2

The Radon Transform

2.1 Definition

For a given function f defined in the plane, which may represent, for instance, the attenuation-coefficient function in a cross section of a sample, the fundamental question of image reconstruction calls on us to consider the value of the integral of f along a typical line $\ell_{t,\theta}$. For each pair of values of t and θ, we will integrate f along a different line. Thus, we really have a new function on our hands, where the inputs are the values of t and θ and the output is the value of the integral of f along the corresponding line $\ell_{t,\theta}$. But even more is going on than that because we also wish to apply this process to a whole variety of functions f. So really we start by selecting a function f. Then, once f has been selected, we get a corresponding function of t and θ. Schematically,

$$\text{input}\, f \mapsto \text{output} \left\{ (t,\ \theta) \mapsto \int_{\ell_{t,\theta}} f\, ds \right\}.$$

This multi-step process is called the *Radon transform*, named for the Austrian mathematician Johann Karl August Radon (1887–1956) who studied its properties. For the input f, we denote by $\mathcal{R}(f)$ the corresponding function of t and θ shown in the schematic. That is, we make the following definition.

Definition 2.1. For a given function f, whose domain is the plane, the Radon transform of f is defined, for each pair of real numbers $(t,\ \theta)$, by

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$$\mathcal{R}f(t,\,\theta) := \int_{\ell_{t,\,\theta}} f\,ds$$
$$= \int_{s=-\infty}^{\infty} f(t\cos(\theta) - s\sin(\theta),\; t\sin(\theta) + s\cos(\theta))\,ds. \tag{2.1}$$

A few immediate observations are that (i) both f and $\mathcal{R}f$ are functions; (ii) f is a function of the Cartesian coordinates x and y while $\mathcal{R}f$ is a function of the polar coordinates t and θ; (iii) for each choice of f, t, and θ, $\mathcal{R}f(t,\,\theta)$ is a number (the value of a definite integral); (iv) in the integral on the right, the variable of integration is s, while the values of t and θ are preselected and so should be treated as "constants" when evaluating the integral.

We can visualize the Radon transform of a given function by treating θ and t as *rectangular coordinates*. We depict the values of the Radon transform according to their brightness on a continuum of grey values, with the value 0 representing the color *black*, 0.5 representing a neutral grey, and the value 1 representing *white*. Such a graph is called a *sinogram* and essentially depicts all of the data generated by the X-ray emission/detection machine for the given slice of the sample. The choice of the term *sinogram* is no doubt suggested by the symmetry $\mathcal{R}f(-t,\,\theta+\pi) = \mathcal{R}f(t,\,\theta)$ as well as by the appearance of the graphs for some simple examples that we will explore next. Sinograms can also be portrayed in color, using an appropriate segment of the rainbow in place of the grey scale.

2.2 Examples

Example 2.2. As a first example, suppose our patient has a small circular tumor with radius 0.05 centered at the point $(0,\,1)$, and suppose the attenuation-coefficient function f has the constant value 10 there. Now take an arbitrary value of θ. In this case, when $t = \sin(\theta)$, the line $\ell_{t,\theta}$ will pass through $(0,\,1)$ and make a diameter of the circular tumor. Since the diameter of this disc is $2\cdot(0.05) = 0.1$, we get $\mathcal{R}f(\sin(\theta),\theta) = 10\cdot(0.1) = 1$. Moreover, for any given θ, the value of $\mathcal{R}f(t,\,\theta)$ will be zero except on the narrow band $\sin(\theta) - 0.05 < t < \sin(\theta) + 0.05$.

Thus, as θ varies from 0 to π, the graph will show a narrow, brighter grey ribbon of width 0.1 centered around $t = \sin(\theta)$. In other words, the graph of $\mathcal{R}f$ in the $(\theta,\,t)$ plane will resemble the graph of the *sine* function. Similarly, the graph in the $(\theta,\,t)$ plane of the Radon transform of a small, bright disc located at $(1,\,0)$ will resemble the graph of the *cosine* function. Figure 2.1 shows the sinograms for these two bright discs. Perhaps these examples helped to motivate the use of the term *sinogram* for the graph of a Radon transform.

In the previous example, we considered an attenuation-coefficient function that had a constant (nonzero) value on a finite region of the plane and the value 0 outside of that region. To attach some terminology to functions of this type, suppose Ω is some finite region in the plane and take f_Ω to be the function that has the value 1 at each point contained in Ω and the

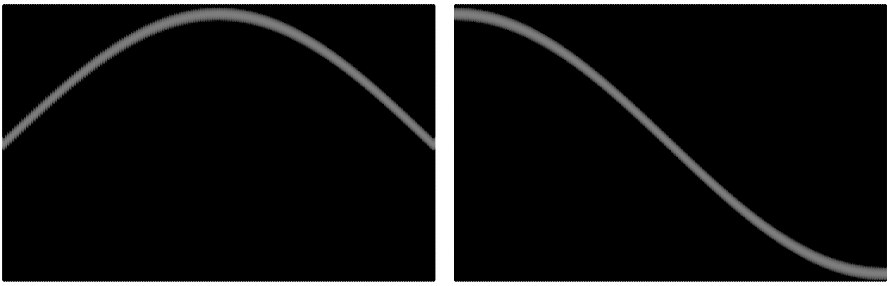

Fig. 2.1. The *sinograms* for small, bright circular discs centered at (0, 1) (left) and at (1, 0) (right) resemble the graphs of the *sine* and *cosine* functions, respectively.

value 0 at each point not in Ω. This function f_Ω is known as the *characteristic function*, or the *indicator function*, of the region Ω. Thus, in our previous example, we looked at 10 times the characteristic function of a small disc.

When the attenuation-coefficient function is a characteristic function f_Ω, its Radon transform is particularly easy to comprehend. Indeed, along any line $\ell_{t,\theta}$, the value of f_Ω will be 0 except when the line is passing through the region Ω, where the value is 1. Thus, for each pair of values of t and θ, the value of $\mathcal{R}f_\Omega(t, \theta)$ *is equal to the length of the intersection of the line* $\ell_{t,\theta}$ *with the region* Ω.

In general, the object we are testing may comprise a collection of "blobs" of various materials, each with its specific attenuation coefficient. As we shall soon see, the Radon transform of the entire collection will be a composite of the transforms of the separate blobs. Consequently, understanding this basic sort of example will play a central role in assessing the accuracy of various approaches to image reconstruction.

Example 2.3. For instance, let's take the region Ω to be the closed circular disc of radius $R > 0$ centered at the origin. Then the characteristic function of Ω is given by

$$f_\Omega(x, y) := \begin{cases} 1 & \text{if } x^2 + y^2 \leq R^2, \\ 0 & \text{otherwise.} \end{cases}$$

If we choose a value of t such that $|t| > R$, then, regardless of the value of θ, the line $\ell_{t,\theta}$ will not intersect the disc Ω. Thus, $f_\Omega = 0$ at every point on $\ell_{t,\theta}$, and, hence, $\mathcal{R}f_\Omega(t, \theta) = 0$ for such t. On the other hand, if $|t| \leq R$, then $\ell_{t,\theta}$ intersects Ω along the segment corresponding to the parameter values $-\sqrt{R^2 - t^2} \leq s \leq \sqrt{R^2 - t^2}$ in the standard parameterization for the line. The length of this segment is $2\sqrt{R^2 - t^2}$. The value of f_Ω is 1 at points in this interval and 0 otherwise. Therefore, for such t, the value of $\mathcal{R}f_\Omega(t, \theta)$ is the same as the length of the segment, namely, $2\sqrt{R^2 - t^2}$.

In summary, we have shown that

$$\mathcal{R}f_\Omega(t, \theta) = \begin{cases} 2\sqrt{R^2 - t^2} & \text{if } |t| \leq R, \\ 0 & \text{if } |t| > R. \end{cases}$$

Example 2.4. Continuing with the theme of characteristic functions, take $\mathcal{S}$ to be the region enclosed by the square whose edges lie along the vertical lines $x = \pm 1$ and the horizontal lines $y = \pm 1$. This square is centered at the origin and has side length 2. Let $f_{\mathcal{S}}$ be the function that takes the value 1 at each point of $\mathcal{S}$ and the value 0 at each point not in $\mathcal{S}$. That is, let

$$f_{\mathcal{S}}(x, y) = \begin{cases} 1 & \text{if } \max\{|x|, |y|\} \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

As discussed above, for each t and each θ, the corresponding value $\mathcal{R}f_{\mathcal{S}}(t, \theta)$, of the Radon transform of $f_{\mathcal{S}}$, will equal the length of the intersection of the line $\ell_{t,\theta}$ and the square region $\mathcal{S}$. There are two values of θ for which this intersection length is easy to see: $\theta = 0$ and $\theta = \pi/2$. In these cases, the lines $\ell_{t,\theta}$ are vertical or horizontal, respectively. Thus, for $\theta = 0$ or $\theta = \pi/2$, we get $\mathcal{R}f(t, \theta) = 2$ when $-1 \leq t \leq 1$ and $\mathcal{R}f(t, \theta) = 0$ when $|t| > 1$. In general, the function $\mathcal{R}f_{\mathcal{S}}(t, \theta)$ will be piecewise linear in t for each fixed value of θ. Figure 2.2 shows these cross sections for several values of θ.

