Preface

The numerous attempts over the last 15–20 years to define a quantum Lie algebra as an elegant algebraic object with a binary "quantum" Lie bracket have not been evidently and widely accepted. Nevertheless, the q-deformations of the enveloping algebras introduced independently by Drinfeld and Jimbo have profoundly impacted the development of both the modern theory of quantum groups and the much older mathematical theory of Hopf algebras. Although the definition of the Drinfeld–Jimbo quantization is not simple, a clear common property unites all of these quantizations, as well as those that appeared later in different multiparameter versions articulated by Reshetikhin, Costantini, Varagnolo, Chin, Musson, and Benkart, with the universal enveloping algebras. Especially, these quantizations as Hopf algebras are generated by skew-primitive semi-invariants. This book is mainly concerned with Hopf algebras possessing this property. Because the action on a semi-invariant is defined by a character, we call such Hopf algebras *character Hopf algebras.*

We treat the character Hopf algebras as universal enveloping algebras of "quantum Lie algebras." The quantum Lie algebra must be an algebraic object located inside a character Hopf algebra. The Cartier–Kostant theorem asserts a category equivalence between Lie algebras (in characteristic zero) and connected co-commutative Hopf algebras. Given this equivalence, a Lie algebra corresponds to the space of primitive elements. This correspondence provides a clear idea to treat the space spanned by skew-primitive elements as a quantum Lie algebra.

To maintain the Cartier–Kostant category equivalence in characteristic $p > 0$, one must consider an additional unary operation $x \mapsto x^p$ on the Lie algebras. Thus, we must consider not only binary operations (brackets) but also operations involving one or various variables. In this manner, we develop the notion of *quantum Lie operation*, a polynomial in noncommutative skew-primitive variables with skew-primitive values. We thus consider the space spanned by the skew-primitive elements and equipped with the quantum Lie operations as a quantum analog of a Lie algebra.

There are many reasons motivating the extension of research to operations that replace the Lie bracket but that depend on greater numbers of variables, for example, operations of n-Lie algebras introduced by V.T. Filippov and then independently appearing under the name “Nambu–Lie algebras” in theoretical research on generalizations of Nambu mechanics.

Another group of problems requiring the generalization of Lie algebras corresponds to research on skew derivations of noncommutative algebras. A noncommutative version of the fundamental Dedekind algebraic independence lemma states that the algebraic structure of a Lie algebra and operators with “inner” action define all algebraic dependencies in ordinary derivations. This result was extended to the field of skew derivations by Chen-Lian Chuang. His fundamental theorem may be interpreted in the same manner, i.e., the algebraic structure and operators with “inner” action define all algebraic dependencies in skew derivations. Hence, the following question arises: Which algebraic structure corresponds to the skew derivation operators? This question requires the consideration of n-ary operations irreducible to bilinear operations.

A third group of problems concerning multivariable generalizations of the Lie bracket appeared in nonassociative algebra. P.O. Miheev and L.V. Sabinin demonstrated that a simply connected local analytic loop is determined by an algebraic system consisting of a series of multilinear operations. These systems are now called *Sabinin algebras*.

This book is intended as an introduction to the mathematics behind the phrase “quantum Lie algebra.” Despite the complexity of the subject, we have attempted to make this exposition accessible to a wide audience. We assume a standard knowledge of linear algebra and some rudimentary knowledge of representation theory. Most of the text will be accessible to graduate students in mathematics who have completed an introductory course in linear algebra.

Chapter 1 is introductory in nature. It contains many basic definitions related to noncommutative algebra that are used in subsequent chapters. Starting with Gauss polynomials and Lyndon–Shirshov standard words, we discuss the foundations of Gröbner–Shirshov theory, which is the basic tool for investigating noncommutative algebras specified by generators and defining relations. In this “combinatorial paradigm,” the Poincaré–Birkhoff–Witt theorem obeys an elegant proof, whereas the concepts of a skew group ring and crossed product can be perfectly analyzed. We then introduce the braid monoid and the permutation group and consider the set of shuffles as a transversal of a direct product of symmetric subgroups. Although representation theory is not used intensively in this book, we formulate the theorems of Maschke and Wedderburn as initial statements without proofs. The concept of a character Hopf algebra is central to this monograph. In the combinatorial paradigm, the free character Hopf algebra plays a crucial role. The notion of a combinatorial rank appears in the analysis of generators for Hopf ideals, which are the defining relations for Hopf algebras. We develop the bracket technique as an important tool for performing calculations that allows one to preserve and apply the intuition of the Lie algebra machinery. Coordinate differential calculi, filtered and associated graded spaces, and specific fundamental concepts from P.M. Cohn theory are developed as tools for further applications. We conclude the chapter with notes that provide the reader with an opportunity to learn more about the subjects we review in the introductory chapter. We have constructed this chapter to be as self-contained as possible. Some arguments are new, and the remaining chapters have not previously appeared in book form.

