2 L^p–L^q-decay estimates for the linear wave equation

For the proof of Theorem 1.1 simple decay properties of solutions to the linear wave equation play an important role (see Chapter 7). The decay rates of L^q-norms are typically of polynomial order in $\mathbb{R}^n$ depending on the space dimension n and on q.
We consider the solution of the linear initial value problem

$$y_{tt} - \Delta y = 0, \tag{2.1}$$

$$y(t=0) = 0, \quad y_t(t=0) = g, \tag{2.2}$$

where $y = y(t,x)$ is a real-valued function, $t \geq 0, x \in \mathbb{R}^n$ and g is assumed to be smooth for the moment.
Let the operator $w(t)$ be defined through

$$(w(t)g)(x) \;:=\; y(t,x).$$

Remark: The assumption $y(t=0) = 0$ is made without loss of generality because the function y_1 defined by

$$y_1(t,x) := \partial_t (w(t)g)(x)$$

solves the initial value problem

$$\partial_t^2 y_1 - \Delta y_1 = 0$$

$$y_1(t=0) = g, \quad \partial_t y_1(t=0) = \partial_t^2 (w(t)g)(t=0) = \Delta w(t=0)g = 0.$$

(Cf. the representation of solutions in Chapter 7 and the considerations in Section 11.5.)

Theorem 2.1 $\exists c = c(n) > 0 \quad \forall g \in C_0^\infty \quad \forall t \geq 0$*:*

$$\begin{array}{llcl} (i) & \|Dw(t)g\|_2 & = & \|g\|_2, \\ (ii) & \|Dw(t)g\|_\infty & \leq & c(1+t)^{-\frac{n-1}{2}} \|g\|_{n,1}. \end{array}$$

Proof: Let $g \in C_0^\infty$. Then $y = w(\cdot)g \in C^\infty([0,\infty) \times \mathbb{R}^n)$ and $D^\alpha w \in C^0([0,\infty), L^2)$ for $\alpha \in \mathbb{N}_0^n$. (Cf. Chapter 3 or the book of R. Leis [98].) c will denote various positive constants at most depending on n.
Multiplying both sides of (2.1) with $y_t(t,\cdot)$ in L^2 (inner product denoted by $\langle \cdot, \cdot \rangle$) and dropping the parameter t, we obtain

$$\begin{aligned} 0 &= \langle y_{tt}, y_t \rangle + \langle \nabla y, \nabla y_t \rangle \\ &= \frac{1}{2}\frac{d}{dt}(\|y_t\|_2^2 + \|\nabla y\|_2^2) \\ &= \frac{1}{2}\frac{d}{dt}\|Dw(t)g\|_2^2. \end{aligned}$$

This proves (i).

(ii) will be proved here for $n = 1$ and $n = 3$ to give some main ideas. For odd space dimensions $n \geq 3$ or even space dimensions see Section 11.5 and the paper of W. von Wahl [187], respectively.

$n = 1$: The solution y is given by d'Alembert's formula:

$$y(t,x) := \frac{1}{2}\int_{x-t}^{x+t} g(r)dr$$

(*Jean Baptiste Le Rond d'Alembert,* 16.11.1717 – 29.10.1783).

We have

$$\begin{aligned} y_t(t,x) &= \frac{1}{2}(g(x+t) + g(x-t)), \\ y_x(t,x) &= \frac{1}{2}(g(x+t) - g(x-t)) \end{aligned}$$

whence it is obvious that y solves the initial value problem (2.1), (2.2). Moreover

$$\forall t \geq 0: \quad \|Dw(t)g\|_\infty \leq \|g\|_\infty \leq c\|g\|_{1,1}$$

by Sobolev's imbedding theorem. This proves (ii) for the case $n = 1$.

Now let $n = 3$: Kirchhoff's formula says that y defined by

$$y(t,x) := \frac{t}{4\pi}\int_{S^2} g(x+tz)dz, \tag{2.3}$$

is the solution, where $S^2 = \partial B(0,1)$ denotes the unit sphere in $\mathbb{R}^3$ *(Gustav Robert Kirchhoff,* 12.3.1824 – 17.10.1887). This is easily checked. From (2.3) we obtain

$$\begin{aligned} y(t=0) &= 0, \\ 4\pi y_t(t,x) &= \int_{S^2} g(x+tz)dz + t\int_{S^2} (\nabla g)(x+tz)z\,dz, \\ y_t(t=0) &= g. \end{aligned}$$

