Block-Decoupling Multivariate Polynomials Using the Tensor Block-Term Decomposition

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Abstract. We present a tensor-based method to decompose a given set of multivariate functions into linear combinations of a set of multivariate functions of linear forms of the input variables. The method proceeds by forming a three-way array (tensor) by stacking Jacobian matrix evaluations of the function behind each other. It is shown that a block-term decomposition of this tensor provides the necessary information to block-decouple the given function into a set of functions with small input-output dimensionality. The method is validated on a numerical example.

Keywords: Multivariate polynomials · Multilinear algebra · Tensor decomposition · Block-term decomposition · Waring decomposition

1 Introduction

1.1 Problem Statement

The problem we study in the current paper is how to decompose a given multivariate vector-valued function $\mathbf{f}(\mathbf{u})$ into a (parametric) representation of the form

$$\mathbf{f}(\mathbf{u}) = \begin{bmatrix} \mathbf{W}_1 \cdots \mathbf{W}_R \end{bmatrix} \begin{bmatrix} \mathbf{g}_1(\mathbf{V}_1^T \mathbf{u}) \\ \vdots \\ \mathbf{g}_R(\mathbf{V}_R^T \mathbf{u}) \end{bmatrix}, \tag{1}$$

where $\mathbf{g}_i(\mathbf{x}_i) : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ map from m_i inputs to n_i outputs, $\mathbf{W}_i \in \mathbb{R}^{N \times n_i}$ and $\mathbf{V}_i \in \mathbb{R}^{M \times m_i}$, with $i = 1, \ldots, R$. Figure 1 is a schematical representation of the proposed structure. The case in which all $\mathbf{g}_i(\mathbf{x}_i)$ are univariate functions is related to the Waring decomposition [1,9] and is discussed in [5]. The current paper considers the case of *block-decoupling* with the internal functions $\mathbf{g}_i(\mathbf{x}_i)$ being multivariate vector-valued functions. It is assumed that the decomposition (1) exists (in the exact sense).

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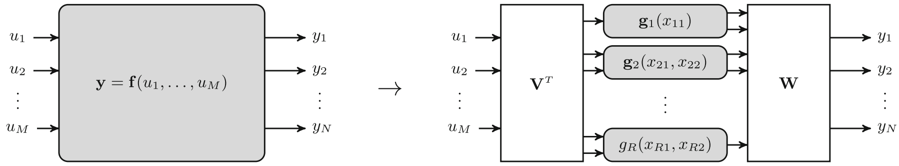


Fig. 1. The block-decoupling problem statement. From the polynomial mapping $\mathbf{y} = \mathbf{f}(\mathbf{u})$ we wish to find $\mathbf{V} = [\mathbf{V}_i]$ and $\mathbf{W} = [\mathbf{W}_i]$ and the mappings $\mathbf{g}_i(\mathbf{x}_i)$ such that $\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{R} \mathbf{W}_i \mathbf{g}_i(\mathbf{V}_i^T \mathbf{u})$.

1.2 When and Why is a Block-Decoupling Favorable?

A block-decoupling (1) is a natural representation of a nonlinear mapping when inherent coupling among some internal variables exists, for instance due to underlying physics. Rather than unraveling the function into univariate branches, solely to be able to decouple the variables, it may be desirable to keep sets of variables together (see Example 1). Moreover, the introduction of (possibly many) internal branches may *increase* the parametric complexity of the function representation, which is undesirable. Therefore, block-decoupling (1) may also contribute to reducing parametric complexity.

Let us look at a simple case where we derive 'manually' from a coupled function its fully decoupled representation. We will see that full decoupling requires the introduction of several branches $g_i(x_i)$. This example serves as a justification to prefer a block-decoupling over full decoupling.

