio P2, i.e., for $j = n_1 + 1, n_1 + 2, \ldots, n$:

$$P^{(j)} = x^{(j)} \bar{p}_2 \tag{2.2.25}$$

Let the following inequalities hold:

$$p_1 \leq \bar{p}_1 \leq \bar{p}_2 \leq p_2 \tag{2.2.26}$$

Assume that the premium rates $\bar{p}_1$ and $\bar{p}_2$ ensure the technical equilibrium in the portfolio P, that is, $\mathbb{E}[Z^{[\mathrm{P}]}] = 0$. Then, $\bar{p}_1$ and $\bar{p}_2$ must be solutions of the following equation:

$$\bar{p}_1 \sum_{j=1}^{n_1} x^{(j)} + \bar{p}_2 \sum_{j=n_1+1}^{n} x^{(j)} - \left[p_1 \sum_{j=1}^{n_1} x^{(j)} + p_2 \sum_{j=n_1+1}^{n} x^{(j)} \right] = 0 \qquad (2.2.27)$$

In particular, if all the sums insured are equal to x, formula (2.2.27) reduces to:

$$\bar{p}_1 n_1 + \bar{p}_2 n_2 = p_1 n_1 + p_2 n_2 \qquad (2.2.28)$$

which can also be written as follows:

$$\bar{p}_1 \frac{n_1}{n} + \bar{p}_2 \frac{n_2}{n} = p_1 \frac{n_1}{n} + p_2 \frac{n_2}{n} \qquad (2.2.29)$$

Thus, the weighted arithmetic mean of the premium rates $\bar{p}_1$ and $\bar{p}_2$ must be equal to the weighted arithmetic mean of the claim probabilities p_1 and p_2, with the same weights.

We note that:

- setting $\bar{p}_1 = p_1$ and $\bar{p}_2 = p_2$, we find the "natural" rating system, with premiums differentiated according to the claim probabilities (see (2.2.12) and (2.2.13));
- setting $\bar{p}_1 = \bar{p}_2$, we find the system with just one premium rate (see (2.2.17));
- to find other rating systems, only the cases such that

$$p_1 < \bar{p}_1 < \bar{p}_2 < p_2 \qquad (2.2.30)$$

  have to be considered.

We note, from Eqs. (2.2.27) and (2.2.28), that the unknowns $\bar{p}_1$ and $\bar{p}_2$ cannot be univocally determined. Then, an additional condition is required, for example, $\bar{p}_1 = \alpha \bar{p}_2$, or $\bar{p}_1 = \bar{p}_2 - \beta$, with $\alpha < 1$, $\beta > 0$, and such that inequalities (2.2.30) are fulfilled.

Clearly, the aim of such a rating system is to keep premium rates differentiated, while charging a "reasonable" premium to risks with a higher claim probability, and then transferring part of the cost to risks with a lower probability.

Remark Although inequalities (2.2.30) are quite reasonable, in principle we could also assume,

$$\bar{p}_1 < p_1 < p_2 < \bar{p}_2 \qquad (2.2.31)$$

that is, aiming to "reward" risks with a low probability, while "penalizing" risks with a high probability.

2.2.6 Rating Systems and Technical Equilibrium

When rating systems other than those constructed by setting the premium rates equal to the claim probabilities are adopted, problems concerning the technical equilibrium may arise. To discuss such problems, we refer, for simplicity, to a portfolio in which all the sums insured are equal to x.

Looking at Eqs. (2.2.23) and (2.2.28), we note that the premium rate $\bar{p}$ and the premium rates $\bar{p}_1, \bar{p}_2$ depend on the sizes n_1 and n_2 that we have assumed for the two sub-portfolios P_1 and P_2. However, when the premiums, based on the premium rate $\bar{p}$ or $\bar{p}_1, \bar{p}_2$, are charged to a group of new applicants for the insurance cover, the actual sizes of the sub-groups of risks with claim probability p_1 and p_2 respectively are unknown. Thus, n_1 and n_2 should only be understood as estimates of the actual numbers of applicants.

Let n_1^*, n_2^* denote the actual sizes of the sub-groups, and $n^* = n_1^* + n_2^*$. If

$$\frac{n_1^*}{n^*} = \frac{n_1}{n} \quad \left(\text{and then } \frac{n_2^*}{n^*} = \frac{n_2}{n}\right) \tag{2.2.32}$$

the technical equilibrium is ensured, as the relative sizes of the actual groups coincide with the estimated relative sizes (see formulae (2.2.23) and (2.2.29)).

Conversely, assume that

$$\frac{n_1^*}{n^*} \neq \frac{n_1}{n} \quad \left(\text{and then } \frac{n_2^*}{n^*} \neq \frac{n_2}{n}\right) \tag{2.2.33}$$

In this case, the technical equilibrium is not achieved. In particular, if

$$\frac{n_1^*}{n^*} < \frac{n_1}{n} \quad \left(\text{and then } \frac{n_2^*}{n^*} > \frac{n_2}{n}\right) \tag{2.2.34}$$

a negative expected result follows. In formal terms, referring to the portfolio P, the following relations hold.

- In the case of one premium rate $\bar{p}$:

$$\begin{aligned}
\text{Total premium income} &= x n^* \bar{p} = x\,(n_1^* \bar{p} + n_2^* \bar{p}) \\
\text{Total expected outgo} &= x\,(n_1^* p_1 + n_2^* p_2) \\
\text{Expected portfolio result} &= x\,\big(n_1^*\,(\bar{p} - p_1) + n_2^*\,(\bar{p} - p_2)\big)
\end{aligned}$$

- In the case of two premium rates $\bar{p}_1, \bar{p}_2$:

$$\begin{aligned}
\text{Total premium income} &= x\,(n_1^* \bar{p}_1 + n_2^* \bar{p}_2) \\
\text{Total expected outgo} &= x\,(n_1^* p_1 + n_2^* p_2) \\
\text{Expected portfolio result} &= x\,\big(n_1^*\,(\bar{p}_1 - p_1) + n_2^*\,(\bar{p}_2 - p_2)\big)
\end{aligned}$$

Example 2.2.1 Two different rating systems, A and B, are defined. Both the systems are constructed by assuming that the number of risks with the lower probability, p_1, is twice the number of risks with the higher probability, p_2, that is, $n_1 = 2n_2$; see Table 2.1.

Table 2.1 Claim probabilities and premium rates

n_1	n_2	p_1	p_2	Rating A	Rating B	
				$\bar{p}$	$\bar{p}_1$	$\bar{p}_2$
4 000	2 000	0.005	0.008	0.006	0.0055	0.007

Table 2.2 Expected outgo, premium income, and expected portfolio result

n_1^*	n_2^*	Expected outgo	Premium income		Expected result	
			Rating A	Rating B	Rating A	Rating B
8 000	4 000	72 000	72 000	72 000	0	0
3 000	3 000	39 000	36 000	37 500	−3 000	−1 500

Table 2.2 shows the total expected outgo, the total premium income, and the expected portfolio result, referred to two actual portfolios P, the first one leading to an equilibrium situation, whilst the second one (for which inequalities (2.2.34) hold) implies an expected outgo greater than the premium income, whatever the rating system adopted, and hence a negative expected result. As regards the portfolio leading to a non-equilibrium situation, the system A obviously implies a higher loss. ❑

A practical problem: is the situation described by inequalities (2.2.34) a likely one? The following points provide an answer to this critical question.

- The (expected) equilibrium at the portfolio level is based on a transfer of money (shares of premiums) from insureds charged with a premium higher than their "true" premium, i.e., the premium resulting from the probability of a claim, to insureds charged with a premium lower than their "true" premium. In the technical language, such a transfer of money is called *solidarity* (among the insureds). In particular, referring for simplicity to the case of one premium rate $\bar{p}$, the generic insured with claim probability p_1 transfers to the pool the amount

$$S_1^{(j)} = x^{(j)}\,\bar{p} - x^{(j)}\,p_1 > 0 \tag{2.2.35}$$

  whereas the pool transfers to the generic insured with claim probability p_2 the amount

$$S_2^{(j)} = x^{(j)}\,\bar{p} - x^{(j)}\,p_2 < 0 \tag{2.2.36}$$

  The amounts $S_1^{(j)}$ and $S_2^{(j)}$ are usually called *solidarity premiums* (positive and negative, respectively).
- Rating systems based on solidarity may cause *self-selection*, as individuals forced to provide solidarity to other individuals can reject the policy, moving to other insurance solutions (or, more generally, risk management actions). The resulting

effect is a portfolio with a (relative) prevalence of risks with the higher claim probability. Thus, from the insurer's point of view, self-selection constitutes *adverse selection*.

- The severity of this self-selection phenomenon depends on how people perceive the solidarity mechanism, as well as on the premium systems adopted by competitors in the insurance market.
- So, in practice, solidarity mechanisms can work provided that they are mandatory (for example, imposed by insurance regulation) or they constitute a common market practice.

2.2.7 From Risk Factors to Rating Classes

The rating system defined by formulae (2.2.17)–(2.2.23) adopts one premium rate $\bar{p}$ versus two claim probabilities p_1, p_2. The underlying rationale can be extended to more general situations.

When we define a "population," we have to adopt a rigorous criterion to decide whether a given "individual" belongs to the population (i.e., is a "member" of the population) or not. For example, the population can be defined as consisting of all males currently alive, born in Italy in the period 1950–1970. Although the definition is rigorous, we are aware that the population consequently defined is rather heterogeneous, in particular with regard to the risk of death. Indeed, individuals can have various ages, can be more or less healthy, can have a more or less risky occupation, etc. Thus, we can recognize various *risk factors*(age, current health conditions, occupation, and so on), which should be taken into account when stating, for example, the individual probability of dying within one year.

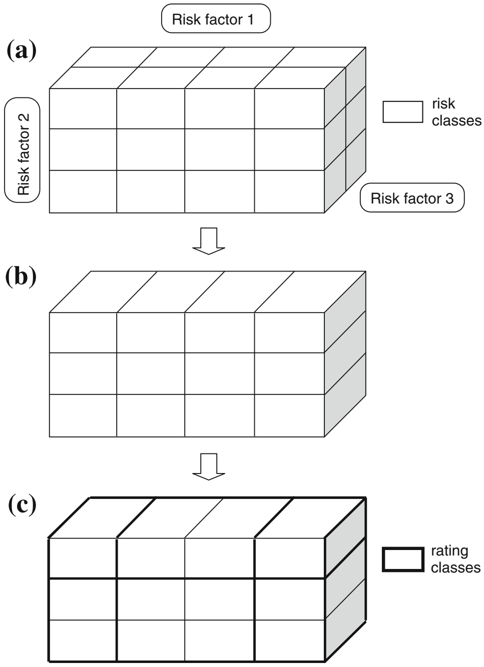
Problems concerning heterogeneity and the use of risk factors in life and non-life insurance calculations will specifically be addressed in Chaps. 3, 4 and 9. Now, we just provide a first insight into the role of risk factors in the pricing procedures.

We assume that each risk factor can take one out of a given (integer) number of "values," either scalar (e.g., the age) or nominal (e.g., the gender). Figure 2.1 refers to a population for which three risk factors have been initially recognized, with 4, 3, and 2 values, respectively (each factor is represented by a coordinate). Thus, the population has been split into $4 \times 3 \times 2 = 24$ *risk classes* (see panel (a)).

In principle, a specific claim probability, and hence a specific premium rate, should be determined for each risk class. However, the resulting tariff structure could be considered too complex, or some premium rates too high. Then, a first simplification could be obtained disregarding one of the risk factors; see Fig. 2.1, panel (b), which shows that risk factor 3 has been disregarded. A further grouping of risk classes is illustrated by panel (c), in which we see the grouping of some values of risk factors 1 and 2. As the final result, the population is split into $3 \times 2 = 6$ *rating classes*.

When two or more risk classes are aggregated into one rating class, some insureds pay a premium higher than their "true" premium, i.e., the premium resulting from the risk classification, while other insureds pay a premium lower than their "true"

Fig. 2.1 From risk factors to rating classes



premium. Thus, the equilibrium inside a rating class relies on a money transfer among individuals belonging to different risk classes. As mentioned above, this transfer is usually called solidarity (among the insureds).

When the rating classes coincide with the risk classes, the rating system is "tailored" on the features of each insured risk (at least to the extent these features can be detected), and no solidarity transfer works. Conversely, the solidarity effect is stronger when the number of rating classes is smaller, compared with the number of risk classes.

Remark Even if the rating classes coincide with the risk classes, a "residual" heterogeneity still affects the insured risks inside each rating class, because of the presence of *unobservable risk factors*; for example, genetic characteristics as regards mortality, personal attitude to cause accidents in car insurance, and so on. Thus, an unavoidable degree of solidarity among insured risks is implied by unobservable risk factors, whatever the number of rating classes.

The residual heterogeneity (and hence the solidarity) can be reduced if the individual claim experience allows the insurer to "learn" about the features of each insured risk. In particular, in non-life insurance rating classes can be defined, for example, accounting for the numbers of claims experienced in the previous years. So, an *individual experience rating* (also called *merit rating* in car insurance) determines an *a-posteriori risk classification*, whereas an *a-priori risk classification*, based on rating factors known in advance, works at policy issue. This topic will be specifically dealt with in Chap. 9.

In the field of private insurance, an extreme case is achieved when one rating class only relates to a large number of underlying risk classes. Outside the area of private insurance, the solidarity principle is commonly applied in social security. In this field, the extreme case arises when the whole national population contribute to fund the benefits, even if only a part of the population itself is eligible to receive benefits; so, the burden of insurance is shared among the community.

2.2.8 Cross-Subsidy: Mutuality and Solidarity

Mutuality and solidarity constitute two forms of *cross-subsidy* among the insureds (or, in general, among the members of a pool). However, some important points should be stressed in order to single out the different features of these forms of cross-subsidy.

First, mutuality is an implication of the pooling process (and, in particular, of the risk transfer to an insurance company), as clearly emerges in Sect. 1.7.4. Conversely, solidarity among the insureds is the straight consequence of the adoption of a rating system with a number of rating classes smaller than the number of risk classes. So, the presence and the magnitude of solidarity effects strictly depend on the tariff structure (see, in particular, the amounts of solidarity premiums, expressed by formulae (2.2.35) and (2.2.36)).

Second, it is worth noting that the mutuality affects the benefit payment phase, so that the "direction" and "measure" of the mutuality effect in a portfolio (or, in general, in a pool of risks) are only known ex-post. Conversely, the solidarity (possibly) affects the premium income phase, and hence its direction and measure are known ex-ante.

Figure 2.2 illustrates cross-subsidy in a pool of insured risks.

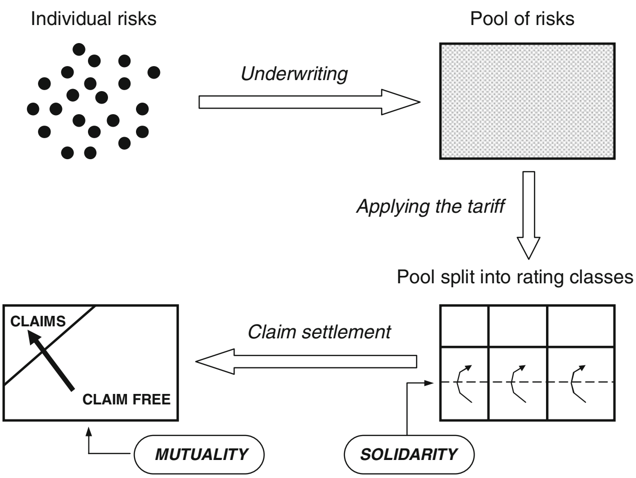


Fig. 2.2 Mutuality and solidarity in a pool of insured risks

2.3 Facing Portfolio Riskiness

Risks inherent in the results obtained by managing a pool of risks have been already discussed in Chap. 1 (see Sect. 1.6). We now turn back on these issues, referring to a portfolio of insured risks. In particular, we focus on the following aspects:

- what are the "components" of the risk inherent in portfolio outgoes (and hence in portfolio results);
- what are the elements of an appropriate toolkit for managing this risk.

2.3.1 Expected Outgo versus Actual Outgo

We consider a portfolio of n basic insurance covers (see Case 2 in Sects. 1.7.2 and 1.7.4), in which all the sums insured are equal to x, and we assume that the portfolio is homogeneous with respect to the claim probability; we denote with p this probability.

Let f denote the observed relative claim frequency, i.e., $f = \frac{k}{n}$, where k is the observed number of claims. If $f = p$, the equilibrium is actually achieved (of course, provided that the premium rate is set equal to p), as the actual outgo, given by nxf, is equal to the expected outgo, nxp, and hence to the premium income. Indeed, we have

$$\sum_{j=1}^{n} P^{(j)} = nxp = nxf \tag{2.3.1}$$

Conversely, we may find that $f \neq p$, and clearly our concern is for the case $f > p$. Figure 2.3 sketches three portfolio stories in which we find that, in various years, $f \neq p$. Reasons underlying this inequality may be quite different in the three stories.

- In Fig. 2.3a, we see that the observed claim frequency randomly fluctuates around the probability, namely around the expected frequency. This possibility is usually denoted as the *risk of random fluctuations*, or the *process risk*.

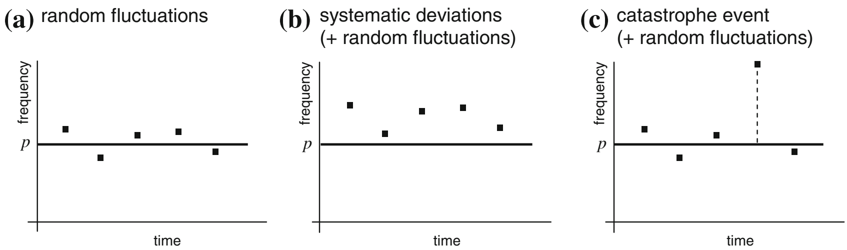


Fig. 2.3 Observed frequency versus probability

- On the contrary, Fig. 2.3b depicts a situation in which, besides random fluctuations, we see "systematic" deviations from the expected frequency; likely, this occurs because the assessment of the probability p does not capture the true nature of the insured risks. This possibility is usually called the *risk of systematic deviations*, or the *uncertainty risk*, referring to the uncertainty in the assessment of the expected frequency.
- In Fig. 2.3c, the effect of a "catastrophe," which causes a huge number of claims in a given year, clearly appears. This possibility is commonly known as the *catastrophe risk*.

2.3.2 Risk Components and Risk Factors

Three *risk components* have been singled out, namely: the risk of random fluctuations, the risk of systematic deviations, and the catastrophe risk.

All the components impact on the monetary results of the portfolio. However, the severity of the impact strongly depends on the portfolio structure, and the portfolio size in particular.

- The severity of the risk of random fluctuations decreases, in relative terms, as the portfolio size increases. This feature is the direct consequence of the risk pooling (see Sect. 1.6.1), and thus is commonly known as the *pooling effect*, or the *diversification via pooling*. Nevertheless, the distribution of the sums insured plays an important role in determining the absolute and the relative portfolio riskiness, as we will see in Sects. 2.3.3 and 2.3.4.
- The severity of the risk of systematic deviations is independent, in relative terms, of the portfolio size (as we will see in Sect. 2.3.11). Indeed, systematic deviations affect the pool as an aggregate. Conversely, the total impact on portfolio results increases as the portfolio size increases.
- The severity of the catastrophe risk can be higher due, for example, to a high concentration of insured risks within a geographic area.

The portfolio size, the distribution of the sums insured, and the geographic concentration are portfolio characteristics which determine a more or less severe impact of risk causes and related components on the total payout. These (and other) characteristics are named *risk factors*.

Remark We note that the expression "risk factors" has a rather broad meaning, as it can be used to denote characteristics of individual risks (see Sect. 2.2.7), as well as to denote portfolio characteristics (as mentioned in the present section). In both the cases, however, a risk factor determines some quantitative features of a monetary result: for example, the probability of a loss for an individual risk, and the probability distribution of the total payout for a portfolio of risks.

In the following sections we focus on the risk of random fluctuations.

Example 2.3.1 The distributions of the sums insured in three portfolios are sketched in Fig. 2.4. We assume that the average sum insured, $\bar{x}$, and the number of risks, n, are

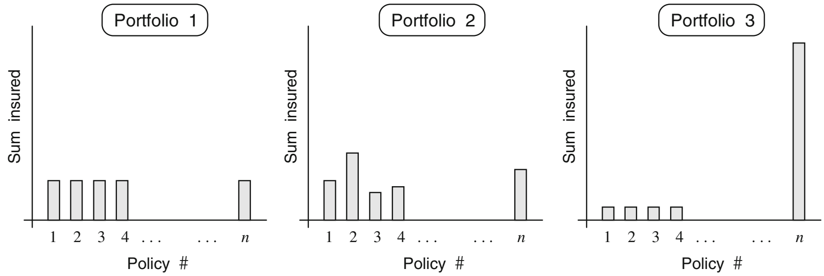


Fig. 2.4 Distribution of the sums insured in three portfolios

the same in the three portfolios. In spite of these common characteristics, intuitively the risk of random fluctuations has an impact which only depends on the size n in Portfolio 1, a more severe impact in Portfolio 2, and an even more severe impact in Portfolio 3. Indeed, the actual total payout in Portfolio 1 only depends on the number of claims in the portfolio itself, whilst it does not depend on which policies are affected by claims. Conversely, in Portfolios 2 and 3 the total payout does depend on which policies are affected by claims, and, in particular, in Portfolio 3 the huge amount insured in one policy can jeopardize the pooling effect. These aspects will be analyzed in Sects. 2.3.3 and 2.3.4. ❑

2.3.3 Risk Assessment

We still refer to a portfolio of n basic insurance covers; for the generic cover, the insurer's random payment is given by

$$X^{(j)} = \begin{cases} x^{(j)} & \text{in the case of claim} \\ 0 & \text{otherwise} \end{cases} \tag{2.3.2}$$

where $x^{(j)}$ is the sum insured.