Moreover

$$4\pi\nabla y(t,x) = t\int_{S^2} (\nabla g)(x+tz)dz,$$

hence

$$\begin{aligned} 4\pi y_{tt}(t,x) &= 2\int_{S^2} (\nabla g)(x+tz)zdz + t\int_{S^2} \nabla\{(\nabla g)(x+tz)z\}zdz \\ &= 3t\int_{B(0,1)} (\Delta g)(x+tz)dz + t^2\int_{B(0,1)} (\nabla\Delta g)(x+tz)zdz, \end{aligned}$$

$$\begin{aligned}4\pi\Delta y(t,x) &= t\int_{S^2}(\Delta g)(x+tz)dz \;=\; t\int_{S^2}\{(\Delta g)(x+tz)z\}zdz\\ &= t^2\int_{B(0,1)}(\nabla\Delta g)(x+tz)zdz \;+\; 3t\int_{B(0,1)}(\Delta g)(x+tz)dz.\end{aligned}$$

This implies

$$y_{tt}-\Delta y \;=\; 0.$$

Now we shall prove (ii).

First let $t\geq 1$:

1.

$$\begin{aligned}-\int_{S^2}g(x+tz)dz &= \int_{S^2}\int_t^\infty \frac{d}{ds}g(x+sz)dsdz = \int_{S^2}\int_t^\infty(\nabla g)(x+sz)zdsdz\\ &= \int_{S^2}\int_t^\infty \frac{s^2}{s^3}(\nabla g)(x+sz)szdsdz\\ &= \int_{|z|>t}|z|^{-3}(\nabla g)(x+z)zdz.\end{aligned}$$

This implies

$$|\int_{S^2}g(x+tz)dz|\leq t^{-2}\int_{|z|>t}|(\nabla g)(x+z)|\,dz\leq t^{-2}\|g\|_{1,1}.$$

2. Analogously one obtains

$$|t\int_{S^2}(\nabla g)(x+tz)zdz|\leq t^{-1}\|g\|_{2,1}$$

and

$$|t\int_{S^2}\nabla g(x+tz)dz|\leq t^{-1}\|g\|_{2,1}.$$

Hence we get for $t\geq 1$:

$$\|Dw(t)g\|_\infty\leq(4\pi\, t)^{-1}\|g\|_{2,1}. \tag{2.4}$$

3. Now let $0\leq t<1$:

$$\begin{aligned}-\int_{S^2}g(x+tz)dz &= \int_{S^2}\int_t^\infty\frac{d}{ds}g(x+sz)dsdz\\ &= -\int_{S^2}\int_t^\infty(s-t)\,\frac{d^2}{ds^2}\,g(x+sz)dsdz\\ &= \int_{S^2}\int_t^\infty\frac{(s-t)^2}{2}\,\frac{d^3}{ds^3}\,g(x+sz)dsdz\\ &= \int_{|z|>t}\frac{(|z|-t)^2}{2|z|^5}\,\sum_{i,j,k=1}^3 z_iz_jz_k(\partial_i\partial_j\partial_k g)(x+z)dz.\end{aligned}$$

This implies

$$|\int_{S^2} g(x+tz)dz| \le \sum_{i,j,k=1} \int_{|z|>t} |\partial_i\partial_j\partial_k g(x+z)|dz \le \|g\|_{3,1}.$$

Analogously for the terms discussed in 2. Thus we have obtained for $0 \le t < 1$:

$$\|Dw(t)g\|_\infty \le c\|g\|_{3,1}. \tag{2.5}$$

(2.4) and (2.5) prove (ii).

Q.E.D.

Remarks: For $g \in W^{n,1}$ there is still a distributional solution y to the initial value problem (2.1), (2.2). Since $W^{n,1}$ is continuously imbedded into L^2 we have

$$y \in C^0([0,\infty), W^{1,2}) \cap C^1([0,\infty), L^2)$$

(see e.g. [98]).
Moreover one can define a trace on $\partial\Omega$ for $g \in W^{1,p}(\Omega), 1 \le p < \infty$, where Ω is a smoothly bounded domain in $\mathbb{R}^n$ (Lipschitz boundary is sufficient); namely, there is a continuous map B,

$$B : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

with

$$Bg = g_{/\partial\Omega} \quad \text{if} \quad g \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$$

(see e.g. the book of H.W. Alt [6]), (*Rudolf Otto Sigismund Lipschitz,* 14.5.1832 – 7.10.1903). Therefore Kirchhoff's formula (2.3) makes sense for $g \in W^{3,1} \hookrightarrow W^{1,2}$ ($\hookrightarrow$ denotes the continuous imbedding).
Thus we obtain the corresponding results for $g \in W^{n,1}$ by approximation with $(g_k)_k \subset C_0^\infty$, expressed in the following theorem.