Example 1. To fully decouple the function $f(u_1, u_2) = u_1^2 u_2$, one needs to introduce three univariate branches. Indeed, it is easy to see that we have

$$u_1^2 u_2 = \frac{1}{6}\left((u_1 + u_2)^3 - (u_1 - u_2)^3\right) - \frac{1}{3}u_2^3,$$

from which we conclude that $f(u_1, u_2) = u_1^2 u_2$ can be fully decoupled as the sum of three univariate functions $g_1(x_1) = 1/6x_1^3$, with $x_1 = u_1 + u_2$, $g_2(x_2) = -1/6x_2^3$ with $x_2 = u_1 - u_2$ and $g_3(x_3) = -1/3x_3^3$ with $x_3 = u_2$. In more complicated cases, full decoupling may require the introduction of more univariate functions $g_i(x_i)$ than block-decoupled vector-valued functions $\mathbf{g}_i(\mathbf{x}_i)$. $\diamondsuit$

2 Method

2.1 Block-Diagonalization of Jacobian Matrices

We assume that $\mathbf{f}(\mathbf{u})$ can be written as in (1). Although we will describe the method for the case that $\mathbf{f}(\mathbf{u})$ is polynomial, the method can easily be generalized to the non-polynomial case, and is applicable as long as the derivatives of $\mathbf{f}(\mathbf{u})$ can be obtained.

The task at hand is to decompose $\mathbf{f}(\mathbf{u})$ into blocks of multivariate functions as in (1). The method generalizes the result of [5] and proceeds by collecting first-order information of $\mathbf{f}(\mathbf{u})$ in a set of sampling points $\mathbf{u}^{(k)}$. The first-order information is obtained from the Jacobian of $\mathbf{f}(\mathbf{u})$, denoted by $\mathbf{J}_\mathbf{f}(\mathbf{u})$ and defined as

$$\mathbf{J}_\mathbf{f}(\mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\mathbf{u}) & \cdots & \frac{\partial f_1}{\partial u_M}(\mathbf{u}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial u_1}(\mathbf{u}) & \cdots & \frac{\partial f_N}{\partial u_M}(\mathbf{u}) \end{bmatrix}. \tag{2}$$

Applying the chain rule of differentiation to $\mathbf{f}(\mathbf{u}) = \sum_{i=1}^R \mathbf{W}_i \mathbf{g}_i(\mathbf{V}_i^T \mathbf{u})$ leads to

$$\mathbf{J}_\mathbf{f}(\mathbf{u}) = \begin{bmatrix} \mathbf{W}_1 \ldots \mathbf{W}_R \end{bmatrix} \begin{bmatrix} \mathbf{J}_{\mathbf{g}_1}(\mathbf{V}_1^T\mathbf{u}) & & \\ & \ddots & \\ & & \mathbf{J}_{\mathbf{g}_R}(\mathbf{V}_R^T\mathbf{u}) \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \vdots \\ \mathbf{V}_R^T \end{bmatrix}, \tag{3}$$

where the $\mathbf{J}_{\mathbf{g}_i}(\mathbf{x}_i)$ are defined similar to (2).

2.2 Computing $\mathbf{W}_i$, $\mathbf{V}_i$ and $\boldsymbol{\mathcal{H}}_i$

From (3) it follows that finding from the Jacobian evaluations $\mathbf{J}_\mathbf{f}(\mathbf{u}^{(k)})$ the matrices $\mathbf{W}_i$, $\mathbf{V}_i$ and the functions $\mathbf{g}_i(\mathbf{x}_i)$, amounts to solving a simultaneous matrix block-diagonalization problem. By evaluating the Jacobian of $\mathbf{f}(\mathbf{u})$ in a set of K sampling points $\mathbf{u}^{(k)}$ we obtain a collection of Jacobian matrices $\mathbf{J}_\mathbf{f}(\mathbf{u}^{(k)})$, $k = 1, \ldots, K$, which are stacked behind each other into the $N \times M \times K$ tensor $\boldsymbol{\mathcal{J}} = \left\{\mathbf{J}_\mathbf{f}(\mathbf{u}^{(1)}), \ldots, \mathbf{J}_\mathbf{f}(\mathbf{u}^{(K)})\right\}$. The recent years have seen an increased research interest in tensor decompositions [2,8], which can be seen as higher-order extensions of well-known matrix decompositions such as the singular value decomposition [6]. The tensor decomposition that will be of interest for the current task is the block-term decomposition (BTD) in rank$(n_i, m_i, \cdot)$-terms [3,4,10,12], as it can be used to compute the simultaneous block-diagonalization of the Jacobian tensor $\boldsymbol{\mathcal{J}}$. The BTD of $\boldsymbol{\mathcal{J}}$ in rank$(n_i, m_i, \cdot)$-terms is the decomposition of $\boldsymbol{\mathcal{J}}$ into

$$\boldsymbol{\mathcal{J}} = \sum_{i=1}^R \boldsymbol{\mathcal{H}}_i \bullet_1 \mathbf{W}_i \bullet_2 \mathbf{V}_i, \tag{4}$$

where $\bullet_i$ denotes the mode-i tensor product, and $\mathbf{W}_i$ and $\mathbf{V}_i$ are defined as above. The $n_i \times m_i \times K$ core tensors $\boldsymbol{\mathcal{H}}_i$ contain in the slices the Jacobians $\mathbf{J}_{\mathbf{g}_i}(\mathbf{x}^{(k)})$, with $\mathbf{x}_i^{(k)} = \mathbf{V}_i^T\mathbf{u}^{(k)}$. Figure 2 gives a graphical overview of the method.

