

Chapter 2

Introduction to the Vertical Derivatives of Horizontal Stress (VDoHS) Rates

Abstract We explain the structure of the force balance equations at the Earth's surface for flat Earth and spherical Earth cases. A key assumption is that at the surface of the Earth, where GPS observations are made, the inter-seismic behavior is purely elastic; that is, the rocks are cold and the rheology is much less complicated than at mid- and lower-crustal depths. Whereas at depth velocities are related to total stresses because ductility comes into play, at the surface inter-seismic displacements are related only to incremental stresses and surface velocities are related only to rates of change of stress. A simple consequence is that the horizontal-component manifestations at the surface of any subsurface deformation source, in the form of the vertical derivatives of horizontal stress (VDoHS) rates, can be deduced from surface observations of horizontal velocity using elasticity theory alone.

Keywords Force balance equations • Earth's surface • Flat Earth • Spherical Earth • Elastic inter-seismic behavior • Surface velocities • Stress rates • Strain rates • VDoHS rates • Green's functions

2.1 Flat Earth Form of the Force Balance Equations at the Earth's Surface

At the Earth's surface, the traction rates $\dot{\sigma}_{xz}$, $\dot{\sigma}_{yz}$, $\dot{\sigma}_{zz}$ are 0 and their vertical derivatives appear in the rate-of-change forms of the force balance equations

$$\frac{\partial \dot{\sigma}_{xx}}{\partial x} + \frac{\partial \dot{\sigma}_{xy}}{\partial y} + \frac{\partial \dot{\sigma}_{xz}}{\partial z} = 0, \quad (2.1)$$

$$\frac{\partial \dot{\sigma}_{xy}}{\partial x} + \frac{\partial \dot{\sigma}_{yy}}{\partial y} + \frac{\partial \dot{\sigma}_{yz}}{\partial z} = 0, \quad (2.2)$$

$$\frac{\partial \dot{\sigma}_{xz}}{\partial x} + \frac{\partial \dot{\sigma}_{yz}}{\partial y} + \frac{\partial \dot{\sigma}_{zz}}{\partial z} + \frac{\partial(\rho g)}{\partial t} = 0. \quad (2.3)$$

With $\dot{\sigma}_{xz}, \dot{\sigma}_{yz}$ being 0 at the surface, the last of these equations reduces to

$$\frac{\partial \dot{\sigma}_{zz}}{\partial z} + \frac{\partial(\rho g)}{\partial t} = 0. \quad (2.4)$$

Neither the vertical derivative of $\dot{\sigma}_{zz}$ nor the rate of change of the gravitational force per unit volume ρg can be obtained using GPS observations, and it is convenient to lump them together as a single rate of force per unit volume term

$$\dot{f}_z = \frac{\partial \dot{\sigma}_{zz}}{\partial z} + \frac{\partial(\rho g)}{\partial t}. \quad (2.5)$$

This term is 0 at the surface in the flat-Earth case. The corresponding terms in the horizontal-component force-balance equations are the vertical derivatives of horizontal stress (VDoHS) rates

$$\dot{f}_x = \frac{\partial \dot{\sigma}_{xz}}{\partial z}, \quad \dot{f}_y = \frac{\partial \dot{\sigma}_{yz}}{\partial z}, \quad (2.6)$$

whose surface values also cannot be obtained directly using GPS observations. The VDoHS rates, being a derivative of the stress rates themselves, are a higher spatial resolution representation of subsurface deformation sources than velocities and strain rates.

The other stress rates in the horizontal-component force-balance equations for an elastic medium, with Lamé constants λ, μ , are expressible in terms of the velocities u_x, u_y, u_z as

$$\dot{\sigma}_{xx} = (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z}, \quad (2.7)$$

$$\dot{\sigma}_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad (2.8)$$

$$\dot{\sigma}_{yy} = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_z}{\partial z}. \quad (2.9)$$

Though the surface value of the vertical derivative of the vertical velocity u_z cannot be obtained directly from GPS observations, we can use the fact that at the surface

$$\dot{\sigma}_{zz} = \lambda \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} + (\lambda + 2\mu) \frac{\partial u_z}{\partial z} = 0. \quad (2.10)$$

