Chapter 2
Boundary and Initial Data

Abstract This chapter introduces the notions of boundary and initial value problems. Some operator notation is developed in order to represent boundary and initial value problems in a compact manner. Familiarity with this notation is essential for understanding the presentation in later chapters. An initial classification of partial differential equations is then developed.

Our starting point here is a simple ordinary differential equation (ODE): find $u(t)$ such that $u' = 2t$. Integrating gives the solution $u = t^2 + C$ where C is the “constant” of integration. To compute C and thus get a unique solution we need to know $u(t)$ at some specific time $t = T$; for example, if $u(0) = 1$ then $C = 1$ and $u = t^2 + 1$.

To build on this, suppose that $u(x, t)$ satisfies the simple PDE

$$u_t = 2t. \tag{2.1}$$

Writing this in the form $\partial_t(u - t^2) = 0$, it is seen that $u(x, t) - t^2$ does not vary with t (but it may vary with x), so the PDE is readily integrated to give the solution

$$u(x, t) = t^2 + A(x), \tag{2.2}$$

where $A(x)$ is an arbitrary “function” of integration. We may regard the PDE (2.1) as an (uncountably) infinite set of ODEs, one for each value of x. The arbitrary function of integration is then seen to be a consequence of requiring a different constant of integration at each value of x. For a unique solution, we must specify an additional initial condition; for example,

$$u(x, 0) = g(x), \tag{2.3}$$

where $g(x)$ is a given function. Putting $t = 0$ in (2.2) and using (2.3) gives a solution that is uniquely defined for all time:

$$u(x, t) = t^2 + g(x). \tag{2.4}$$

D.F. Griffiths et al., *Essential Partial Differential Equations*, Springer
Undergraduate Mathematics Series, DOI 10.1007/978-3-319-22569-2_2

This combination of PDE (2.1) and initial condition (2.3) is referred to as an *initial value problem* (IVP).

Applying this logic to the solution of the test equation (**pde.10**), we deduce that the function $u(x, t)$ in Example 1.1 will uniquely solve the PDE if we augment (**pde.10**) by an additional initial condition, say,

$$u(x, 0) = g(x) \tag{2.5}$$

together with an additional "boundary condition", say,

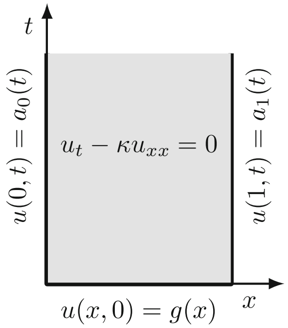
$$u(0, t) = a(t). \tag{2.6}$$

Insisting on continuity of u at the origin requires $u(0, 0) = g(0) = a(0)$ and we then obtain the unique solution

$$u(x, t) = x^2t^2 + g(x) + a(t) - g(0).$$

The combination of PDE (**pde.10**) with initial condition (2.5) and boundary condition (2.6) is referred to as an *initial–boundary value problem* (IBVP) or simply as a *boundary value problem* (BVP). Further insight into these issues may be found in the exercises at the end of the chapter.

Let us move on to consider the heat equation (**pde.4**). For a unique solution, we will need an initial condition and two boundary conditions. For example, it will be shown in Chap. 7 that the following BVP has a uniquely defined solution $u(x, t)$:



$$\left.\begin{aligned} u_t - \kappa u_{xx} = 0 &\quad \text{in } (0,1)\times(0,T] \\ u(x,0) = g(x) &\quad \text{for all } x \in [0,1] \\ u(0,t) = a_0(t);\ u(1,t) = a_1(t) &\quad t > 0, \end{aligned}\right\} \tag{2.7}$$

where $g(x)$, $a_0(t)$ and $a_1(t)$ are given functions. Two boundary conditions are needed because of the second derivative with respect to x. Those used in (2.7), where the value of u is specified, are known as Dirichlet boundary conditions. It is not necessary for the values at corners to be uniquely defined. For instance, it is not necessary for $\lim_{x\to 0} g(x)$ to equal $\lim_{t\to 0} a_0(t)$. This might model the situation where one end of an initially "hot" bar of material is plunged into an ice bath. A solution of the heat equation with this property is given in Exercise 2.3.