2.3 Uniqueness

A lack of *global uniqueness* of the BTD can be expected because one can introduce nonsingular transformations $\mathbf{S}_i$ and $\mathbf{T}_i$ in the R terms of (4) to obtain the (equivalent) decomposition $\boldsymbol{\mathcal{J}} = \sum_{i=1}^R \left(\boldsymbol{\mathcal{H}}_i \bullet_1 \mathbf{T}_i^{-1} \bullet_2 \mathbf{S}_i^{-1}\right) \bullet_1 (\mathbf{W}_i\mathbf{T}_i) \bullet_2 (\mathbf{V}_i\mathbf{S}_i)$.

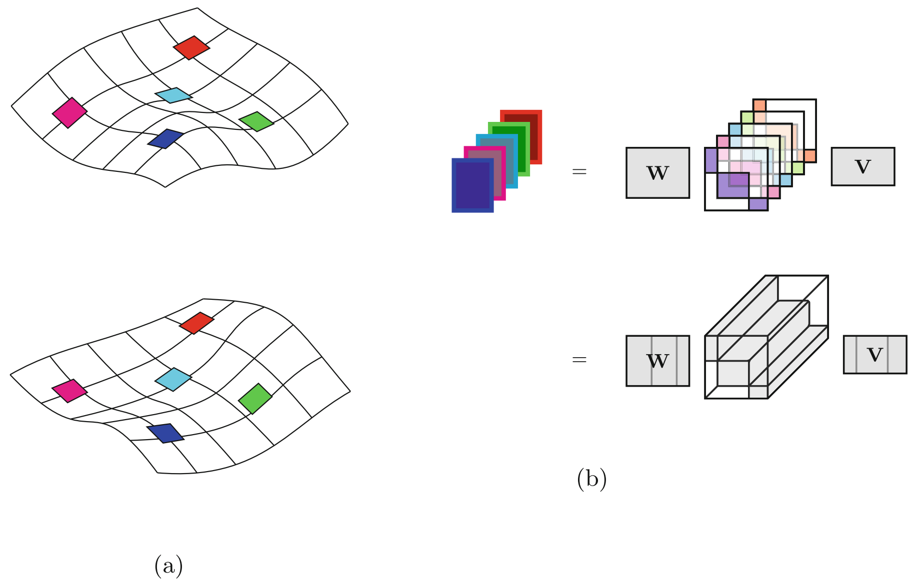


Fig. 2. Visual representation of the decomposition method. From the first-order information of $\mathbf{f}(\mathbf{u})$ a tensor consisting of Jacobian matrices is constructed. The block-term decomposition of this tensor results in the factors $\mathbf{V}_i$, $\mathbf{W}_i$ and the core tensors $\boldsymbol{\mathcal{H}}_i$ from which the decoupling of $\mathbf{f}(\mathbf{u})$ can be found.