We assume that:

- the portfolio is homogeneous with respect to the claim probability, denoted with p;
- claims and hence random numbers $X^{(j)}$ are independent each other.

Let $P^{(j)}$ denote the expected value of $X^{(j)}$ (namely, the equivalence premium according to the realistic basis), thus

$$P^{(j)} = \mathbb{E}[X^{(j)}] = x^{(j)}\, p \tag{2.3.3}$$

Moving to the portfolio level, we denote with $X^{[P]}$ the total payment

$$X^{[P]} = \sum_{j=1}^{n} X^{(j)} \tag{2.3.4}$$

whose expected value, denoted by $P^{[P]}$, is given by

$$P^{[P]} = \mathbb{E}[X^{[P]}] = p \sum_{j=1}^{n} x^{(j)} = \sum_{j=1}^{n} P^{(j)} \tag{2.3.5}$$

Our first aim is to quantify the portfolio riskiness, in order to determine an appropriate safety loading. In general, a basic information about riskiness is obviously provided by the variance of the total payment.

For the generic insured risk, the variance of the random payment is given by

$$\mathbb{V}\mathrm{ar}[X^{(j)}] = (x^{(j)})^2\, p\,(1-p) \tag{2.3.6}$$

Then, for the total payment, thanks to the independence assumption, we find

$$\mathbb{V}\mathrm{ar}[X^{[P]}] = \sum_{j=1}^{n} \mathbb{V}\mathrm{ar}[X^{(j)}] = p\,(1-p) \sum_{j=1}^{n} (x^{(j)})^2 \tag{2.3.7}$$

It is interesting to analyze the link between the variance of the total payment and the structure of the portfolio itself, in terms of the sums insured. We denote with $\bar{x}$ the average sum insured, namely

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x^{(j)} \tag{2.3.8}$$

and with $\bar{x}^{(2)}$ the second moment of the distribution of the sums insured, that is,

$$\bar{x}^{(2)} = \frac{1}{n} \sum_{j=1}^{n} (x^{(j)})^2 \tag{2.3.9}$$

Finally, we denote with v the variance of the distribution of the sums insured

$$v = \frac{1}{n} \sum_{j=1}^{n} (x^{(j)} - \bar{x})^2 \tag{2.3.10}$$

which can also be expressed as follows:

$$v = \bar{x}^{(2)} - (\bar{x})^2 \tag{2.3.11}$$

From relations (2.3.7)–(2.3.11), it follows that

$$\mathbb{V}\mathrm{ar}[X^{[\mathrm{P}]}] = np\,(1-p)\,\bar{x}^{(2)} = np\,(1-p)\left(v + (\bar{x})^2\right) \tag{2.3.12}$$

Thus, for a given portfolio size n and a given average sum insured $\bar{x}$ (and hence a given value of $(\bar{x})^2$), the variance of the total payment is lower when the variance of the sums insured, v, is lower. In particular, we find the minimum variance $\mathbb{V}\mathrm{ar}[X^{[\mathrm{P}]}]$ when $v = 0$, that is, when all the policies have the same sum insured. Note that, in this case, the actual total payment (and hence the actual portfolio result) only depends on the number of claims in the portfolio, whilst it does not depend on which policies are affected by claims (see also Example 2.3.1).

2.3.4 The Risk Index

As shown in Sect. 1.6.1, an interesting insight into the riskiness of a pool of risks (and thus a portfolio of insured risks, in particular) is given by the coefficient of variation of the total payment, $X^{[\mathrm{P}]}$. The coefficient of variation provides a measure of relative riskiness, i.e., riskiness related to the expected value of the total payment. As already mentioned, the coefficient of variation is also called, in the actuarial literature, the *risk index*. We will denote it with ρ (reference to the portfolio payment $X^{[\mathrm{P}]}$ is understood). Hence,

$$\rho = \mathbb{CV}[X^{[\mathrm{P}]}] = \frac{\sqrt{\mathbb{V}\mathrm{ar}[X^{[\mathrm{P}]}]}}{\mathbb{E}[X^{[\mathrm{P}]}]} = \frac{\sigma^{[\mathrm{P}]}}{P^{[\mathrm{P}]}} \tag{2.3.13}$$

where $\sigma^{[\mathrm{P}]}$ denotes the standard deviation of the total payment.

We now analyze some aspects of the link between the risk index and the portfolio structure. We still refer to the portfolio defined in Sect. 2.3.3.

From Eqs. (2.3.5) and (2.3.7), we find

$$\rho = \sqrt{\frac{1-p}{p}}\,\frac{\sqrt{\sum_{j=1}^{n}(x^{(j)})^2}}{\sum_{j=1}^{n}x^{(j)}} = \sqrt{\frac{1-p}{np}}\,\frac{\sqrt{\bar{x}^{(2)}}}{\bar{x}} \tag{2.3.14}$$

From (2.3.14) we note that, for a given portfolio size n and a given average sum insured $\bar{x}$, the risk index ρ is higher when $\bar{x}^{(2)}$ is higher, and thus the variance v of the distribution of the sums insured is higher (see the conclusions after formula (2.3.12)).

Example 2.3.2 Tables 2.3, 2.4 and 2.5 refer to three portfolios, all with the same average sum insured, $\bar{x} = 1\,000$; in all the portfolios, the claim probability is $p = 0.005$. However, the three portfolios have different sizes, or structures in terms of sums insured. Various typical values (among which the risk index) summarize the total payment and the inherent risk.

By comparing the results in Table 2.3 to those in Table 2.4, we clearly perceive the magnitude of the pooling effect. Conversely, by comparing results in Table 2.3 to those in Table 2.5, we can see the effect of heterogeneity in the sums insured. ❑

What can we say, in general terms, about the range of values of the risk index ρ, for a given portfolio size n and a given claim probability p? First, it can be proved that

$$\frac{\sqrt{n}}{n} \leq \frac{\sqrt{\sum_{j=1}^{n} (x^{(j)})^2}}{\sum_{j=1}^{n} x^{(j)}} \leq 1 \tag{2.3.15}$$

Then, from these inequalities, it follows that

$$\sqrt{\frac{1-p}{np}} \leq \rho \leq \sqrt{\frac{1-p}{p}} \tag{2.3.16}$$

As regards the lower bound, we have already shown that it is actually reached if (and only if) all sums insured are equal (see Sect. 1.6.1). As regards the upper bound, note that, if one sum insured "diverges" (ceteris paribus), we have:

$$\frac{\sqrt{\sum_{j=1}^{n} (x^{(j)})^2}}{\sum_{j=1}^{n} x^{(j)}} \to 1 \tag{2.3.17}$$

and hence

$$\rho \to \sqrt{\frac{1-p}{p}} \tag{2.3.18}$$

Table 2.3 Portfolio A

Number of policies	Sum insured	Typical values
100 000	1 000	
		$\bar{x} = 1\,000$
		$v = 0$
		$P^{[P]} = \mathbb{E}[X^{[P]}] = 500\,000$
		$\sigma^{[P]} = \sqrt{\mathbb{V}\mathrm{ar}[X^{[P]}]} = 22\,304$
		$\rho = \dfrac{\sigma^{[P]}}{P^{[P]}} = 0.0446$

Table 2.4 Portfolio B

Number of policies	Sum insured	Typical values
10 000	1 000	
		$\bar{x} = 1\,000$
		$v = 0$
		$P^{[P]} = \mathbb{E}[X^{[P]}] = 50\,000$
		$\sigma^{[P]} = \sqrt{\mathbb{V}\text{ar}[X^{[P]}]} = 7\,053$
		$\rho = \dfrac{\sigma^{[P]}}{P^{[P]}} = 0.1411$

Table 2.5 Portfolio C

Number of policies	Sum insured	Typical values
70 000	500	
25 000	1 000	
5 000	8 000	
		$\bar{x} = 1\,000$
		$v = 2\,625\,000$
		$P^{[P]} = \mathbb{E}[X^{[P]}] = 500\,000$
		$\sigma^{[P]} = \sqrt{\mathbb{V}\text{ar}[X^{[P]}]} = 42\,467$
		$\rho = \dfrac{\sigma^{[P]}}{P^{[P]}} = 0.0849$

In more practical terms, when just one sum insured is extremely high if compared to the other sums, the advantage provided by the portfolio size vanishes, so that the riskiness of the portfolio is roughly equal to the riskiness of a portfolio consisting of just one policy (see also Example 2.3.1).

Hence, we can conclude stating that the relative riskiness reduces as the portfolio size increases, provided that each individual position (and the related contribution to the riskiness) becomes negligible in respect of the overall portfolio.

2.3.5 The Probability Distribution of the Total Payment

More information about the riskiness of a portfolio can be achieved via the probability distribution of the total payment $X^{[P]}$. Deriving this probability distribution is, in general, a rather complex problem. Then, we restrict our attention to a particular case, and to the use of approximations.

We assume that our portfolio, which consists of n independent risks, is homogeneous with respect to both the probability, p, and the sum insured, x. Hence, the random total payment can be expressed as follows:

$$X^{[P]} = Kx \tag{2.3.19}$$

where K denotes the random number of claims in the portfolio.

Thanks to the hypothesis of independence, K has a binomial distribution, thus

$$K \sim \mathrm{Bin}(n, p) \tag{2.3.20}$$

and hence

$$\mathbb{P}[X^{[P]} = kx] = \mathbb{P}[K = k] = \binom{n}{k} p^k \, (1-p)^{n-k}; \quad k = 0, 1, \dots, n \tag{2.3.21}$$

In order to get more tractable calculation procedures, various approximations to the binomial distribution can be used. In particular, for a large size n and small probability p, the Poisson distribution can be adopted. Thus, we can assume

$$K \sim \mathrm{Pois}(\lambda) \tag{2.3.22}$$

and hence

$$\mathbb{P}[X^{[P]} = kx] = e^{-\lambda} \, \frac{\lambda^k}{k!}; \quad k = 0, 1, \dots \tag{2.3.23}$$

with

$$\lambda = np \quad \text{(=expected number of claims in the portfolio)} \tag{2.3.24}$$

Further, the normal distribution provides an approximation, which relies on the Central Limit Theorem. Then,

$$X^{[P]} \sim \mathcal{N}(P^{[P]}, \sigma^{[P]}) \tag{2.3.25}$$

where

$$P^{[P]} = \mathbb{E}[X^{[P]}] = n \, x \, p \tag{2.3.26}$$

$$\sigma^{[P]} = \sqrt{\mathbb{V}\mathrm{ar}[X^{[P]}]} = x \, \sqrt{np \, (1-p)} \tag{2.3.27}$$

Hence

$$\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} \sim \mathcal{N}(0, 1) \tag{2.3.28}$$

So, we have, for example,

$$\mathbb{P}\left[z_1 < \frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} \le z_2 \right] = \Phi_{\mathcal{N}(0,1)}(z_2) - \Phi_{\mathcal{N}(0,1)}(z_1) \tag{2.3.29}$$

where $\Phi_{\mathscr{N}(0,1)}(z)$ denotes the cumulative distribution function, namely

$$\Phi_{\mathscr{N}(0,1)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} \mathrm{d}u \tag{2.3.30}$$

The normal approximation can also be adopted in more general cases, e.g., for portfolios of insured risks with various sums insured and/or various probabilities of claim.

The goodness of some approximations is briefly discussed via numerical examples, in the Appendix of this chapter.

Some interesting results can be achieved looking at how the risk index enters probabilities concerning the total payment $X^{[\mathrm{P}]}$. For example, consider the following probability:

$$\psi_\delta = \mathbb{P}\left[(1-\delta)\, P^{[\mathrm{P}]} < X^{[\mathrm{P}]} \le (1+\delta)\, P^{[\mathrm{P}]}\right] \tag{2.3.31}$$

(see Fig. 2.5). The probability on the right-hand side of (2.3.31) can be expressed in terms of the risk index ρ. Indeed, we find

$$\psi_\delta = \mathbb{P}\left[-\delta\,\frac{1}{\rho} < \frac{X^{[\mathrm{P}]} - P^{[\mathrm{P}]}}{\sigma^{[\mathrm{P}]}} \le \delta\,\frac{1}{\rho}\right] \tag{2.3.32}$$

and then:

$$\psi_\delta = \Phi\left(\delta\,\frac{1}{\rho}\right) - \Phi\left(-\delta\,\frac{1}{\rho}\right) \tag{2.3.33}$$

where Φ denotes the cumulative distribution function of the standardized random variable $\frac{X^{[\mathrm{P}]}-P^{[\mathrm{P}]}}{\sigma^{[\mathrm{P}]}}$. From (2.3.33) we see that, for any given value of δ, the lower is ρ the higher is ψ_δ. Thus, the "concentration" increases as the risk index decreases, e.g., because the size of the portfolio increases (see also Table 2.6 in Example 2.3.3).

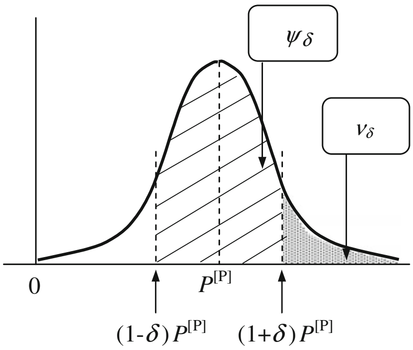


Fig. 2.5 Probability distribution of the random payment $X^{[\mathrm{P}]}$

Table 2.6 The concentration around the expected value

n	ψ_δ		
	$\delta = 0.10$	$\delta = 0.05$	$\delta = 0.01$
100	0.06983	0.03495	0.00699
1 000	0.17737	0.08924	0.01788
10 000	0.61920	0.33877	0.06984

Focussing on "downside" payments is clearly of great interest when assessing the riskiness of a portfolio. To this purpose, probabilities like

$$\pi(t) = \mathbb{P}\left[X^{[P]} > P^{[P]} + t\right] \tag{2.3.34}$$

should be addressed; t represents a critical "threshold," which expresses the insurer's capability to meet the total payment. For example, consider the probability ν_δ defined as follows:

$$\nu_\delta = \pi\left(\delta P^{[P]}\right) = \mathbb{P}\left[X^{[P]} > (1+\delta)\, P^{[P]}\right] \tag{2.3.35}$$

in which the threshold t is expressed in terms of the expected value $P^{[P]}$ (see Fig. 2.5). We find:

$$\nu_\delta = \mathbb{P}\left[\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} > \delta\,\frac{1}{\rho}\right] = 1 - \Phi\left(\delta\,\frac{1}{\rho}\right) \tag{2.3.36}$$

It is easy to understand that, for any given δ, the probability ν_δ decreases as ρ decreases, e.g., because the size of the pool increases.

If we assume, in particular, the normal approximation to the distribution of $X^{[P]}$, we find

$$\nu_\delta = \frac{1 - \psi_\delta}{2} \tag{2.3.37}$$

(See Table 2.7 in Example 2.3.3 for a numerical illustration).

Table 2.7 The probability of "downside" payments

n	ν_δ		
	$\delta = 0.10$	$\delta = 0.05$	$\delta = 0.01$
100	0.46509	0.48253	0.49651
1 000	0.41132	0.45538	0.49106
10 000	0.19040	0.33062	0.46508

Example 2.3.3 We refer to a portfolio, which consists of n independent risks, homogeneous with respect to both the sum insured and the claim probability p. We assume $p = 0.005$. The normal approximation has been used for the numerical evaluations. Table 2.6 illustrates the concentration, in terms of the probability (2.3.31), for some values of δ and various pool sizes. On the other hand, Table 2.7 shows the probability of "downside" payments. ❑

2.3.6 The Safety Loading

In this section we show how to calculate the safety loading consistently with the portfolio riskiness. So, a practical feature of the risk index will clearly emerge.

Refer to the portfolio of n basic insurance covers, described in Sect. 2.3.3. Let $m^{(j)}$ denote the (explicit) safety loading for risk j, and $\Pi^{(j)}$ the premium including the safety loading, that is,

$$\Pi^{(j)} = P^{(j)} + m^{(j)} \tag{2.3.38}$$

where $P^{(j)} = x^{(j)}\,p$ (see Eq. (2.3.3)).

Moving to the portfolio level, let $\Pi^{[P]}$ denote the total premium income

$$\Pi^{[P]} = \sum_{j=1}^{n} \Pi^{(j)} \tag{2.3.39}$$

which can also be expressed as

$$\Pi^{[P]} = P^{[P]} + m^{[P]} \tag{2.3.40}$$

with obvious meaning of the symbol $m^{[P]}$.

The portfolio result, $Z^{[P]}$, is then defined as follows:

$$Z^{[P]} = \Pi^{[P]} - X^{[P]} \tag{2.3.41}$$

We obviously have:

$$\mathbb{E}[Z^{[P]}] = m^{[P]} \tag{2.3.42}$$

$$\mathbb{V}\mathrm{ar}[Z^{[P]}] = \mathbb{V}\mathrm{ar}[X^{[P]}] \tag{2.3.43}$$

We consider the event $Z^{[P]} < 0$, that is, the event $X^{[P]} > P^{[P]} + m^{[P]}$. According to the notation defined by (2.3.34), the probability of this event, namely the *probability of loss*, is denoted as follows:

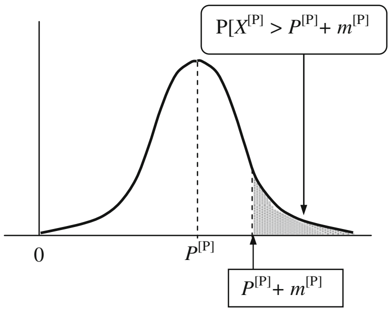


Fig. 2.6 The probability distribution of the random payment $X^{[P]}$

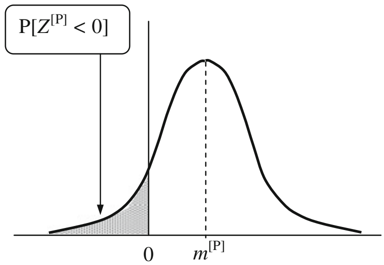


Fig. 2.7 The probability distribution of the random result $Z^{[P]}$

$$\pi(m^{[P]}) = \mathbb{P}\left[X^{[P]} > P^{[P]} + m^{[P]}\right] \tag{2.3.44}$$

Clearly, the probability of loss should be kept reasonably low via an appropriate choice of the (total) safety loading $m^{[P]}$.

Figures 2.6 and 2.7 show the probability distributions of the random payment $X^{[P]}$ and the portfolio result $Z^{[P]}$, respectively (the probability distributions are assumed to be continuous, so that the behavior of the density functions is displayed).

Note that, in the present setting of the problem, the safety loading $m^{[P]}$ is the only parameter whose value can be chosen to lower the probability of a loss (i.e., a negative value of $Z^{[P]}$). Clearly, the effect of a change in this parameter (see Fig. 2.8) is a shift in the probability distribution of $Z^{[P]}$ (see Fig. 2.9).

From (2.3.44), we have

$$\pi(m^{[P]}) = \mathbb{P}\left[\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} > \frac{m^{[P]}}{\sigma^{[P]}}\right] = 1 - \Phi\left(\frac{m^{[P]}}{\sigma^{[P]}}\right) \tag{2.3.45}$$

where Φ denotes the cumulative distribution function of the random number $\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}}$, with expected value equal to 0 and standard deviation equal to 1.

Let ε denote the accepted probability of loss. We want to find $m^{[P]}$ such that

$$\pi(m^{[P]}) = \varepsilon \tag{2.3.46}$$

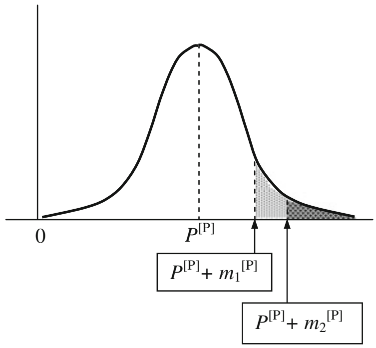


Fig. 2.8 The probability distribution of $X^{[P]}$: probability of exceeding two different levels of safety loading

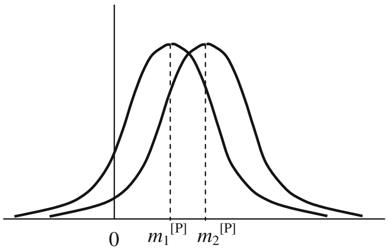


Fig. 2.9 The probability distribution of $Z^{[P]}$: safety loading as a shift parameter of the random result

that is

$$1 - \Phi\left(\frac{m^{[P]}}{\sigma^{[P]}}\right) = \varepsilon \tag{2.3.47}$$

and then

$$m^{[P]} = \sigma^{[P]}\, \Phi^{-1}(1 - \varepsilon) \tag{2.3.48}$$

Finally, we find that the required safety loading per unit of expected value, namely the *safety loading rate*, is given by

$$\frac{m^{[P]}}{P^{[P]}} = \frac{\sigma^{[P]}}{P^{[P]}}\, \Phi^{-1}(1 - \varepsilon) \tag{2.3.49}$$

that is

$$\frac{m^{[P]}}{P^{[P]}} = \rho\, \Phi^{-1}(1 - \varepsilon) \tag{2.3.50}$$

Thus, for a given accepted probability ε, the lower is the risk index ρ, the lower is the safety loading rate.