Figure 2.3 shows the full sinogram for the function $f_{\mathcal{S}}$. Note the symmetry inherited from that of the square region $\mathcal{S}$. This example will play an important role in the image reconstruction algorithms examined in Chapter 9.

Example 2.5. For an example that is not a characteristic function, let f be the function defined by

$$f(x, y) := \begin{cases} 1 - \sqrt{x^2 + y^2} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{if } x^2 + y^2 > 1. \end{cases} \tag{2.2}$$

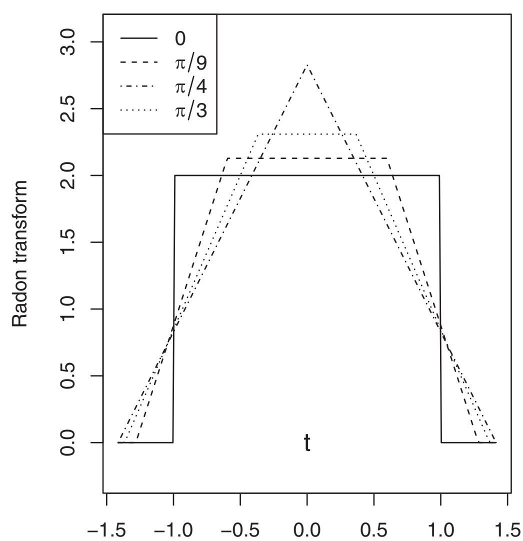

Fig. 2.2. Cross sections at $\theta = 0$, $\theta = \pi/9$, $\theta = \pi/4$, and $\theta = \pi/3$ of the sinogram of the characteristic function of the basic square.

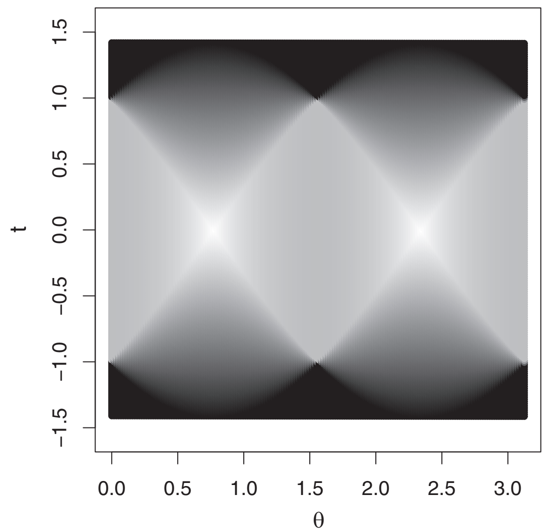

Fig. 2.3. The sinogram for the characteristic function of the square $\mathcal{S} = \{(x, y) : |x| \leq 1, \text{ and } |y| \leq 1\}$.

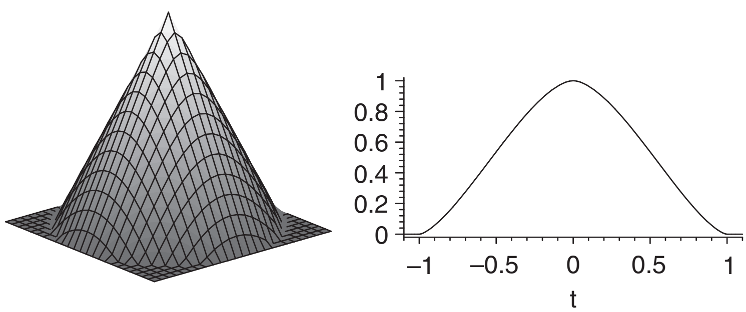

Fig. 2.4. The figure shows the cone defined in (2.2) and the graph of its Radon transform for any fixed value of θ.

The graph of f is a cone, shown in Figure 2.4. We have already observed that, on the line $\ell_{t,\theta}$, we have

$$x^2 + y^2 = (t\cos(\theta) - s\sin(\theta))^2 + (t\sin(\theta) + s\cos(\theta))^2 = t^2 + s^2.$$

It follows that, on the line $\ell_{t,\theta}$, the function f is given by

$$\begin{aligned} &f(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta)) \\ &:= \begin{cases} 1 - \sqrt{t^2+s^2} & \text{if } t^2+s^2 \leq 1, \\ 0 & \text{if } t^2+s^2 > 1. \end{cases} \end{aligned} \tag{2.3}$$

From this, we see that the value of $\mathcal{R}f(t, \theta)$ depends only on t and not on θ and that $\mathcal{R}f(t, \theta) = 0$ whenever $|t| > 1$. For a fixed value of t such that $|t| \leq 1$, the condition $t^2 + s^2 \leq 1$ will be satisfied provided that $s^2 \leq 1 - t^2$. Thus, for any value of θ and for t

such that $|t| \leq 1$, we have

$$\begin{aligned} & f(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta)) \\ & := \begin{cases} 1 - \sqrt{t^2 + s^2} & \text{if } -\sqrt{1-t^2} \leq s \leq \sqrt{1-t^2}, \\ 0 & \text{otherwise;} \end{cases} \end{aligned} \tag{2.4}$$

whence

$$\int_{\ell_{t,\theta}} f\, ds = \int_{s=-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \left(1 - \sqrt{t^2 + s^2}\right)\, ds\,. \tag{2.5}$$

This integral requires a trigonometric substitution for its evaluation. Sparing the details for now, we have

$$\int_{s=-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \left(1 - \sqrt{t^2 + s^2}\right)\, ds = \sqrt{1-t^2} - \frac{1}{2}t^2 \ln\left(\frac{1 + \sqrt{1-t^2}}{1 - \sqrt{1-t^2}}\right). \tag{2.6}$$

In conclusion, we have shown that the Radon transform of this function f is given by

$$\mathcal{R}f(t,\ \theta) := \begin{cases} \sqrt{1-t^2} - \frac{1}{2}t^2 \ln\left(\frac{1+\sqrt{1-t^2}}{1-\sqrt{1-t^2}}\right) & \text{if } -1 \leq t \leq 1, \\ 0 & \text{if } |t| > 1. \end{cases} \tag{2.7}$$

In this case, where $\mathcal{R}f$ is independent of θ, the value of $\mathcal{R}f(t,\ \theta)$ corresponds to the area under an appropriate vertical cross section of the cone defined by $z = f(x,\ y)$. Several of these cross sections are visible in Figure 2.4.

2.3 Some properties of $\mathcal{R}$

Suppose that two functions f and g are both defined in the plane. Then so is the function $f + g$. Since the integral of a sum of two functions is equal to the sum of the integrals of the functions separately, it follows that we get, for every choice of t and θ,

$$\begin{aligned} \mathcal{R}(f+g)(t,\ \theta) &= \int_{s=-\infty}^{\infty} (f+g)(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta))\, ds \\ &= \int_{s=-\infty}^{\infty} \{f(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta)) \\ &\qquad + g(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta))\}\, ds \end{aligned}$$

$$
\begin{aligned}
&= \int_{s=-\infty}^{\infty} f(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta))\, ds \\
&\qquad + \int_{s=-\infty}^{\infty} g(t\cos(\theta) - s\sin(\theta),\ t\sin(\theta) + s\cos(\theta))\, ds \\
&= \mathcal{R}f(t,\ \theta) + \mathcal{R}g(t,\ \theta).
\end{aligned}
$$

In other words, $\mathcal{R}(f + g) = \mathcal{R}f + \mathcal{R}g$ as functions.

Similarly, when a function is multiplied by a constant and then integrated, the result is the same as if the function were integrated first and then that value multiplied by the constant; i.e., $\int \alpha f = \alpha \int f$. In the context of the Radon transform, this means that $\mathcal{R}(\alpha f) = \alpha \mathcal{R}f$.

We now have proven the following proposition.

Proposition 2.6. *For two functions f and g and any constants α and β,*

$$
\mathcal{R}(\alpha f + \beta g) = \alpha \mathcal{R}f + \beta \mathcal{R}g. \tag{2.8}
$$

In the language of linear algebra, we say that the Radon transform is a *linear transformation*; that is, the Radon transform $\mathcal{R}$ maps a linear combination of functions to the same linear combination of the Radon transforms of the functions separately. We also express this property by saying that "$\mathcal{R}$ preserves linear combinations."