Theorem 2.2 $\exists c = c(n) > 0 \quad \forall g \in W^{n,1} \quad \forall t \ge 0$*:*

$$\begin{array}{lrcl} (i) & \|Dw(t)g\|_2 & = & \|g\|_2, \\ (ii) & \|Dw(t)g\|_\infty & \le & c(1+t)^{-\frac{n-1}{2}}\|g\|_{n,1}. \end{array}$$

In other words, the operator T_t, defined by

$$T_t g := Dw(t)g$$

maps as follows:

$$T_t : W^{n,1} \longrightarrow L^\infty \quad \text{with norm} \quad M_0 \le c(1+t)^{-\frac{n-1}{2}},$$

$$T_t \,:\, L^2 \longrightarrow L^2 \quad \text{with norm} \quad M_1 = 1.$$

By interpolation we obtain

$$T_t \,:\, [W^{n,1}, L^2]_\theta \longrightarrow [L^\infty, L^2]_\theta, \quad 0 \le \theta \le 1,$$

$$\text{with norm} \quad M_\theta \le c M_0^{1-\theta} M_1^\theta, \quad c = c(\theta, n).$$

The interpolation spaces $[\cdot,\cdot]_\theta$ are described in Appendix A. We have

$$1 \le q_0, q_1 \le \infty \quad \Rightarrow \quad [L^{q_0}, L^{q_1}]_\theta = L^{q_\theta}, \tag{2.6}$$

where q_θ is defined by the relation

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular we get

$$[L^\infty, L^2]_\theta = L^{q_\theta} \quad \text{with} \quad q_\theta = \frac{2}{\theta}.$$

The proof of (2.6) is not very difficult after having given an appropriate meaning to $[\cdot,\cdot]_\theta$. This is possible for example in general Banach spaces. The proof uses the Three-Line-Theorem of J. Hadamard (*Jacques Hadamard,* 8.12.1865 – 17.10.1963). The interpolation of $W^{n,1}$ and L^2 is much more difficult. For this purpose Besov spaces and Bessel potential spaces are used (*Friedrich Wilhelm Bessel,* 22.7.1784 – 17.3.1846). We refer the reader to Appendix A for a survey and to the books of Bergh & Löfström [11] and H. Triebel [181] for details. One special result suitable for our purposes is:

$$W^{N,p_\theta} \hookrightarrow [W^{n,1}, L^2]_\theta \quad \text{if} \quad N > (1-\theta)n,$$

where

$$\frac{1}{p_\theta} + \frac{1}{q_\theta} = 1$$

defines p_θ (see Theorem A.10 in Appendix A).

Remark: For $\theta \in \{0,1\}$ we may allow $N = (1-\theta)n$.

Thus we obtain the following theorem on the L^p–L^q-decay of solutions to the linear wave equation.

Theorem 2.3 *Let* $2 \le q \le \infty$, $1/p + 1/q = 1$, $N_p > n(1-2/q)$. *Then*

$$\exists c = c(q,n) > 0 \quad \forall g \in W^{N_p,p} \quad \forall t \ge 0 : \quad \|Dw(t)g\|_q \le c(1+t)^{-\frac{n-1}{2}(1-\frac{2}{q})}\|g\|_{N_p,p}.$$

Remarks: $N_p = n(1-2/q)$ is possible if $q \in \{2,\infty\}$.

Since

$$N_p \cdot p > n(1-2/q)p = n(2-p)$$

we have

$$W^{N_p,p} \hookrightarrow L^2$$

and hence

$$Dw(\cdot)g \in C^0([0,\infty), L^2).$$

If $(g_m)_m \subset C_0^\infty$ converges to g in $W^{N_p,p}$, then $(Dw(t)g_m)_m$ converges in L^2 to $Dw(t)g$. Several sharper results for solutions to linear wave equations are contained in Section 11.5 and in the paper of W. v. Wahl [187] respectively. Another method of proving L^p–L^q-decay estimates (at least for $q < \infty$) is to use the Fourier representation of the solution (*Jean-Baptiste-Joseph Fourier,* 21.3.1768 – 16.5.1830). This has been carried out by H. Pecher in [138] and the result is essentially expressed in Lemma 11.16 in Section 11.7.

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