Alternative, equally viable, BVPs would be obtained by replacing one or both boundary conditions in (2.7) by conditions on the x derivative, for example,

$$u(0,t) = a_0(t); \quad u_x(1,t) = a_1(t) \quad \text{for all } t > 0, \tag{2.8}$$

where the boundary condition at $x = 1$ is known as a Neumann condition, or

$$u(0,t) = a_0(t); \quad \alpha u(1,t) + \beta u_x(1,t) = a_1(t) \quad \text{for all } t > 0, \tag{2.9}$$

where the boundary condition at $x = 1$ is known as a Robin condition. Note that choosing $\alpha = 1$ and $\beta = 0$ reduces this to a Dirichlet condition and the combination $\alpha = 0$ and $\beta = 1$ leads to a Neumann boundary condition.

Note however that the following combination of boundary condition does not make sense:

$$u(0,t) = a_0(t); \quad u_{xx}(1,t) = a_1(t) \quad \text{for all } t > 0,$$

since the value of $u_{xx}(1,t)$ provided by the boundary condition would generally conflict with the value obtained from the PDE as $x \to 1$. This leads to the general rule of thumb that the order of derivatives appearing in boundary conditions must be lower than the highest order derivative terms appearing in the PDE.

Turning to the wave equation (**pde.5**), we need *two* initial conditions because of the second derivative with respect to t, for example,

$$\left.\begin{array}{ll} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,1) \times (0,T] \\ u(x,0) = f_1(x);\ u_t(x,0) = f_2(x) & \text{for all } x \in [0,1] \\ u(0,t) = g_0(t);\ u(1,t) = g_1(t) & \text{for all } t > 0. \end{array}\right\} \tag{2.10}$$

Replacing the Dirichlet boundary conditions by either Neumann or Robin boundary conditions would also lead to legitimate BVPs. In this case, for $u(x,t)$ to be a continuous function, the initial and boundary conditions need to be equal where they meet so that $f_1(0) = g_0(0)$ and $f_1(1) = g_1(0)$.

In our final example in this section we consider Laplace's equation (**pde.3**) on a domain Ω in two dimensions which has a boundary that we will denote by $\partial\Omega$ (Fig. 2.1). In order to obtain a unique solution it is necessary to specify a condition at every point on this boundary.

Since we have a second-order PDE the possible types of boundary condition are Dirichlet, Neumann and Robin and these take the form:

- For a Dirichlet boundary condition, the value of u is specified

$$u(\vec{x}) = g_D(\vec{x}) \quad \text{for all } \vec{x} \in \partial\Omega. \tag{2.11a}$$

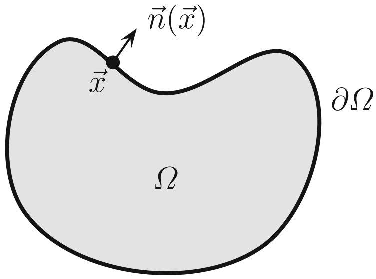


Fig. 2.1 A domain Ω in $\mathbb{R}^2$ with boundary $\partial\Omega$. Also shown is the outward normal vector $\vec{n}(\vec{x})$ at a point $\vec{x} \in \partial\Omega$

- For a Neumann boundary condition, the value of the (outward) normal derivative of u is specified, that is

$$\frac{\partial u}{\partial n}(\vec{x}) = g_N(\vec{x}) \quad \text{for all } \vec{x} \in \partial\Omega. \tag{2.11b}$$

  The outward normal derivative of $u(x, t)$ is the rate of change of u with distance moved in the normal direction with any other independent variables, such as tangential displacement and time, being held fixed. For a unit normal vector $\vec{n}$ it is the component of the gradient ∇u in the normal direction, which can be written as $\partial_n u = u_n = \vec{n} \cdot \nabla u$.
- For a Robin boundary condition, a linear combination of the value of u and its (outward) normal derivative is specified, that is

$$\alpha u(\vec{x}) + \beta\frac{\partial u}{\partial n}(\vec{x}) = g_R(\vec{x}) \quad \text{for all } \vec{x} \in \partial\Omega, \tag{2.11c}$$

  where α and β are usually constant, but in some situations could depend on $\vec{x}$ or even u.
- Finally, we could also mix the three types by partitioning $\partial\Omega$ into nonoverlapping pieces so that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N \cup \partial\Omega_R$ and then specify a boundary condition of Dirichlet, Neumann and Robin type on $\partial\Omega_D$, $\partial\Omega_N$ and $\partial\Omega_R$, respectively.