From this we obtain the standard result relating vertical strain to areal strain

$$\frac{\partial u_z}{\partial z} = -\frac{\lambda}{(\lambda + 2\mu)} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right). \quad (2.11)$$

The resulting expressions for $\dot{\sigma}_{xx}, \dot{\sigma}_{yy}$ at the surface are

$$\dot{\sigma}_{xx} = \frac{2\mu}{1-\nu} \frac{\partial u_x}{\partial x} + \frac{2\mu\nu}{1-\nu} \frac{\partial u_y}{\partial y}, \quad (2.12)$$

$$\dot{\sigma}_{yy} = \frac{2\mu\nu}{1-\nu} \frac{\partial u_x}{\partial x} + \frac{2\mu}{1-\nu} \frac{\partial u_y}{\partial y}, \quad (2.13)$$

where $\nu = \frac{\lambda}{2(\lambda + \mu)}$ is the Poisson's ratio.

These expressions, like that for $\dot{\sigma}_{xy}$, involve only horizontal derivatives of the horizontal velocities u_x, u_y , and at the surface the horizontal-component force-balance equations become

$$\frac{\partial}{\partial x} \left[\frac{2\mu}{1-\nu} \frac{\partial u_x}{\partial x} + \frac{2\mu\nu}{1-\nu} \frac{\partial u_y}{\partial y} \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \dot{f}_x = 0, \quad (2.14)$$

$$\frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[\frac{2\mu\nu}{1-\nu} \frac{\partial u_x}{\partial x} + \frac{2\mu}{1-\nu} \frac{\partial u_y}{\partial y} \right] + \dot{f}_y = 0. \quad (2.15)$$

What we have done is convert the horizontal-component force-balance equations at the Earth's surface into 2-dimensional second-order partial differential equations for the horizontal velocities u_x, u_y in terms of the horizontal VDoHS rates surface quantities $\dot{f}_x, \dot{f}_y$. Somewhat trivially, if the values of u_x, u_y were known at every point in a region on the surface of the Earth, then the values of $\dot{f}_x, \dot{f}_y$ could be determined by differentiating the u_x, u_y values. Also, boundary values of u_x, u_y would be known around the perimeter of the region. Of more interest is that the converse problem completely specifies the horizontal velocities. That is, if the $\dot{f}_x, \dot{f}_y$ values are given everywhere in a surface region, along with appropriate boundary conditions around the perimeter of the region, then the 2-dimensional partial differential equations can be straightforwardly solved to obtain the u_x, u_y values throughout the region. A feature of the solution is that the u_x, u_y values will be much smoother and more widely distributed spatially than the $\dot{f}_x, \dot{f}_y$ values, whereas the strain rates

$$\dot{e}_{xx} = \frac{\partial u_x}{\partial x}, \quad \dot{e}_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad \dot{e}_{yy} = \frac{\partial u_y}{\partial y} \quad (2.16)$$

have intermediate smoothness. In other words, $\dot{f}_x, \dot{f}_y$ values are fundamentally much better than u_x, u_y values at providing high spatial resolution surface images of subsurface sources, and they are also better than strain rates.

Equations 2.14 and 2.15 for the flat-Earth case are simply the equations for plane stress, which apply on any planar surface, where tractions acting on the surface are zero (e.g., Malvern 1969; Turcotte and Schubert 2002; Bower 2009). The only feature not commonly encountered in the engineering literature is the explicit inclusion of the VDoHS-rate source term, following straightforwardly from the original forms (Eqs. 2.1 and 2.2) of the force balance equations. A standard engineering application of plane stress is to thin elastic sheets, and in the Earth sciences, variants of this approach involving viscous rheologies are often used to model the lithosphere (e.g., England and McKenzie 1982, 1983; Flesch et al. 2001). Such applications make the assumption that horizontal strain rates (Eq. 2.16) are independent of depth within the thin sheet. As we are considering only behavior at the Earth's surface, we are making no such assumption; in fact, below the Earth's surface there is no restriction on the nature of the deformation, and, furthermore, the rheology is presumed to be elastic only at the surface.