Example 2.7. Consider the function

$$
f(x,\ y) := \begin{cases} 0.5 & \text{if } x^2 + y^2 \leq 0.25, \\ 1.0 & \text{if } 0.25 < x^2 + y^2 \leq 1.0, \\ 0 & \text{otherwise.} \end{cases}
$$

This a linear combination of the characteristic functions of two circular discs. Namely, $f = f_{\Omega_1} - (0.5)f_{\Omega_2}$, where Ω_1 and Ω_2 are the discs of radii 1 and 0.5, respectively, centered at the origin.

Using property (2.8), along with the computation in Example 2.3, it follows that

$$
\begin{aligned}
\mathcal{R}f(t,\ \theta) &= \mathcal{R}(f_{\Omega_1})(t,\ \theta) - (0.5)\mathcal{R}(f_{\Omega_2})(t,\ \theta) \\
&= \begin{cases} 2\sqrt{1 - t^2} - \sqrt{(0.25) - t^2} & \text{if } |t| \leq 0.5, \\ 2\sqrt{1 - t^2} & \text{if } (0.5) < |t| \leq 1, \\ 0 & \text{if } |t| > 1. \end{cases}
\end{aligned}
$$

Figure 2.5 shows the graph of this attenuation-coefficient function alongside a graph of the cross section of its Radon transform corresponding to any fixed value of θ and $-1 \leq t \leq 1$.

What happens to the Radon transform if we modify a function either by shifting it or by re-scaling it? That is, suppose we know the Radon transform of a function f, and now look at the functions $g(x,\ y) = f(x - a,\ y - b)$, where a and b are some real numbers, and $h(x,\ y) = f(cx,\ cy)$, where $c > 0$ is a positive scaling factor.

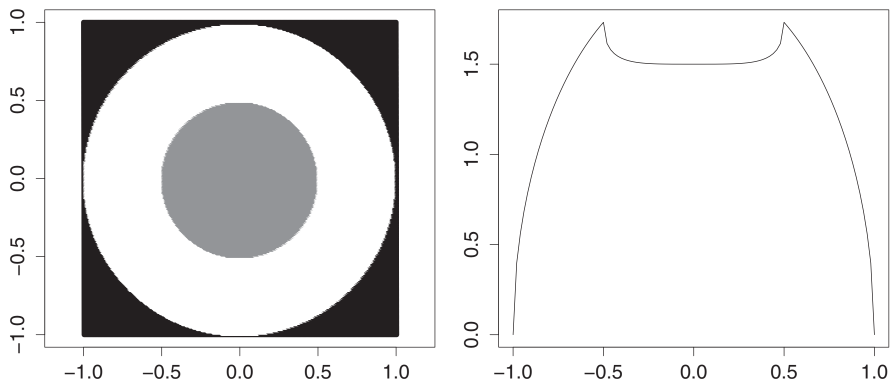

Fig. 2.5. The figure shows the attenuation function defined in Example 2.7 alongside the graph of its Radon transform for any fixed value of θ.

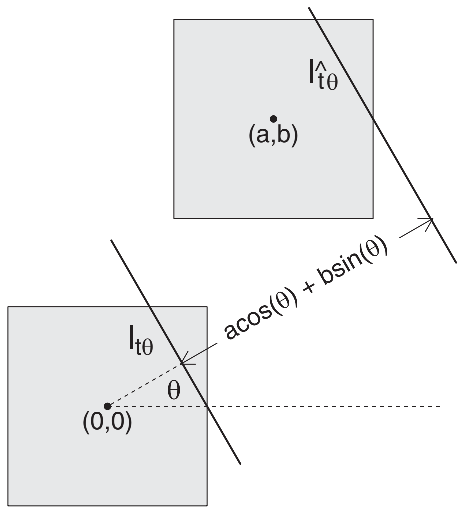

Fig. 2.6. The shifting property of $\mathcal{R}$: The lines $\ell_{t,\theta}$ and $\ell_{t+a\cos(\theta)+b\sin(\theta),\theta}$ intersect congruent regions, centered at $(0, 0)$ and (a, b), respectively, along segments of the same length.

In the first case, we obtain the graph of g by shifting the graph of f by a units in the x direction and b units in the y direction. It follows that if we take any line $\ell_{t,\,\theta}$ and shift it by just the right amount, we will get a line $\ell_{\widehat{t},\,\theta}$ with the property that $\mathcal{R}g(\widehat{t},\,\theta) = \mathcal{R}f(t,\,\theta)$. What is the correct shift in the value of t? Well, when $t = 0$, then $\ell_{0,\,\theta}$ passes through the origin, while the parallel line $\ell_{a\cos(\theta)+b\sin(\theta),\,\theta}$ passes through $(a,\,b)$, as we saw in Exercise 12 of Chapter 1. So, the relationship between $\ell_{0,\,\theta}$ and the graph of f is the same as that between $\ell_{a\cos(\theta)+b\sin(\theta),\,\theta}$ and the graph of g. In other words, the correct shift is $\widehat{t} = t + a\cos(\theta) + b\sin(\theta)$. Figure 2.6 illustrates this correspondence.

In the case of the function $h(x,\,y) = f(cx,\,cy)$, we can think of the domain of h as a $(1/c)$-times scale model of the corresponding domain for f. Thus, to compute the Radon transform of h for a given choice of t and θ, we first have to scale the value of t by the factor c, in order to locate a parallel line that intersects a similar part of the domain of f. Then multiply

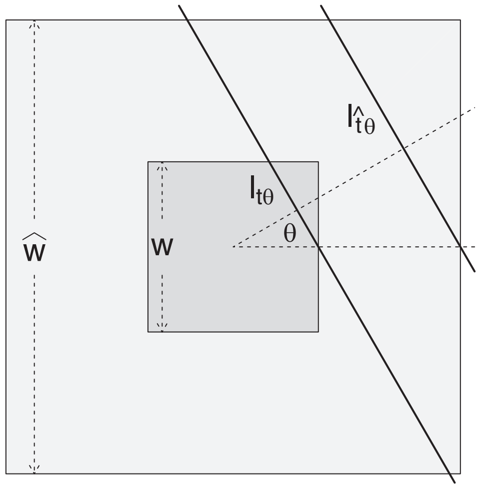

Fig. 2.7. The scaling property of $\mathcal{R}$: With $c = \hat{w}/w = \hat{t}/t$, the length of the intersection of $\ell_{t,\theta}$ with the smaller square region is equal to $1/c$ times the length of the intersection of $\ell_{\hat{t},\theta}$ with the larger square region.

the corresponding value of the Radon transform of f by $1/c$ to get back to the scale of h. That is, for given values of t and θ, we get $\mathcal{R}h(t,\,\theta) = (1/c)\cdot\mathcal{R}f(ct,\,\theta)$. This relationship is shown in Figure 2.7, in which f and h are taken to be the characteristic functions of two square regions.

We summarize these statements in a proposition.

Proposition 2.8. *Let the function f be defined in the plane, let a and b be arbitrary real numbers, and let $c > 0$ be a positive real number. Define the function g by $g(x,\,y) = f(x-a,\,y-b)$ and the function h by $h(x,\,y) = f(cx,\,cy)$. Then, for all real numbers t and θ,*

$$\mathcal{R}g(t,\,\theta) = \mathcal{R}f(t - a\cos(\theta) - b\sin(\theta),\,\theta) \text{ and} \tag{2.9}$$

$$\mathcal{R}h(t,\,\theta) = (1/c)\cdot\mathcal{R}f(ct,\,\theta)\,. \tag{2.10}$$

Example 2.9. We already know the Radon transform for the characteristic function of a disc of radius $R > 0$ centered at the origin. So now suppose Ω is the disc of radius $R > 0$ centered at $(a,\,b)$, with characteristic function f_Ω. It follows from (2.9) that the Radon transform of f_Ω is given by

$$\mathcal{R}f_\Omega(t,\,\theta) = \begin{cases} 2\sqrt{R^2 - \widehat{t}^2} & \text{if } |\widehat{t}| \le R, \\ 0 & \text{if } |\widehat{t}| > R, \end{cases}$$

where $\widehat{t} := t - a\cos(\theta) - b\sin(\theta)$. This certainly looks difficult to compute, but in practice we will use a digital computer, so there is no sweat for us!