Example 2.3.4 Tables 2.8, 2.9 and 2.10 refer to the portfolio structures described by Tables 2.3, 2.4 and 2.5, respectively. The normal approximation has been used to evaluate the probabilities, namely it has been assumed:

$$\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} \sim \mathcal{N}(0, 1) \tag{2.3.51}$$

The analysis of the results in the three tables leads, of course, to conclusions strictly related to those presented in Example 2.3.2. Now, the effect of risk pooling (compare Tables 2.8 and 2.9) and the effect of heterogeneity in the sums insured (compare Tables 2.8 and 2.10) clearly appears in terms of the safety loading rate $\frac{m^{[P]}}{P^{[P]}}$. Note, in particular, the huge values of this rate in Portfolio B when a very low probability of loss is assumed as the target. So, the need for tools other than the safety loading clearly emerges. ❑

Table 2.8 Safety loading-Portfolio A

$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{m^{[P]}}{\sigma^{[P]}}$	$\pi(m^{[P]})$
64 200	0.1284	2.880	0.002
57 550	0.1151	2.580	0.005

Table 2.9 Safety loading-Portfolio B

$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{m^{[P]}}{\sigma^{[P]}}$	$\pi(m^{[P]})$
20 312	0.4063	2.880	0.002
18 195	0.3639	2.580	0.005
6 420	0.1284	0.910	0.181

Table 2.10 Safety loading-Portfolio C

$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{m^{[P]}}{\sigma^{[P]}}$	$\pi(m^{[P]})$
122 300	0.2446	2.880	0.002
109 550	0.2191	2.580	0.005
64 200	0.1284	1.512	0.065

2.3.7 Capital Allocation and Beyond

The outcome of the total payment $X^{[P]}$ can be higher than the amount of premiums, even when these include an appropriate safety loading. In order to manage this risk, the insurer can assign to the portfolio a fund which consists of shareholders' capital (and, as such, may derive from previous profits, or from the issue of shares). This action is usually referred to as the *capital allocation*. Hence, the purpose of the allocation is to protect the insurance company against possible negative results produced by the portfolio.

Let M denote the amount of capital allocated to the portfolio. Figure 2.10 illustrates the use of resources available to the insurer, in order to face the portfolio total payment, and the results corresponding to the possible outcomes of the payment itself.

In particular, the event $X^{[P]} > P^{[P]} + m^{[P]} + M$ means the *portfolio default*, or *ruin*. We note that both the safety loading $m^{[P]}$ and the capital M are variables whose values can be chosen to lower the *probability of default*, namely the probability:

$$\pi(m^{[P]} + M) = \mathbb{P}[X^{[P]} > P^{[P]} + m^{[P]} + M] = \mathbb{P}[Z^{[P]} < -M] \tag{2.3.52}$$

Remark We note that, while "profit" and "loss" are related to the amount $\Pi^{[P]}$ of premiums (and hence to the safety loading $m^{[P]}$), the default situation also involves the allocated capital M. Further, the capital M (and the relevant cost) must be considered to define the "creation of value", (from the shareholders' perspective) which will be addressed in Sect. 2.3.9.

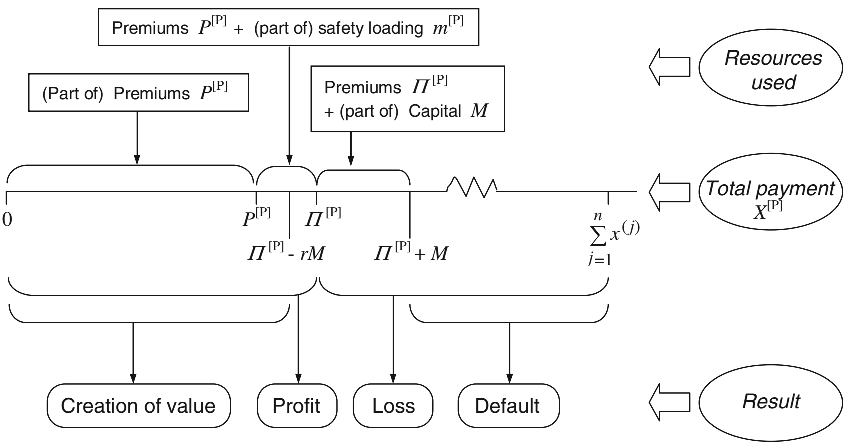


Fig. 2.10 Facing the total payment

Fig. 2.11 The probability distribution of the random payment $X^{[P]}$

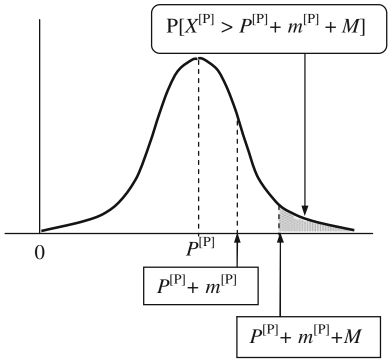
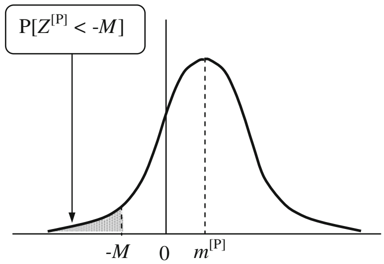


Fig. 2.12 The probability distribution of the random result $Z^{[P]}$



If the total safety loading $m^{[P]}$ has been already stated, the following problem should be considered: find the amount M such that:

$$\pi(m^{[P]} + M) = \alpha \tag{2.3.53}$$

where α is an assigned low probability (see Figs. 2.11 and 2.12). Of course, we have:

$$M = -VaR_{\alpha}[X^{[P]}] \tag{2.3.54}$$

From (2.3.52) we have:

$$\pi(m^{[P]} + M) = \mathbb{P}\left[\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}} > \frac{m^{[P]} + M}{\sigma^{[P]}}\right] = 1 - \Phi\left(\frac{m^{[P]} + M}{\sigma^{[P]}}\right) \tag{2.3.55}$$

where Φ denotes the cumulative distribution function of the random number $\frac{X^{[P]} - P^{[P]}}{\sigma^{[P]}}$, with expected value equal to 0 and standard deviation equal to 1. Thus, the target expressed by (2.3.53) can also be written as follows:

$$1 - \Phi\left(\frac{m^{[P]} + M}{\sigma^{[P]}}\right) = \alpha \tag{2.3.56}$$

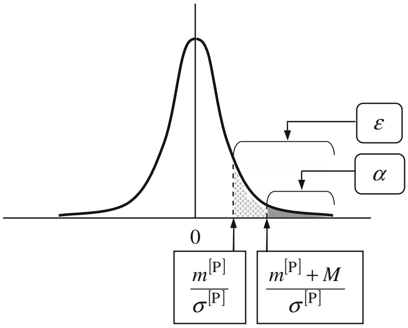


Fig. 2.13 The standardized probability distribution of the random payment

and hence

$$\frac{m^{[P]}+M}{\sigma^{[P]}} = \Phi^{-1}(1-\alpha) \tag{2.3.57}$$

(see Fig. 2.13).

We note that, setting $M = 0$, we trivially find formula (2.3.48), with $\alpha = \varepsilon$. Conversely, for a given probability α (and a given standard deviation $\sigma^{[P]}$, which is univocally determined by the portfolio structure), Eq. (2.3.57) can be solved with respect to the total amount $m^{[P]} + M$. In other terms, if the safety loading is not yet stated, both the amounts $m^{[P]}$ and M can be chosen in order to achieve the target probability.

The unit-free index

$$s = \frac{m^{[P]}+M}{\sigma^{[P]}} \tag{2.3.58}$$

is sometimes called the *relative stability index*. From (2.3.55), we see that the higher is s, the lower is the ruin probability. To raise s, the following actions can be taken:

1. raise the safety loading $m^{[P]}$;
2. raise the allocated capital M;
3. reduce $\sigma^{[P]}$ via appropriate reinsurance arrangements (thus affecting the portfolio structure, in terms of sums insured), and, in particular, by choosing the "retention" level (we will deal with these concepts in Sects. 2.4 and 2.5).

As the insurer can choose (at least in principle) the safety loading, the amount of allocated capital, and the retention level, these quantities are called *decision variables*. However, the following aspects should be stressed. Action 1 affects the premiums, and hence is bounded by market constraints. Conversely, action 2 has constraints at the company level because capital is a limited resource.

As regards action 3, whatever reinsurance arrangements may be chosen, the related cost obviously affects the resources available to the portfolio, in particular reducing the expected profit $m^{[P]}$. As both numerator and denominator of the stability index are affected (see (2.3.58)), the effect is not univocally determined in general.

Table 2.11 Capital allocation and safety loading-Portfolio A

M	$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{M}{\Pi^{[P]}}$	s	$\pi(m^{[P]}+M)$
10 000	50 000	0.100	0.018	2.6901	0.0036
15 000	50 000	0.100	0.027	2.9143	0.0018
20 000	50 000	0.100	0.036	3.1385	0.0009

Table 2.12 Capital allocation and safety loading-Portfolio B

M	$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{M}{\Pi^{[P]}}$	s	$\pi(m^{[P]}+M)$
10 000	5 000	0.100	0.182	2.1268	0.0167
13 200	5 000	0.100	0.240	2.5805	0.0050
10 000	8 200	0.164	0.172	2.5805	0.0050

Table 2.13 Capital allocation and safety loading-Portfolio C

M	$m^{[P]}$	$\frac{m^{[P]}}{P^{[P]}}$	$\frac{M}{\Pi^{[P]}}$	s	$\pi(m^{[P]}+M)$
10 000	50 000	0.100	0.018	1.4129	0.0788
60 000	50 000	0.100	0.109	2.5902	0.0048
35 000	75 000	0.150	0.061	2.5902	0.0048

Example 2.3.5 Tables 2.11, 2.12 and 2.13 refer to the portfolio structures described by Tables 2.3, 2.4 and 2.5, respectively.

In particular, from Tables 2.12 and 2.13 the important role of the capital allocation clearly appears, especially when very high safety loading rates should otherwise be applied, because of either the size of the portfolio or its structure, in order to keep low the probability of default. ❑

2.3.8 Solvency

As seen above, the event $Z^{[P]} < -M$ represents the portfolio default, or ruin. Conversely, when $M + Z^{[P]} \geq 0$ the insurer is able to meet the total payment by using the premiums and, possibly, (part of) the allocated capital, that is, the insurer is *solvent*. Hence, a *solvency requirement* can be expressed as follows:

$$\mathbb{P}[M + Z^{[P]} \geq 0] = 1 - \alpha \tag{2.3.59}$$

where α is the accepted default probability (see Eq. (2.3.53)).

Equation (2.3.59) can be solved with respect to M. The solution (see (2.3.57), for given values of $m^{[\mathrm{P}]}$ and $\sigma^{[\mathrm{P}]}$) provides the capital requirement for solvency purposes.

It is worth noting that, in the ordinary language, the term "solvency" is often used in a not well-defined sense. Commonly, it is used to denote the capability of an agent to pay the amounts when these fall due. It is apparent that this definition does not fit obvious actuarial requirements. Indeed, in the insurance activity, the capability cannot be meant in a deterministic sense (which leads to the concept of "absolute solvency"): actually, the total amount due could be equal to the sum of all sums insured with the policies in force at a given time, if all the insureds claim at that time. Hence, the insurance business needs a definition of solvency in a probabilistic sense, as witnessed in particular by Eq. (2.3.59).

2.3.9 Creating Value

We now return to the choice between action 1 (raise $m^{[\mathrm{P}]}$) and action 2 (raise M), aiming to lower the probability of default (or to achieve an assigned target probability α). First, we note that allocating capital implies a cost to the shareholders, whereas raising the safety loading leads to a higher cost to the policyholders.

Let r denote the (annual) rate which quantifies the opportunity cost of the shareholders' capital. Thus, the cost of allocating the amount M is given by $r\,M$. However, the common definition of a profit (or a loss) is only based on the comparison between actual revenues and costs (see Sect. 1.3.2). Thus, for the portfolio we are addressing, we have

$$
\begin{aligned}
\Pi^{[\mathrm{P}]} < X^{[\mathrm{P}]} \quad &\Leftrightarrow \quad Z^{[\mathrm{P}]} < 0 \quad \Rightarrow \quad \text{loss} \\
\Pi^{[\mathrm{P}]} > X^{[\mathrm{P}]} \quad &\Leftrightarrow \quad Z^{[\mathrm{P}]} > 0 \quad \Rightarrow \quad \text{profit}
\end{aligned}
$$

(note that the only cost accounted for is given by the payment for claims, $X^{[\mathrm{P}]}$, as, in our simplified setting, expenses are disregarded). Conversely, if we want to assess the portfolio result also allowing for the cost of capital allocation, the total amount of premiums, $\Pi^{[\mathrm{P}]}$, has to be compared to $X^{[\mathrm{P}]} + rM$. A new concept then arises, namely the *value creation*.

Remark As mentioned in Sect. 1.3.2, various meanings can be attributed to the word "value" and hence to the expression "value creation." Here we are referring to value creation as the positive difference between the revenues and the costs associated to all of the production factors, hence including the cost of the capital invested in the business. In this sense, value creation is a synonym to (positive) "economic earnings." Thus, we are referring to value creation from the shareholders' perspective.

We then have, for our portfolio (see also Fig. 2.10):

$$\begin{array}{rcccl}
\Pi^{[P]} < X^{[P]} & \Leftrightarrow & Z^{[P]} < 0 & \Rightarrow & \text{loss and value destruction} \\
X^{[P]} < \Pi^{[P]} < X^{[P]} + rM & \Leftrightarrow & 0 \le Z^{[P]} < r\,M & \Rightarrow & \text{profit and value destruction} \\
\Pi^{[P]} = X^{[P]} + rM & \Leftrightarrow & Z^{[P]} = r\,M & \Rightarrow & \text{profit and no value} \\
\Pi^{[P]} > X^{[P]} + rM & \Leftrightarrow & Z^{[P]} > r\,M & \Rightarrow & \text{profit and value creation}
\end{array}$$

In order to compare strategies which consist in mixing action 1 and action 2, we have to move to expected values. Thus, we have to replace $X^{[P]}$ with its expected value $\mathbb{E}[X^{[P]}] = P^{[P]}$. Noting that $\Pi^{[P]} = P^{[P]} + m^{[P]}$, we find, in terms of expected values:

$$m^{[P]} < rM \quad \Leftrightarrow \quad \text{value destruction} \tag{2.3.60}$$

$$m^{[P]} = rM \quad \Leftrightarrow \quad \text{no value} \tag{2.3.61}$$

$$m^{[P]} > rM \quad \Leftrightarrow \quad \text{value creation} \tag{2.3.62}$$

Example 2.3.6 We refer to portfolio B and assume $\alpha = 0.005$ as the target probability; hence, an amount $M + m^{[P]} = 18\,200$ is required (see Table 2.12). Further, we assume $r = 0.08$. Table 2.14 illustrates some situations of value creation (Value > 0), value destruction (Value < 0), and "equilibrium" (Value $= 0$), respectively. ❑

Whatever the target probability, the equation

$$m^{[P]} = rM \tag{2.3.63}$$

defines the borderline between value creation and value destruction. Conversely, for a given target probability, we have

$$m^{[P]} + M = \text{const.} \tag{2.3.64}$$

(represented by a "level line") as it results from Eq. (2.3.57). See Fig. 2.14; we note, in particular, that the higher is r the smaller is the region of value creation.

It is worth stressing that both value creation and solvency are two important goals for any insurance business (and, more in general, for any organization; see

Table 2.14 Value creation versus value destruction-Portfolio B

M	$m^{[P]}$	rM	Value $m^{[P]} - rM$
10 000	8 200	800	7 400
15 000	3 200	1 200	2 000
16 852	1 348	1 348	0
18 000	200	1 440	−1 240

Fig. 2.14 Capital allocation; value creation versus value destruction

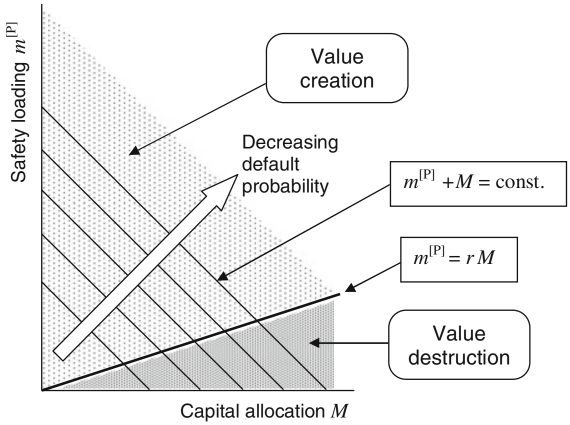


Table 2.15 Value creation and default probability-Portfolio B

M	rM	Value $m^{[P]} - rM$	Default prob. $\pi(m^{[P]} + M)$
10 000	800	1 700	0.063
15 700	1 256	1 244	0.005
20 000	1 600	900	0.001
35 000	2 800	−300	≈0

Sect. 1.3.2). Clearly, for any given portfolio (and a given amount of safety loading), the two targets require opposite actions: a higher amount of capital improves the solvency level, while reducing the value creation.

Example 2.3.7 We still refer to portfolio B, and assume 5 % as the safety loading rate, so that $m^{[P]} = 2\,500$. The opportunity cost of capital is $r = 0.08$. Table 2.15 illustrates value creation and default probability as functions of the capital allocation. ❑

2.3.10 Risk Management and Risk Analysis: Some Remarks

Various issues dealt with in the previous sections of this chapter can be properly placed in the framework of insurance risk management, and in particular can be interpreted as risk management actions.

Pricing the insurance product, which in our setting simply reduces to calculate an appropriate safety loading, aims at loss prevention and loss reduction (see Sect. 1.3.5). In a more general setting, also product design (and, in particular, the design of various policy conditions) contributes to loss prevention and loss reduction.

Capital allocation is the action aiming at loss financing via retention (see Sect. 1.3.5). More precisely, the shareholders' capital allocated to a portfolio constitutes the tool for funding possible future losses.

Like other business entities, insurers can finance potential losses via risk transfer. In the following sections, we will first focus on traditional risk transfers, namely via reinsurance arrangements (Sects. 2.4 and 2.5). Then, alternative risk transfers (Sect. 2.6), and in particular the transfer to capital markets, will be analyzed in the framework of loss financing actions.

Enterprise Risk Management (ERM), as a methodological framework, has provided important contributions to risk analysis and risk assessment. Nevertheless, it should be stressed that the earliest contribution to risk quantification can be traced back to the eighteenth century. In 1786 Johannes Tetens first addressed the analysis of the process risk inherent in a life insurance portfolio. Tetens showed that the risk in absolute terms increases as the portfolio size n increases, whereas the risk in respect of each insured decreases in proportion to $\sqrt{n}$. This feature of the risk pooling process has been described in Sect. 1.6.1 (in particular, see Examples 1.6.1 and 1.6.2), and Sect. 2.3.4 (see Example 2.3.2).

In a modern theoretical perspective, Tetens' ideas constitute a pioneering contribution to the *individual risk theory*. Note that the term "individual" recalls the nature of the approach, which starts from the description of the individual risks $X^{(j)}$ (in Case 2, the amount $x^{(j)}$ of the potential loss, and the relevant probability $p^{(j)}$), and leads to the construction of the probability distribution (or, at least, some typical values) of the total payment $X^{[\mathrm{P}]}$. According to the terminology commonly used in the ERM context, the adoption of this method is called the *bottom-up approach*.

The *collective risk theory*, whose origin can be traced back to the seminal contribution by Filip Lundberg, dated 1909, directly focuses on the characteristics of the total payment $X^{[\mathrm{P}]}$. In the ERM context, this approach is usually called the *top-down approach*. Well-known implementations lead, for instance, to the calculation of the VaR and the TailVaR (see Sect. 1.5.4), and to various solvency requirements according to a dynamic perspective (as we will see in Sect. 2.7).

2.3.11 The "Uncertainty Risk"

We refer, as in Sects. 2.3.3 and 2.3.4, to a portfolio of n basic insurance covers, all with the same probability of claim. Further, we assume that all the policies have the same sum insured x. We denote simply with X the random payment for the generic policy.

Unlike the previous sections, we now suppose that p does not necessarily represent the "correct" estimate of the claim probability. If p is not a correct estimate of this probability, situations like the one displayed in Fig. 2.3b, and thus involving systematic deviations, can occur.

To make explicit our awareness, we can express uncertainty about the estimate of the claim probability through a random quantity $\tilde{p}$, to which a probability distribution

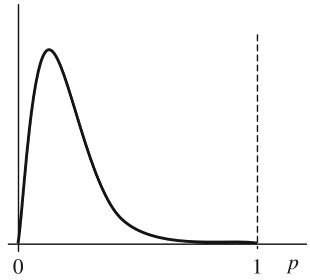


Fig. 2.15 The pdf of a Beta distribution

should be assigned. We now denote with p the generic outcome of the random quantity $\tilde{p}$.