2.2 1-D Expressions

In the 1-dimensional case the surface u_x, u_y and $\dot{f}_x, \dot{f}_y$ values have no dependence on the horizontal y coordinate; that is, they are functions of x alone. The horizontal-component force-balance equations at the Earth's surface become the simple second-order ordinary differential equations

$$\frac{d}{dx} \left[\frac{2\mu}{1-\nu} \frac{du_x}{dx} \right] + \dot{f}_x = 0, \quad (2.17)$$

$$\frac{d}{dx} \left[\mu \frac{du_y}{dx} \right] + \dot{f}_y = 0. \quad (2.18)$$

The first equation is the dip-slip equation involving u_x and $\dot{f}_x$, and has the same form as the second equation, which is the strike-slip equation involving u_y and $\dot{f}_y$, except that the shear modulus μ is replaced by the composite modulus $2\mu/(1-\nu)$. We will focus on the strike-slip equation, and drop the subscripts y to simplify the expressions.

Let us suppose that the region of interest is between $x = 0$ and $x = L$, and that the end values of velocity are $u(0) = 0$ and $u(L) = U$. Then in the absence of a VDoHS rates (rate of force per unit volume) term $\dot{f}$, the solution to the second-order ordinary differential equation has the linear form $u(x) = u_0(x) \equiv Ux/L$ with the modulus μ presumed to have a uniform value; the associated strain rate is a constant

value corresponding to $\frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} \equiv U/L$. The general solution for the case of uniform μ is

$$u(x) = u_0(x) + \int_0^L G(x|x_0) \dot{f}(x_0) dx_0, \quad (2.19)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + \int_0^L \frac{d}{dx} G(x|x_0) \dot{f}(x_0) dx_0. \quad (2.20)$$

Here $G(x|x_0) = G(x_0|x)$ is the Green's function with zero end values $G(0|x_0) = 0$ and $G(L|x_0) = 0$ that satisfies

$$\frac{d}{dx} \left[\mu \frac{dG(x|x_0)}{dx} \right] + \delta(x - x_0) = 0. \quad (2.21)$$

Note that in this form of the differential equation $\delta(x - x_0)$ (which is 0 everywhere except at $x = x_0$) corresponds to $\dot{f}$, while $G(x|x_0)$ corresponds to u and has linear values $G(x|x_0) = x(L - x_0)/(\mu L)$ between $x = 0$ and $x = x_0$ and $G(x|x_0) = x_0(L - x)/(\mu L)$ between $x = x_0$ and $x = L$. Furthermore, $\frac{\partial u}{\partial x} = dG(x|x_0)/dx$ has the constant values $\frac{\partial u}{\partial x} = (L - x_0)/(\mu L)$ between $x = 0$ and $x = x_0$ and $\frac{\partial u}{\partial x} = -x_0/(\mu L)$ between $x = x_0$ and $x = L$. The Green's function is a limiting case where the VDoHS rates term is concentrated at a single point. Even in such a case, where the deformation source would be at the surface, the associated velocity and strain-rate functions have non-zero values over the whole domain. In general, actual deformation sources lie below the surface and lead to spatially spread-out values of VDoHS rates.

2.3 Structure of 2-D Expressions

For the general 2-dimensional problem of obtaining surface values of the horizontal velocities u_x, u_y as functions of $\mathbf{x} = (x, y)$ from the horizontal VDoHS rates values $\dot{f}_x, \dot{f}_y$ at positions $\mathbf{x}^0 = (x_0, y_0)$ in a surface region S , there are corresponding solutions involving Green's functions: for $\alpha = x, y$

$$u_\alpha(\mathbf{x}) = u_\alpha^0(\mathbf{x}) + \int_S \sum_{\gamma=x,y} G_\alpha^\gamma(\mathbf{x}|\mathbf{x}^0) \dot{f}_\gamma(\mathbf{x}^0) dS(\mathbf{x}^0). \quad (2.22)$$