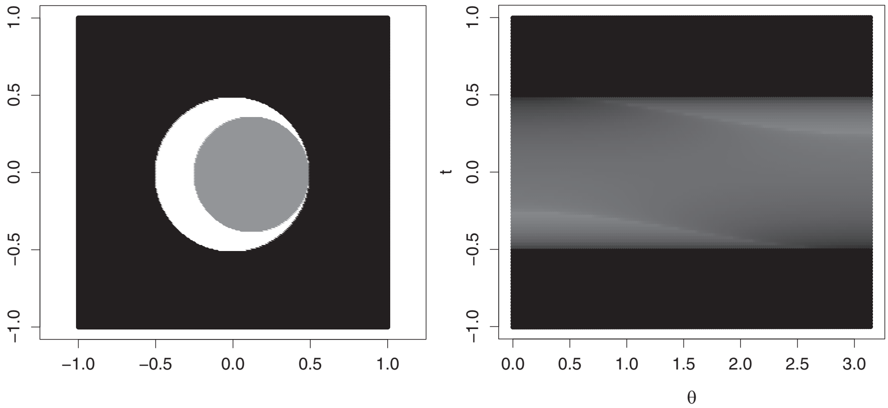

Fig. 2.8. The figure shows the graph of the attenuation-coefficient function A defined in (2.11), alongside a sinogram of its Radon transform $\mathcal{R}A(t,\,\theta)$.

Example 2.10. To combine several properties of the Radon transform in one example, consider a crescent-shaped region inside the circle $x^2+y^2=1/4$ and outside the circle $(x-1/8)^2+y^2=9/64$. Assign density 1 to points in the crescent, density $1/2$ to points inside the smaller disc, and density 0 to points outside the larger disc. Thus, the attenuation function is

$$A(x,\,y) := \begin{cases} 1 & \text{if } x^2+y^2 \le 1/4 \text{ and } (x-1/8)^2+y^2 > 9/64; \\ 0.5 & \text{if } (x-1/8)^2+y^2 \le 9/64; \\ 0 & \text{if } x^2+y^2 > 1/4. \end{cases} \tag{2.11}$$

To break this example into pieces, take Ω_1 to be the closed disc of radius $1/2$ centered at the origin and Ω_2 to be the closed disc of radius $3/8$ centered at $(1/8,\,0)$. Then the attenuation function $A(x,\,y)$ just described can be written as $A(x,\,y) = f_{\Omega_1}(x,\,y) - (1/2)\cdot f_{\Omega_2}(x,\,y)$. It follows from the property (2.8) that, for all t and all θ, $\mathcal{R}A(t,\,\theta) = \mathcal{R}f_{\Omega_1}(t,\,\theta) - (1/2)\cdot \mathcal{R}f_{\Omega_2}(t,\,\theta)$. The second of these two Radon transforms can be computed, as we just discussed, by shifting a Radon transform that we already know, using the property (2.9).

Figure 2.8 shows the graph of this attenuation-coefficient function alongside a graph of its Radon transform in the $(t,\,\theta)$ plane. Figure 2.9 shows graphs of the Radon transform for the angles $\theta=0$ and $\theta=\pi/3$.

2.4 Phantoms

The fundamental question of image reconstruction asks whether a picture of an attenuation-coefficient function can be generated from the values of the Radon transform of that function. We will see eventually that the answer is "*Yes*," if all values of the Radon transform are available. In practice, though, only a finite set of values of the Radon transform are measured

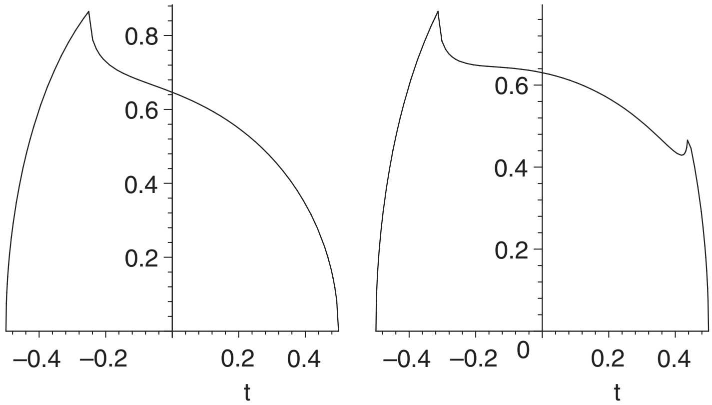

Fig. 2.9. For the function A defined in (2.11), the figure shows graphs of its Radon transform $\mathcal{R}A(t, 0)$ (left) and $\mathcal{R}A(t, \pi/3)$ (right), for $-1/2 \le t \le 1/2$.

by a scanning machine, so our answer becomes "*Approximately yes*." Consequently, the nice solution that works in the presence of full information will splinter into a variety of approximation methods that can be implemented when only partial information is at hand.

One method for testing the accuracy of a particular image reconstruction algorithm, or for comparing algorithms, is simply to apply each algorithm to data taken from an actual human subject. The drawback of this approach is that usually we don't know exactly what we ought to see in the reconstructed image. That is what we are trying to find out by creating an image in the first place. But without knowing what the real data are, there is no way to determine the accuracy of any particular image. To get around this, we can apply algorithms to data taken from a physical object whose internal structure is known. That way, we know what the reconstructed image ought to look like and we can recognize inaccuracies in a given algorithm or identify disparities between different algorithms. Nonetheless, this approach can be misleading. Although the internal structure of the object is known, there may be errors in the data that were collected to represent the object. In turn, these errors may lead to errors in the reconstructed image. We will not be able to distinguish these flaws from errors caused by the algorithm itself. To resolve this dilemma, Shepp and Logan, in [49], introduced the concept of a *mathematical phantom*. This is a simulated object, or test subject, whose structure is entirely defined by mathematical formulas. Thus, *no errors occur in collecting the data* from the object. When an algorithm is applied to produce a reconstructed image of the phantom, *all inaccuracies are due to the algorithm*. This makes it possible to compare different algorithms in a meaningful way.

Since measurement of the Radon transform of an object forms the basis for creating a CT image of the object, it makes sense to use phantoms for which the Radon transform is known exactly. We can then test a proposed algorithm by seeing how well it handles the data from such a phantom. For example, we have computed the Radon transform of a circular disc of constant density centered at the origin. Using the linearity of $\mathcal{R}$, along with the shifting and rescaling formulas, we can now compute the Radon transform of any collection of discs, each having constant density, with any centers and radii. More generally,

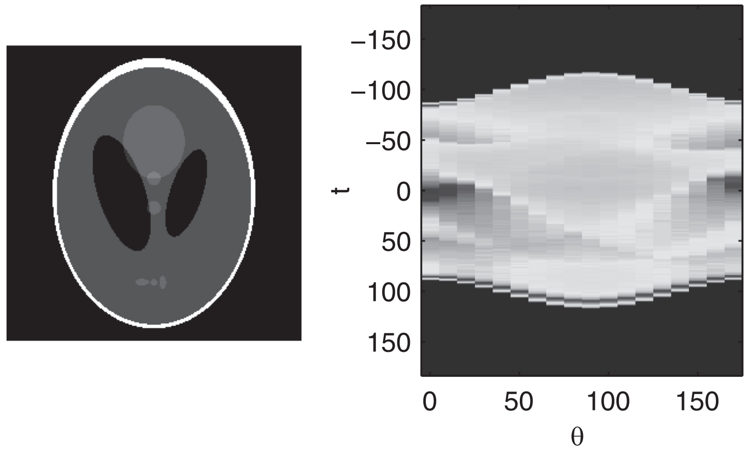

Fig. 2.10. The Shepp–Logan phantom is used as a mathematical facsimile of a brain for testing image reconstruction algorithms. This version of the phantom is stored in MATLAB^R. On the right is a sinogram of the phantom's Radon transform, with θ in increments of $\pi/18$ ($10°$).

the boundary of an arbitrary ellipse is defined by a quadratic expression in x and y. So, as with the circle, determining the intersection of any line $\ell_{t,\theta}$ with an ellipse amounts to finding the difference between two roots of a quadratic equation. In this way, we can calculate exactly the Radon transform of a phantom composed of an assortment of elliptical regions, each having a constant density. Shepp and Logan [49] developed just such a phantom, shown in Figure 2.10.

The Shepp–Logan phantom is composed of eleven ellipses of various sizes, eccentricities, locations, and orientations. (The MATLAB^R version shown here does not include an ellipse that models a blood clot in the lower right near the boundary.) The densities are assigned so that they fall into the ranges typically encountered in a clinical setting. This phantom serves as a useful model of an actual slice of brain tissue and has proven to be a reliable tool for testing reconstruction algorithms.