Note that whenever Laplace's equation is solved in $\Omega \subset \mathbb{R}^2$ the boundary $\partial\Omega$ is one-dimensional. When solving the equation in $\mathbb{R}^3$ the boundary is two-dimensional, so that there are two tangential derivatives and one normal derivative at every point on the boundary.

2.1 Operator Notation

Before embarking on a more detailed study of PDEs we introduce some notation that will enable us to write PDEs, boundary conditions and BVPs in a compact fashion much as linear algebraic equations are commonly expressed using matrices and vectors.

A calligraphic font ($\mathcal{L}$, $\mathcal{M}$, $\mathcal{B}$, ...) will be used for symbols that denote differential operators in the *space variables* only. For example, defining

$$\mathcal{L}u(x,t) = -\kappa u_{xx}(x,t), \quad (x,t) \in (0,1) \times (0,T)$$

would allow us to write the heat equation (**pde.4**) as

$$u_t + \mathcal{L}u = 0.$$

Similarly, defining the boundary condition operator

$$\mathcal{B}u(x,t) = \begin{cases} u(0,t) & \text{for } t > 0, x = 0 \\ u_x(1,t) & \text{for } t > 0, x = 1, \end{cases} \tag{2.12}$$

allows the conditions (2.8) to be expressed as

$$\mathcal{B}u = f(x,t), \quad t > 0, \ x = 0, 1,$$

where $f(0,t) = a_0(t)$ and $f(1,t) = a_1(t)$. Then, by defining

$$\mathscr{L}u(x,t) = \begin{cases} u_t(x,t) + \mathcal{L}u(x,t) & \text{for } (x,t) \in (0,1) \times (0,T) \\ \mathcal{B}u(x,t) & \text{for } (x,t) \in \{0,1\} \times (0,T) \\ u(x,0) & \text{for } t = 0, x \in [0,1] \end{cases} \tag{2.13}$$

and

$$\mathscr{F}(x,t) = \begin{cases} 0 & \text{for } (x,t) \in (0,1) \times (0,T) \\ f(x,t) & \text{for } (x,t) \in \{0,1\} \times (0,T) \\ g(x) & \text{for } t = 0, x \in [0,1] \end{cases} \tag{2.14}$$

the BVP (2.7) could be written in compact form, so that

$$\mathscr{L}u = \mathscr{F}. \tag{2.15}$$

In the sequel, symbols in script font ($\mathscr{L}$, $\mathscr{M}$, $\mathscr{F}$, ...) will be reserved for statements of BVPs.

Example 2.1
Consider the BVP defined by Laplace's equation in the unit square $0 \le x, y \le 1$ with

(a) Dirichlet boundary conditions on the vertical edges: $u(0,y) = \cos \pi y$ and $u(1,y) = y - 1$ for $0 < y < 1$.

(b) A Neumann condition[1] on the lower edge: $-u_y(x,0) = 1$ for $0 < x < 1$.
(c) A Robin condition on the upper edge: $u_y(x,1) + u(x,1) = 0$ for $y = 1$, $0 < x < 1$.

Define suitable forms for $\mathscr{L}$ and $\mathscr{F}$ so that it can be expressed as $\mathscr{L}u = \mathscr{F}$.