The uniqueness properties of BTD are discussed in [3,4], however, the case $\text{rank}(n_i, m_i, \cdot)$ is not included. It is expected that uniqueness conditions along the lines of [4] can be obtained for the $\text{rank}(n_i, m_i, \cdot)$ case, but this is beyond the scope of the current paper. During numerical experiments (using Tensorlab [11]) we have not encountered uniqueness issues—it seems safe to claim that cases with relatively small R are not problematic. In terms of decomposition (1), the effects of rotational ambiguities due to $\mathbf{S}_i$ and $\mathbf{T}_i$ are easy to understand as well. Let us consider the $R = 1$ case $\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{g}(\mathbf{V}^T\mathbf{u})$, in which we insert $\mathbf{S}^T$ and $\mathbf{T}$ and their inverses as $\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{T}\mathbf{T}^{-1}\mathbf{g}(\mathbf{S}^{-T}\mathbf{S}^T\mathbf{V}^T\mathbf{u}) = \widetilde{\mathbf{W}}\widetilde{\mathbf{g}}(\widetilde{\mathbf{V}}^T\mathbf{u})$, where $\widetilde{\mathbf{W}} = \mathbf{W}\mathbf{T}$, $\widetilde{\mathbf{V}}^T = \mathbf{S}^T\mathbf{V}^T$ and $\widetilde{\mathbf{g}}(\mathbf{x}) = \mathbf{T}^{-1}\mathbf{g}(\mathbf{S}^{-T}\mathbf{x})$. Both representations are equivalent, and the factors $\mathbf{V}$ and $\mathbf{W}$ can only be obtained up to linear transformations. The internal function $\mathbf{g}(\mathbf{x})$ has undergone both a change of input variables due to $\mathbf{S}^{-T}$ as well as a linear transformation at the output due to $\mathbf{T}^{-1}$, but the identified $\widetilde{\mathbf{g}}$ is still polynomial of the same degree as the true $\mathbf{g}$.

2.4 Recovering the Coefficients of $\mathbf{g}_i(\mathbf{x}_i)$

A parameterization of the internal functions $\mathbf{g}_i(\mathbf{x})$ can be obtained using interpolation. Since the internal functions $\mathbf{g}_i(\mathbf{x})$ are polynomial, the coefficients of $\mathbf{g}_i(\mathbf{x})$ can be obtained from solving a system of linear equations. We will illustrate the main idea by means of a simple example, from which a general method can easily be derived.

Example 2. Consider a function $\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{g}(\mathbf{V}^T\mathbf{u})$ with $R = 2$, $m_1 = 2$, $m_2 = 1$, $n_1 = n_2 = 1$ that maps from M inputs to N outputs. Furthermore assume that $g_{11}(x_{11}, x_{12})$, $g_{12}(x_{11}, x_{12})$ and $g_2(x_2)$ are polynomial of (total) degree two. Then $\mathbf{f}(\mathbf{u})$ can then be parameterized as

$$\mathbf{f}(\mathbf{u}) = \mathbf{W} \underbrace{\left[\begin{array}{c|c|c} 1\ x_{11}\ x_{12}\ x_{11}^2\ x_{11}x_{12}\ x_{12}^2 & & \\ & 1\ x_{11}\ x_{12}\ x_{11}^2\ x_{11}x_{12}\ x_{12}^2 & \\ & & 1\ x_2\ x_2^2\ x_2^3 \end{array}\right]}_{\mathbf{G}(\mathbf{V}^T\mathbf{u})} \left[\begin{array}{c} \mathbf{c}_{11} \\ \hline \mathbf{c}_{12} \\ \hline \mathbf{c}_2 \end{array}\right],$$

illustrating how the coefficients $\mathbf{c}_{11}$, $\mathbf{c}_{12}$ and $\mathbf{c}_2$ appear linearly in the expression. For each of the operating points $\mathbf{u}^{(k)}$ the above expression can be obtained. We stack them on top of each other into an overdetermined (assuming $K \gg 1$) system of linear equations in the coefficients $\mathbf{c}_{11}$, $\mathbf{c}_{12}$ and $\mathbf{c}_2$ as

$$\begin{bmatrix} \mathbf{f}(\mathbf{u}^{(1)}) \\ \vdots \\ \mathbf{f}(\mathbf{u}^{(K)}) \end{bmatrix} = \begin{bmatrix} \mathbf{W} & & \\ & \ddots & \\ & & \mathbf{W} \end{bmatrix} \begin{bmatrix} \mathbf{G}(\mathbf{V}^T\mathbf{u}^{(1)}) \\ \vdots \\ \mathbf{G}(\mathbf{V}^T\mathbf{u}^{(K)}) \end{bmatrix} \begin{bmatrix} \mathbf{c}_{11} \\ \mathbf{c}_{12} \\ \mathbf{c}_2 \end{bmatrix}. \qquad \diamondsuit$$

2.5 Algorithm Summary

The complete block-decoupling procedure is summarized as follows.