2.5 Designing phantoms

In this section, we will explore how to design our own phantoms. Each phantom will be a collection of either elliptical regions or square regions, with a constant attenuation-coefficient function on each region. We will find exact formulas for the Radon transforms of these phantoms. This will allow us to plot both the phantom and its sinogram. Indeed, Figures 2.1, 2.3, 2.5, and 2.8 were all created using the templates developed in this section. Like the Shepp–Logan phantom, the phantoms we create here can be used as mathematical models for testing the different image reconstruction algorithms discussed in this book.

2.5.1 Plotting an elliptical region

We begin with the problem of designing a phantom made up of elliptical regions. Every ellipse in the xy-plane is determined by the values of five parameters: the lengths a and b of the semi-major and semi-minor axes, the coordinates (x_0, y_0) of the center point of the ellipse, and the angle ϕ of rotation of the axes of the ellipse away from the horizontal and vertical coordinate axes. We can compute coordinates relative to the new center and the rotated framework as

$$\widehat{x} = (x - x_0)\cos(\phi) + (y - y_0)\sin(\phi)\,, \text{ and}$$
$$\widehat{y} = (y - y_0)\cos(\phi) - (x - x_0)\sin(\phi)\,.$$

Then the general formula for the resulting ellipse is given by

$$\frac{\widehat{x}^2}{a^2} + \frac{\widehat{y}^2}{b^2} = 1\,. \tag{2.12}$$

The characteristic function of the region inside the ellipse has value 1 at those points (x, y) for which the expression on the left-hand side of (2.12) is less than or equal to 1, and the value 0 otherwise.

Next, we assign to each region an attenuation-coefficient function that is some constant multiple of the characteristic function for the region. Thus, each elliptical region in the phantom is defined by six values: the five already mentioned along with the attenuation coefficient δ assigned to the region. Our phantom is determined by the sum of the attenuation-coefficient functions of the different regions that make up the phantom. When two regions overlap, the attenuation coefficient on the overlap will be the sum of the individual attenuation coefficients. For instance, we can fashion a skull, as it were, by assigning an attenuation coefficient of 1.0 to a large elliptical region and an attenuation coefficient of, say, -0.9 to a slightly smaller elliptical region inside it. The net effect will be to have a shell with attenuation coefficient 1.0 between the two ellipses (the skull) and a coefficient of $0.1 = 1.0 - 0.9$ inside the shell, where we can place the remaining elements of the phantom.

Example with *R* 2.11. Figure 2.11 shows a phantom composed of seven elliptical regions each defined by six values $(a, b, x_0, y_0, \phi, \delta)$, as just described. In *R*, we encode the phantom as a matrix, where each elliptical region defines one row, like so.

```
## each ellipse is a 6-D vector [a,b,x0,y0,phi,greyscale]
p1=c(.7,.8,0,0,0,1)
p2=c(.65,.75,0,0,0,-.9)
p3=c(.15,.2,0,.4,0,.5)
p4=c(.25,.15,-.25,.25,2.37,.2)
p5=c(.25,.15,.25,.25,.79,.2)
p6=c(.08,.25,0,-.3,.5,.65)
p7=c(.05,.05,.5,-.3,0,.8)
#combine into a matrix with one ellipse in each row
P=matrix(c(p1,p2,p3,p4,p5,p6,p7),byrow=T,ncol=6)
```

Fig. 2.11. A phantom of elliptical regions; and its sinogram.

Notice the small bright slivers in the top half of the phantom, each formed by the intersection of two of the interior ellipses. These slivers have the attenuation coefficient $0.8 = 0.5 + 0.2 + 1.0 - 0.9$. (Don't forget that both of these ellipses are inside the two ellipses that define the skull.)

To plot the phantom, we need to create a grid of points in the *xy*-plane that form the locations of the "pixels" in the picture. Here, we first define lists of x and y values in the interval $[-1,\ 1]$. Then we replicate the full list of x values paired with each y value.

```
##define a K-by-K grid of points in the square [-1,1]x[-1,1]
K=100 #larger K gives better resolution
yval=seq(-1,1,2/K)
grid.y=double((K+1)^2)
for (i in 1:(K+1)^2){
grid.y[i]=yval[floor((i-1)/(K+1))+1]}
xval=seq(-1,1,2/K)
grid.x=rep(xval,K+1)
```

Next, we define a procedure, or function, that checks each point in the grid and adds the attenuation coefficients for those regions that contain it. We apply this function to our phantom to determine the color value for each grid point. Finally, we plot each grid point using a plot character of the assigned color. (The plot character `pch=15` in *R* is an open square that can be filled with any specified color.)

```
##procedure to compute color value at each grid point
phantom.proc=function(x,y,M){
phantom.1=matrix(double(nrow(M)*length(x)),nrow(M),length(x))
for (i in 1:nrow(M)){
x.new=x-M[i,3]
y.new=y-M[i,4]
phantom.1[i,]=ifelse(M[i,2]^2*(x.new*cos(M[i,5])
+y.new*sin(M[i,5]))^2+M[i,1]^2*(y.new*cos(M[i,5])
```

```
-x.new*sin(M[i,5]))^2-M[i,1]^2*M[i,2]^2<0,M[i,6],0)}
colorvec=colsums(phantom.1)
list(colorvec=colorvec)}#output is "_$colorvec"
# apply the procedure to our phantom P
P.new=phantom.proc(grid.x,grid.y,P)
#create the picture
par(mar=c(0,0,0,0))#removes margins
plot(grid.x,grid.y,pch=15,col=gray(P.new$colorvec))
```

2.5.2 The Radon transform of an ellipse

We now turn our attention to computing the Radon transform of a phantom composed of elliptical regions, each having a constant attenuation coefficient.

To begin with something basic, let $\alpha > 0$ be some positive constant and let $\mathcal{E}_0$ be the closed region bounded by the ellipse with equation $x^2 + \alpha^2 y^2 = \alpha^2$. The characteristic function of $\mathcal{E}_0$ is then

$$f_{\mathcal{E}_0}(x, y) := \begin{cases} 1 & \text{if } x^2 + \alpha^2 y^2 \leq \alpha^2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

To determine the Radon transform of $f_{\mathcal{E}_0}$, choose real numbers t and θ, with $0 \leq \theta < \pi$. We give the line $\ell_{t,\theta}$ its usual parameterization:

$$x = t\cos(\theta) - s\sin(\theta)\,;\ \ y = t\sin(\theta) + s\cos(\theta)\,;\ \ -\infty < s < \infty\,.$$

Plugging these expressions for x and y into the equation for the boundary ellipse and reorganizing the terms yields a quadratic equation for the parameter s. Specifically, let

$$\begin{aligned} A &= \left(\sin^2(\theta) + \alpha^2\cos^2(\theta)\right)\,, \\ B &= t\sin(2\theta)\left(\alpha^2 - 1\right)\,, \text{ and} \\ C &= t^2(\cos^2(\theta) + \alpha^2\sin^2(\theta)) - \alpha^2\,. \end{aligned} \tag{2.14}$$

Then we wish to find the roots of $A\,s^2 + B\,s + C = 0$. (Notice that, if $\alpha = 1$, so that the ellipse is the unit circle, then we get simply $s^2 + t^2 - 1 = 0$, whence, $s = \pm\sqrt{1 - t^2}$. This agrees with our earlier example of a circular region.)

Since $f_{\mathcal{E}_0}$ is the characteristic function of $\mathcal{E}_0$, the value of $\mathcal{R}(f_{\mathcal{E}_0})(t, \theta)$ is equal to the difference between the two roots of this quadratic, in the case where those roots are real numbers, and 0 otherwise. With some persistence, we can express the discriminant of the quadratic as

$$4\alpha^2\left(\sin^2(\theta) + \alpha^2\cos^2(\theta) - t^2\right)\,.$$

Thus, we get that

$$\mathcal{R}(f_{\mathcal{E}_0})(t,\theta) = \frac{2\alpha\sqrt{\sin^2(\theta) + \alpha^2\cos^2(\theta) - t^2}}{\sin^2(\theta) + \alpha^2\cos^2(\theta)}\,, \tag{2.15}$$

whenever $t^2 \leq \sin^2(\theta) + \alpha^2\cos^2(\theta)$, and $\mathcal{R}(f_{\mathcal{E}_0})(t,\theta) = 0$ otherwise. (Again, for $\alpha = 1$, the result is what we got before for the characteristic function of the unit disc.)