Here we define the BVP terms directly without first defining spatial differential operators $\mathcal{L}$ and $\mathcal{B}$:

$$\mathscr{L}u(x,y) = \begin{cases} u_{xx}(x,y) + u_{yy}(x,y) & \text{for } 0 < x, y < 1 \\ u(0,y) & \text{for } x = 0, 0 < y < 1 \\ u(1,y) & \text{for } x = 1, 0 < y < 1 \\ -u_y(x,0) & \text{for } y = 0, 0 < x < 1 \\ u_y(x,1) + u(x,1) & \text{for } y = 1, 0 < x < 1 \end{cases}$$

and

$$\mathscr{F}(x,y) = \begin{cases} 0 & \text{for } 0 < x, y < 1 \\ \cos \pi y & \text{for } x = 0, 0 < y < 1 \\ y - 1 & \text{for } x = 1, 0 < y < 1 \\ 1 & \text{for } y = 0, 0 < x < 1 \\ 0 & \text{for } y = 1, 0 < x < 1. \end{cases}$$

It can be observed that boundary values have not been specified at the corners of the domain as these do not affect the solution in the interior. An example of a problem with a discontinuity is given in Exercise 1.10. ◊

2.2 Classification of Boundary Value Problems

We will categorise BVPs (that is, PDEs and associated initial or boundary conditions) into those that are linear and those that are nonlinear in the next two sections. A formal definition of a well-posed boundary value problem is the subject of the final section.

2.2.1 Linear Problems

Linear BVPs have a number of useful properties, some of which will be investigated in this section. Our first goal is to identify which problems are linear.

[1] The outward normal direction on $y = 0$ is in the direction of $\vec{n} = (0, -1)$.

Definition 2.2 (*Linearity*) An operator $\mathcal{L}$ is *linear* if for any two functions u and v and any $\alpha \in \mathbb{R}$ the following two properties are satisfied:

(a) $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$;
(b) $\mathcal{L}(\alpha u) = \alpha \mathcal{L}(u)$.

An operator that does not satisfy these conditions is said to be *nonlinear*. We will explore various kinds of nonlinearity in the next section.

Example 2.3 Show that the BVP defined by (2.13)–(2.15) is linear.

The spatial operator $\mathcal{L}u(x,t) = -\kappa u_{xx}(x,t)$ is linear since

$$\begin{aligned} \mathcal{L}(u+v) &= -\kappa(u+v)_{xx} \\ &= -\kappa u_{xx} - \kappa v_{xx} = \mathcal{L}u + \mathcal{L}v \\ \mathcal{L}(\alpha u) &= -\kappa(\alpha u)_{xx} \\ &= -\alpha\kappa u_{xx} = \alpha \mathcal{L}u. \end{aligned}$$

Similarly, the boundary condition operator $\mathcal{B}$ satisfies

$$\begin{aligned} \mathcal{B}(u+v) &= \begin{cases} u+v \\ (u+v)_x \end{cases} = \begin{cases} u+v & \text{for } t>0, x=0 \\ u_x+v_x & \text{for } t>0, x=1, \end{cases} \\ &= \mathcal{B}u + \mathcal{B}v \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(\alpha u) &= \begin{cases} \alpha u \\ (\alpha u)_x \end{cases} = \begin{cases} \alpha u & \text{for } t>0, x=0 \\ \alpha u_x & \text{for } t>0, x=1, \end{cases} \\ &= \alpha \mathcal{B}u \end{aligned}$$

so it is also linear. Note that the same approach shows that conventional Dirichlet/Neumann/Robin boundary conditions are always linear. This means that linearity of a BVP normally depends only on the linearity of the PDE component of the problem.[2] ♢

Also, from the definition of $\mathscr{L}$ in (2.13), we see that

$$\mathscr{L}(u+v) = \begin{cases} (u+v)_t + \mathcal{L}(u+v) \\ \mathcal{B}(u+v) \\ u+v \end{cases} = \begin{cases} (u_t + \mathcal{L}u) + (v_t + \mathcal{L}v) \\ \mathcal{B}u + \mathcal{B}v \\ u+v \end{cases}$$

[2] Nonlinear boundary conditions such as $u_n = \mathrm{e}^u$ on $\partial\Omega$ are certainly possible, but will not be considered here.

so that $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$. (The proof that $\mathscr{L}(\alpha u) = \alpha \mathscr{L}u$ is left as an exercise.) We conclude that the given BVP for the heat equation (**pde.4**) is linear.