1. Evaluate the Jacobian matrix $\mathbf{J_f}(\mathbf{u})$ in a set of K sampling points $\mathbf{u}^{(k)}$, $k = 1, \ldots, K$ (Sects. 2.1 and 2.2).
2. Stack the Jacobian matrices into an $N \times M \times K$ tensor $\boldsymbol{\mathcal{J}}$ (Sect. 2.2).
3. Compute the rank$(n_i, m_i, \cdot)$ block-term decomposition of $\boldsymbol{\mathcal{J}}$, resulting in the factors $\mathbf{W}_i$, $\mathbf{V}_i$ and the core tensors $\boldsymbol{\mathcal{H}}_i$ (Sect. 2.2).
4. Recover the coefficients of the internal functions $\mathbf{g}_i(\mathbf{x}_i)$ by solving a linear system (Sect. 2.4).

3 Numerical Example

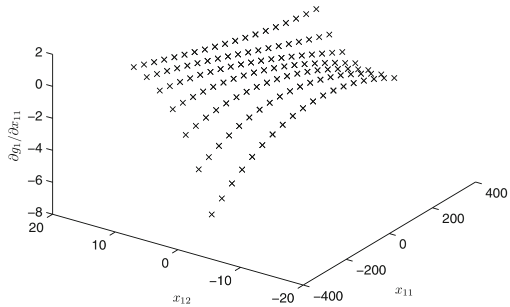
We will now illustrate the method by means of a numerical example.

Example 3. We assume that a multivariate vector-valued function $\overline{\mathbf{f}}(\mathbf{s})$ is given that has an underlying representation of the form (1)

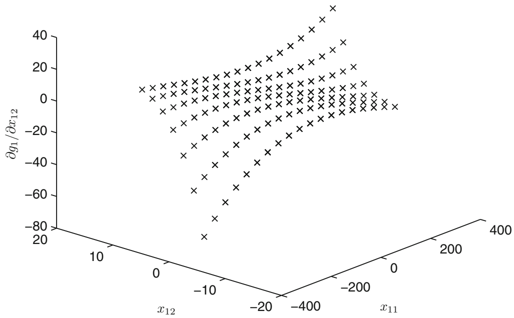
$$\overline{\mathbf{f}}(\mathbf{u}) = \overline{\mathbf{W}}\overline{\mathbf{g}}(\overline{\mathbf{V}}^T\mathbf{u}) = \overline{\mathbf{w}}_1\overline{g}_1(\overline{\mathbf{V}}_1^T\mathbf{u}) + \overline{\mathbf{w}}_2\overline{g}_2(\overline{\mathbf{v}}_2^T\mathbf{u}), \tag{5}$$

with $\overline{\mathbf{V}} = \left[\overline{\mathbf{V}}_1 \middle| \overline{\mathbf{v}}_2\right]$ and $\overline{\mathbf{W}} = \left[\overline{\mathbf{w}}_1 \middle| \overline{\mathbf{w}}_2\right]$ as

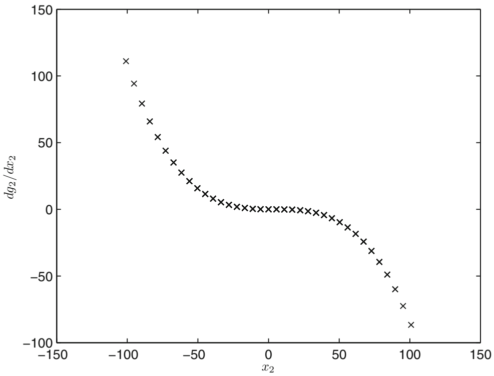
$$\overline{\mathbf{V}} = \left[\begin{array}{rr|r} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 3 & -1 & 0 \\ -1 & 1 & 3 \end{array}\right], \quad \overline{\mathbf{W}} = \left[\begin{array}{r|r} 0 & 1 \\ 1 & 3 \\ -1 & 2 \\ 3 & 0 \end{array}\right], \tag{6}$$



(a) $\partial g_1(x_{11}, x_{12})/\partial x_{11}$



(b) $\partial g_1(x_{11}, x_{12})/\partial x_{12}$



(c) $dg_2(x_2)/dx_2$

Fig. 3. Jacobians of $g_1(x_{11}, x_{12})$ and $g_2(x_2)$, obtained from the core tensors $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$, which were computed using the BTD (with $\mathbf{x}_i = \mathbf{V}_i^T \mathbf{u}$).

and

$$\begin{aligned}\bar{g}_1(x_{11}, x_{12}) &= x_{11}^3 x_{12} - 2x_{11}^3 - x_{11}^2 x_{12} + 4x_{12}^2,\\ \bar{g}_2(x_2) &= x_2^4 - 2x_2^3 + 3x_2^2,\end{aligned} \tag{7}$$

in which the 'true' representation is denoted by barred symbols.