Next, consider a more general ellipse, with equation $x^2/a^2 + y^2/b^2 = 1$, and let $\mathcal{E}_1$ be the closed region bounded by this ellipse. Notice that the new region $\mathcal{E}_1$ is a re-scaling by the factor b of the region $\mathcal{E}_0$ bounded by the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$, where $\alpha = a/b$. That is, the point $(x,\ y)$ lies on the boundary ellipse for $\mathcal{E}_1$ if, and only if, the point $(x/b,\ y/b)$ lies on the boundary ellipse for $\mathcal{E}_0$. It follows that the characteristic function of the region $\mathcal{E}_1$, denoted by $f_{\mathcal{E}_1}$, satisfies the condition

$$f_{\mathcal{E}_1}(x,\ y) = f_{\mathcal{E}}(x/b,\ y/b).$$

Now apply the property (2.10), with $c = 1/b$, to the formula (2.15) above, with $\alpha = a/b$. This gives us the Radon transform of $f_{\mathcal{E}_1}$. Namely,

$$\mathcal{R}(f_{\mathcal{E}_1})(t,\theta) = \frac{2ab\sqrt{b^2\sin^2(\theta) + a^2\cos^2(\theta) - t^2}}{b^2\sin^2(\theta) + a^2\cos^2(\theta)}\,, \tag{2.16}$$

whenever $t^2 \leq b^2\sin^2(\theta) + a^2\cos^2(\theta)$, and $\mathcal{R}(f_{\mathcal{E}_1})(t,\theta) = 0$ otherwise.

At this point, we have computed the Radon transform of the characteristic function of any ellipse centered at the origin with its major and minor axes lying on the x- and y-coordinate axes. Our next step is to consider an ellipse that is centered at the origin but whose axes lie at an angle from the coordinate axes.

For this, let $\mathcal{E}_1$ denote, as above, the closed region bounded by the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, and let $\mathcal{E}_\phi$ denote the closed region obtained by rotating $\mathcal{E}_1$ counterclockwise about the origin by an angle ϕ, with $0 \leq \phi \leq \pi$. It follows that the point $(x,\ y)$ lies in the region $\mathcal{E}_\phi$ if, and only if, the point $(x\cos(\phi) + y\sin(\phi),\ -x\sin(\phi) + y\cos(\phi))$ lies in the region $\mathcal{E}_1$. Thus, the characteristic functions of the two regions are related by the equation

$$f_{\mathcal{E}_\phi}(x,\ y) = f_{\mathcal{E}_1}(x\cos(\phi) + y\sin(\phi),\ -x\sin(\phi) + y\cos(\phi))\,.$$

Moreover, for every pair of real numbers t and θ, the intersection of the line $\ell_{t,\theta}$ with the region $\mathcal{E}_\phi$ has the same length as that of the line $\ell_{t,\theta-\phi}$ with the region $\mathcal{E}_1$. Hence, as in Exercise 7 below, we see that

$$\mathcal{R}f_{\mathcal{E}_\phi}(t,\ \theta) = \mathcal{R}f_{\mathcal{E}_1}(t,\ \theta - \phi)\,, \text{for all } t,\ \theta \in \mathbb{R}.$$

In other words, to compute the Radon transform of $f_{\mathcal{E}_\phi}$, we substitute $\theta - \phi$ for θ in formula (2.16) above.

Finally, we can apply the shifting property (2.9) to obtain the formula for the Radon transform of an elliptical region whose center is located at (x_0, y_0), not necessarily at the origin. In this modification, we replace the value of t by $\hat{t} = t - x_0 \cos(\theta) - y_0 \sin(\theta)$.

To sum up these findings, let $\mathcal{E}$ be the closed region in the xy-plane bounded by the ellipse with center at the point (x_0, y_0) and semi-axes of lengths a and b making an angle of ϕ with the horizontal (x-axis) and vertical (y-axis), respectively. Next, for real numbers t and θ, let $\hat{\theta} = \theta - \phi$ and let $\hat{t} = t - x_0 \cos(\theta) - y_0 \sin(\theta)$. Then, the Radon transform of the characteristic function $f_{\mathcal{E}}$ is given by

$$\mathcal{R}(f_{\mathcal{E}})(t,\theta) = \frac{2ab\sqrt{b^2\sin^2(\hat{\theta}) + a^2\cos^2(\hat{\theta}) - \hat{t}^2}}{b^2\sin^2(\hat{\theta}) + a^2\cos^2(\hat{\theta})}, \tag{2.17}$$

whenever $\hat{t}^2 \leq b^2\sin^2(\hat{\theta}) + a^2\cos^2(\hat{\theta})$, and $\mathcal{R}(f_{\mathcal{E}})(t,\theta) = 0$ otherwise. Keep in mind that a computer will evaluate all of this for us!

Remark 2.12. The reader might protest that we could have started with the general formula (2.12) for an ellipse in the plane. After all, computing the Radon transform of the characteristic function of the region inside this ellipse just amounts to finding the distance between two roots of a quadratic equation. This would have led us to the formula (2.17) all at once. This is a fair objection, but it is still fun to solve a simpler quadratic first, and then use the properties of the Radon transform to generalize, isn't it?

Example with *R* 2.13. Figure 2.11 shows the sinogram of the phantom defined in Example 2.11. The sinogram was produced using *R*, following a process similar to that used for the image of the phantom itself. This time, we create a grid of values of t and θ corresponding to the lines $\ell_{t,\theta}$ in the "scan" of the phantom, like so.

```
## define values of t and theta for our X-rays
tau=0.02 #space betw/ x-rays
Nangle=180 #no. of angles
Nrays=(1+2/tau)*(Nangle) # total no. of X-rays
tval=seq(-1,1,tau)#for t betw/ -1, 1
thetaval=(pi/Nangle)*(0:(Nangle-1))
## now make a "t, theta" grid:
grid.t=rep(tval,length(thetaval))
grid.theta=double(Nrays)
for (i in 1:Nrays){
grid.theta[i]=thetaval[1+floor((i-1)/length(tval))]}
```

Then we define a procedure that applies formula (2.17) to each elliptical region in the phantom. The results are added together to yield the Radon transform of the full phantom at each point (t, θ). The values of the Radon transform are interpreted as color values in the picture. Finally, we apply this procedure to our particular phantom and plot the resulting sinogram.

```
##this procedure computes the Radon transform
#ellipse parameters stored as rows of matrix E
radon.proc=function(theta,t,E){
tmp=matrix(double(nrow(E)*length(theta)),nrow(E),length
   (theta))
for (i in 1:nrow(E)){
theta.new=theta-E[i,5]
t.new=(t-E[i,3]*cos(theta)-E[i,4]*sin(theta))/E[i,2]
v1=sin(theta.new)^2+(E[i,1]/E[i,2])^2*cos(theta.new)^2-t.
   new^2
v2=ifelse(sin(theta.new)^2
+(E[i,1]/E[i,2])^2*cos(theta.new)^2-t.new^2>0,1,0)
v3=sqrt(v1*v2)
v4=sin(theta.new)^2+(E[i,1]/E[i,2])^2*cos(theta.new)^2
tmp[i,]=E[i,1]*E[i,6]*(v3/v4)}
radvec=colSums(tmp)
list(radvec=radvec)}
##apply the procedure to the phantom P
rp7=radon.proc(grid.theta,grid.t,P)
### plot the sinogram
plot(grid.theta,grid.t,pch=15,col=gray(rp7$radvec))
```

2.5.3 Plotting a square region

We have seen that we can compute the Radon transform of an attenuation-coefficient function that is constant on the region inside an ellipse and zero outside the ellipse. For a region bounded by a polygon, it is a simple matter to determine the point of intersection of any given line $\ell_{t,\theta}$ with each segment of the polygon. By comparing these points, we can determine which segments of $\ell_{t,\theta}$ lie inside the polygonal region. This comparison is easier if the region is convex. Here, we will consider only the example of a square. The attenuation coefficient function under consideration will be a constant multiple of the characteristic function of the region inside the square.

A square in general position in the plane is defined by four parameters: the coordinates (x_0, y_0) of the center of the square, the side length w, and the angle of counterclockwise rotation ϕ from the horizontal. The region inside the square is the set $\{(x, y) \mid \max\{u(x, y), v(x, y)\} < w/2\}$, where

$$u(x, y) = |(x - x_0)\cos(\phi) + (y - y_0)\sin(\phi)| \text{ and}$$
$$v(x, y) = |-(x - x_0)\sin(\phi) + (y - y_0)\cos(\phi)|\ .$$

This region will be assigned a constant attenuation coefficient, which is interpreted as a color value in the image of the phantom.

