

# Preface

‘Where shall I begin, please your Majesty?’ he asked. ‘Begin at the beginning,’ the King said, gravely, and go on till you come to an end; then stop.’

—Lewis Carroll, *Alice in Wonderland*

Traditional approaches to continuum modeling usually focus either on the mechanics of materials or on the dynamics of fluids and gases. This book is designed to treat multi-physics aspects in a unified way in terms of local and global balance equations, switching frequently the attention of the reader from molecular diffusion to hot fluids, from the elastic behavior of a material to chemical reactions or heat conduction in porous media.

The book can be used in the classroom as auxiliary material for a semester course in advanced topics in continuum modeling for first-year graduate students in applied mathematics or in computational science and engineering. We take as starting point the fact that, quite often, the mathematics student, after having completed his undergraduate education in mathematical sciences, feels the need to describe the surrounding Nature in terms of well-understood mathematical objects (equations, algebraic relations, logical implications, ...) that he/she can master based on known axioms and theorems. The acquired power of mathematical abstraction and generalization needs now to be challenged in a setup where mathematics can potentially be relevant to science (outside mathematics itself), society, or to technology.

Responding to the scientific needs of such student, we are interested in linking the problem of balance laws and transport fluxes to modern challenges, particularly in materials science and population dynamics in chemistry and biology. In this sense, we are very much driven by the spirit of Hilbert’s 6th problem.

*Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.... Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones....; see [Hil02].*

Instead of aiming to an axiomatic construction of mathematical modeling, we rather wish to discover new models, which potentially lead to new mathematics, especially in the analysis and numerical approximation of partial differential equations.

The material presented here was taught in the framework of a Continuum Mechanics course at the Eindhoven University of Technology during 2012–2015 as well as in the Mastermath program at the University of Utrecht during 2014–2015. We used it both as self-study and in the classroom. Each chapter contains many exercises. On purpose, we do not provide complete solutions to the exercises. We strongly believe that the student needs to take the challenge to face them without being biased by the way of thinking of the teacher. We encourage the students to brainstorm together on possible solution strategies. One must be aware of the fact that, strange as it may seem, a modeling problem usually admits multiple good solutions. The quality of the solutions depends very much not only on the assumptions one is ready to rely on, but also on the length and space scales the problem solver is taking into account. Instead of listing complete solutions, we provide hints whenever necessary. On the other hand, the book contains a wealth of practical examples and thought experiments that are discussed in detail, which finally are turned into mathematical models with increasing complexity.

Prerequisites include basic knowledge of calculus, ordinary, and partial differential equations and physics.

We aim at training the student to think in terms of balance laws for extensive quantities. Such ability often turns out to be useful in the everyday life of an applied mathematician not only to help his or her efforts of translating the real world in mathematical terms, but also in studying the rigorous mathematical analysis and numerical approximation of the constructed models.

The main objective is to develop modeling skills in at least a threefold direction:

1. derivation of global (integral) laws together with the associated local (differential) equations;
2. design of constitutive laws;
3. modeling boundary processes.

The strength of the presentation lies in the use of many practical, often research-driven, examples. A few of them are worked out in detail, while others are left as exercises. The spectrum of applications is broad ranging from coupled flow, diffusion, and reaction in porous media or cooling of a hot cup of coffee, to traffic issues and energy harvesting from geothermal wells.

To introduce the idea of measuring physical quantities, we need concepts like units, dimensions, characteristic scales. Chapter 1 briefly introduces the reader to these concepts by means of a few examples referring to traffic of cars on motorways, droplets rolling on surfaces, and to the voice of the dragon. Key working tools are dimension renormalization, non-dimensionalization, and scaling.

In Chap. 2, we present the basic equations of continuum physics both in global and local formulations. The local balance laws together with the global Clausius-Duhem inequality define the class of admissible thermodynamic

processes. Boundary conditions and discontinuities appear as prominent issues. We give a few examples of constitutive equations for the stress tensor and for the transport fluxes. A set of worked-out practical modeling scenarios is emphasized.

Chapter 3 introduces the concept of flux of matter in the context of conservation laws. The attention moves to presenting two averaging techniques for guessing the structure of two conceptually distinct transport fluxes. First, the concern falls on the derivation of the Darcy's law for two particular cases of microstructures: (i) a periodic array of cells and (ii) an array of randomly distributed cells. Then a possible derivation for the structure of the thermo-diffusion flux is given. The considerations extend immediately to the cross-diffusion case.

Each chapter ends with "Notes and comments". This is the place where we add remarks, open problems (often at the research level) as well as additional references.

As target audience we have in mind graduate students in pure and applied mathematics as well as applied mathematicians. The modeling way of thinking promoted in this framework is particularly useful to those working on multi-physics multi-scale problems, where tailored modeling is needed to address specific questions whose answers cannot always be found in textbooks. Although these notes are primarily meant for mathematics students, more theoretically oriented research and development engineers could also be interested in following such problem-based approach, especially if they need to design boundary conditions and/or have to handle situations with multiple scale issues.

## Note to the Instructors

The materials presented here are thought to complement lecture notes in continuum modeling or in continuum mechanics. The content of the three chapters can be covered in a mini-course of approximately nine lectures (cca. two lectures and one instruction per chapter), accompanied with homework exercises and project-based assignments. Section 2.7 contains a collection of modeling exercises inspired by practical examples. They can be used in the classroom for working in groups during instructions.

To get a better understanding of the topics and also to acquire more knowledge, the reader must also consult standard textbooks like [Sed59, TM05, Gur81, vdV09, Cha76, EGK08]. For more information on porous media, a good book is [Bea88], while for mixture theory, we refer to [M68]. A *visually nice* perspective on (applied) partial differential equations connected to modeling is given in [Mar07]. Instructive exercises on vector calculus and kinematics can be found in [Spe80], e.g. More focused references are given inside the chapters.

Continuum Modeling

An Approach Through Practical Examples

Muntean, A.

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