As regards the probability distribution of $\tilde{p}$, we can, for example, choose a Beta distribution (see Fig. 2.15), the parameters of which are usually denoted with α, β. Thus,

$$\tilde{p} \sim \text{Beta}(\alpha, \beta) \tag{2.3.65}$$

Hence, for the random quantity $\tilde{p}$, we have:

$$\mathbb{E}[\tilde{p}] = \frac{\alpha}{\alpha + \beta} \tag{2.3.66}$$

$$\mathbb{V}\text{ar}[\tilde{p}] = \frac{\alpha\,\beta}{(\alpha + \beta)^2\,(\alpha + \beta + 1)} \tag{2.3.67}$$

When uncertainty about the claim probability is accounted for, the expected value of X, conditional on any value p of $\tilde{p}$ is given by

$$\mathbb{E}[X|p] = x\,p \tag{2.3.68}$$

Conversely, the quantity

$$\mathbb{E}[X|\tilde{p}] = x\tilde{p} \tag{2.3.69}$$

is a random amount, as it is a function of $\tilde{p}$. Its expectation, according to the Beta distribution assigned to $\tilde{p}$, is given by

$$\mathbb{E}_{\text{Beta}}\big[\mathbb{E}[X|\tilde{p}]\big] = \mathbb{E}_{\text{Beta}}[x\tilde{p}] = x\,\frac{\alpha}{\alpha + \beta} \tag{2.3.70}$$

Note that, in the uncertainty framework, formula (2.3.70) expresses the unconditional expected value, namely $\mathbb{E}[X]$. For the variance of the random amount $\mathbb{E}[X|\tilde{p}]$ we find

$$\mathbb{V}\text{ar}_{\text{Beta}}\big[\mathbb{E}[X|\tilde{p}]\big] = \mathbb{V}\text{ar}_{\text{Beta}}[x\tilde{p}] = x^2\,\frac{\alpha\,\beta}{(\alpha + \beta)^2\,(\alpha + \beta + 1)} \tag{2.3.71}$$

In the presence of uncertainty, the variance of X, conditional on any value p of $\tilde{p}$, is given by

$$\mathbb{V}\text{ar}[X|p] = x^2\, p\,(1-p) \tag{2.3.72}$$

while

$$\mathbb{V}\text{ar}[X|\tilde{p}] = x^2\, \tilde{p}\,(1-\tilde{p}) \tag{2.3.73}$$

is a random quantity. Its expectation, according to the Beta distribution assigned to $\tilde{p}$, is given by

$$\mathbb{E}_{\text{Beta}}\big[\mathbb{V}\text{ar}[X|\tilde{p}]\big] = x^2\, \mathbb{E}_{\text{Beta}}[\tilde{p}\,(1-\tilde{p})] \tag{2.3.74}$$

It can be proved that $\mathbb{E}_{\text{Beta}}[\tilde{p}\,(1-\tilde{p})] = \frac{\alpha\,\beta}{(\alpha+\beta)(\alpha+\beta+1)}$, so that

$$\mathbb{E}_{\text{Beta}}\big[\mathbb{V}\text{ar}[X|\tilde{p}]\big] = x^2\, \frac{\alpha\,\beta}{(\alpha+\beta)(\alpha+\beta+1)} \tag{2.3.75}$$

Moving to the portfolio level, we now address the total payment $X^{[P]}$. When uncertainty in the claim probability is allowed for, the expected value $\mathbb{E}[X^{[P]}|p]$ and the variance $\mathbb{V}\text{ar}[X^{[P]}|p]$ must be meant as conditional on the generic value p of the random quantity $\tilde{p}$, as for the corresponding typical values of X. Further, we have:

$$\mathbb{E}[X^{[P]}|\tilde{p}] = n\,\mathbb{E}[X|\tilde{p}] \tag{2.3.76}$$

and for the variance

$$\mathbb{V}\text{ar}[X^{[P]}|\tilde{p}] = n\,\mathbb{V}\text{ar}[X|\tilde{p}] \tag{2.3.77}$$

Expected value and variance, as given by (2.3.76) and (2.3.77) respectively, are random quantities. We have:

$$\mathbb{E}[X^{[P]}] = \mathbb{E}_{\text{Beta}}\big[n\,\mathbb{E}[X|\tilde{p}]\big] = n\,x\,\frac{\alpha}{\alpha+\beta} \tag{2.3.78}$$

Note that (2.3.78) expresses the unconditional expected value of $X^{[P]}$.

As regards the variance of $X^{[P]}$, first it should be stressed that the independence among the individual random claims must be meant only conditional on any given value of the probability p. Then, in the presence of uncertainty about this probability, namely when the random quantity $\tilde{p}$ is addressed, the unconditional variance of $X^{[P]}$ cannot be expressed as the sum of the individual unconditional variances. Conversely, it can be proved that the unconditional variance of $X^{[P]}$ can be expressed as follows:

$$\begin{aligned}\mathbb{V}\mathrm{ar}[X^{[P]}] &= \mathbb{V}\mathrm{ar}_{\mathrm{Beta}}\big[\mathbb{E}[X^{[P]}|\tilde{p}]\big] + \mathbb{E}_{\mathrm{Beta}}\big[\mathbb{V}\mathrm{ar}[X^{[P]}|\tilde{p}]\big] \\ &= \mathbb{V}\mathrm{ar}_{\mathrm{Beta}}\big[n\,\mathbb{E}[X|\tilde{p}]\big] + \mathbb{E}_{\mathrm{Beta}}\big[n\,\mathbb{V}\mathrm{ar}[X|\tilde{p}]\big]\end{aligned} \tag{2.3.79}$$

Hence, from (2.3.71) and (2.3.75) we have:

$$\mathbb{V}\mathrm{ar}[X^{[P]}] = n^2\,x^2\,\frac{\alpha\,\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)} + n\,x^2\,\frac{\alpha\,\beta}{(\alpha+\beta)(\alpha+\beta+1)} \tag{2.3.80}$$

Finally, for the (unconditional) coefficient of variation, namely the risk index, after some manipulations we find

$$\mathbb{CV}[X^{[P]}] = \frac{\sqrt{\mathbb{V}\mathrm{ar}[X^{[P]}]}}{\mathbb{E}[X^{[P]}]} = \sqrt{\frac{\beta}{\alpha\,(\alpha+\beta+1)} + \frac{1}{n}\,\frac{\beta\,(\alpha+\beta)}{\alpha\,(\alpha+\beta+1)}} \tag{2.3.81}$$

Hence, we have

$$\lim_{n\to\infty} \mathbb{CV}[X^{[P]}] = \sqrt{\frac{\beta}{\alpha\,(\alpha+\beta+1)}} > 0 \tag{2.3.82}$$

Note that, on the contrary, when no uncertainty is allowed for, the risk index tends to 0 when the pool size n diverges (see (1.6.14)). In more practical terms, this means that:

- the process risk (namely, the risk of random fluctuations) is a *diversifiable* risk, and the diversification is achieved by increasing the portfolio size, and is referred to as *diversification via pooling*;
- the uncertainty risk (namely, the risk of systematic deviations) is an *undiversifiable* risk, as its (relative) magnitude is independent of the portfolio size.

(see also Sects. 2.3.1 and 2.3.2).

Example 2.3.8 We assume, for the random quantity $\tilde{p}$, the Beta distribution with the following parameters:

$$\alpha = 4; \quad \beta = 796 \tag{2.3.83}$$

Hence, from (2.3.66) and (2.3.67), we find:

$$\begin{aligned}\mathbb{E}[\tilde{p}] &= 0.005 \\ \mathbb{V}\mathrm{ar}[\tilde{p}] &= 7.754 \times 10^{-9}\end{aligned}$$

Let us now assume the following parameters

$$\alpha = 2; \quad \beta = 398 \tag{2.3.84}$$

Table 2.16 The coefficient of variation $\mathbb{CV}[X^{[\mathrm{P}]}]$

n	$p = 0.005$	$\alpha = 4,\ \beta = 796$	$\alpha = 2,\ \beta = 398$
10	4.461	4.486	4.511
100	1.411	1.495	1.575
1 000	0.446	0.669	0.834
10 000	0.141	0.518	0.718
...	...	...	...
∞	0.000	0.498	0.704

In this case, we have:

$$\mathbb{E}[\tilde{p}] = 0.005$$
$$\mathbb{V}\mathrm{ar}[\tilde{p}] = 3.094 \times 10^{-8}$$

Note that, while keeping the same expected value, we now have a higher variance, which clearly expresses a higher degree of uncertainty about the claim probability.

Table 2.16 shows the risk index, namely $\mathbb{CV}[X^{[\mathrm{P}]}]$, for various portfolio sizes n; the cases of no uncertainty (i.e., a fixed value of p) and uncertainty expressed by the parameters specified by (2.3.83) and (2.3.84) respectively are considered. The results are self-evident: the undiversifiable part of the risk clearly appears when uncertainty is explicitly introduced into the valuations. ❑

2.4 Reinsurance: The Basics

2.4.1 General Aspects

The reinsurance is the traditional risk transfer from an insurer (the *cedant*) to another insurer (the *reinsurer*). From a technical point of view, the main aim of the reinsurance transfer is to find protection against the portfolio ruin (and the insurer's ruin, as well). Further aims of reinsurance will be addressed in Sect. 2.5.4.

The basic idea underlying any *reinsurance form* (or *arrangement*) is to split the portfolio random payment, $X^{[\mathrm{P}]}$, as follows:

$$X^{[\mathrm{P}]} = X^{[\mathrm{ret}]} + X^{[\mathrm{ced}]} \tag{2.4.1}$$

where:

- the random amount $X^{[\mathrm{ced}]}$ is the *ceded* part of the total payment; this amount will be paid by the reinsurer to the cedant;

- the random amount $X^{[\mathrm{ret}]}$ is the *retained* part of the total payment, hence it is the net payment of the cedant.

A *reinsurance premium* is paid by the cedant to the reinsurer, as the price of the possible reinsurer's intervention.

How to define the two terms on the right-hand side of (2.4.1)? The two following approaches can be adopted.

1. In principle, the simplest way to define the splitting consists of assigning a retention function Γ, which works at the portfolio level, so that
$$X^{[\mathrm{ret}]} = \Gamma(X^{[\mathrm{P}]}) \tag{2.4.2}$$
In some cases, the retained payment can also depend on other quantities, e.g., the total number of claims, K, in the portfolio, thus
$$X^{[\mathrm{ret}]} = \Gamma(X^{[\mathrm{P}]}, K) \tag{2.4.3}$$
Anyway, this approach relies on the definition of the splitting on a *portfolio basis*, and then leads to a *global reinsurance* arrangement.
2. As the random payment is the sum of the payments related to the various risks, namely $X^{[\mathrm{P}]} = \sum_{j=1}^{n} X^{(j)}$, we can split each $X^{(j)}$ by defining a retention function γ, so that
$$X^{(j)[\mathrm{ret}]} = \gamma(X^{(j)}) \tag{2.4.4}$$
Then, the retained total payment is given by
$$X^{[\mathrm{ret}]} = \sum_{j=1}^{n} X^{(j)[\mathrm{ret}]} \tag{2.4.5}$$
In some cases, a set of retention functions $\gamma^{(j)}, j = 1, 2, \ldots, n$, must be defined, instead of a single function γ. Anyhow, this approach requires the splitting on a *policy basis*, hence leading to an *individual reinsurance* arrangement.

We now describe an implementation of approach 1. Another implementation of this approach will be presented in Sect. 2.5.3.

2.4.2 Stop-Loss Reinsurance

Stop-loss reinsurance provides a "direct" protection against the portfolio default, or ruin, as it directly refers to the portfolio total payment. The reinsurer gets the reinsurance premium $\Pi^{[\mathrm{reins}]}$ and pays the part of $X^{[\mathrm{P}]}$ which exceeds a stated amount, Λ, the *stop-loss retention*, or *priority*. The priority is commonly expressed in terms of the total premium income $\Pi^{[\mathrm{P}]}$ (and usually $\Lambda > \Pi^{[\mathrm{P}]}$).

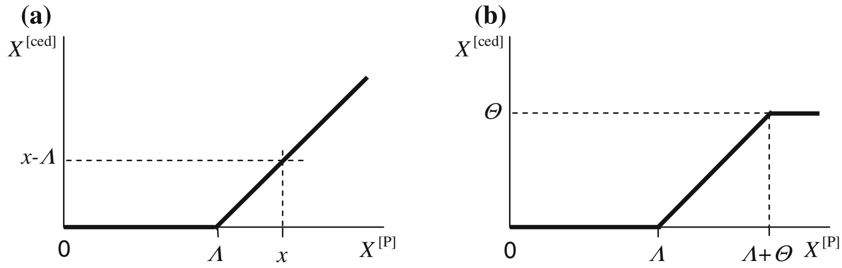


Fig. 2.16 The reinsurer's payment

The cedant's retention and the reinsurer's payment are then given by:

$$X^{[\mathrm{ret}]} = \begin{cases} X^{[\mathrm{P}]} & \text{if } X^{[\mathrm{P}]} \le \Lambda \\ \Lambda & \text{if } X^{[\mathrm{P}]} > \Lambda \end{cases} \tag{2.4.6a}$$

$$X^{[\mathrm{ced}]} = \begin{cases} 0 & \text{if } X^{[\mathrm{P}]} \le \Lambda \\ X^{[\mathrm{P}]} - \Lambda & \text{if } X^{[\mathrm{P}]} > \Lambda \end{cases} \tag{2.4.6b}$$

Figure 2.16a shows the reinsurer's intervention.

An *upper limit*, Θ, to reinsurer's intervention can be stated. In this case, the cedant's retention and the reinsurer's payment are respectively given by:

$$X^{[\mathrm{ret}]} = \begin{cases} X^{[\mathrm{P}]} & \text{if } X^{[\mathrm{P}]} \le \Lambda \\ \Lambda & \text{if } \Lambda < X^{[\mathrm{P}]} < \Lambda + \Theta \\ X^{[\mathrm{P}]} - \Theta & \text{if } X^{[\mathrm{P}]} \ge \Lambda + \Theta \end{cases} \tag{2.4.7a}$$

$$X^{[\mathrm{ced}]} = \begin{cases} 0 & \text{if } X^{[\mathrm{P}]} \le \Lambda \\ X^{[\mathrm{P}]} - \Lambda & \text{if } \Lambda < X^{[\mathrm{P}]} < \Lambda + \Theta \\ \Theta & \text{if } X^{[\mathrm{P}]} \ge \Lambda + \Theta \end{cases} \tag{2.4.7b}$$

Figure 2.16b shows the reinsurer's intervention.

Note that Eqs. (2.4.6) and (2.4.7) constitute two implementations of the general scheme expressed by Eqs. (2.4.1) and (2.4.2).

When dealing with reinsurance arrangements, the portfolio loss, L, rather than the portfolio result $Z^{[\mathrm{P}]}$, is often referred to. The loss of the cedant is given, in the absence of reinsurance, by:

$$L = X^{[\mathrm{P}]} - \Pi^{[\mathrm{P}]} \tag{2.4.8}$$

Clearly, $L = -Z^{[\mathrm{P}]}$.

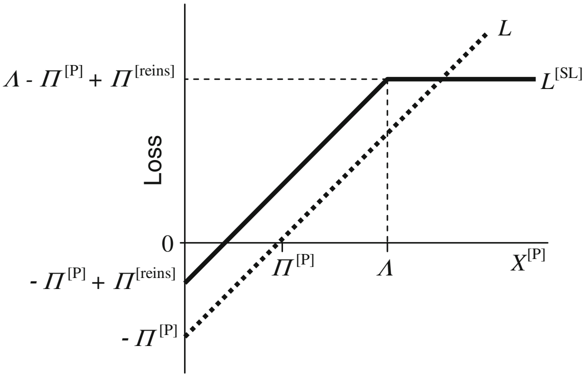


Fig. 2.17 The cedant's loss

If a stop-loss reinsurance works (without an upper limit, and hence with $X^{[ced]}$ defined by Eq. (2.4.6b)), the loss, $L^{[SL]}$, is given by:

$$L^{[SL]} = X^{[P]} - \Pi^{[P]} + \Pi^{[reins]} - X^{[ced]} = \begin{cases} L + \Pi^{[reins]} & \text{if } X^{[P]} \leq \Lambda \\ \Lambda - \Pi^{[P]} + \Pi^{[reins]} & \text{if } X^{[P]} > \Lambda \end{cases} \tag{2.4.9}$$

(see Fig. 2.17). Note that, in the presence of reinsurance, the portfolio outgo also includes the reinsurance premium, and thus is given by $X^{[P]} + \Pi^{[reins]}$, whereas the income is given by $\Pi^{[P]} + X^{[ced]}$.

As the stop-loss reinsurance directly refers to the portfolio loss, it represents in theory the best solution to portfolio protection. However, in practice, it should be noted that this reinsurance form implies a potentially dangerous exposure of the reinsurer, related to the tail of the probability distribution of $X^{[P]}$ (especially if no upper limit is stated). This means that a very high safety loading should be included into the premium $\Pi^{[reins]}$, possibly making this reinsurance cover extremely expensive. Hence, it is mainly used as an ingredient in a reinsurance programme (see Sect. 2.5.6), after other reinsurance covers have been implemented to protect the portfolio.

2.4.3 *From Portfolios to Contracts*

We now move to individual reinsurance arrangements, whose parameters are thus defined at a contract level (rather than a portfolio level), still referring to the "basic" insurance cover.

A *reinsurance policy* at a contract level is defined as

$$\underline{a} = (a^{(1)}, a^{(2)}, \ldots, a^{(n)}) \tag{2.4.10}$$

where $a^{(j)}$ $(0 < a^{(j)} \leq 1)$ is the share retained of the jth contract, i.e., the *retained proportion*.

For any given reinsurance policy $\underline{a}$, relation (2.4.4) becomes:

$$X^{(j)[\text{ret}]} = a^{(j)}\, X^{(j)} = \begin{cases} a^{(j)}\, x^{(j)} & \text{in the case of claim} \\ 0 & \text{otherwise} \end{cases} \tag{2.4.11}$$

and hence we have:

$$\mathbb{E}[X^{(j)[\text{ret}]}] = a^{(j)}\, \mathbb{E}[X^{(j)}] = a^{(j)}\, P^{(j)} \tag{2.4.12}$$

$$\mathbb{V}\text{ar}[X^{(j)[\text{ret}]}] = (a^{(j)})^2\, \mathbb{V}\text{ar}[X^{(j)}] \leq \mathbb{V}\text{ar}[X^{(j)}] \tag{2.4.13}$$

where $P^{(j)}$ denotes the equivalence premium (relying on a realistic basis).

Shares of premiums and, hence, safety loadings (namely, expected profits) are ceded to the reinsurer. For $j = 1, 2, \dots, n$, let $\Pi^{(j)[\text{ret}]}$ and $m^{(j)[\text{ret}]}$ denote the retained share of premium (including the safety loading) and safety loading respectively. Clearly,

$$m^{(j)[\text{ret}]} = \Pi^{(j)[\text{ret}]} - a^{(j)}\, P^{(j)} \tag{2.4.14}$$

In particular, if

$$\Pi^{(j)[\text{ret}]} = a^{(j)}\, \Pi^{(j)} \tag{2.4.15}$$

it follows that

$$m^{(j)[\text{ret}]} = a^{(j)}\, \Pi^{(j)} - a^{(j)}\, P^{(j)} = a^{(j)}\, m^{(j)} \tag{2.4.16}$$

However, the ceded share can be different from $(1 - a^{(j)})m^{(j)}$, and, in particular:

- it can be lower, if
  - the reinsurer grants a reward to the cedant for the underwriting work (namely, a *reinsurance commission*);
  - the reinsurer accepts a lower safety loading thanks to a larger portfolio size;
- it can be either lower or higher because the reinsurer adopts a technical basis different from the one adopted by the ceding company, and hence a different premium.

Example 2.4.1 Assume that, for the policy 1 in the portfolio, the sum insured is $x^{(1)} = 1\,000$, and the probability of claim (assessed by the cedant) is $p^{(1)} = 0.01$; the safety loading is 10 % of the equivalence premium $P^{(1)} = 10$, and thus $m^{(1)} = 1$. Hence, $\Pi^{(1)} = 11$. Let $a^{(1)} = 0.70$ be the retained share of the risk.

First, assume that the reinsurer agrees on the technical basis, i.e., on $p^{(1)} = 0.01$, and 10 % as the safety loading, and is willing to obtain a proportional share of the safety loading. Thus, for the ceding company we have $m^{(1)[\mathrm{ret}]} = 0.7$, so that $\Pi^{(1)[\mathrm{ret}]} = 7.7$, thus resulting proportional to $\Pi^{(1)}$ according to the retention share.

Second, suppose that the reinsurer still agrees on the technical basis, but is willing to leave to the cedant a share of the safety loading higher than 70 %, say 80 %. Hence, we find

$$\Pi^{(1)[\mathrm{ret}]} = 0.70\, P^{(1)} + 0.80\, m^{(1)} = 7.8$$

Finally, assume that the reinsurer does not agree on the technical basis. In particular, the reinsurer accepts a safety loading equal to 10 % of the equivalence premium, whilst evaluates the claim probability as $\tilde{p}^{(1)} = 0.012$. Thus, according to the reinsurer's judgement, the equivalence premium should be $\tilde{P}^{(1)} = 12$, and the premium including the safety loading should be $\tilde{\Pi}^{(1)} = 13.2$. If the reinsurer is willing to obtain a proportional share of $\tilde{\Pi}^{(1)}$, namely $0.30 \times 13.2 = 3.96$, the cedant retains

$$\Pi^{(1)[\mathrm{ret}]} = \Pi^{(1)} - 0.30\, \tilde{\Pi}^{(1)} = 11 - 3.96 = 7.04$$

and thus

$$m^{(1)[\mathrm{ret}]} = \Pi^{(1)[\mathrm{ret}]} - 0.70\, P^{(1)} = 7.04 - 7 = 0.04$$

❑

To assess the effect of reinsurance on the portfolio riskiness, we have to look at the retained total payment, $X^{[\mathrm{ret}]}$, and some related typical values, in particular the index defined by (2.3.58).