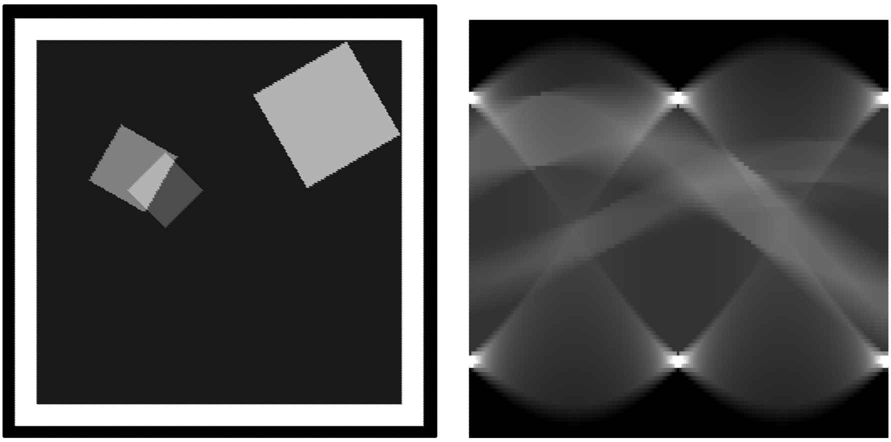

Fig. 2.12. A phantom of square regions; and its sinogram.

Example with *R* 2.14. To create a phantom, we select a set of squares, each determined by five parameters, as just described. In *R*, we can create a matrix in which each row contains the parameters of one square region. As with the ellipses, we examine each point in a grid of "pixels" and add up the attenuation coefficients of all square regions that contain that point. Here is an *R* script that generates the phantom shown in Figure 2.12.

```
##Phantom of squares
##define a grid of points in the square [-1,1]x[-1,1]
K=256 #larger K gives better resolution
yval=seq(-1,1,2/K)
grid.y=double((K+1)^2)
for (i in 1:(K+1)^2){
grid.y[i]=yval[floor((i-1)/(K+1))+1]}
xval=seq(-1,1,2/K)
grid.x=rep(xval,K+1)
##define a phantom using squares, each with 5 parameters
#center (x0, y0);side length w;rotation phi;density
#phantom is a matrix; one row per square
S1=c(0,0,1.9,0,1)
S2=c(0,0,1.7,0,-.9)
S3=c(.5,.5,.5,pi/6,.4)
S4=c(-.25,.15,.25,pi/4,.2)
S5=c(-.4,.25,.3,pi/3,.4)
S=matrix(c(S1,S2,S3,S4,S5),byrow=T,ncol=5)
##check each grid point to see if it is
##inside each square; #add the color values
#use the output "square.out" as the color vector
phantom.square=function(x,y,E){
phantom.1=matrix(double(nrow(E)*length(y)),nrow(E),
```

```
  length(y))
for (i in 1:nrow(E)){
u=abs((x-E[i,1])*cos(E[i,4])+(y-E[i,2])*sin(E[i,4]))
v=abs(-1*(x-E[i,1])*sin(E[i,4])+(y-E[i,2])*cos(E[i,4]))
phantom.1[i,]=E[i,5]*(u<.5*E[i,3])*(v<.5*E[i,3])}
square.out=colSums(phantom.1)
list(square.out=square.out)}
#apply to specific phantom
sq1=phantom.square(grid.x,grid.y,S)$square.out
## plotting the phantom
plot(grid.x,grid.y,pch=20,col=gray(sq1),xlab="",ylab="")
```

2.5.4 The Radon transform of a square region

Now we want to compute the Radon transform of the characteristic function of a square region. As with the ellipses, we'll consider a basic square first. Then we can apply the scaling, shifting, and rotational properties of the Radon transform to solve our problem for a general square.

As in Example 2.4, let $\mathcal{S}$ to be the square region whose edges lie along the lines $x = \pm 1$ and $y = \pm 1$, and let $f_{\mathcal{S}}$ denote the characteristic function of $\mathcal{S}$. For each t and each θ, we wish to compute the length of the intersection of the line $\ell_{t,\,\theta}$ with $\mathcal{S}$. As we observed in Example 2.4,

$$\mathcal{R}f_{\mathcal{S}}(t,\,0) = \mathcal{R}f_{\mathcal{S}}(t,\,\pi/2) = \begin{cases} 2 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| > 1. \end{cases}$$

More generally, $\mathcal{R}f_{\mathcal{S}}(t,\,\theta)$ will be piecewise linear in t, as illustrated in Figure 2.2. To generate the sinogram, it is helpful to think of $\mathcal{S}$ as the intersection of two infinite bands – the vertical band $V = \{(x,\,y) \,:\, |x| \leq 1\}$ and the horizontal band $H = \{(x,\,y) \,:\, |y| \leq 1\}$. Suppose θ satisfies $0 < \theta < \pi$ and $\theta \neq \pi/2$. In this case, each of the lines $\ell_{t,\,\theta}$ passes through both of the bands V and H. Indeed, using the standard parameterization of $\ell_{t,\,\theta}$, we can think of the parameter s as representing time as we travel along the line. Thus, the line crosses the edges of the band V at the times $s_1 = (t\cos(\theta) - 1)/\sin(\theta)$ and $s_2 = (t\cos(\theta) + 1)/\sin(\theta)$. (Note that $\sin(\theta) > 0$ due to our choice of θ.) The lesser of these two time values is the time when $\ell_{t,\,\theta}$ enters the band V, while the larger time value is the time when $\ell_{t,\,\theta}$ exits V. Let's denote the entry time by s_{V1} and the exit time by s_{V2}. Similarly, $\ell_{t,\theta}$ crosses the edges of the horizontal band H at the times $s_3 = (-t\sin(\theta)+1)/\cos(\theta)$ and $s_4 = (-t\sin(\theta)-1)/\cos(\theta)$. The entry and exit times are given by $s_{H1} = \min\{s_3,\,s_4\}$ and $s_{H2} = \max\{s_3,\,s_4\}$, respectively. Now, for the line $\ell_{t,\,\theta}$ to be inside $\mathcal{S}$, it must be inside both bands V and H simultaneously. For that to happen, both entry times must come before both exit times. (Otherwise, the line would leave one of the bands before it entered the other.) That is, the line $\ell_{t,\,\theta}$ will intersect the region $\mathcal{S}$ if, and only if, $\max\{s_{V1},\,s_{H1}\} \leq \min\{s_{V2},\,s_{H2}\}$. Then the length of the intersection is just the difference $\min\{s_{V2}, s_{H2}\} - \max\{s_{V1}, s_{H1}\}$. (This is because the standard parameterization is executed at unit speed.) It is straightforward for *R* to compute these values for each pair (t, θ) included in our CT scan of $\mathcal{S}$. The sinogram for the characteristic function of this basic square is shown in Figure 2.3.

For a square in general position, with center at (x_0, y_0), side length w, and rotation ϕ from the coordinate axes, we calculate entry and exit times as above, but we replace t with $(2/w)\cdot(t - x_0\cos(\theta) - y_0\sin(\theta))$ and θ with $\theta - \phi$. (The factor $(2/w)$ is the ratio of the side lengths of the basic square examined above and the new square.) Then we multiply the answer by $(w/2)$, to rescale to our new square. Finally, we multiply by the attenuation coefficient assigned to the region.

Example with *R* 2.15. To produce the sinogram shown in Figure 2.12, first create a grid of t and θ values, as in Example 2.13. Then we devise a procedure to compute the corresponding values of the Radon transform.

```
##Radon transform for a square region
## assume: values of t and theta are defined
## procedure to compute "entry" and "exit" times
radon.square=function(theta,t,E){
R1=matrix(double(length(theta)*nrow(E)),nrow(E),length
  (theta))
for (i in 1:nrow(E)){
theta.new=theta-E[i,4]
t.new=(t-E[i,1]*cos(theta)-E[i,2]*sin(theta))*(2/E[i,3])
v1=ifelse(theta.new==0,-1*(abs(t.new)<1),
(t.new*cos(theta.new)-1)/sin(theta.new))
v2=ifelse(theta.new==0,1*(abs(t.new)<1),
(t.new*cos(theta.new)+1)/sin(theta.new))
vvmax=ifelse(v1-v2>0,v1,v2)
vvmin=ifelse(v1-v2>0,v2,v1)
h1=ifelse(theta.new==pi/2,-1*(abs(t.new)<1),
(1-t.new*sin(theta.new))/cos(theta.new))
h2=ifelse(theta.new==pi/2,1*(abs(t.new)<1),
-1*(t.new*sin(theta.new)+1)/cos(theta.new))
hhmax=ifelse(h1-h2>0,h1,h2)
hhmin=ifelse(h1-h2>0,h2,h1)
entryval=ifelse(vvmin-hhmin>0,vvmin,hhmin)
exitval=ifelse(vvmax-hhmax>0,hhmax,vvmax)
R1[i,]=(0.5)*E[i,5]*E[i,3]*ifelse(exitval-entryval>0,
(exitval-entryval),0)}
radvec=colSums(R1)
radvec.sq=radvec/max(radvec)#normalizes color vector
list(radvec.sq=radvec.sq)}
## apply radon.square to phantom matrix S
rsq1=radon.square(grid.theta,grid.t,S)
```

```
#plot the sinogram
plot(grid.theta,grid.t,pch=20,
col=gray(rsq1$radvec.sq),xlab=expression(theta),ylab="t")
```

Using the ideas from Examples 2.11, 2.13, 2.14, and 2.15, create some interesting phantoms and sinograms of your own. Use your imagination and have some fun!