Other examples of linear BVPs are associated with the classical second-order PDEs: the wave equation (**pde.5**), Laplace's equation (**pde.3**), as well as the Black–Scholes equation (**pde.8**). As shown in the next definition and the subsequent theorems, a linear BVP leads to a "principle of superposition" which allows us to combine solutions together.

Definition 2.4 (*Homogeneous BVP*) Suppose that $\mathscr{L}$ is a *linear* operator associated with the BVP $\mathscr{L}u = \mathscr{F}$, then the *homogeneous* BVP is the corresponding problem $\mathscr{L}u = 0$. To generate a homogeneous BVP, any terms that are independent of u must be removed from the PDE and all boundary and initial data must be set to zero.

The statements of the following theorems may be familiar from studies of ordinary differential equations and linear algebra.

Theorem 2.5
Suppose that u_1 and u_2 are any two solutions of a homogeneous boundary value problem $\mathscr{L}u = 0$, then any linear combination $v = \alpha u_1 + \beta u_2$, with constants α, β, is also a solution.

Proof

$$\mathscr{L}v = \mathscr{L}(\alpha u_1 + \beta u_2) = \alpha \underbrace{\mathscr{L}u_1}_{0} + \beta \underbrace{\mathscr{L}u_2}_{0} = 0. \qquad \square$$

Theorem 2.6 *Suppose that u_* is a "particular" solution of the linear boundary value problem $\mathscr{L}u = \mathscr{F}$, and that v is a solution of the associated* homogeneous *problem, then $w = u_* + v$ is also a solution of the BVP $\mathscr{L}u = \mathscr{F}$.*

Proof

$$\mathscr{L}(w) = \mathscr{L}(u_* + v) = \underbrace{\mathscr{L}(u_*)}_{\mathscr{F}} + \underbrace{\mathscr{L}(v)}_{0} = \mathscr{F}. \qquad \square$$

The superposition principle will prove to be invaluable in Chap. 8, where we will construct analytic solutions to BVPs like (2.7).

Theorem 2.7 *(Uniqueness) A linear boundary value problem $\mathscr{L}u = \mathscr{F}$ will have a unique solution if, and only if, $v = 0$ is the only solution of the homogeneous problem $\mathscr{L}v = 0$.*

Proof We suppose that there are two solutions, u_1 and u_2. Hence $\mathscr{L}u_1 = \mathscr{F}$ and $\mathscr{L}u_2 = \mathscr{F}$. Subtracting these equations from each other and using the linearity of $\mathscr{L}$ gives

$$\mathscr{L}v = 0, \qquad v = u_1 - u_2.$$

Thus, if $v = 0$ is the only solution of the homogeneous problem, we must have $u_1 = u_2$. $\square$

When the homogeneous problem has a nontrivial solution v, then αv, for any constant α, is also a solution. Hence $u_2 = u_1 + \alpha v$ and there are therefore an infinite number of solutions corresponding to different choices of α. This is typical of linear problems: they have either no solutions, one solution or an infinite number of solutions.

Example 2.8 Consider the solution of the wave equation $u_{xx} - u_{yy} = 0$ and Laplace's equation $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < 1$, $0 < y < 1/2$ with the boundary conditions given in Fig. 2.2 (with n a positive integer).

The wave equation has a particular solution $u_* = \sin n\pi x \cos n\pi y$ that also satisfies the boundary conditions. However, it is readily shown that

$$v(x, y) = \sin(2m\pi x)\sin(2m\pi y)$$

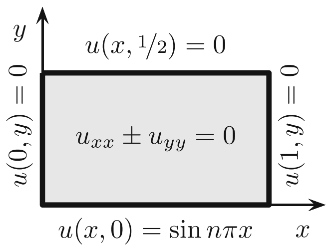
solves the corresponding *homogeneous* BVP for any integer m and so the wave equation has the nonunique solutions

$$u(x, y) = u_*(x, y) + \alpha v(x, y)$$

for each constant α and each integer m. In contrast, it is readily checked that

$$u(x, y) = \sin n\pi x \, \frac{\sinh n\pi(1 - y)}{\sinh n\pi}$$

solves the BVP for Laplace's equation. In fact, this is the only solution. It will be shown in Chap. 7 that a solution of Laplaces equation always has its maximum and minimum values on the boundary. This ensures that the only solution of the corresponding homogeneous BVP is zero. ♢



$$u(x,0) = \sin n\pi x, \quad 0 < x < 1$$
$$u(x, 1/2) = 0, \quad 0 < x < 1$$
$$u(0,y) = u(1,y) = 0, \quad 0 < y < 1/2.$$

Fig. 2.2 Domain and boundary conditions for Example 2.8

2.2.2 Nonlinear Problems

Not all of the PDEs that are listed in Sect. 1.1 are linear and the theorems given in the previous section do not apply to these.