The retained total payment is defined by (2.4.5). Then, we have

$$\mathbb{E}[X^{[\mathrm{ret}]}] = \mathbb{E}\left[\sum_{j=1}^{n} X^{(j)[\mathrm{ret}]}\right] = \sum_{j=1}^{n} a^{(j)}\, P^{(j)} \tag{2.4.17}$$

and (assuming the independence among the insured risks)

$$\mathbb{V}\mathrm{ar}[X^{[\mathrm{ret}]}] = \sum_{j=1}^{n} \mathbb{V}\mathrm{ar}[X^{(j)[\mathrm{ret}]}] = \sum_{j=1}^{n} (a^{(j)})^2\, \mathbb{V}\mathrm{ar}[X^{(j)}] \tag{2.4.18}$$

Let $\sigma^{[\mathrm{ret}]}$ denote the standard deviation of the total payment, that is,

$$\sigma^{[\mathrm{ret}]} = \sqrt{\mathbb{V}\mathrm{ar}[X^{[\mathrm{ret}]}]} \tag{2.4.19}$$

Further, let $m^{[\mathrm{ret}]}$ denote the retained safety loading (and hence the retained expected profit):

$$m^{[\text{ret}]} = \sum_{j=1}^{n} m^{(j)[\text{ret}]} \tag{2.4.20}$$

Then, we have:

$$s^{[\text{ret}]} = \frac{m^{[\text{ret}]} + M}{\sigma^{[\text{ret}]}} \tag{2.4.21}$$

From (2.4.21) we can see that, in the presence of a reinsurance arrangement, the probability of default depends on:

- the effect of reinsurance on the variability of the total payout, expressed by $\sigma^{[\text{ret}]}$;
- the retained share of the total expected profit, expressed by $m^{[\text{ret}]}$.

Note that, in particular, we have:

$$m^{[\text{ret}]} < m^{[\text{P}]} \quad \text{and} \quad \sigma^{[\text{ret}]} < \sigma^{[\text{P}]} \tag{2.4.22}$$

The probability of default, $\pi(m^{[\text{ret}]} + M)$, is then given by:

$$\pi(m^{[\text{ret}]} + M) = \mathbb{P}[X^{[\text{ret}]} > P^{[\text{ret}]} + m^{[\text{ret}]} + M] = 1 - \Phi\left(\frac{m^{[\text{ret}]} + M}{\sigma^{[\text{ret}]}}\right) = 1 - \Phi(s^{[\text{ret}]}) \tag{2.4.23}$$

(see Eq. (2.3.55))

To quantify the probability of default, and then to determine an appropriate capital allocation, we need to refer to specific reinsurance policies $\underline{a} = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})$, and to the rules adopted for splitting the safety loading (see Example 2.4.1 in particular).

2.4.4 Two Reinsurance Arrangements

The *quota-share reinsurance* is defined by the following policy:

$$\underline{a} = (a, a, \ldots, a); \quad 0 < a < 1 \tag{2.4.24}$$

namely, the same retention share is applied to all the individual risks. The effect on the sums insured is illustrated by Fig. 2.18 which shows that, in relative terms, all the sums insured are reduced in the same proportion.

For the standard deviation of the portfolio payment, we immediately find:

$$\sigma^{[\text{ret}]} = a\sigma^{[\text{P}]} \tag{2.4.25}$$

whereas the retained profit is given by

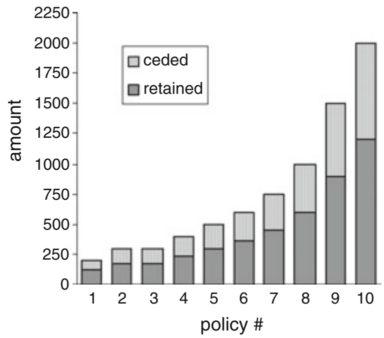


Fig. 2.18 Quota-share reinsurance

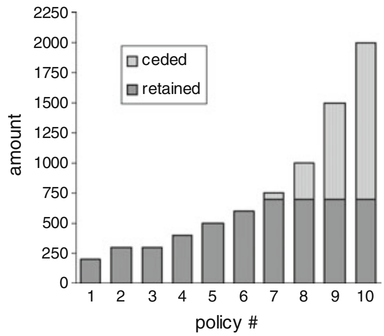


Fig. 2.19 Surplus reinsurance

$$m^{[\mathrm{ret}]} = a m^{[\mathrm{P}]} \tag{2.4.26}$$

if the reinsurer and the cedant agree on a proportional sharing.

A *surplus reinsurance* arrangement is defined by the retention, $x^{[\mathrm{ret}]}$, in terms of the sum insured. The amount $x^{[\mathrm{ret}]}$ is commonly called the *retention line*. For the generic risk, whose sum insured is $x^{(j)}$, the splitting (see (2.4.4)) is determined as follows:

- the amount $\min\{x^{(j)}, x^{[\mathrm{ret}]}\}$ is retained;
- the amount $\max\{0, x^{(j)} - x^{[\mathrm{ret}]}\}$, i.e., the *surplus*, is ceded.

Hence, the reinsurance policy $\underline{a}$ is defined as follows:

$$a^{(j)} = \frac{\min\{x^{(j)}, x^{[\mathrm{ret}]}\}}{x^{(j)}} = \min\left\{1, \frac{x^{[\mathrm{ret}]}}{x^{(j)}}\right\}; \quad j = 1, 2, \ldots, n \tag{2.4.27}$$

Figure 2.19 illustrates the effect of the surplus reinsurance, namely the "leveling" of sums insured.

Intuitively, a higher efficiency is expected from surplus reinsurance, thanks to the leveling effect. It is worth recalling (see Sect. 2.3.4, and formula (2.3.17) in particular) that, as a consequence of a huge sum insured, the diversification via pooling tends to disappear. Clearly, the surplus reinsurance can mitigate this dangerous effect by leveling (at least to some extent) the sums insured. On the contrary, according to the quota-share arrangement there is no leveling, as all the sums insured are reduced in the same proportion.

Remark We note that, comparing the effects of quota-share and surplus reinsurance is, to some extent, similar to comparing the effects of fixed-percentage deductible and fixed-amount deductible, discussed in Sect. 1.3.6.

2.4.5 Examples

We address the following aspects of reinsurance policies by using numerical examples:

- first, we discuss the effects of quota-share and surplus reinsurance, in terms of the retained expected profit, the standard deviation of the portfolio payment, and the resulting probability of default $\pi(m^{[\mathrm{ret}]} + M)$ (as given by formula (2.4.23)); see Example 2.4.2;
- then, we compare various combinations of surplus reinsurance and capital allocation, in terms of the retained expected profit and the standard deviation of the portfolio payment, for a fixed level of probability of default; see Example 2.4.3.

Example 2.4.2 We refer to portfolio C, described in Example 2.3.2 (see Table 2.5). We assume what follows:

- safety loading rate $\frac{m^{[\mathrm{P}]}}{P^{[\mathrm{P}]}} = 0.10$;
- allocated capital $M = 10\,000$;
- retained share of premiums (and hence expected profit) equal to retained share of sums insured.

See Tables 2.17 and 2.18.

Some comments can help in understanding the higher effectiveness of the surplus reinsurance compared to the quota-share arrangement.

The same amount of retained expected profit, namely $m^{[\mathrm{ret}]} = 45\,000$, is achieved with $a = 0.90$ and $x^{[\mathrm{ret}]} = 6\,000$; however, in the quota-share reinsurance the standard deviation is higher ($\sigma^{[\mathrm{ret}]} = 38\,220$ versus $\sigma^{[\mathrm{ret}]} = 33\,271$), and hence the probability of default is higher ($\pi(m^{[\mathrm{ret}]}+M) = 0.075$ versus $\pi(m^{[\mathrm{ret}]}+M) = 0.049$). A similar situation holds for $a = 0.75$ and $x^{[\mathrm{ret}]} = 3\,000$.

Finally, we note that the same probability of default, $\pi(m^{[\mathrm{ret}]} + M) = 0.004$, is achieved in the quota-share with $a = 0.157$, and the surplus reinsurance with $x^{[\mathrm{ret}]} = 1\,500$; however, the latter arrangement leaves a much higher expected profit ($m^{[\mathrm{ret}]} = 33\,750$ versus $m^{[\mathrm{ret}]} = 7\,865$). ❑

Table 2.17 Quota-share reinsurance-Portfolio C

a	$m^{[\text{ret}]}$	$\sigma^{[\text{ret}]}$	$s^{[\text{ret}]}$	$\pi(m^{[\text{ret}]}+M)$
1.000	50 000	42 467	1.413	0.079
0.900	45 000	38 220	1.439	0.075
0.750	37 500	31 850	1.491	0.068
0.157	7 865	6 680	2.674	0.004

Table 2.18 Surplus reinsurance-Portfolio C

$x^{[\text{ret}]}$	$m^{[\text{ret}]}$	$\sigma^{[\text{ret}]}$	$s^{[\text{ret}]}$	$\pi(m^{[\text{ret}]}+M)$
≥8 000	50 000	42 467	1.413	0.079
6 000	45 000	33 271	1.653	0.049
5 000	42 500	28 867	1.819	0.034
3 000	37 500	20 864	2.277	0.011
1 500	33 750	16 353	2.675	0.004

Example 2.4.3 We refer to portfolio B, described in Example 2.3.2 (see Table 2.4), which consists of 10 000 risks, all with $x = 1\,000$ as the sum insured and $p = 0.005$ as the claim probability. We focus on some combinations of retention line $x^{[\text{ret}]}$ and allocated capital M, leading to the same probability of default $\pi(m^{[\text{ret}]}+M) = 0.005$, and hence to the same value $s^{[\text{ret}]} = 2.5805$. Thus,

$$\frac{m^{[\text{ret}]}+M}{\sigma^{[\text{ret}]}} = 2.5805$$

We assume that the safety loading rate $\frac{m^{[\text{P}]}}{P^{[\text{P}]}} = 0.10$ is adopted, which leads to $m^{[\text{P}]} = 5\,000$ (see Table 2.12). Then, we find:

$$m^{[\text{ret}]} = m^{[\text{P}]}\,\frac{\min\{x^{[\text{ret}]}, 1\,000\}}{1\,000} = 5\,\min\{x^{[\text{ret}]}, 1\,000\}$$

Further, we have:

$$\begin{aligned}\sigma^{[\text{ret}]} &= \sqrt{10\,000\,(\min\{x^{[\text{ret}]}, 1\,000\})^2\,p\,(1-p)} \\ &= 100\,\min\{x^{[\text{ret}]}, 1\,000\}\sqrt{p\,(1-p)} = 7.053\,\min\{x^{[\text{ret}]}, 1\,000\}\end{aligned}$$

so that we find:

$$M = 13.2\,\min\{x^{[\text{ret}]}, 1\,000\}$$

This formula can be generalized (although referring still to the particular portfolio structure, with x as the sum insured for all the risks) as follows:

$$M = \kappa\,\min\{x^{[\text{ret}]}, x\} \tag{2.4.28}$$

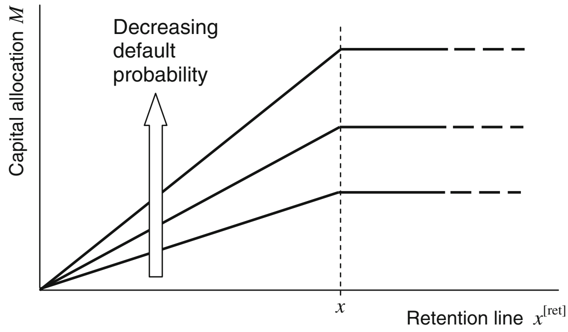


Fig. 2.20 Capital allocation versus surplus reinsurance

Table 2.19 Capital allocation versus surplus reinsurance-Portfolio B

M	$x^{[\text{ret}]}$	$m^{[\text{ret}]}$	$\sigma^{[\text{ret}]}$	Value $m^{[\text{ret}]} - rM$
13 200	≥1 000	5 000	7 053	3 944
6 600	500	2 500	3 527	1 972
2 640	200	1 000	1 411	789
1 320	100	500	705	394

where the coefficient κ depends, in particular, on the target probability of default. Figure 2.20 illustrates the relation (2.4.28), for various target probabilities; x denotes the sum insured (in the numerical example $x = 1\,000$).

Table 2.19 illustrates the effects of some choices of retention line and capital allocation (all the combinations leading to the same result in terms of the probability of default, that is, 0.005). We note that, the lower the retention line $x^{[\text{ret}]}$ (i.e., the higher the cession to the reinsurer), the lower is the need for both the capital allocation M and the safety loading $m^{[\text{ret}]}$, but, at the same time, the smaller is the value creation ($r = 0.08$ has been assumed). ❑

2.4.6 *Optimal Reinsurance Policy*

We consider the following problem: find the reinsurance policy

$$\underline{a} = (a^{(1)}, a^{(2)}, \ldots, a^{(n)})$$

which implies the lowest probability of default, out of the set of reinsurance policies leading to the same amount of retained expected profit $m^{[\text{ret}]}$. It is worth noting that the results reported below hold in general situations, namely are not restricted to the "basic" insurance cover we have so far addressed.

The problem we are attacking is a problem of constrained optimization. In formal terms, let $\hat{m}^{(j)}$ denote the safety loading of the jth risk ceded in the case of zero retention (that is, if $a^{(j)} = 0$). As seen in Sect. 2.4.3, we can have $\hat{m}^{(j)} \lesseqgtr m^{(j)}$. Assume that, for any value of $a^{(j)}$ $(0 \leq a^{(j)} \leq 1)$, the ceded safety loading is $(1 - a^{(j)})\, \hat{m}^{(j)}$. Then, we have

$$m^{(j)[\mathrm{ret}]} = m^{(j)} - (1 - a^{(j)})\, \hat{m}^{(j)} \tag{2.4.29}$$

and, for the total retained safety loading:

$$m^{[\mathrm{ret}]} = m^{[\mathrm{P}]} - \sum_{j=1}^{n} (1 - a^{(j)})\, \hat{m}^{(j)} \tag{2.4.30}$$

Consider the index $s^{[\mathrm{ret}]}$, defined by (2.4.21), and the probability of default, given by (2.4.23). Note that, under the constraint

$$m^{[\mathrm{ret}]} + M = \text{constant} \tag{2.4.31}$$

we have

$$\min_{\underline{a}}\{\sigma^{[\mathrm{ret}]}\} \Rightarrow \max_{\underline{a}}\{s^{[\mathrm{ret}]}\} \Rightarrow \min_{\underline{a}}\{\pi(m^{[\mathrm{ret}]} + M)\} \tag{2.4.32}$$

where

$$\sigma^{[\mathrm{ret}]} = \sqrt{\sum_{j=1}^{n} (a^{(j)})^2\, (\sigma^{(j)})^2} \tag{2.4.33}$$

with $(\sigma^{(j)})^2 = \mathbb{V}\mathrm{ar}[X^{(j)}]$

Hence, the optimization problem is as follows:

$$\min_{\underline{a}} \sum_{j=1}^{n} (a^{(j)})^2\, (\sigma^{(j)})^2 \tag{2.4.34}$$

subject to:

$$\begin{cases} \displaystyle\sum_{j=1}^{n} (1 - a^{(j)})\, \hat{m}^{(j)} = A \\ 0 \leq a^{(j)} \leq 1;\ j = 1, 2, \dots, n \end{cases}$$

We note that the optimization problem is parametric, as its solution depends on the parameter A.

It is possible to prove that the optimal solution is given by:

$$a^{(j)} = \min\left\{1,\ B\,\frac{\hat{m}^{(j)}}{(\sigma^{(j)})^2}\right\} \tag{2.4.35}$$

where the parameter B depends, in particular, on the value assigned to A, and hence on the amount of ceded expected profit: the lower is the ceded expected profit, the higher is B and then the retention.

We now return to the "basic" insurance cover, and assume the same claim probability p for all the n risks. Hence, for $j = 1, 2, \ldots, n$, we have

$$(\sigma^{(j)})^2 = (x^{(j)})^2\, p\,(1-p) \tag{2.4.36}$$

Moreover, we assume that, for $j = 1, 2, \ldots, n$, the quantity $\hat{m}^{(j)}$ is proportional to the sum insured $x^{(j)}$:

$$\hat{m}^{(j)} = \alpha\, x^{(j)} \tag{2.4.37}$$

Note that relation (2.4.37) holds, in particular, if:

1. $m^{(j)} = \beta P^{(j)} = \beta p\, x^{(j)}$,
   and
2. $\hat{m}^{(j)} = m^{(j)}$.

From (2.4.35) it follows that

$$a^{(j)} = \min\left\{1,\ B\,\frac{\alpha}{x^{(j)}\, p\,(1-p)}\right\} \tag{2.4.38}$$

and, in monetary terms:

$$a^{(j)}\, x^{(j)} = \min\left\{x^{(j)},\ B\,\frac{\alpha}{p\,(1-p)}\right\} \tag{2.4.39}$$

The amount $B\,\frac{\alpha}{p\,(1-p)}$ is independent of j, so that we can write:

$$a^{(j)}\, x^{(j)} = \min\left\{x^{(j)}, x^{[\mathrm{ret}]}\right\} \tag{2.4.40}$$

Hence, the solution of the constrained optimization problem (2.4.34) is given by the surplus reinsurance.

It is worth noting that, conversely, if a surplus reinsurance arrangement is adopted, the probability of default is minimized, subject to the loss of expected profit related to the value of A implied by the retention level $x^{[\mathrm{ret}]}$.

2.5 Reinsurance: Further Aspects

2.5.1 Reinsurance Arrangements

Reinsurance arrangements can be classified according to several criteria. In particular, the classification into global reinsurance arrangements (that is, on a portfolio basis) and individual arrangements (on a policy basis) has been mentioned in Sect. 2.4 (see also Fig. 2.21).

When a reinsurance arrangement is defined on a *policy basis*, the relevant parameters concern the individual risks (for example, the share a in the quota-share reinsurance, the retention line $x^{[\text{ret}]}$ in the surplus reinsurance). Another reinsurance arrangement belonging to this category, the so-called Excess-of-Loss reinsurance, will be described in Sect. 2.5.2.

The parameters of reinsurance arrangements defined on a *portfolio basis* relate to quantities concerning the portfolio total payment (for example, the priority Λ and the upper limit Θ in the stop-loss reinsurance). Another reinsurance arrangement belonging to this category, the so-called catastrophe reinsurance, will be described in Sect. 2.5.3.

According to another criterion, reinsurance arrangements can be classified into proportional and non-proportional arrangements (see Fig. 2.21).

In a *proportional* reinsurance arrangement, claims and premiums are divided between the cedant and the reinsurer in the ratio of their shares in the reinsurance contract. Hence, the sharing of claims is determined when the reinsurance arrangement is defined. Quota-share and surplus reinsurance belong to this category.

In a *non-proportional* reinsurance arrangement, the rule for the sharing of claims is stated when the reinsurance contract is defined, but the actual sharing of claims is determined depending on the severity of each claim, or the number of claims in the portfolio, or the total portfolio payment. Examples are given by the stop-loss, catastrophe, and XL reinsurance.

Fig. 2.21 Reinsurance arrangements

Level	PROPORTIONAL	NON-PROPORTIONAL
POLICY	Quota-share Surplus	Excess-of-loss (XL)
PORTFOLIO		Stop-loss Catastrophe

2.5.2 Random Claim Sizes: XL Reinsurance

Other features of the reinsurance arrangements we have already dealt with, namely quota-share and surplus reinsurance, emerge when moving to individual risks more general than those related to the basic insurance cover, in particular by allowing for random claim sizes. For example, we can refer to the risks described as Cases 3d (A fire in a factory) and 3e (Car driver's liability) in Sect. 1.2.4. Further, the specific role of the Excess-of-Loss reinsurance emerges if we allow for random claim sizes.

Let us refer to the jth risk in the portfolio. An example of the (continuous) probability distribution of the generic kth claim, $X_k^{(j)}$, is provided, in terms of the related density function, by Fig. 2.22; $x_{\max}^{(j)}$ represents the maximum possible outcome.

In a quota-share arrangement, with *retention share* a for all the risks in the portfolio, the retained amount is defined as follows:

$$X_k^{(j)[\mathrm{ret}]} = a\, X_k^{(j)} \tag{2.5.1}$$

In a surplus reinsurance arrangement, with $x^{[\mathrm{ret}]}$ as the retention line, we have (assuming $x^{[\mathrm{ret}]} < x_{\max}^{(j)}$):

$$X_k^{(j)[\mathrm{ret}]} = \frac{x^{[\mathrm{ret}]}}{x_{\max}^{(j)}}\, X_k^{(j)} \tag{2.5.2}$$

We note that, while in a quota-share arrangement the retained share is trivially equal to a for all the risks in the portfolio, according to the surplus reinsurance the retained share is $\frac{x^{[\mathrm{ret}]}}{x_{\max}^{(j)}}$, and hence depends on $x_{\max}^{(j)}$ which is specific to each insured risk. Figures 2.23 and 2.24 show the retention (and the reinsurer's intervention), in a surplus arrangement, depending on the relation between the amount $x_{\max}^{(j)}$ and a given retention line $x^{[\mathrm{ret}]}$. Both the arrangements can be classified as proportional reinsurance, because, whatever the amount $X_k^{(j)}$, the retained share (either a or $\frac{x^{[\mathrm{ret}]}}{x_{\max}^{(j)}}$) is known at the time the reinsurance contract is written.

