

Preface

Nonlinear mappings appear throughout mathematics, and their range of applications is immense, including the theory of differential equations, the theory of probability, the theory of dynamical systems, mathematical biology, and statistical physics. Most of the simplest nonlinear operators are quadratic. Even in a one-dimensional setting, the behavior of such operators reveals their complicated structure. If one considers multidimensional analogues of quadratic operators, then the situation becomes more complicated, i.e., the investigation of the dynamical behavior of such operators is very difficult.

The history of quadratic stochastic operators and their dynamics can be traced back to Bernstein's work [18]. The continuous time dynamics of this type of operator was considered by Lotka [134] and Volterra [252]. Quadratic stochastic operators are an important source of analysis in the study of dynamical properties and for modeling in various fields such as mathematical economics, evolutionary biology, population and disease dynamics, and the dynamics of economic and social systems.

Unfortunately, up to now, there have been no books devoted to the dynamics of quadratic stochastic operators. This omission in the literature gave us the motivation to write a systematic book about such operators.

The general objectives of this book are: (i) to give the first systematic presentation of both analytical and probabilistic techniques used in the study of the dynamics of quadratic stochastic operators and corresponding processes; (ii) to establish a connection between the dynamics of quadratic stochastic operators with the theory of Markov processes; and (iii) to give a systematic introduction to noncommutative or quantum analogues of quadratic stochastic operators and processes.

The book addresses the most fundamental questions in the theory of quadratic stochastic operators: dynamics, constructions, regularity, and the connection with stochastic processes. This connection means that the dynamics of such operators can be treated as certain Markov or quadratic processes. This interpretation allows us to use the methods of stochastic processes for a better understanding of the limiting behavior of the dynamics of quadratic operators.

Below we provide an overview of the main topics discussed in this book and explain why they have been selected.

The starting point of our book is to introduce a quadratic stochastic operator $V : S(X, F) \rightarrow S(X, F)$ defined on the set of all probability measures $S(X, F)$ on (X, F) and to present some motivations to study such operators. The next step is to define and study stochastic processes that are related to the quadratic stochastic operators in the same way as Markov processes are related to linear transformations. After this, it is natural to develop analytic methods for such processes. The last step is to generalize the theory of quadratic stochastic operators and processes to different algebraic structures, including von Neumann algebras. Such quadratic operators are called quantum quadratic stochastic operators (q.q.s.o.s). In this direction, we study the asymptotic properties of dynamical systems generated by q.q.s.o.s. Moreover, we also investigate Markov and quantum quadratic stochastic processes associated with q.q.s.o.s.

An essential feature of our exposition is the first systematic presentation of both the classical and quantum theory of quadratic stochastic operators and processes. We combine analytical and probabilistic tools to get a better insight into the dynamics of both classical and quantum quadratic operators. Moreover, we use several methods from the theory of noncommutative probability, matrix analysis, etc.

Now we discuss the structure of the book in more detail. The book is divided into eight chapters; at the end of each chapter, we give some comments and references related to the chapter.

The first chapter is an introduction where we collect some models, which can be described by quadratic stochastic operators.

Chapter 2 is devoted to quadratic stochastic operators (q.s.o.s) defined on a finite-dimensional simplex. In this chapter, we essentially deal with asymptotical stability (or regularity) condition for such operators. Moreover, we show how the dynamics of q.s.o.s are related to some Markov processes. Some relations between the regularity of a q.s.o. and the corresponding Markov process are investigated.

In Chap. 3, we introduce quadratic stochastic processes (q.s.p.s) and give examples of such processes. Note that these quadratic processes naturally arise in the study of certain models with interactions, where interactions are described by quadratic stochastic operators. Furthermore, this chapter contains a construction of nontrivial examples of q.s.p.s. Given a q.s.p., one can associate two kinds of processes, which are called *marginal processes*. One of them is a Markov process. We prove that marginal processes uniquely define q.s.p.s. The weak ergodicity of q.s.p.s is also studied in terms of the marginal processes.

In Chap. 4, we develop analytical methods for q.s.p.s. We follow the lines of Kolmogorov's [121] paper. Namely, we will derive partial differential equations with delaying argument, for quadratic processes of types A and B, respectively.

In the previous chapters, we are considering classical (i.e., commutative) quadratic operators. These operators are defined over commutative algebras. However, such operators do not cover the case of quantum systems. Therefore, in Chap. 5 we introduce a noncommutative analogue of a q.s.o., which is called a *quantum quadratic stochastic operator* (q.q.s.o.). We show that the set of q.q.s.o.s

is weakly compact. By means of q.q.s.o.s, one can define a nonlinear operator, which is called a *quadratic operator*. We also study the asymptotical stability of the dynamics of quadratic operators.

Chapter 6 is devoted to quantum quadratic stochastic operators (q.q.s.o.s) acting on the algebra of 2×2 matrices $M_2(\mathbb{C})$. Positive, trace-preserving maps arise naturally in quantum information theory (see, e.g., [199]) and in other situations where one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system which it interacts with. Therefore, we first describe quadratic operators with a Haar state (invariant with respect to the trace). Then q.q.s.o.s with the Kadison–Schwarz property are characterized. By means of such a description, we provide an example of a positive q.q.s.o., which is not a Kadison–Schwarz operator. On the other hand, this characterization is related to a separability condition, which plays an important role in quantum information [17]. We also examine the stability of the dynamics of quadratic operators associated with q.q.s.o.s given on $M_2(\mathbb{C})$.

In Chap. 7, we investigate a class of q.q.s.o.s defined on the commutative algebra ℓ^∞ . We define the notion of a Volterra quadratic operator and study its properties. It is proved that such operators have infinitely many fixed points and the set of Volterra operators forms a convex compact set. In addition, its extreme points are described. Furthermore, we study certain limit behaviors of such operators and give some more examples of Volterra operators for which their trajectories do not converge. Finally, we define a compatible sequence of finite-dimensional Volterra operators and prove that any power of this sequence converges in the weak topology. Note that in the finite-dimensional setting such operators have been studied by many authors (see, for example, [74, 252]).

In Chap. 8, we define a quantum (noncommutative) analogue of quadratic stochastic processes. In our case, such a process is defined on a von Neumann algebra. In this chapter, we essentially study the ergodic principle for these processes. From a physical point of view, this principle means that for sufficiently large values of time a system described by the process does not depend on the initial state of the system.

This book is not intended to contain a complete discussion of the theory of quadratic operators, but primarily relates to the asymptotic stability of such operators and associated processes. Moreover, it reflects the interests of the authors in key aspects of this theory. There are many omitted topics that naturally fit into the purview of quadratic operators. However, we have tried to collect the existing references on quadratic stochastic operators. Some of these are discussed in the separate sections entitled “Comments and References.”

This book is suitable as a textbook for an advanced undergraduate/graduate level course or summer school in quantum dynamical systems. It can also be used as a reference book by researchers looking for interesting problems to work on, or useful techniques and discussions of particular problems. It also includes the latest developments in the fields of quadratic dynamical systems, Markov processes, and

quantum stochastic processes. Researchers at all levels are likely to find the book inspiring and useful.

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