Example 2.9 The inviscid Burgers' equation (**pde.2**), which we write as $u_t + \mathcal{L}(u) = 0$ with $\mathcal{L}(u) = uu_x$ satisfies

$$\begin{aligned}\mathcal{L}(\alpha u) &= (\alpha u)(\alpha u)_x \\ &= \alpha^2 uu_x \neq \alpha \mathcal{L}(u)\end{aligned}$$

and violates the second condition in Definition 2.2. This shows that this PDE is *nonlinear*. The KdV equation (**pde.9**) can be shown to be nonlinear using the same argument. ◊

It is sometimes useful to classify the degree of nonlinearity of a PDE or associated BVP. A standard classification is as follows:

Linear: The PDE should satisfy Definition 2.2. In effect, all coefficients of u and any of its derivatives must depend only on the independent variables $t, x, y, \ldots$.
Semi-linear: The coefficients of the *highest derivatives* of u do not depend on u or any derivatives of u.
Quasi-linear: The coefficients of the highest derivatives of u depend only on lower derivatives of u.

Otherwise the PDE is *fully nonlinear*. Some examples are listed below.

$$\begin{aligned}u_{xxx} - 4u_{xxyy} + u_{yyzz} &= f(x, y, z) && : \text{“linear”} \\ u_x^2 u_{tt} - \tfrac{1}{2}u_{xxxxx} &= 1 - u^2 && : \text{“semi-linear”} \\ u_{tt}u_{xxx} + u_x u_{ttt} &= f(u, x, t) && : \text{“quasi-linear”} \\ \exp(u_{xtt}) - u_{xt}u_{xxx} + u^2 &= 0 && : \text{“fully nonlinear”.}\end{aligned}$$

Additional nonlinearity "classification" exercises are given in Exercise 2.8.

We will see in Chap. 9 that the character of nonlinear *first-order* PDEs is completely governed by the nature of the nonlinearity, so it is important to classify the nonlinearity correctly. This can be readily achieved if the first-order PDE is written in the "additive form",

$$au_t + bu_x + cu = f, \tag{2.16}$$

where a and b are functions of t, x, u, u_t and u_x; c is a function of t, x and u, and f is a function of t and x. The classification is then immediate:

Linear: If a, b and c depend only on t and x, and not on u or any of its derivatives, then the PDE is *linear*.
Semi-linear: If a and b do not depend on u or any of its derivatives, but c depends on u, then the PDE is *semi-linear*.

Quasi-linear: If a and/or b depend on u but not on any derivatives of u, then the PDE is *quasi-linear*.

Otherwise the PDE is *fully nonlinear*. Some examples are listed below.

$$\begin{aligned}
2\cos(xt)u_t - x\mathrm{e}^t u_x - 9u &= \mathrm{e}^t \sin x && : \text{“linear”} \\
x\cos(t)u_t + tu_x + u^2 \cdot u &= \frac{x}{t} u \sin(u) && : \text{“semi-linear”} \\
uu_t + u^2 u_x + u &= \mathrm{e}^x && : \text{“quasi-linear”} \\
u_t + \tfrac{1}{2}u_x^2 - u &= \cos(xt) && : \text{“fully nonlinear”.}
\end{aligned}$$

2.2.3 Well-Posed Problems

So far so good. Well-conceived boundary value problems typically have unique solutions. However, an unfortunate complication is that such BVPs can still be inordinately sensitive to the problem data (for example, the boundary data). Thus, even though such problems might have practical applications, they are much too demanding for a textbook at this level. Accordingly our aim is to filter out such BVPs and focus on those that are relatively well behaved.