In the second chapter, we demonstrate that every character Hopf algebra has a PBW basis. Our proof intensively uses the coalgebraic structure, distinct from the known Lusztig's method, which uses the algebraic structure only. Because the coproduct may not differ between a polynomial with a zero value and a polynomial with a skew-primitive value, in establishing linear independence, we automatically obtain important information regarding the skew-primitive polynomials.

In the third chapter, we review possible quantum deformations of the universal enveloping algebras of Kac–Moody algebras. To this end, we associate a class $\mathfrak{A}$ with a given Kac–Moody algebra $\mathfrak{g}$. The class $\mathfrak{A}$ consists of all character Hopf algebras defined by the same number of relations and with the same degrees as $\mathfrak{g}$ has. $\mathfrak{A}$ contains all known quantizations of $\mathfrak{g}$. We demonstrate that Hopf algebras from $\mathfrak{A}$ have the so-called triangular decomposition as coalgebras. If the generalized Cartan matrix A of $\mathfrak{g}$ is indecomposable, then up to a finite number of exceptional cases, the algebraic structure is solely defined by one "continuous" parameter q related to the symmetrization of A and one "discrete" parameter $\mathfrak{m}$ related to the modular symmetrizations of A.

In the fourth chapter, consistent with the main concept of the book, we treat the skew-primitive polynomials as quantum Lie operations. We discuss linearization and specialization processes and criteria for a polynomial to be classified as a quantum Lie operation. We also classify multilinear quantum Lie operations in two, three, and four variables. Although generally a bilinear bracket there does not exists as an operation, a binary bracket exists that is an important and effective tool for the investigation. Specifically, all quantum Lie operations can be expressed in terms of that bracket. The bracket becomes a quantum operation only if characters that define the action of group-like elements satisfy a multiplicative skew-symmetry condition. In this case, the quantum Lie algebra transforms into a color Lie algebra.

The fifth chapter focuses on multilinear quantum Lie operations involving more than four variables. We establish a necessary and sufficient existence condition and the number of linearly independent operations that may exist and define the principle n-linear operation which by permutations of variables spans the space of all n-linear operations. The symmetric operations pose an opposite property, namely, in the context of permutations of variables, they do not change their values up to a scalar factor. We deduce that there are precisely $(n-2)!$ linearly independent symmetric generic quantum Lie operations and at least one principle generic n-linear operation. Although this chapter does not require specialized knowledge, it demands persistence from the reader.

The main goal of the sixth chapter is a detailed construction of free braided Hopf algebra and shuffle braided Hopf algebra on the tensor space of a given braided space. We define a Nichols algebra as a subalgebra of the shuffle braided Hopf algebra generated by the given braided space. All calculations are performed within the braid monoid but not in the braid group; therefore, the constructions remain valid for a noninvertible braiding. We then consider braided Hopf algebras that appear in the Radford decomposition of character Hopf algebras and discuss filtrations.

As previously mentioned, numerous definitions had been proposed for the binary quantum analog of a Lie algebra. It is likely that only the Gurevich–Manin generalization up to Lie τ-algebras represents a completely successful definition. In the seventh chapter we consider this generalization and its particular cases, specifically, Lie superalgebras and color Lie algebras. The PBW theorem for Lie τ-algebras transforms into a coalgebra isomorphism between universal enveloping algebras of Lie τ-algebras defined within the same braided space. We establish a τ-Friedrichs criterion and consider subalgebras of free Lie τ-algebra.

In the field of nonassociative algebras, there are known generalizations of Lie algebras with nonassociative envelopes. Many of these well-known generalizations involve only one or two operations. In the eighth chapter, we consider nonassociative primitive polynomials as operations for nonassociative Lie theory similar to how we considered skew-primitive polynomials as operations for quantum Lie theory. I.P. Shestakov and U.U. Umirbaev discovered infinitely many independent operations of that type. The proof constructed in this chapter demonstrates that Shestakov–Umirbaev primitive operations together with the commutator form a complete set of nonassociative Lie operations.

I am grateful to all who have offered suggestions or made corrections to the manuscript. I am pleased to express my thanks to Ivan Shestakov, Cristian Vay, Ualbai Umirbaev, Zbigniew Oziewicz, Robert Yamaleev, Mayra Lorena Díaz Sosa, David Tinoco Varela, José Luis Garza Rivera, Alma Virginia Lara Sagahón, Angélica Espinoza Godínez, Rodolfo Alvarado Cervantes, Alejandro Andrade Álvarez, and Ricardo Paramont Hernández García for valuable discussions and comments. Finally, I offer a special expression of thanks to my advisor, Leonid Bokut', who initiated me on the path toward understanding modern noncommutative algebra.

Acknowledgements The author was supported by PAPIIT, project IN 112913, and PIAPI of the FES-Cuautitlán, project VC06, UNAM, México.

Cuautitlán Izcalli, Mexico Vladislav Kharchenko
June 2014

http://www.springer.com/978-3-319-22703-0

Quantum Lie Theory
A Multilinear Approach
Kharchenko, V.
2015, XIII, 302 p., Softcover
ISBN: 978-3-319-22703-0