Taking $\dot{e}_{\alpha\beta}^0(\mathbf{x})$ to be the horizontal strain rates associated with the velocity solution $u_\alpha^0(\mathbf{x})$ that satisfies the boundary conditions in the absence of VDoHS rate terms, the full expression for the horizontal strain rates for $\alpha, \beta = x, y$ is

$$\dot{e}_{\alpha\beta}(\mathbf{x}) = \dot{e}_{\alpha\beta}^0(\mathbf{x}) + \int_S \sum_{\gamma=x,y} \frac{1}{2} \left(\frac{\partial G_\alpha^\gamma(\mathbf{x}|\mathbf{x}^0)}{\partial x_\beta} + \frac{\partial G_\beta^\gamma(\mathbf{x}|\mathbf{x}^0)}{\partial x_\alpha} \right) \dot{f}_\gamma(\mathbf{x}^0) dS(\mathbf{x}^0). \quad (2.23)$$

Thus, on one hand, differentiating horizontal velocities at the surface gives horizontal strain rates (Eq. 2.16) and VDoHS rates (Eqs. 2.14–2.15). On the other hand, we can use integration to go from surface VDoHS rates to horizontal strain rates (Eq. 2.23) and velocities (Eq. 2.22). As always, integration is the better-behaved process in practical applications involving numerical computations and discrete sampling.

2.4 Spherical Polar Coordinate Form of the Force Balance Equations at the Earth's Surface

In practice, we use the spherical polar coordinate version of the above equations. With ϕ being longitude, θ being latitude, and r being radius, the general rate-of-change forms of the force balance equations are

$$\frac{1}{r \cos \theta} \left(\frac{\partial \dot{\sigma}_{\phi\phi}}{\partial \phi} - 2\dot{\sigma}_{\phi\theta} \sin \theta \right) + \frac{1}{r} \frac{\partial \dot{\sigma}_{\phi\theta}}{\partial \theta} + \frac{\partial \dot{\sigma}_{\phi r}}{\partial r} + \frac{3\dot{\sigma}_{\phi r}}{r} = 0, \quad (2.24)$$

$$\frac{1}{r \cos \theta} \left(\frac{\partial \dot{\sigma}_{\phi\theta}}{\partial \phi} + (\dot{\sigma}_{\phi\phi} - \dot{\sigma}_{\theta\theta}) \sin \theta \right) + \frac{1}{r} \frac{\partial \dot{\sigma}_{\theta\theta}}{\partial \theta} + \frac{\partial \dot{\sigma}_{\theta r}}{\partial r} + \frac{3\dot{\sigma}_{\theta r}}{r} = 0, \quad (2.25)$$

$$\frac{1}{r \cos \theta} \left(\frac{\partial \dot{\sigma}_{\phi r}}{\partial \phi} - \dot{\sigma}_{\theta r} \sin \theta \right) + \frac{1}{r} \frac{\partial \dot{\sigma}_{\theta r}}{\partial \theta} + \frac{\partial \dot{\sigma}_{rr}}{\partial r} + \frac{1}{r} (2\dot{\sigma}_{rr} - \dot{\sigma}_{\phi\phi} - \dot{\sigma}_{\theta\theta}) - \frac{\partial(\rho g)}{\partial t} = 0. \quad (2.26)$$

At the Earth's surface, the traction rates $\dot{\sigma}_{\phi r}, \dot{\sigma}_{\theta r}, \dot{\sigma}_{rr}$ are 0 and the rate-of-change forms of the force balance equations become

$$\frac{1}{r \cos \theta} \left(\frac{\partial \dot{\sigma}_{\phi\phi}}{\partial \phi} - 2\dot{\sigma}_{\phi\theta} \sin \theta \right) + \frac{1}{r} \frac{\partial \dot{\sigma}_{\phi\theta}}{\partial \theta} + \dot{f}_\phi = 0, \quad (2.27)$$

$$\frac{1}{r \cos \theta} \left(\frac{\partial \dot{\sigma}_{\phi\theta}}{\partial \phi} + (\dot{\sigma}_{\phi\phi} - \dot{\sigma}_{\theta\theta}) \sin \theta \right) + \frac{1}{r} \frac{\partial \dot{\sigma}_{\theta\theta}}{\partial \theta} + \dot{f}_\theta = 0, \quad (2.28)$$

$$\dot{f}_r - \frac{1}{r} (\dot{\sigma}_{\phi\phi} + \dot{\sigma}_{\theta\theta}) = 0. \quad (2.29)$$