An overview of the situation can be obtained by considering a generic *linear* BVP written in the notation introduced in Sect. 2.1, namely,

$$\mathscr{L}u = \mathscr{F}. \tag{2.17}$$

Suppose that we now make a “small” change $\delta\mathscr{F}$ to the data $\mathscr{F}$ and we denote the subsequent change to the solution by δu. Thus,

$$\mathscr{L}(u + \delta u) = \mathscr{F} + \delta\mathscr{F}. \tag{2.18}$$

Then, since $\mathscr{L}$ is a linear operator, (2.17) may be subtracted from (2.18) to give

$$\mathscr{L}(\delta u) = \delta\mathscr{F}, \tag{2.19}$$

so we see that δu satisfies the same BVP as u with $\mathscr{F}$ replaced by $\delta\mathscr{F}$. We now get to a definition which is the crux of the issue.

Definition 2.10 (*Well-posed BVP*) A boundary value problem which has a *unique* solution that varies *continuously* with the initial and boundary data is said to be *well posed*. A problem that is not well posed is said to be *ill posed*.

In the context of (2.19) this means that δu should be "small" whenever $\delta\mathscr{F}$ is "small" in the sense that there are norms[3] $\|\cdot\|_a$, $\|\cdot\|_b$ and a constant C that does not depend on u, $\mathscr{F}$ or $\delta\mathscr{F}$ so that

$$\|\delta u\|_a \leq C\|\delta\mathscr{F}\|_b \tag{2.20}$$

which must hold for all admissible choices of $\delta\mathscr{F}$. By choosing $\delta\mathscr{F} = -\mathscr{F}$ we deduce that $\delta u = -u$ and (2.20) then implies that

$$\|u\|_a \leq C\|\mathscr{F}\|_b. \tag{2.21}$$

This reflects the general approach for linear problems: beginning with the homogeneous problem, the change in the data from zero to $\mathscr{F}$ causes the solution to change from zero to u.

The good news here is that the BVPs defined at the start of the chapter, namely, (2.7), (2.10) and Laplace's equation with either Dirichlet, Neumann or mixed conditions are well posed in the sense of satisfying this definition.[4] The bad news is that boundary value problems which have a unique solution are not automatically well posed. A pathological example is given next.

Example 2.11 Consider the BVP obtained by setting $\kappa = -1$ in (**pde.4**), so that

backward heat equation
$$u_t + u_{xx} = 0, \qquad (\textbf{pde}.\textbf{11})$$

and subjecting it to the initial data $u(x, 0) = 0$.

This BVP has the unique solution, $u(x, t) = 0$. However, if we make *tiny* changes in the initial data to say $u(x, 0) = 10^{-99}\cos(nx)$, then the unique solution changes to

$$u(x, t) = 10^{-99}\mathrm{e}^{n^2 t}\cos(nx).$$

The ratio of solution to data (initial value) is $u(x, t)/u(0, x) = \exp(n^2 t)$. This can be made as large as we wish, even for very small values of t, by taking a large enough value of n. This happens in spite of the fact that the size of the change in initial data would probably be subatomic in any practical example—it reflects the fact that "anti-diffusive" behaviour violates the second law of thermodynamics and is something like having time running backwards! ◊

Our second example extends the PDE from Example 2.8 and has a rearranged boundary condition.

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[3]The examples of ill-posed problems that we shall give are clear cut without the need to specify precisely which norms are used.

[4]This proof is deferred to Chap. 7.

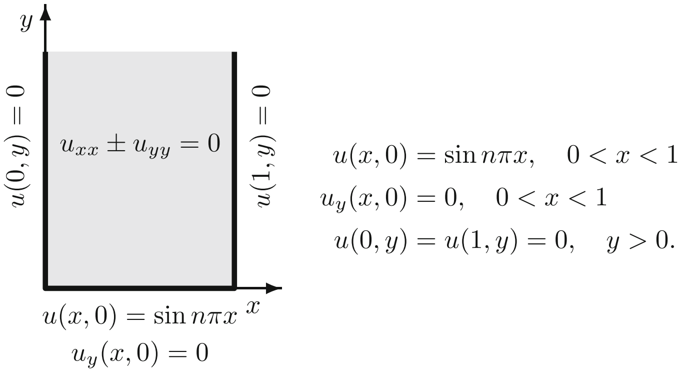

Fig. 2.3 Domain and boundary conditions for Example 2.12

Example 2.12 Consider the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ and the wave equation $u_{xx} - u_{yy} = 0$ in the semi-infinite strip with the boundary conditions given in Fig. 2.3 (with n a positive integer).