2.6 The domain of $\mathcal{R}$

As we can see from the definition (2.1), the Radon transform $\mathcal{R}f$ of a function f is defined provided that the integral of f along $\ell_{t,\theta}$ exists for every pair of values of t and θ. Each of these integrals is ostensibly an improper integral evaluated on an infinite interval. Thus, in general, the function f must be integrable along every such line, as discussed in greater detail in Appendix A.

In the context of medical imaging, the function f represents the density or attenuation-coefficient function of a slice of whatever material is being imaged. Thus, the function has *compact support*, meaning that there is some finite disc outside of which the function has the value 0. In this case, the improper integrals $\int_{\ell_{t,\theta}} f\,ds$ become regular integrals over finite intervals. The only requirement, then, for the existence of $\mathcal{R}f$ is that f be integrable over the finite disc on which it is supported. This will be the case, for instance, if f is piecewise continuous on the disc.

For a wealth of information about the Radon transform and its generalizations, as well as an extensive list of references on this topic, see the monograph [24] and the book [15]. A translation of Radon's original 1917 paper ([43]) into English is included in [15].

2.7 The attenuated Radon transform

The Radon transform is the foundation of our study of computerized tomography when the data we are analyzing come from X-rays that are transmitted externally to the patient or other subject of interest. There are several other forms of tomography, though, in which the data arise from signals that are *emitted from within* the patient. In these so-called *emission tomography* modalities, a radioactive isotope is injected into the patient. The isotope tends to concentrate at sites where a pathology may be present. Thus, we would like to determine the location and distribution of the isotope within the body. Two main variations of emission tomography are *single photon emission computerized tomography*, or SPECT, and *positron emission tomography*, or PET. (A joke: The patient enters the doctor's office and is taken to the examination room. While the patient is waiting, two siamese cats, a schnauzer, and a guinea pig come in, sniff about for a while, and leave. A few moments later, the doctor enters, states that the exam is now concluded, and hands the patient a hefty bill for services rendered. Aghast, the patient exclaims,"But you haven't even examined me!" The doctor replies, "Nonsense! You had both a CAT scan and a PET scan. What more do you want?")

In the case of SPECT, an isotope that emits individual photons, such as iodine–131, is used. When a photon hits the external detector, a device called a *collimator* determines the direction or line of the photon's path. PET uses isotopes that emit positrons, such as carbon–11. Each positron annihilates with a nearby electron to form two γ-rays of known energy and traveling in opposite directions. The simultaneous detection of these γ-rays at opposite detectors determines the line along which the emission took place. In both situations, the count of either photons or γ-rays recorded by each detector corresponds to the measurement of the concentration of the isotope along some portion of the line on which the detector lies. This amounts to knowing the value of the Radon transform along part of that line. The two readings for an opposing pair of positrons combine to give the value of the Radon transform for a full line.

The analysis is complicated by the fact that the medium through which the photons or γ-rays travel, such as the patient's brain perhaps, causes the emitted particles to lose energy. That is, the medium causes attenuation, just as it does for externally transmitted X-rays. The difference is that, with the CT scan based on X-rays, the attenuation coefficient of the medium at each point is the unknown function we wish to find. With emission tomography, it is the unknown location and concentration of the isotope that we wish to determine; but to do that we also have to consider the unknown attenuation coefficient of the medium. To be specific, suppose a photon is emitted at the point (x_0, y_0) and travels along the line $\ell_{t,\theta}$ until it hits the detector. Thus, the photon will pass through all points along $\ell_{t,\theta}$ between (x_0, y_0) and the detector. If (x_0, y_0) corresponds to the parameter value $s = s_0$ in the standard parameterization of $\ell_{t,\theta}$, then the photon will pass through all points of the form $(x, y) = (t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))$, for $s \geq s_0$. Denote by $\mu(x, y)$ the attenuation coefficient of the medium at the point (x, y). From Beer's law, it follows that the photon will encounter an attenuation of $\mathcal{A}_\mu(x_0, y_0, t, \theta) = \exp\left[-\int_{s\geq s_0} \mu(x,y)\,ds\right]$, where the integral is evaluated along the portion of $\ell_{t,\theta}$ corresponding to parameter values $s \geq s_0$. Letting $f(x, y)$ denote the (unknown) concentration of radioactive isotope at the point (x, y), we now make the following definition.

Definition 2.16. For a given function f, whose domain is the plane, and a given function μ, the *attenuated Radon transform* of f relative to μ is defined, for each pair of real numbers (t, θ), by

$$\mathcal{R}_\mu f(t, \theta) := \int_{\ell_{t,\theta}} \mathcal{A}_\mu(x, y, t, \theta) f(x,y)\, ds, \tag{2.18}$$

with $\mathcal{A}_\mu$ as just described. If the attenuation of the medium is negligible, so that $\mu(x, y) = 0$, then $\mathcal{A}_\mu = 1$ and $\mathcal{R}_\mu f = \mathcal{R}f$. In general, though, both functions f and μ are unknown. We will not consider these variations on the fundamental question of image reconstruction any further here. More information can be found in the books by Deans [15], Natterer [39], and Kuchment [32].

2.8 Exercises

1. The line $\ell_{1/2,\ \pi/6}$ has the standard parameterization

$$x = \frac{\sqrt{3}}{4} - \frac{s}{2} \text{ and } y = \frac{1}{4} + \frac{\sqrt{3}}{2}s\,, \text{ for } -\infty < s < \infty.$$

   (a) Find the values of s at which this line intersects the unit circle.

   (b) Now, define f by $f(x, y) = \begin{cases} x, & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{if } x^2 + y^2 > 1. \end{cases}$ Compute $\mathcal{R}f(1/2, \pi/6)$.

2. Evaluate the integral $\int_{s=-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \left(1 - \sqrt{t^2 + s^2}\right) ds$ from (2.5).

3. As in Example 2.4 above, consider the function

$$f(x, y) := \begin{cases} 1 & \text{if } |x| \leq 1 \text{ and } |y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (That is, f has the value 1 inside the square where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and the value 0 outside this square.)

   (a) Sketch the graph of the function $\mathcal{R}f(t, 0)$, the Radon transform of f corresponding to the angle $\theta = 0$. Then find a formula for $\mathcal{R}f(t, 0)$ as a function of t.

   (b) Sketch the graph of the function $\mathcal{R}f(t, \pi/4)$, the Radon transform of f corresponding to the angle $\theta = \pi/4$. Then find a formula for $\mathcal{R}f(t, \pi/4)$ as a function of t.

   (c) For θ with $0 \leq \theta \leq \pi/4$, find relationships, if any, between $\mathcal{R}f(t, \theta)$, $\mathcal{R}f(t, \pi/2-\theta)$, $\mathcal{R}f(t, \theta + \pi/2)$, and $\mathcal{R}f(t, \pi - \theta)$.

4. Show that, for all choices of t and θ and all suitable functions f, $\mathcal{R}f(t,\ \theta) = \mathcal{R}f(-t,\ \theta + \pi)$. (This symmetry is one reason that the graph of the Radon transform is called a *sinogram*.)

5. With f as in Example 2.7, find a single disc whose characteristic function g satisfies $\mathcal{R}g(0,\ \theta) = \mathcal{R}f(0,\ \theta)$, for all θ. Is $\mathcal{R}g(t,\ 0) = \mathcal{R}f(t,\ 0)$ for any values of t with $0 < t < 1$? (*Hint:*Look at the graphs of the Radon transforms.)

6. Provide a rigorous proof of Proposition 2.8 using the definition of the Radon transform as an integral.

7. ($\mathcal{R}$ and rotation.) For a function f defined in the plane and a real number ϕ, define a function g, for all real numbers x and y, by

$$g(x,\ y) = f(x\cos(\phi) + y\sin(\phi),\ -x\sin(\phi) + y\cos(\phi))\,.$$

Thus, the graph of g is a counterclockwise rotation by the angle ϕ of the graph of f. Prove that, for all real numbers t and θ,

$$\mathcal{R}g(t,\ \theta) = \mathcal{R}f(t,\ \theta - \phi).$$

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