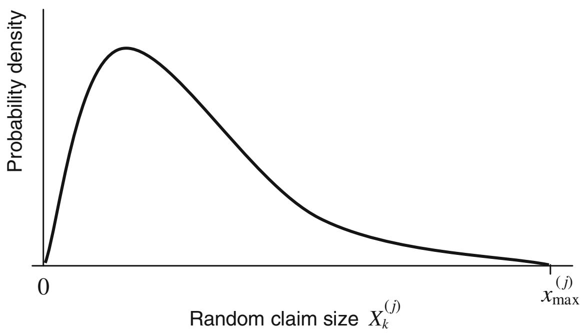


Fig. 2.22 Probability density of the random payment in a claim

Fig. 2.23 The retained payment of the cedant in surplus reinsurance; no upper limit; $x^{(j)}_{\max} = 4\,x^{[\mathrm{ret}]}$

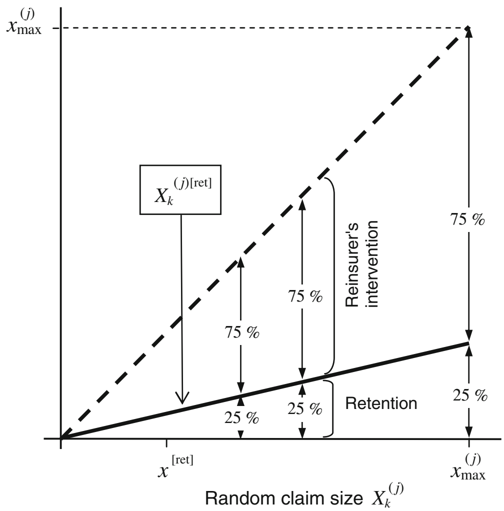
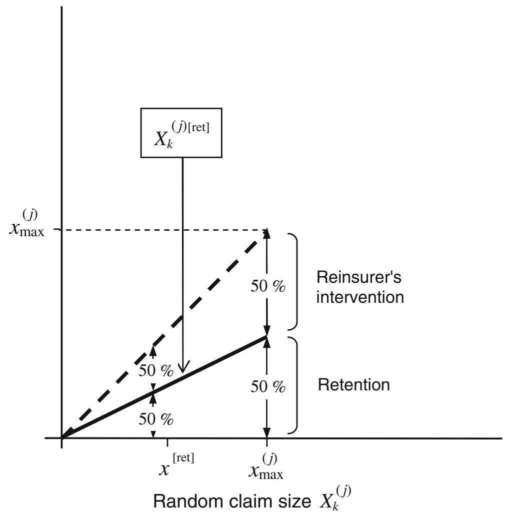


Fig. 2.24 The retained payment of the cedant in surplus reinsurance; no upper limit; $x^{(j)}_{\max} = 2\,x^{[\mathrm{ret}]}$



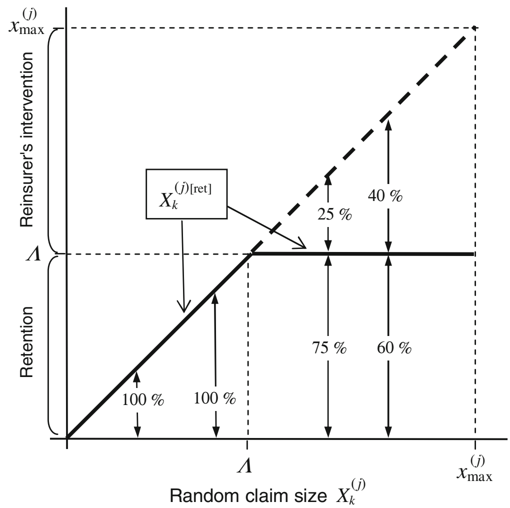
The retention and the reinsurer's intervention in the *Excess-of-Loss* reinsurance (briefly, XL reinsurance) are defined as follows:

$$X_k^{(j)[\mathrm{ret}]} = \min\{X_k^{(j)}, \Lambda\} \tag{2.5.3a}$$

$$X_k^{(j)[\mathrm{ced}]} = \max\{X_k^{(j)} - \Lambda, 0\} \tag{2.5.3b}$$

where Λ denotes the *deductible*. The analogy with the deductible in a generic risk transfer is apparent (see Sect. 1.3.6, and Eqs. (1.3.4)).

Fig. 2.25 The retained payment of the cedant in XL reinsurance (no upper limit)



We note that, in this simple XL arrangement, the reinsurer pays the whole amount beyond the deductible, net of the deductible itself, namely no upper limit has been stated. The retained share decreases as the claim size $X_k^{(j)}$ increases; see Fig. 2.25. Indeed, from (2.5.3a) we have:

$$\frac{X_k^{(j)[\mathrm{ret}]}}{X_k^{(j)}} = \min\left\{1, \frac{\Lambda}{X_k^{(j)}}\right\} \tag{2.5.4}$$

As the retained share depends on the amount $X_k^{(j)}$ and hence is not known at the time of issue of the reinsurance contract, the result is a non-proportional reinsurance.

Assume, conversely, that the upper limit of the reinsurance cover is set to $h\,\Lambda$ (with h an integer number, $h \geq 2$). For a generic claim with random size $X_k^{(j)}$, possible situations are as follows:

1. if $X_k^{(j)} \leq \Lambda$, then the insurer totally retains the claim amount;
2. if $\Lambda < X_k^{(j)} \leq h\,\Lambda$, then the XL cover exhausts the cession;
3. if $X_k^{(j)} > h\,\Lambda$, then the insurer has still to cede $X_k^{(j)} - h\,\Lambda$, through a second XL cover (or possibly more XL covers), with another reinsurer (or even with the first reinsurer, however, according to a technical basis usually different from the one used in the first cover).

Hence, the cession is split into two (or more) *layers*: the first layer covers the interval $(\Lambda, h\,\Lambda)$, whereas the interval $(h\,\Lambda, X_k^{(j)})$ can be covered by a further XL reinsurance (or more than one XL). See Fig. 2.26, where it has been assumed $h = 3$.

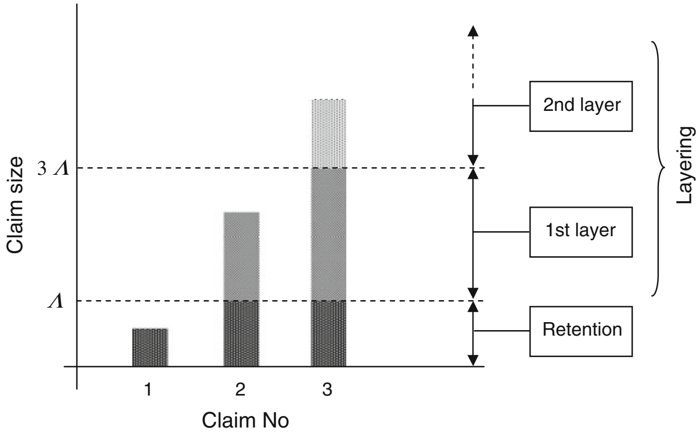


Fig. 2.26 Layering in XL reinsurance

2.5.3 Catastrophe Reinsurance

The *Catastrophe reinsurance* (briefly, *Cat-XL*) is a non-proportional reinsurance arrangement at a portfolio level. Its aim is to protect the portfolio (and the insurance company) against the risk that a single accident (that is, a "catastrophe") causes a huge number of claims in the portfolio itself. For example,

- in a generic portfolio, a high number of claims can occur because of a disaster (hurricane, earthquake, and so on);
- in "a group insurance," a number of insureds can suffer body injuries owing to a single accident in the workplace (explosion, fire, collapse, and so on); see, for example, Cases 3b (Disability benefits; one-year period) and 3c (Disability benefits; multi-year period) in Sect. 1.7.2.

A *catastrophe* is usually defined in terms of a given (minimum) number of claims, c, within a time interval of a given (maximum) duration, for example, 48 h. In formal terms, let K denote the random number of claims, $X^{[P]}$ the consequent total payment (before reinsurer's intervention); the reinsurer will intervene only if $K \geq c$.

There are various definitions of the Cat XL structure. We just focus on the two following definitions.

First, the Cat XL arrangement can be defined on a claim number basis. Let λ denote the deductible in terms of number of claims. Then, the cedant's retention and the reinsurer's intervention are respectively given by:

$$X^{[\mathrm{ret1}]} = \min\left\{X^{[P]}, \frac{\lambda}{K} X^{[P]}\right\} \tag{2.5.5a}$$

$$X^{[\mathrm{ced1}]} = \max\left\{0, \frac{K-\lambda}{K} X^{[P]}\right\} \tag{2.5.5b}$$

Note that, according to this definition, if $X^{[P]}$ is large then $X^{[ret1]}$ is large. Thus, the reinsurance arrangement is effective if individual claims have approximately the same amount, and hence the total payment $X^{[P]}$ mainly depends on the number of claims. Otherwise, effectiveness can be gained via a preliminary surplus or XL reinsurance.

Another definition of the Cat XL arrangement is based on the amount $X^{[P]}$ of the total payment. Let Λ denote the deductible (in monetary terms). Then:

$$X^{[ret2]} = \min\{X^{[P]}, \Lambda\} \tag{2.5.6a}$$

$$X^{[ced2]} = \max\{X^{[P]} - \Lambda, 0\} \tag{2.5.6b}$$

Example 2.5.1 Consider the Cat XL reinsurance defined by Eqs. (2.5.5), with $c = 5$, and $\lambda = 8$. According to the outcome of the number of claims, K, we have the following situations:

$$K = \underbrace{\overbrace{1, 2, 3, 4}^{\text{no cat}},\ \ 5, 6, 7, 8,}_{\text{no reinsurer's intervention}}\quad \underbrace{9, 10, 11, \ldots}_{\text{reinsurer's intervention}}$$

Move to the Cat XL arrangement defined by Eqs. (2.5.6), still with $c = 5$, and with $\Lambda = 1\,200$. Then, we have

$$K = \overbrace{1, 2, 3, 4}^{\text{no cat}},\qquad \underbrace{5, 6, 7, 8, 9, 10, 11, \ldots}_{\text{possible reinsurer's intervention, depending on } X^{[P]}}$$

Consider the following cases:

(a) $K = 10,\ X^{[P]} = 1\,000$;
(b) $K = 10,\ X^{[P]} = 5\,000$.

In case (a), according to the first Cat XL arrangement we have:

$$X^{[ret1]} = \frac{8}{10} 1\,000 = 800, \quad X^{[ced1]} = \frac{2}{10} 1\,000 = 200$$

whereas the second arrangement yields:

$$X^{[ret2]} = 1\,000, \quad X^{[ced2]} = 0$$

In case (b), the first arrangement leads to:

$$X^{[ret1]} = \frac{8}{10} 5\,000 = 4\,000, \quad X^{[ced1]} = \frac{2}{10} 5\,000 = 1\,000$$

while the second yields:

$$X^{[\mathrm{ret2}]} = 1\,200, \quad X^{[\mathrm{ced2}]} = 3\,800$$

❑

Example 2.5.2 Consider a portfolio of n basic risks, and assume $x^{(j)} = x$, for $j = 1, 2, \ldots, n$. Then:

$$X^{[\mathrm{P}]} = Kx$$

The two Cat XL arrangements respectively lead to:

$$X^{[\mathrm{ret1}]} = \min\left\{Kx, \frac{\lambda}{K}\, K\, x\right\} = x\, \min\{K, \lambda\}$$
$$X^{[\mathrm{ret2}]} = \min\{K\, x,\, \Lambda\}$$

If we define

$$\lambda = \frac{\Lambda}{x}$$

then we have:

$$X^{[\mathrm{ret2}]} = \min\{K\, x, \lambda\, x\} = x\, \min\{K, \lambda\}$$

and hence:

$$X^{[\mathrm{ret1}]} = X^{[\mathrm{ret2}]}$$

Then, in a portfolio homogeneous in terms of sums insured, the two Cat XL arrangements lead to the same retention. ❑

2.5.4 Purposes of Reinsurance

Although, from a strictly actuarial point of view, it is apparent that reinsurance arrangements aim to keep the portfolio riskiness at a level acceptable by the insurance company, resorting to reinsurance can have various purposes. Some considerations follow:

1. As regards the reduction of the portfolio riskiness, it should be noted that reinsurance arrangements mainly aim at reducing the impact of random fluctuations and catastrophic events. In fact, the reinsurance company is willing to take the ceded risks as it can achieve a higher pooling effect and hence an improved diversification of risks (see Sects. 2.3.1 and 2.3.2). From the point of view of the cedant, more insurance implies:

   - a lower capital allocation;
   - an increased underwriting capacity.

Conversely, risks affected by possible systematic deviations could be rejected by reinsurers, as these deviations affect the pool as an aggregate, and the total impact on portfolio results increases as the portfolio size increases. Notwithstanding, the reinsurer can take the risk of systematic deviations, with the proviso that a further transfer of this risk can be worked out. We will address this issue in Sect. 2.6.

2. The cedant company can benefit from technical advice provided by the reinsurer. In particular:

   - the reinsurer, thanks to specific experience, can suggest statistical bases and inform about market features for new insurance products;
   - as regards in-force portfolios, the reinsurer can provide the cedant with an update of statistical bases (which is more effective if a quota-share arrangement works, as this allows the reinsurer to monitor all claims pertaining to the reinsured portfolio).

3. Reinsurance can have a "financing" role, thanks to a sharing of policy and portfolio expenses between the cedant and the reinsurer.

2.5.5 Insurance–Reinsurance Networks

Figure 2.27 illustrates an *insurance–reinsurance network*. Following the paths marked by solid arrows, we first find an example of *direct insurance* (or *primary insurance*): insurer X directly takes risks from clients A1, A2, …, An. Hence, X works in the insurance market. Then, we find examples of *cession*: insurer X cedes risks to Y and Z; for example, policies implying a huge exposure are only partially accepted by Y, so that the residual portions are ceded to Z. Thus, companies Y and Z provide company X with *reinsurance*. Finally, company Y cedes to W part of the risks, in particular taken from X; this reinsurance transaction is called *retrocession*.

Further examples can be found following the paths marked by dashed arrows. First, we find another example of direct insurance: insurer Y directly takes risks from clients B1, B2, …, Bm. Note that company Y works both in primary insurance and in reinsurance as well, as it takes risks ceded by company X. The relationship between X and Y is twofold, as Y also cedes risks to company Y. Finally, we note that companies X and Y share a risk ceded by client B1, and this constitutes an example of *coinsurance*.

Reinsurance arrangements can be stated on various bases, for the cedant and the reinsurer respectively:

1. facultative/facultative (briefly, *facultative*);
2. obligatory/obligatory (briefly, *obligatory*);
3. facultative/obligatory (briefly, *facob*).

According to an arrangement of type 1, if an insurer is willing to cede a risk to a reinsurer, then the reinsurer can decide to accept the risk itself. Usually, this

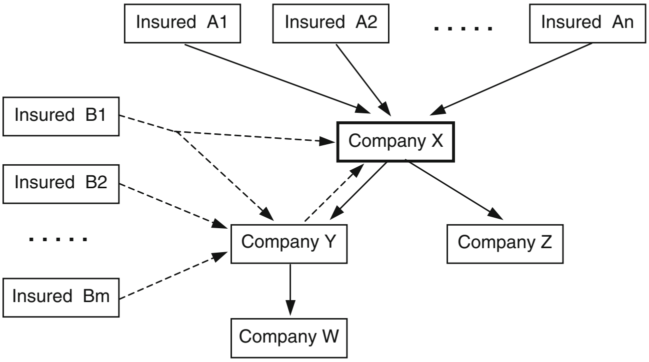


Fig. 2.27 Insurance, coinsurance, and reinsurance: a network

arrangement concerns the cession of single risks, in particular those involving huge exposures.

Types 2 and 3 require that a reinsurance contract, usually called a *treaty*, has been written by the cedant and the reinsurer. In particular, in an arrangement of type 2 the insurer is obliged to cede portions (as defined in the treaty) of the risks underwritten, and the reinsurer is obliged to accept them. In type 3, the insurer can decide to cede risks and, if so, the reinsurer is obliged to take them.

2.5.6 Reinsurance Treaties. Reinsurance Programmes

A *reinsurance treaty* concerns all the aspects of a reinsurance arrangement, in particular:

- the time interval of the reinsurance cover;
- the reinsurance form (stop-loss, quota-share, XL, and so on);
- the *limitations* of the reinsurance cover (priority, upper limit, deductibles, retention lines, and so on);
- the technical bases for the calculation of the reinsurance premiums, and the conditions concerning the premium payment.

Limitations to a reinsurance cover can be classified into "vertical" and "horizontal" limitations. *Horizontal limitations* refer to the total reinsurer's payment related to the cover interval; an example is provided by the upper limit in the stop-loss reinsurance (see Sect. 2.4.2).

Vertical limitations concern the reinsurer's payment related either to each single claim or to each single policy. An example of vertical limitations concerning each single claim is provided by the *layering* in the XL arrangement (see Sect. 2.5.2).

A *reinsurance programme* combines several reinsurance treaties, possibly supplemented by facultative reinsurance when needed (for example, in relation to single huge exposures), and can involve various reinsurers. Resorting to reinsurance programmes is more common in non-life insurance, because of the random size of the claims and, hence, the higher riskiness.

Usually, reinsurance programmes are designed on a class-by-class basis, namely separate reinsurance programmes concern, for example, fire insurance, third-party liability, domestic property, and so on. Notwithstanding, reinsurance programmes can include special treaties arranged to cover risks, although belonging to various classes, in specific geographic areas, for example exposed to the risk of hurricanes or earthquakes.

Applying a reinsurance programme to each individual risk within a portfolio determines a progressive reduction of the cedant's exposure, and hence of the default probability. Figure 2.28 illustrates the effect, at a policy level, of an XL reinsurance followed by a quota-share reinsurance. Figure 2.29, conversely, illustrates the effects on the portfolio exposure, for which a stop-loss arrangement supplements the reinsurance covers at a policy level.

Combining quota-share and surplus arrangements provides basic examples of reinsurance programmes. Assume the retention share a for the quota-share, and the retention line $x^{[\mathrm{ret}]}$ for the surplus. We have, for the jth risk, the following results:

- a quota-share "followed" by a surplus reinsurance leads to the retention

$$x^{(j)[\mathrm{ret1}]} = \min\{a\, x^{(j)}, x^{[\mathrm{ret}]}\} \tag{2.5.7}$$

- a surplus "followed" by a quota-share leads to the retention

$$x^{(j)[\mathrm{ret2}]} = a \, \min\{x^{(j)}, x^{[\mathrm{ret}]}\} \tag{2.5.8}$$

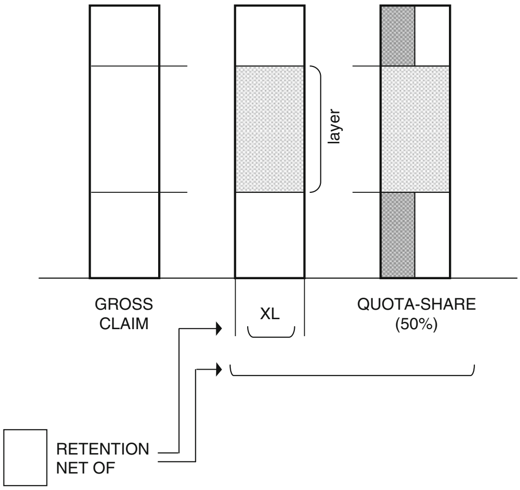


Fig. 2.28 Applying a reinsurance programme; effects at policy level

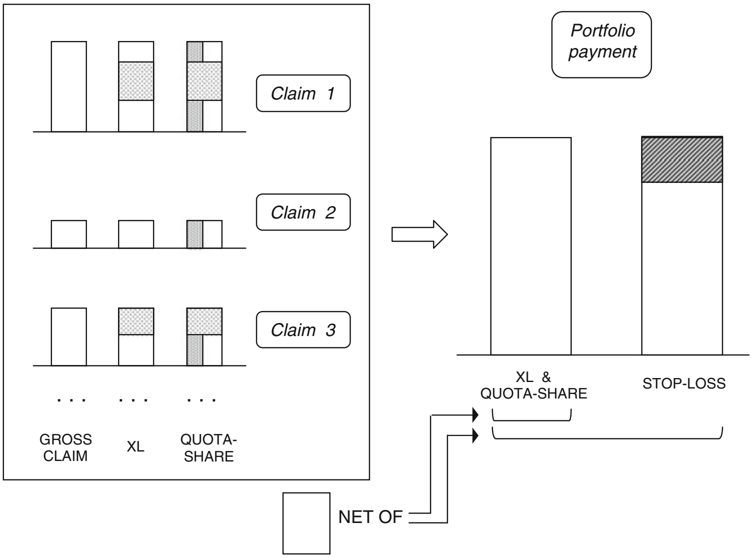


Fig. 2.29 Applying a reinsurance programme; effects at portfolio level

2.6 Alternative Risk Transfers

2.6.1 Some Preliminary Ideas

The (traditional) *insurance–reinsurance process* can be split into two basic steps (see Fig. 2.30):

1. the *insurance step*, which consists of transferring risks from organizations (individuals, families, firms, institutions, and so on) to an insurance company, and whose effects are
   a. building up a pool;
   b. reducing the relative riskiness (caused by random fluctuations);
2. the *reinsurance step*, which consists of transferring risks from the insurance company (the cedant) to the reinsurer, and whose effects are
   a. building up larger pools;
   b. a further reduction of the relative riskiness (caused by random fluctuations).

However, risk components other than random fluctuations can affect insurers' and reinsurers' results, namely systematic deviations and catastrophic events. As regards the latter, larger pools can improve diversification, for instance thanks to an increased variety of geographical locations of insured risks. As regards the former, the relative impact of systematic deviations is independent of the pool size (and the absolute

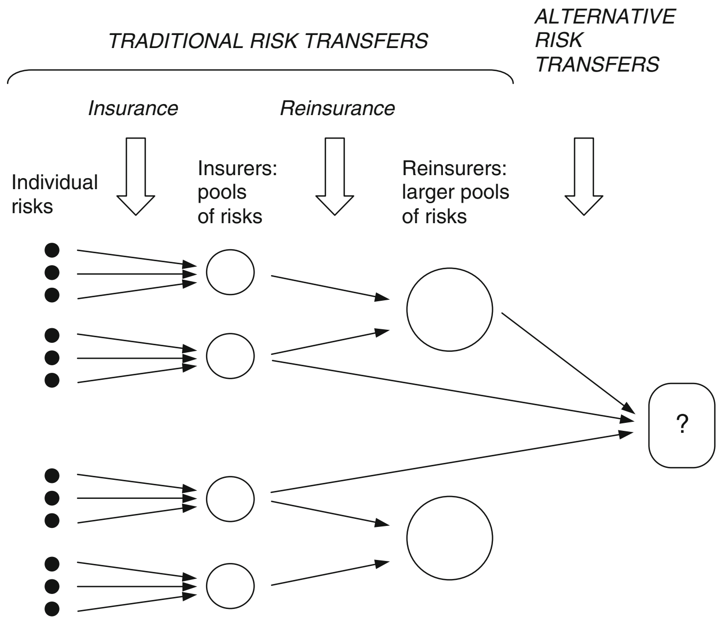


Fig. 2.30 The insurance–reinsurance process

impact increases as the pool size increases). Thus, risk transfer arrangements other than the traditional reinsurance, namely Alternative Risk Transfers (*ART*),

- are needed for transferring (at least to some extent) the risk of systematic deviations;
- can help in managing the catastrophe risk (lowering the cost of reinsurance, and/or the need for capital allocation).