Now the wave equation has the unique solution

$$u(x, y) = \sin n\pi x \cos n\pi y$$

in which its magnitude (amplitude) is the same as that of the data.[5] Laplace's equation, however, has the solution

$$u(x, y) = \sin n\pi x \cosh n\pi y$$

which, for any $y > 0$ can be made as large as we wish by taking a suitably large value of n. Hence this type of BVP for Laplace's equation is ill posed.

A well-posed problem could be recovered by replacing one of the conditions applied at $y = 0$ by a condition such as $u(x, y) \to 0$ as $y \to \infty$ for all $x \in (0, 1)$.

Exercises

2.1 For each of the cases (a)–(f) in Exercise 1.3, can you determine the functions A and/or B using the initial condition $u(x, 0) = f(x)$, where f is some given function? Give expressions for A and/or B wherever they can be determined.

2.2 For each of the cases (a)–(f) in Exercise 1.3, can you determine the functions A and/or B using the alternative "initial condition" $u(x, 1) = g(x)$, where g is some given function?

[5] Note that we cannot conclude that this problem is well posed since we would need to consider all possible choices of the data in order to make that claim.

2.3 Let the function g be defined by

$$g(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and suppose that $u(x,t)$ is defined by (1.2). (See also Exercise 1.6.) Use the result that

$$\int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \tfrac{1}{2}\sqrt{\pi}$$

to show that $u(0,t) = \frac{1}{2}$ for $t > 0$ and $u(x,0) = 1$ for $x > 0$.

2.4 By following the example in Sect. 2.1, define suitable forms for $\mathscr{L}$ and $\mathscr{F}$ so the BVP (2.10) can be written as $\mathscr{L}u = \mathscr{F}$.

2.5 Show that $\mathscr{L}$ defined by (2.13) satisfies $\mathscr{L}(\alpha u) = \alpha\mathscr{L}u$.

2.6 Show that $\mathscr{L}$ defined in Example 2.1 is a linear operator.

2.7* Consider the backward heat equation (see Example 2.11) for $u(x,t)$, corresponding to having a negative thermal diffusivity coefficient $\kappa = -1$:

$$u_t + u_{xx} = 0.$$

Confirm that, for constant values of A and T a solution, for any $t < T$, is given by

$$u(x,t) = \frac{AT^{1/2}}{(T-t)^{1/2}}\exp\left(-\frac{x^2}{4(T-t)}\right).$$

Use this to show that solutions can exist with, initially, $|u(0,x)| \le \varepsilon$ for any $\varepsilon > 0$ but which become infinite in value after any given subsequent time. Deduce that the backward heat equation is not well posed for $t > 0$ when subjected to initial conditions at $t = 0$.

2.8 Determine the order and categorise the following PDEs by linearity or degree of nonlinearity.

(a) $u_t - (x^2 + u)u_{xx} = x - t$.
(b) $u^2 u_{tt} - \frac{1}{2}u_x^2 + (uu_x)_x = \mathrm{e}^u$.
(c) $u_t - u_{xx} = u^3$.
(d) $(u_{xy})^2 - u_{xx} + u_t = 0$.
(e) $u_t + u_x - u_y = 10$.

2.9 Categorise the following second-order PDEs by linearity or degree of nonlinearity.

(a) $u_t + u_{tx} - u_{xx} + u_x^2 = \sin u$.
(b) $u_x + u_{xx} + u_y + u_{yy} = \sin(xy)$.
(c) $u_x + u_{xx} - u_y - u_{yy} = \cos(xyu)$.
(d) $u_{tt} + xu_{xx} + u_t = f(x, t)$.
(e) $u_t + uu_{xx} + u^2u_{tt} - u_{tx} = 0$.

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