Here we have followed the definition of $\dot{f}_z$ in the flat-Earth formulation by introducing a combined rate of force per unit volume term

$$\dot{f}_r = \frac{\partial \dot{\sigma}_{rr}}{\partial r} - \frac{\partial(\rho g)}{\partial t}. \quad (2.30)$$

Unlike $\dot{f}_z$ in the flat-Earth case, this term is not zero at the surface. The terms corresponding to $\dot{f}_x, \dot{f}_y$ in the other, horizontal-component, force-balance equations are the apparent surface sources

$$\dot{f}_\phi = \frac{\partial \dot{\sigma}_{\phi r}}{\partial r}, \quad \dot{f}_\theta = \frac{\partial \dot{\sigma}_{\theta r}}{\partial r}. \quad (2.31)$$

Like $\dot{f}_x, \dot{f}_y$, they are vertical derivatives of horizontal shear (VDoHS) rates, and their surface values cannot be obtained directly using GPS observations.

The other spherical-Earth stress rates $\dot{\sigma}_{\phi\phi}, \dot{\sigma}_{\phi\theta}, \dot{\sigma}_{\theta\theta}$ in the horizontal-component force-balance equations for an elastic medium, with Lamé constants λ, μ , are expressible, like their flat-Earth counterparts, in terms of strain rates as

$$\dot{\sigma}_{\phi\phi} = (\lambda + 2\mu)\dot{e}_{\phi\phi} + \lambda\dot{e}_{\theta\theta} + \lambda\dot{e}_{rr}, \quad (2.32)$$

$$\dot{\sigma}_{\phi\theta} = 2\mu\dot{e}_{\phi\theta}, \quad (2.33)$$

$$\dot{\sigma}_{\theta\theta} = \lambda\dot{e}_{\phi\phi} + (\lambda + 2\mu)\dot{e}_{\theta\theta} + \lambda\dot{e}_{rr}. \quad (2.34)$$

In turn the strain rates are expressible in terms of the velocities u_ϕ, u_θ, u_r as

$$\dot{e}_{\phi\phi} = \frac{1}{r \cos \theta} \left(\frac{\partial u_\phi}{\partial \phi} - u_\theta \sin \theta + u_r \cos \theta \right), \quad (2.35)$$

$$\dot{e}_{\phi\theta} = \frac{1}{2} \left[\frac{1}{r \cos \theta} \left(\frac{\partial u_\theta}{\partial \phi} + u_\phi \sin \theta \right) + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} \right], \quad (2.36)$$

$$\dot{e}_{\theta\theta} = \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right), \quad (2.37)$$

$$\dot{e}_{rr} = \frac{\partial u_r}{\partial r}. \quad (2.38)$$

Again following the flat-Earth formulation, the surface value of the radial derivative of the radial velocity u_r cannot be obtained directly from GPS observations, but we can use the fact that at the surface

$$\dot{\sigma}_{rr} = \lambda \dot{e}_{\phi\phi} + \lambda \dot{e}_{\theta\theta} + (\lambda + 2\mu) \dot{e}_{rr} = 0. \quad (2.39)$$

From this we obtain that

$$\dot{e}_{rr} = -\frac{\lambda}{(\lambda + 2\mu)} (\dot{e}_{\phi\phi} + \dot{e}_{\theta\theta}). \quad (2.40)$$

The resulting expressions for $\dot{\sigma}_{\phi\phi}, \dot{\sigma}_{\theta\theta}$ at the surface are

$$\dot{\sigma}_{\phi\phi} = \frac{2\mu}{1-\nu} \dot{e}_{\phi\phi} + \frac{2\mu\nu}{1-\nu} \dot{e}_{\theta\theta}, \quad (2.41)$$

$$\dot{\sigma}_{\theta\theta} = \frac{2\mu\nu}{1-\nu} \dot{e}_{\phi\phi} + \frac{2\mu}{1-\nu} \dot{e}_{\theta\theta}. \quad (2.42)$$

These expressions, like that for $\dot{\sigma}_{\phi\theta}$, involve the horizontal velocities u_ϕ, u_θ and only their horizontal derivatives, but, unlike in the flat-Earth case, the expressions for $\dot{\sigma}_{\phi\phi}, \dot{\sigma}_{\theta\theta}$ also involve the radial velocity u_r .