The sampling points $\mathbf{u}^{(k)}$ are generated by combining for each of the four inputs $u_1, \ldots, u_4$ seven equidistant points in the interval $[-2, 2]$, such that $K = 7^4$. We sample the Jacobian $\mathbf{J_f}(\mathbf{u})$ in the $K = 2401$ sampling points and stack the Jacobian matrices $\mathbf{J_f}(\mathbf{u}^{(k)})$, $k = 1, \ldots, K$ into the tensor $\boldsymbol{\mathcal{J}}$.

Tensorlab [11] is used to compute the BTD with core tensor dimensions $1 \times 2 \times K$ and $1 \times 1 \times K$, from which we obtain the factors $\mathbf{V} = \left[\mathbf{V}_1 \middle| \mathbf{v}_2\right]$ and $\mathbf{W} = \left[\mathbf{w}_1 \middle| \mathbf{w}_2\right]$ as

$$\mathbf{V} = \left[\begin{array}{rr|r} 7.5051 & -5.2297 & -3.1489 \\ -8.2850 & 14.7523 & 6.2978 \\ 15.7901 & -19.9820 & 0.0000 \\ -0.7799 & 9.5226 & -9.4467 \end{array}\right], \qquad \mathbf{W} = \left[\begin{array}{r|r} 0.0000 & 1.4249 \\ -9.8728 & 4.2748 \\ 9.8728 & 2.8499 \\ -29.6183 & 0.0000 \end{array}\right]. \tag{8}$$

Notice that the factors $\mathbf{V}$ and $\mathbf{W}$ do not exactly correspond to the underlying factors $\overline{\mathbf{V}}$ and $\overline{\mathbf{W}}$, but they are equal up to a similarity transformation. For the vectors $\mathbf{v}_2$, $\mathbf{w}_1$ and $\mathbf{w}_2$ this means that they are equal to the underlying ones up to scaling. The core tensors $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ contain in their frontal slices the Jacobians of $g_1(x_{11}, x_{12})$ and $g_2(x_2)$, for each of the K operating points, *i.e.*, $\mathbf{x}_i^{(k)} = \mathbf{V}_i^T \mathbf{u}^{(k)}$. Figure 3 is a graphical representation obtained by plotting the entries in the fibers of $\boldsymbol{\mathcal{H}}_i$ versus $\mathbf{x}_i^{(k)} = \mathbf{V}_i^T \mathbf{u}^{(k)}$.

We then compute the coefficients of the recovered $g_1(x_{11}, x_{12})$ and $g_2(x_2)$ from the solution of a Vandermonde-like linear system as in Sect. 2.4 (resulting in a norm-wise error on the residual of 2.1207×10^{-7}). From the recovered $\mathbf{V}_1$, $\mathbf{v}_2$, $\mathbf{w}_1$ and $\mathbf{w}_2$, and the internal functions $g_1(x_{11}, x_{12})$ and $g_2(x_2)$ we reconstruct the function $\mathbf{f}(\mathbf{u}) = \mathbf{w}_1 g_1(\mathbf{V}_1^T \mathbf{u}) + \mathbf{w}_2 \mathbf{g}_2(\mathbf{v}_2^T \mathbf{u})$ with a relative norm-wise error on the coefficients of 2.7562×10^{-10} comparing to $\overline{\mathbf{f}}(\mathbf{u})$. $\diamondsuit$

4 Conclusions and Perspectives

We have presented a method to decouple a given set of multivariate polynomials into linear combinations of multivariate polynomials with smaller dimensionality, acting on linear forms of the input variables. By considering the first-order information of the given function in a set of sampling points, we have shown that the problem reduces to the simultaneous block-diagonalization of a set of Jacobian matrices. The block-term tensor decomposition is used to compute the decomposition. The method is illustrated on a numerical example.

Ongoing work is concerned with applying the block-decoupling method to nonlinear block-oriented system identification, where we investigate how to unravel from a black-box nonlinear state-space model the nature of the static nonlinearities [7]. Other open questions include how the decoupling method can be used to simplify or approximate a given multivariate vector-valued function, and how uncertainty on the function $\mathbf{f}(\mathbf{u})$ can be taken into account.

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