In the following sections we will focus on ART in life insurance and reinsurance.

2.6.2 Securitization and the Role of Capital Markets

A (simplified) classification of ARTs in the context of life insurance and reinsurance, hence aiming at the transfer of biometric risks (mortality, longevity, and possibly disability) is sketched in Fig. 2.31. We note that two basic categories can be identified.

First, risks arising from contingent payments (the benefits provided by the insurance policies) can be packaged into securities traded on the capital market. The transaction is usually called *securitization*. Given the link of the payoff of the securities (see below) with the insurer's payments, the expression *insurance-linked securities* (briefly ILS) is commonly adopted. More specifically, when biometric risks are con-

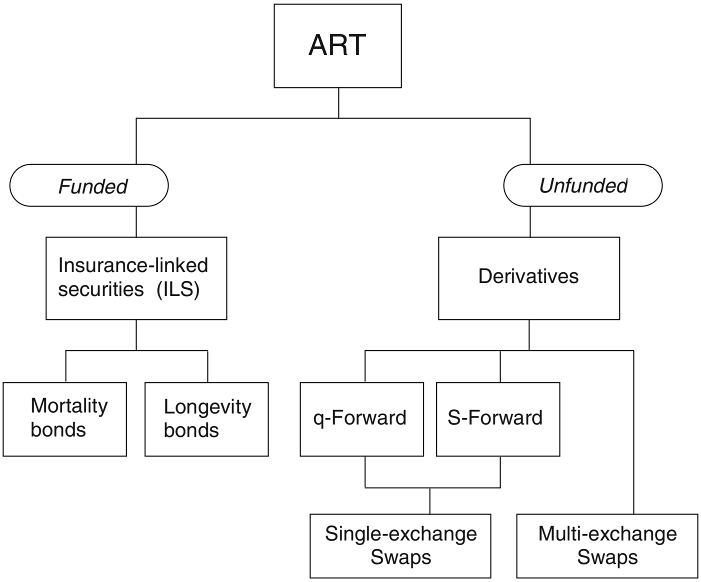


Fig. 2.31 Alternative Risk Transfers for biometric risks

cerned, the expressions *mortality-linked securities* and *longevity-linked securities* are frequently used. Examples of ILS follow.

- *Mortality bonds* are used to (partially) transfer the risk of a mortality higher than expected, then implying an amount of death benefits paid by an insurer (or reinsurer) larger than expected. To this purpose, the issuer of the mortality bond (the insurer or the reinsurer) pays reduced coupons and/or a reduced principal at maturity if the mortality in a given population, called the *reference population*, is higher than a stated benchmark, possibly owing to epidemics or natural disasters. Mortality bonds are typically short-term (3–5 years). More details are provided in Sect. 2.6.4.
- *Longevity bonds* aim at (partially) transferring the risk of a longevity higher than expected, hence implying an amount of survival benefits, e.g., life annuities, paid by an insurer (or reinsurer) larger than expected. The issuer of the longevity bond (the insurer or the reinsurer) pays reduced coupons (and possibly a reduced principal at maturity) if the longevity in the reference population is higher than a stated benchmark. The longevity bonds are typically long-term bonds (20 or more years), because:
  - the longevity risk reveals over a long period of time;
  - the insurer (or reinsurer) needs to offset benefit payments throughout long durations, as the insurance products usually involved are life annuities which are payable lifelong.

In the framework of ILS, the following securities can also be placed:

- *cat-bonds*, for transferring the risk of huge benefit payments due to some catastrophic event (earthquake, flood, etc.).

An Alternative Risk Transfer belonging to this category is called a *funded ART*, as the transaction starts with selling the securities to investors. Investing in ILS basically relies on a diversification target, assuming that the yield provided by an ILS is (reasonably) uncorrelated with the performance of most of other securities traded on the capital market. However, it should be stressed that a counterparty risk arises for the investor, because of possible default of the bond issuer.

We note that ILS structured as described above implement a hedging strategy denoted as approach 1b in Sect. 1.3.9. Indeed, the higher is the benefit payout, the smaller is the payoff of the bond (either in terms of coupons, or principal at maturity, or both).

It should be stressed that the experienced mortality (or longevity) which is compared to the agreed benchmark is the one observed in the reference population, and not in the specific insurance portfolio for which the hedging strategy is implemented. Then, a *basis risk* arises, because of possible imperfect hedging due to different mortality patterns in the population and the portfolio respectively. When a reference population is considered in defining the ART, the risk transfer is denoted as *index-based*, as the population mortality is usually expressed by an appropriate index. Conversely, in the case the actual portfolio mortality is compared to the benchmark mortality, the risk transfer is called *indemnity-based*. Clearly, an index-based transfer is preferred by investors, as population mortality data are collected and the index calculated by independent analysts.

The same argument, as regards possible imperfect hedging, also applies to the derivatives described below.

Remark Motivations other than a risk transfer can underly a securitization transaction. A securitization can consists in packaging a pool of assets (in particular intangible assets) or, more generally, a cash-flow stream into securities traded on the capital market. The aim of such a securitization transaction is to raise liquidity by selling future flows. In the insurance and reinsurance context, the specific aim can be the recovery of acquisition costs (especially in life insurance) or expected profits.

Second, specific derivatives, with mortality (or longevity, or disability) in a given population as the underlying, can be used to face the biometric risks. Examples are as follows:

- The *q-forward* (the letter q usually denotes a probability of dying) is a contract according to which an amount linked to the observed mortality rate in the reference population at a given future date (the maturity of the contract) will be exchanged at maturity in return for an amount linked to a benchmark mortality rate agreed at the time the contract is written.
- The *S-forward* (the letter S usually denotes the survival function, as we will see in Sect. 3.9.1, and hence a probability of being alive) is a contract according to which an amount linked to the observed survival rate in the reference population

at a given future date (the maturity) will be exchanged at maturity in return for an amount linked to a benchmark survival rate agreed at the time the contract is written.

We note that the q-forward and the S-forward realize a *single-exchange swap*.

- In general terms, a *(multi-exchange) swap* is a derivative according to which two counterparties periodically exchange cash flows. If the underlying is the mortality (or longevity) in the reference population, the swap can be thought as a sequence of q-forwards or S-forwards in which all the benchmark mortality (or longevity) rates are stated at the time the swap contract is written.

An Alternative Risk Transfers belonging to this category is called an *unfunded ART*, as no security is issued and sold.

2.6.3 Organizing a Securitization Transaction

The organizational aspects of a securitization transaction are rather complex. Figure 2.32 sketches a simple design for a life insurance deal, focussing on the main agents involved. The transaction starts in the insurance market where policies underwritten give rise to the cash flows which are securitized (at least in part). The insurer then sells the right to some cash flows to a *Special Purpose Vehicle* (*SPV*), which is a financial entity established to link the insurer to the capital market. Securities backed by the chosen cash flows are issued by the SPV, which raises money from the capital market. Such funds are (at least partially) acknowledged to the insurer.

According to the specific features of the transaction, further items may be added to the structure. For example, a fixed interest rate could be paid to investors, so that intervention by a Swap counterparty is required; see Fig. 2.33.

As it has been pointed out above, some counterparty risk is originated by the securitization transaction. This is due to a possible default of the insurer with respect to the obligations assumed against the SPV, as well as of policyholders in respect of the insurer, for example, in the form of lapses which affect the securitized cash-flow stream. To reduce such default risks, some form of credit enhancement may

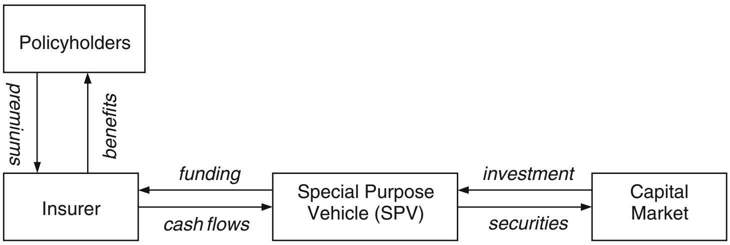


Fig. 2.32 The securitization process in life insurance: a basic structure

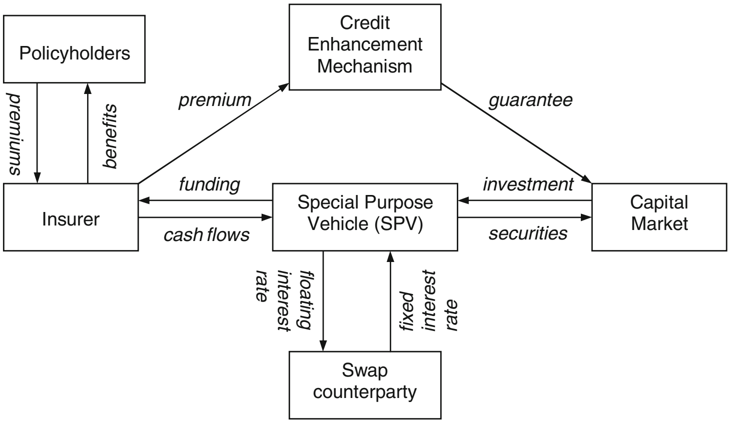


Fig. 2.33 The securitization process in life insurance: a more complex structure

be introduced, both internal (e.g., transferring to the SPV a higher value of cash flows than those required by the actual size of the securities) and external, through intervention of a specific entity (issuing, for example, credit insurance, letters of credit, and so on); see again Fig. 2.33. Further counterparty risk emerges from the other parties involved, similarly to any financial transaction. Note that intervention by a third financial institution may anyhow result in an increase of the rating of the securities.

Further details of the securitization transaction concern services for payments provided by external bodies, investment banks trading the securities on the market, and so on. Since we are interested on the main technical aspects of the securitization process, we do not go deeper into these topics (which, anyhow, do play an important role for the success of the overall transaction).

2.6.4 An Example: The Mortality Bonds

Insurance-linked securities (ILS) have been briefly addressed in Sect. 2.6.2. We recall that, when biometric risks are concerned, the payoff of an ILS is contingent on mortality or longevity in a given reference population; this is obtained, in particular, by embedding some derivatives whose underlying is a mortality/longevity index assessed on the given population. As already mentioned, these securities may serve two opposite purposes: to hedge mortality higher than expected, or survivorship higher than expected. In the former case, we refer to them as *mortality bonds*, in the latter as *longevity bonds*. We restrict the terminology to “bond,” without making explicit reference (in the name) to the derivative which is included in the security (which could be option-like, swap-like, or other) because we are more interested on hedging opportunities rather than on the organizational aspects of the deal (of course, we are anyhow aware of the importance that such aspects play from a practical point of view, but their discussion goes beyond the aim of this section).

As the purpose of mortality bonds is to hedge the risk of a mortality in excess of what expected, possibly owing to epidemics or natural disasters, typically a short position on the bond may offset liabilities of an insurer/reinsurer dealing with life insurances. We stress that mortality bonds are typically short-term bonds (3–5 years) and they are linked to a mortality index expressing the frequency of death observed in the reference population in a given period. Some thresholds are set at bond issue. If the mortality index outperforms a threshold, then either the principal or the coupon are reduced.

We now describe some possible structures for mortality bonds. In what follows, 0 is the time of issue of the bond and T its maturity. Further, S_t denotes the principal of the bond at time t, and C_t the coupon due at time t. Finally, with I_t we denote the mortality index at time t years from bond issue ($t = 0, 1, \dots, T$). Some structures are described in Examples 2.6.1 and 2.6.2.

Example 2.6.1 The bond aims at protecting against high mortality experienced throughout the whole lifetime of the bond itself. This is obtained by reducing the principal at maturity. Albeit just some ages could be considered in detecting situations of high mortality, it is reasonable to address a range of ages. Further, the index should account for mortality over the whole lifetime of the bond. So the following quantities represent possible examples of mortality index:

$$I_T = \max_{t=1,2,\dots,T} \{q(t)\} \tag{2.6.1}$$

$$I_T = \frac{\sum_{t=1}^{T} q(t)}{T} \tag{2.6.2}$$

where $q(t)$ is the observed annual mortality rate averaged over the reference population in year t.

At maturity, the principal paid back to investors is

$$S_T = S_0 \times \begin{cases} 1 & \text{if } I_T \le \lambda' I_0 \\ \Phi(I_T) & \text{if } \lambda' I_0 < I_T \le \lambda'' I_0 \\ 0 & \text{if } I_T > \lambda'' I_0 \end{cases} \tag{2.6.3}$$

where $I_0 = q(0)$, λ' and λ'' are two parameters (stated under bond conditions), with $1 \le \lambda' < \lambda''$, and $\Phi(I_T)$ is a proper decreasing function, such that $\Phi(\lambda' I_0) = 1$ and $\Phi(\lambda'' I_0) = 0$. For example

$$\Phi(I_T) = \frac{\lambda'' I_0 - I_T}{(\lambda'' - \lambda') I_0} \tag{2.6.4}$$

The coupon is independent of the experienced mortality. In particular, it can be given by

$$C_t = S_0\,(i_t + r) \tag{2.6.5}$$

where i_t is the market interest rate at time t, and r is an extra-yield rewarding investors for taking the mortality risk. ❑

While the cash flows related to the bond described in Example 2.6.1 try to match the flows in the life insurance portfolio just at the end of a period of some years, an alternative design of the mortality bond can be conceived to provide a match on a yearly basis.

Example 2.6.2 Assume that the coupon is given by

$$C_t = S_0 \times \begin{cases} i_t + r & \text{if } I_t \leq \Lambda'_t \\ (i_t + r)\,\Psi(I_t) & \text{if } \Lambda'_t < I_t \leq \Lambda''_t \\ 0 & \text{if } I_t > \Lambda''_t \end{cases} \tag{2.6.6}$$

where Λ'_t, Λ''_t set two mortality thresholds. For example,

$$\Lambda'_t = \lambda'\,\mathbb{E}[D_t] \tag{2.6.7}$$

$$\Lambda''_t = \lambda''\,\mathbb{E}[D_t] \tag{2.6.8}$$

where $1 \leq \lambda' < \lambda''$, and $\mathbb{E}[D_t]$ is the expected number of deaths in the reference population (according to a specified mortality assumption). In this structure, the mortality index I_t should express the number of deaths in year $(t-1, t)$. The function $\Psi(I_t)$ should then be decreasing; for example:

$$\Psi(I_t) = \frac{\Lambda''_t - I_t}{\Lambda''_t - \Lambda'_t} \tag{2.6.9}$$

As in (2.6.5), the rate r in (2.6.6) is the extra-yield rewarding investors for the mortality risk inherent in the payoff of the bond. Note that, in this structure, the principal at maturity can be assumed independent of the experienced mortality, for example

$$S_T = S_0 \tag{2.6.10}$$

❑

2.7 The Time Dimension

In this section we focus on how to assess and manage portfolio riskiness according to a multi-year perspective.

2.7.1 General Aspects

Insurance contracts with durations longer than one year have been addressed in Chap. 1; see, for example, Cases 4a (The need for resources at retirement), and 4b (Early death of an individual) in Sect. 1.7.4. Nonetheless, in the present chapter, for the sake of simplicity, we have mainly focussed on one-year insurance covers; see, for example, Sects. 2.3 and 2.4, in which the "basic" insurance cover, namely the Case 2 (Possible loss with fixed amount), has been referred to.

However, a one-year (or, more in general, a one-period) insight into the management of an insurance portfolio, whatever the policy term, can provide us just with a static perspective. Conversely, a number of problems of practical interest can be properly defined and solved only allowing for a sequence of periods, that is, according to a dynamic perspective. The evolution throughout time of the portfolio fund, which originates from premium income and claim payment, and the related capital allocation policies constitute important examples of a perspective involving the "time dimension."

When defining a multi-period analysis of a portfolio (or an insurance company), various approaches are available. For simplicity, we assume that all the policies in the portfolio have the same policy term r. In Figs. 2.34, 2.35 and 2.36 various policy generations are represented with the aid of a coordinate system that has the calendar time as abscissa and the duration as ordinate. The solid part of each line represents the part of the related generation accounted for according to the various approaches.

A *run-off* analysis only addresses the "in-force" portfolio, namely the policies already written. Thus, the portfolio is assumed to be "closed" to new entries, and hence no future business is accounted for. See Fig. 2.34.

Conversely, according to a *going-concern* approach the portfolio is assumed "open," and hence also future business is allowed for. See Fig. 2.35. Of course, such an approach requires an estimate of the numbers of policies written in the future years.

The *break-up* (or *wind-up*) approach, on the contrary, consists of analyzing the insurer's capability of meeting all the obligations assuming that the insurance company has to stop all business within a very short period (say, one year). Figure 2.36 refers to this approach.

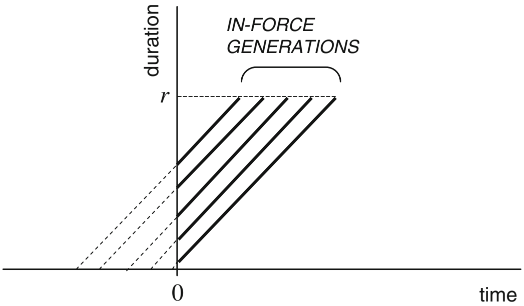


Fig. 2.34 *Run-off* of a portfolio

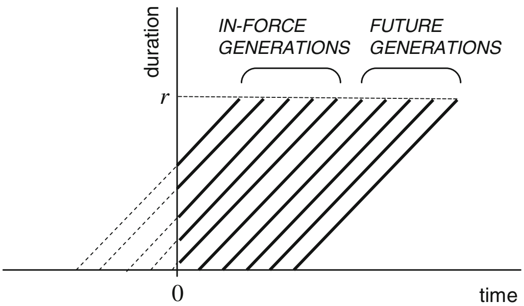


Fig. 2.35 A *going-concern* portfolio

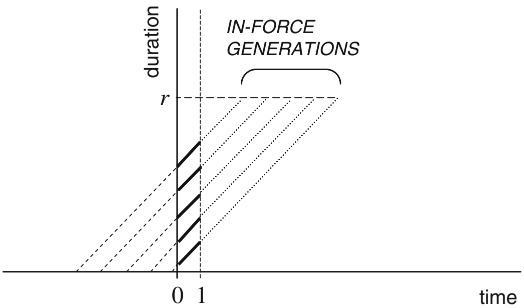


Fig. 2.36 *Break-up* of a portfolio

2.7.2 Premiums, Payments, Portfolio Fund

Consider a portfolio consisting of n one-year policies providing the “basic” insurance cover, namely the cover related to Case 2 (Possible loss with fixed amount). According to a going-concern approach, we assume a time horizon of T years.

As regards the first year, let $\Pi_0^{[P]}$ denote the premium income (including safety loading) at the beginning of the year, i.e., at time 0. Such an amount is assumed to be known. Further, let $X_1^{[P]}$ denote the total random payment, that is,

$$X_1^{[P]} = \sum_{j=1}^{n} X^{(j)} \tag{2.7.1}$$

We assume that, at the beginning of each future year, the insurer underwrites new policies, which constitute a generation of the same type of the first one (possibly, however, with a variable size).

So, we generalize the one-year portfolio model by defining, for $t = 1, 2, \dots, T$, the following quantities:

$\Pi_{t-1}^{[P]}$ = premium income at time $t-1$, i.e., at the beginning of year t
$X_t^{[P]}$ = total payment in year t

The annual portfolio result, $Z_t^{[P]}$, referred at the end of the year, can be defined as follows:

- if we disregard the time value of money, we have

$$Z_t^{[P]} = \Pi_{t-1}^{[P]} - X_t^{[P]} \tag{2.7.2}$$

- conversely, if we assume that all the claims are paid at the end of the year of occurrence, and that i is the return on premium investment, we have

$$Z_t^{[P]} = \Pi_{t-1}^{[P]}\,(1+i) - X_t^{[P]} \tag{2.7.3}$$

According to the second assumption, the *portfolio fund* (or *surplus*), $F_t^{[P]}$, $t = 1, 2, \dots$, is defined as follows:

$$F_t^{[P]} = \sum_{h=1}^{t} Z_h^{[P]}\,(1+i)^{t-h} = \sum_{h=0}^{t-1} \Pi_h^{[P]}(1+i)^{t-h} - \sum_{h=1}^{t} X_h^{[P]}\,(1+i)^{t-h} \tag{2.7.4}$$

With the (provisional) assumption

$$F_0^{[P]} = 0 \tag{2.7.5}$$

we then find:

$$F_t^{[P]} = F_{t-1}^{[P]}\,(1+i) + Z_t^{[P]};\quad t = 1, 2, \dots \tag{2.7.6}$$

namely

$$F_t^{[P]} = \left(F_{t-1}^{[P]} + \Pi_{t-1}^{[P]}\right)(1+i) - X_t^{[P]};\quad t = 1, 2, \dots \tag{2.7.7}$$

From recursion (2.7.6), it clearly appears that, as regards the annual results, the hypothesis underlying the definition of $F_t^{[P]}$ is the accumulation of profits (and possibly losses) in the portfolio fund.