In fact, u_r , which makes no contribution to $\dot{\sigma}_{\phi\theta}$, makes exactly the same contribution to $\dot{\sigma}_{\phi\phi}$ and $\dot{\sigma}_{\theta\theta}$: $\frac{2\mu(1+\nu)}{1-\nu} \frac{u_r}{r}$. This happens because u_r makes the same contribution u_r/r , involving surface expansion or contraction, to $\dot{e}_{\phi\phi}$ and $\dot{e}_{\theta\theta}$. At the surface the horizontal-component force-balance equations become

$$\frac{1}{r \cos \theta} \left(\frac{\partial}{\partial \phi} \left[\frac{2\mu}{1-\nu} \dot{e}_{\phi\phi} + \frac{2\mu\nu}{1-\nu} \dot{e}_{\theta\theta} \right] - 4\mu \dot{e}_{\phi\theta} \sin \theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} [2\mu \dot{e}_{\phi\theta}] + \dot{f}_\phi = 0, \quad (2.43)$$

$$\frac{1}{r \cos \theta} \left(\frac{\partial}{\partial \phi} [2\mu \dot{e}_{\phi\theta}] + 2\mu (\dot{e}_{\phi\phi} - \dot{e}_{\theta\theta}) \sin \theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{2\mu\nu}{1-\nu} \dot{e}_{\phi\phi} + \frac{2\mu}{1-\nu} \dot{e}_{\theta\theta} \right] + \dot{f}_\theta = 0. \quad (2.44)$$

The contributions from u_r in these two equations are $\frac{1}{r \cos \theta} \frac{\partial}{\partial \phi} \left[\frac{2\mu(1+\nu)}{1-\nu} \frac{u_r}{r} \right]$ and $\frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{2\mu(1+\nu)}{1-\nu} \frac{u_r}{r} \right]$ respectively. That is, the contribution that u_r makes to $\dot{\sigma}_{\phi\phi}$ and $\dot{\sigma}_{\theta\theta}$ performs the role of a force-rate potential in the horizontal-component force-balance equations at the surface.

The surface force-balance equations are straightforwardly derived in weak form, which is used when solving the equations using finite elements. For the spherical case the relevant part of the function that the variational principle is applied to is the surface integral

$$\begin{aligned}
& \iint \left[\frac{1}{2} (\dot{\sigma}_{\phi\phi} \dot{e}_{\phi\phi} + 2\dot{\sigma}_{\phi\theta} \dot{e}_{\phi\theta} + \dot{\sigma}_{\theta\theta} \dot{e}_{\theta\theta}) - (\dot{f}_{\phi} u_{\phi} + \dot{f}_{\theta} u_{\theta} + \dot{f}_r u_r) \right] r^2 \cos \theta d\phi d\theta \\
&= \iint \left[\frac{\mu v}{1-v} (\dot{e}_{\phi\phi} + \dot{e}_{\theta\theta})^2 + \mu (\dot{e}_{\phi\phi}^2 + 2\dot{e}_{\phi\theta}^2 + \dot{e}_{\theta\theta}^2) - (\dot{f}_{\phi} u_{\phi} + \dot{f}_{\theta} u_{\theta} + \dot{f}_r u_r) \right] r^2 \cos \theta d\phi d\theta \\
&= \iint \left[\frac{\mu v}{1-v} \left(\frac{1}{r \cos \theta} \left[\frac{\partial u_{\phi}}{\partial \phi} - u_{\theta} \sin \theta + u_r \cos \theta \right] + \frac{1}{r} \left[\frac{\partial u_{\theta}}{\partial \theta} + u_r \right] \right)^2 \right. \\
&\quad + \mu \left(\left\{ \frac{1}{r \cos \theta} \left[\frac{\partial u_{\phi}}{\partial \phi} - u_{\theta} \sin \theta + u_r \cos \theta \right] \right\}^2 + \frac{1}{2} \left\{ \frac{1}{r \cos \theta} \left[\frac{\partial u_{\theta}}{\partial \phi} + u_{\phi} \sin \theta \right] + \frac{1}{r} \frac{\partial u_{\phi}}{\partial \theta} \right\}^2 \right. \\
&\quad \left. \left. + \left\{ \frac{1}{r} \left[\frac{\partial u_{\theta}}{\partial \theta} + u_r \right] \right\}^2 \right) - (\dot{f}_{\phi} u_{\phi} + \dot{f}_{\theta} u_{\theta} + \dot{f}_r u_r) \right] r^2 \cos \theta d\phi d\theta.
\end{aligned} \tag{2.45}$$