If the portfolio fund takes, for some t, a negative value, a *default* (or *ruin*) situation occurs. To lower the probability of such an event, shareholders' capital should be allocated to the portfolio, in particular at time $t = 0$. If M_0 denotes the (initial) allocation, the portfolio fund process must be redefined as follows:

$$F_t^{[P]} = M_0 \, (1+i)^t + \sum_{h=1}^{t} Z_h^{[P]} \, (1+i)^{t-h} \tag{2.7.8}$$

which implies

$$F_0^{[P]} = M_0 \tag{2.7.9}$$

in recursions (2.7.6) and (2.7.7).

2.7.3 Solvency and Capital Requirements

As seen in Sect. 2.3.8, the insurer's solvency should be meant in a probabilistic sense, namely as the capability of meeting, with an assigned (high) probability, the random payments as described by a probabilistic model (which specifies the claim probability and, as regards more general insurance covers, the probability distribution of the claim size, interest rates, expenses, and so on).

The following quantities must be stated:

- the probability of meeting the random payments (say 0.99, or 0.995, …);
- the quantity representing the insurer's solvency level; for example, the portfolio fund $F_t^{[P]}$ can be addressed; if, at time t, we have $F_t^{[P]} < 0$, then the portfolio is in the default state;
- the time horizon which the concept of solvency is referred to (say 2 years, or 5 years, …).

Note that the time horizon must be chosen, as we are working in a multi-year framework.

In formal terms, the following equation expresses the solvency requirement, when the fund $F_t^{[P]}$ is addressed to check the solvency:

$$\mathbb{P}[F_1^{[P]} \geq 0 \cap F_2^{[P]} \geq 0 \cap \cdots \cap F_T^{[P]} \geq 0] = 1 - \alpha \tag{2.7.10}$$

where $1 - \alpha$ denotes the stated probability of meeting the random payments (and hence α denotes the accepted default probability).

In order to achieve the stated probability $1-\alpha$, Eq. (2.7.10) has to be solved with respect to M_0, which enters the definition of the portfolio fund $F_t^{[P]}$ via Eq. (2.7.8).

The following equation represents an alternative solvency requirement:

$$\mathbb{P}[F_T^{[P]} \geq 0] = 1 - \alpha \tag{2.7.11}$$

For a given probability α, Eq. (2.7.11) expresses a requirement weaker than that expressed by (2.7.10) (trivially, if $T > 1$). Note, however, that temporary negative values of the portfolio fund $F_t^{[P]}$ are feasible only if capital outside the portfolio is available and can be used for an immediate reinstatement of the fund. Thus, requirement (2.7.11) should not be adopted when referring to the whole insurance company.

Remark Solvency concepts described above generalize ideas presented in Sect. 2.3.8, referring to the one-period model. In particular, we note that, given the expression (2.7.8), requirements expressed by (2.7.10) and (2.7.11) can be interpreted as generalizations of the solvency requirement (2.3.59).

To achieve a required degree of solvency $1 - \alpha$, Eq. (2.7.10), or (2.7.11) must be solved with respect to capital allocation M_0. In practice, numerical methods based on Montecarlo simulation must be adopted to solve those equations. The simulation procedure consists in generating a sample of paths of $F_t^{[P]}$, for $t = 1, 2, \ldots, T$. Then, the probability $\mathbb{P}[F_1^{[P]} \geq 0 \cap F_2^{[P]} \geq 0 \cap \cdots \cap F_T^{[P]} \geq 0]$ can be estimated via the sample frequency

$$\frac{\text{number of paths with } F_t^{[P]} \geq 0 \ \text{ for } t = 0, 1, \ldots, T}{\text{number of simulations}} \tag{2.7.12}$$

whereas the probability $\mathbb{P}[F_T^{[P]} \geq 0]$ can be estimated via

$$\frac{\text{number of paths with } F_T^{[P]} \geq 0}{\text{number of simulations}} \tag{2.7.13}$$

Example 2.7.1 We refer to a portfolio initially consisting of $n = 10\,000$ one-year policies, all with sum insured $x = 1\,000$, and claim probability $p = 0.005$. Assuming a safety loading rate equal to 10 %, we have a premium income $\Pi_0^{[P]} = 55\,000$. Further, we assume a time horizon of $T = 5$ years, and suppose that at the beginning of each future year a new generation, with the same size and structure of the first one, enters the portfolio. Finally, we assume an initial capital allocation $M_0 = 10\,000$.

Figure 2.37 illustrates 50 paths of the portfolio fund. It has been assumed that times of claim occurrence and payment are uniformly distributed over each year. Time value of money has been disregarded (that is, setting $i = 0$). Moreover, the construction of the statistical distribution of the portfolio fund $F_5^{[P]}$, relying on the simulated paths, is sketched.

Finally, Figs. 2.38 and 2.39 show the statistical distribution of the fund $F_1^{[P]}$ and $F_5^{[P]}$, respectively. In particular, it is interesting to note the higher dispersion of the fund at time $t = 5$. Further, both statistical distributions reveal a positive frequency of negative values of the portfolio fund. Clearly, risk management actions should be taken (e.g., a higher capital allocation) if these frequencies seem to be too high. ❑

Example 2.7.2 To provide an example of capital allocation effects on the solvency degree, we still refer to the portfolio described in Example 2.7.1. Table 2.20 shows some probabilities related to the behavior of the portfolio fund $F_t^{[P]}$. Of course, all the

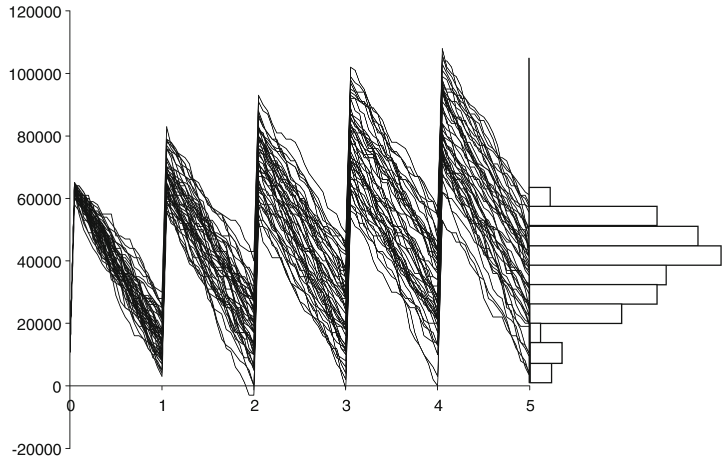


Fig. 2.37 50 paths of the portfolio fund

Fig. 2.38 Statistical distribution of $F_1^{[P]}$ (5 000 simulations)

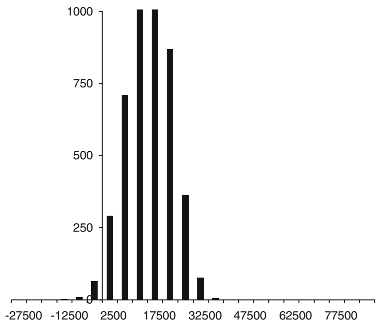


Fig. 2.39 Statistical distribution of $F_5^{[P]}$ (5 000 simulations)

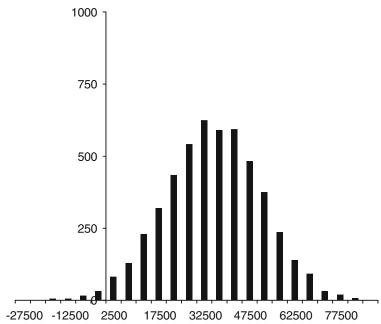


Table 2.20 Probabilities concerning the non-negativity of the portfolio fund

M_0	$\mathbb{P}[F_1^{[P]} \geq 0]$	$\mathbb{P}[F_5^{[P]} \geq 0]$	$\mathbb{P}[F_1^{[P]} \geq 0 \cap \cdots \cap F_5^{[P]} \geq 0]$
0	0.7844	0.9454	0.6954
5 000	0.9264	0.9710	0.8606
10 000	0.9848	0.9894	0.9518
14 000	0.9970	0.9928	0.9788

probabilities depend on the initial capital allocation M_0, and, in particular, increase as M_0 increases.

If we choose, according to the criterion expressed by Eq. (2.7.11), a solvency degree $1 - \alpha = 0.99$, the required capital allocation is $M_0 = 10\,000$: indeed $\mathbb{P}[F_5^{[P]} \geq 0] \approx 0.99$. Conversely, this allocation implies a lower solvency degree if the criterion expressed by (2.7.10) is adopted: in fact, we find $\mathbb{P}[F_1^{[P]} \geq 0 \cap F_2^{[P]} \geq 0 \cap \cdots \cap F_5^{[P]} \geq 0] \approx 0.95$.

Finally, we note that if M_0 is equal to 0, or anyhow is small, compensations among period results are possible, as we can realize by comparing $\mathbb{P}[F_1^{[P]} \geq 0]$ to $\mathbb{P}[F_5^{[P]} \geq 0]$. ❑

2.7.4 Generalizing the Model

The model described above can be generalized in various ways. We just outline some ideas. For example, we can assume that:

1. policies are issued throughout each year according to a time-uniform stream; this implies a time-continuous premium income; the premium income cumulated up to time t, $\Pi^{[P]}(t)$, is given by

$$\Pi^{[P]}(t) = \Pi^{[P]}\, t \tag{2.7.14}$$

   where $\Pi^{[P]}$ denotes the annual income, assumed constant over time;
2. each (one-year) policy can claim one or more times over the year;
3. each claim has a random size.

Note that, thanks to assumptions 2 and 3 a more realistic representation of claims in a portfolio is achieved. In the time-continuous setting, it is usual to define, for any real t ($t \geq 0$), the following quantities:

$$\begin{aligned} K(t) &= \text{number of claims up to time } t \\ X^{[P]}(t) &= \text{total payment cumulated up to time } t \end{aligned}$$

The quantity $K(t)$, as a function of t, is called the *claim number process*, whereas $X^{[P]}(t)$ is called the *aggregate claim process* (see Figs. 2.40 and 2.41).

If we disregard the time value of money (namely, if we assume $i = 0$), the *portfolio fund process*, $F^{[P]}(t)$, can be defined as follows:

$$F^{[P]}(t) = M_0 + \Pi^{[P]}(t) - X^{[P]}(t) \tag{2.7.15}$$

(see Fig. 2.42).

Fig. 2.40 The claim number process

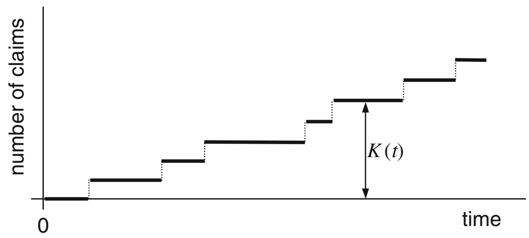


Fig. 2.41 The aggregate claim process

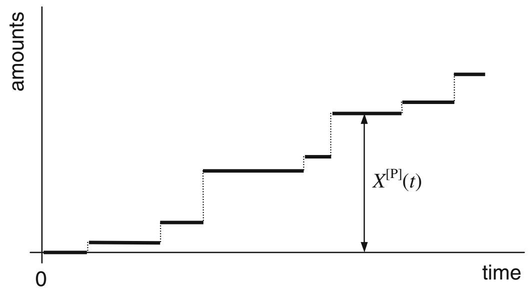
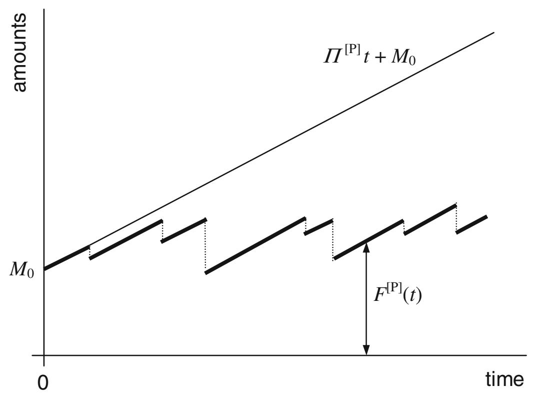


Fig. 2.42 The portfolio fund process



2.7.5 Solvency and Capital Flows

Also capital allocation strategies, aiming at solvency, can be redesigned in a more general context. We still assume that the amount M_0 represents the initial capital allocation. Then, we assume that capital flows can take place in various anniversaries, with the following purposes:

- to protect the portfolio against possible default (see Figs. 2.43 and 2.44);
- to release capital exceeding a reasonable solvency target (see Fig. 2.45).

Note that this more general setting can be properly represented in terms of a *barrier model*: the two barriers provide thresholds which suggest capital release and, respectively, capital allocation to reinstate the portfolio solvency.

Remark Simulations of real-world portfolios require a significant computation time, especially when a multi-year framework is involved. Hence, alternative approaches leading to feasible formulae, which can approximate the relevant results, can be very useful in insurance practice. In particular, the so-called *short-cut formulae* express the required capital, for example, M_0, as a function of some known quantities (e.g., the total amount of insured benefits, the total amount of premiums, etc.) and a set of parameters which should reflect the risk profile of the portfolio (or the insurance company). Formulae of this type are proposed, for instance, by the supervisory authorities.

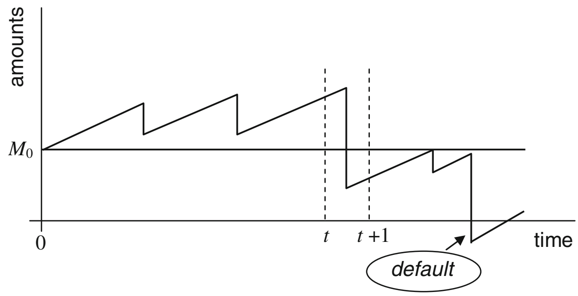


Fig. 2.43 Portfolio fund process incurring in default

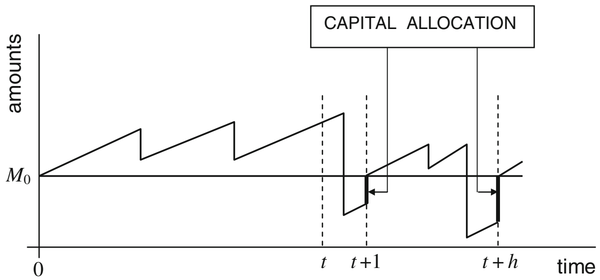


Fig. 2.44 Portfolio fund process with further capital allocations

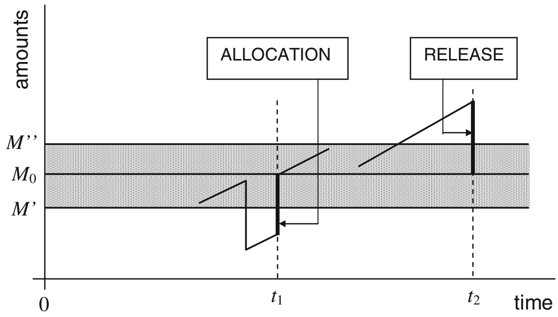


Fig. 2.45 Portfolio fund process with capital flows according to a "barrier" model

2.8 References and Suggestions for Further Reading

Also in this section, as in Sect. 1.8, we only cite textbooks dealing with general aspects of risks and insurance. Studies specifically devoted to non-life insurance, life insurance, and post-retirement solutions will be cited in the relevant sections of the following chapters.

Chapters 6, 9 and 15 in Bellis et al. (2003) focus on managing risks, the need for capital and solvency issues, respectively.

The textbook by Booth et al. (2005) deals with various technical and financial aspects of life and non-life insurance and pension funds. All the important topics of risk theory are presented in the book by Daykin et al. (1994), which provides a significant bridge between theory and insurance practice.

Quantitative tools, and in particular statistical models, used in non-life insurance are described by Hossack et al. (1983).

The object of Carter (2004) is to explain the fundamental principles and practice of non-life reinsurance. A more technical presentation of reinsurance issues is provided by Daykin et al. (1994).

The transfer of risks to capital markets via insurance-linked securities is dealt with by Barrieu and Albertini (2009). In Aspinwall et al. (2009), longevity bonds are in particular addressed.

IAA (2004) proposes a classification of insurer's risks, and the relevant applications in solvency assessment procedures. Cruz (2009) collects contributions which aim at defining an ERM framework in insurance and reinsurance. An extensive presentation of solvency issues, with specific reference to a number of supervisory systems, is given by Sandström (2006).

Finally, Haberman (1996) provides extensive information about the early history of risk theory and insurance mathematics and technique up to 1919.

Appendix

As noted in Sect. 2.3.5, various approximations to the (exact) probability distribution of the total random payment $X^{[P]}$ can be adopted. Whatever the approximating distribution may be, the goodness of the approximation must be carefully assessed, especially with regard to the right tail of the distribution itself, as this tail quantifies the probability of large payments.

The following examples can provide some ideas about the degree of approximation obtained by using the Poisson (see (2.3.22)–(2.3.24)) and the Normal approximation (see (2.3.25)–(2.3.30)) to the binomial distribution (given by (2.3.21)).

Assume the following data:

- individual loss: $x^{(j)} = 1$, for $j = 1, \dots, n$;
- probability: $p = 0.005$;
- pool sizes: $n = 100$, $n = 500$, $n = 5\,000$.

The (exact) binomial distribution and the normal approximation have been adopted for $n = 500$ and $n = 5\,000$; the (exact) binomial distribution and the Poisson approximation have been used for $n = 100$. Tables 2.21, 2.22 and 2.23 and Figs. 2.46 and 2.47 show numerical results.

Table 2.21 Right tails of Binomial distribution and Normal approximation

$n = 500$; $\mathbb{E}[X^{[P]}] = 2.5$			$n = 5\,000$; $\mathbb{E}[X^{[P]}] = 25$		
$\mathbb{P}[X^{[P]} > k]$			$\mathbb{P}[X^{[P]} > k]$		
k	Binomial	Normal	k	Binomial	Normal
5	0.04160282	0.056471062	30	0.136121887	0.158048811
6	0.013944069	0.013238288	35	0.022173757	0.022480517
7	0.004135437	0.002164124	40	0.001983179	0.001316908
8	0.001097966	0.000244022	45	0.000101743	3.03545 E–05
9	0.000263551	1.88389 E–05	50	3.13201 E–06	2.68571 E–07
10	5.76731 E–05	9.90663 E–07	55	6.02879 E–08	8.9912 E–10
…	…	…	…	…	…

Table 2.22 Binomial distribution and Poisson approximation

$n = 100$; $\mathbb{E}[X^{[P]}] = 0.5$		
$\mathbb{P}[X^{[P]} = k]$		
k	Binomial	Poisson
0	0.605770436	0.60653066
1	0.304407255	0.30326533
2	0.075719392	0.075816332
3	0.012429649	0.012636055
4	0.001514668	0.001579507
5	0.000146139	0.000157951
6	1.16275 E–05	1.31626 E–05
7	7.84624 E–07	9.40183 E–07
8	4.58355 E–08	5.87614 E–08
9	2.35447 E–09	3.26452 E–09
10	1.07667 E–10	1.63226 E–10
…	…	…

Table 2.23 Right tails of Binomial distribution and Poisson approximation

$n = 100$; $\mathbb{E}[X^{[P]}] = 0.5$		
$\mathbb{P}[X^{[P]} > k]$		
k	Binomial	Poisson
3	0.001673268	0.001752
4	0.000158599	0.000172
5	1.24604 E–05	1.42 E–05
6	8.32926 E–07	1.00 E–06
7	4.83022 E–08	6.22 E–08
…	…	…

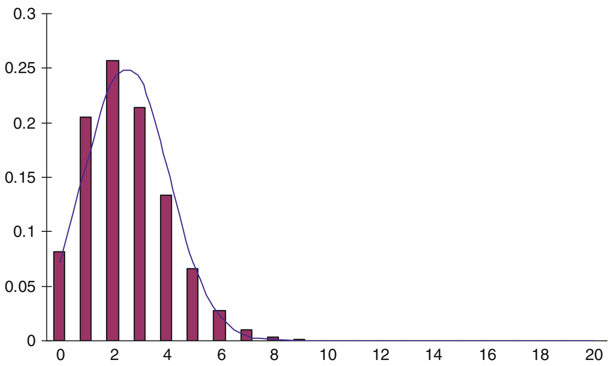


Fig. 2.46 Probability distribution of the random payment ($n = 500$). Binomial distribution and normal approximation

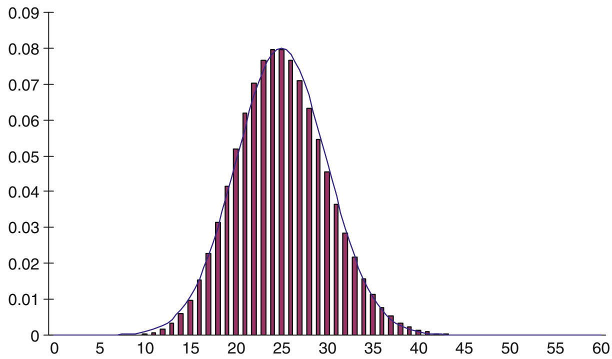


Fig. 2.47 Probability distribution of the random payment ($n = 5\,000$). Binomial distribution and normal approximation

The following aspects should be stressed. In relation to portfolio sizes $n = 500$ and $n = 5\,000$, the normal approximation tends to underestimate the right tail of the payment distribution (see Table 2.21). Conversely, the Poisson distribution provides a good approximation to the exact distribution, also for $n = 100$ (see Tables 2.22 and 2.23); unlike the normal approximation, the Poisson model tends to overestimate the right tail, so that a prudential assessment of the payment follows.

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