The corresponding surface integral in the flat-Earth case is

$$\begin{aligned}
& \iint \left[\frac{1}{2} (\dot{\sigma}_{xx} \dot{e}_{xx} + 2\dot{\sigma}_{xy} \dot{e}_{xy} + \dot{\sigma}_{yy} \dot{e}_{yy}) - (\dot{f}_x u_x + \dot{f}_y u_y + \dot{f}_z u_z) \right] dx dy \\
&= \iint \left[\frac{\mu v}{1-v} (\dot{e}_{xx} + \dot{e}_{yy})^2 + \mu (\dot{e}_{xx}^2 + 2\dot{e}_{xy}^2 + \dot{e}_{yy}^2) - (\dot{f}_x u_x + \dot{f}_y u_y + \dot{f}_z u_z) \right] dx dy \\
&= \iint \left[\frac{\mu v}{1-v} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right)^2 + \mu \left(\left\{ \frac{\partial u_x}{\partial x} \right\}^2 + \frac{1}{2} \left\{ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right\}^2 + \left\{ \frac{\partial u_y}{\partial y} \right\}^2 \right) \right. \\
&\quad \left. - (\dot{f}_x u_x + \dot{f}_y u_y + \dot{f}_z u_z) \right] dx dy.
\end{aligned} \tag{2.46}$$

The equations involving $\dot{f}_{\phi}, \dot{f}_{\theta}, \dot{f}_r$ are obtained by applying the variational principle with respect to increments in $u_{\phi}, u_{\theta}, u_r$ respectively. Similarly, the equations involving $\dot{f}_x, \dot{f}_y, \dot{f}_z$ in the flat-Earth case are obtained by applying the variational principle with respect to increments in u_x, u_y, u_z . The primary difference between the spherical and flat-Earth cases is that the radial velocity u_r appears in the horizontal-component force-balance equations involving $\dot{f}_{\phi}, \dot{f}_{\theta}$. As we have explained, its role is rather like a force term or more strictly a force potential.

2.5 Summary and Discussion

We derive the plane stress form of the force balance equations at the Earth's surface for the flat Earth case. The VDoHS rates are surface terms and in these equations they perform the role played by source terms in plane stress theory. The most important aspect of VDoHS rates is that they provide substantially higher spatial resolution surface images of subsurface deformation sources than the more typically used velocities or strain rates. Furthermore, we present the structure of the 1-D

forms of these force balance equations, followed by corresponding expressions for the general 2-D case. It is also important to note that velocities and strain rates can be expressed in terms of the VDoHS rates using Green's functions, enabling integration of VDoHS rates to obtain more robust strain rates that are not subject to the same problems arising from differentiation of velocities. We also derive the spherical polar form of the formulation, which demonstrates that the primary difference between a flat Earth and a spherical Earth formulation is that in the spherical case the radial component of velocity appears as an additional source like term similar to VDoHS rates.

As discussed previously, we assume that at the surface of the Earth, where GPS observations are made, the inter-seismic behavior is purely elastic. At the surface inter-seismic displacements and velocities are related to only incremental stresses and the rates of change of stress respectively, except in instances of surface breaking frictional plasticity (e.g., Rudnicki and Rice 1975; McClay and Ellis 1987; Huismans et al. 2005; Dempsey et al. 2012). In practice, as we discuss in Chap. 4, finite sampling limits the ability to distinguish between non-elastic signals and the elastic response to subsurface sources that are appreciably shallower than the spacing of GPS observations.

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