Chapter 4
Analytic Methods in the Theory of Quadratic Stochastic Processes

In this chapter we are going to develop analytical methods for q.s.p.s. We will follow the lines of Kolmogorov's [121] paper. Namely, we will derive partial differential equations with delaying argument for quadratic processes of type (A) and (B), respectively.

4.1 Quadratic Processes with a Finite Set of States

In this chapter, we will assume that quadratic stochastic processes are homogeneous per unit time, i.e. $P_{ij,k} = P_{ij,k}^{[t,t+1]}$ for all $t \geq 1$.

We are going to consider two cases with respect to the type of the q.s.p.

First we consider quadratic processes of type (A). Let $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ be a q.s.p. of type (A). Suppose that the functions $P_{ij,k}^{[s,t]}$ are continuous with respect to the variables s and t and the functions $P_{ij,k}^{[s,t]}$ are differentiable with respect to s and t with $t > s+1$. Then for $t > s + 2$, using (3.5), we have

$$\begin{aligned} P_{ij,k}^{[s,t+h]} - P_{ij,k}^{[s,t]} &= \sum_{m,l} P_{ij,m}^{[s,t-1]} P_{ml,k}^{[t-1,t+h]} x_l^{(t-1)} - \sum_{m,l} P_{ij,m}^{[s,t-1]} P_{ml,k}^{[t-1,t]} x_l^{(t-1)} \\ &= \sum_{m,l} P_{ij,m}^{[s,t-1]} \big(P_{ml,k}^{[t-1,t+h]} - P_{ml,k}^{[t-1,t]}\big) x_l^{(t-1)}. \end{aligned} \tag{4.1}$$

Assume

$$a_{ml,k}(t) = \lim_{h \to 0+} \frac{P_{ml,k}^{[t-1,t+h]} - P_{ml,k}}{h}, \tag{4.2}$$

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provided that the limit exists. Dividing both sides of equality (4.1) by h and passing to the limit as $h \to 0$, we get the first system of differential equations

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l} a_{ml,k}(t) x_l^{(t-1)} P_{ij,m}^{[s,t-1]}, i, j, k = 1, \cdots, n. \tag{4.3}$$

By (3.5) we rewrite the equation (4.3) as follows

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q} a_{ml,k}(t) x_r^{(0)} x_q^{(0)} P_{rq,l}^{[0,t-1]} P_{ij,m}^{[s,t-1]}. \tag{4.4}$$

Similarly for $t > s + 2$ one gets

$$\begin{aligned} & P_{ij,k}^{[s,t]} - P_{ij,k}^{[s+h,t]} \\ & \quad = \sum_{m,l} P_{ij,m}^{[s,s+1+h]} P_{ml,k}^{[s+1+h,t]} x_l^{(s+1+h)} - \sum_{m,l} P_{ij,m}^{[s+h,s+1+h]} P_{ml,k}^{[s+1+h,t]} x_l^{(s+1+h)} \\ & \quad = \sum_{m,l} \left(P_{ij,m}^{[s,s+1+h]} - P_{ij,m}^{[s,s+1]}\right) x_l^{(s+1+h)} P_{ml,k}^{[s+1+h,t]} x_l^{(t-1)}. \end{aligned}$$

Here we have used the equality $P_{ij,m}^{[s+h,s+1+h]} = P_{ij,m}^{[s,s+1]}$. Dividing both sides of this equality by h and passing to the limit as $h \to 0$, one finds the second system of partial differential equations

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = -\sum_{m,l} a_{ij,m}(s+1) x_l^{(s+1)} P_{ml,k}^{[s+1,t]}, i, j, k = 1, \cdots, n. \tag{4.5}$$

Again due to (3.5) the Eq. (4.5) can be rewritten as follows

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = -\sum_{m,l,r,q} a_{ij,m}(s+1) x_r^{(0)} x_q^{(0)} P_{rq,l}^{[0,s+1]} P_{ml,k}^{[s+1,t]}. \tag{4.6}$$

Now let us derive a differential equation for $x_k^{(t)}$. Since for $t > 2$

$$x_k^{(t+h)} = \sum_{i,j}^{m} P_{ij,k}^{[t-1,t+h]} x_i^{(t-1)} x_j^{(t-1)}$$

and

$$x_k^{(t)} = \sum_{i,j}^{m} P_{ij,k}^{[t-1,t]} x_i^{(t-1)} x_j^{(t-1)},$$

then

$$x_k^{(t+h)} - x_k^{(t)} = \sum_{i,j=1}^{m} (P_{ij,k}^{[t-1,t+h]} - P_{ij,k}^{[t-1,t]}) x_i^{(t-1)} x_j^{(t-1)},$$

and dividing both sides of this equality by h and passing to the limit as $h \to 0$, we obtain the following system of differential equations

$$\dot{x}_k^{(t)} = \sum_{i,j}^{m} a_{ij,k}(t) x_i^{(t-1)} x_j^{(t-1)}, \quad k = 1, \cdots, n. \tag{4.7}$$

So, we have proved the following theorem.

Theorem 4.1.1 *Let $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ be a q.s.p. of type (A). Then it satisfies the partial differential equations (4.4) and (4.6).*

Now assume that $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ is a q.s.p. of type (B), and

$$\tilde{a}_{ml,k}(t) = \lim_{h \to 0+} \frac{\tilde{P}_{ml,k}^{[t-1,t+h]} - \tilde{P}_{ml,k}}{h}, \tag{4.8}$$

provided that the limit exists. For $t > s + 2$, due to (3.7), one has

$$\begin{aligned}
\tilde{P}_{ij,k}^{[s,t+h]} - \tilde{P}_{ij,k}^{[s,t]} &= \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,t-1]} \tilde{P}_{ir,q}^{[s,t-1]} \tilde{P}_{lq,k}^{[t-1,t+h]} x_m^{(s)} x_r^{(s)} \\
&\quad - \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,t-1]} \tilde{P}_{ir,q}^{[s,t-1]} \tilde{P}_{lq,k}^{[t-1,t]} x_m^{(s)} x_r^{(s)} \\
&= \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,t-1]} \tilde{P}_{ir,q}^{[s,t-1]} \big(\tilde{P}_{lq,k}^{[t-1,t+h]} - \tilde{P}_{lq,k}^{[t-1,t]}\big) x_m^{(s)} x_r^{(s)}.
\end{aligned}$$

Then dividing both sides of this equality by h and passing to the limit as $h \to 0$, we find the first system of differential equations

$$\frac{\partial \tilde{P}_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q} \tilde{a}_{lq,k}(t) x_m^{(s)} x_r^{(s)} \tilde{P}_{im,l}^{[s,t-1]} \tilde{P}_{jr,q}^{[s,t-1]}, i, j, k = 1, \cdots, n. \tag{4.9}$$

For $t > s + 2$, from

$$\tilde{P}_{ij,k}^{[s,t]} = \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,s+1+h]} \tilde{P}_{ir,q}^{[s,s+1+h]} \tilde{P}_{lq,k}^{[s+1+h,t]} x_m^{(s)} x_r^{(s)}$$

$$\tilde{P}_{ij,k}^{[s+h,t]} = \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,s+1+h]} \tilde{P}_{ir,q}^{[s,s+1+h]} \tilde{P}_{lq,k}^{[s+1+h,t]} x_m^{(s+h)} x_r^{(s+h)}$$

one gets

$$\tilde{P}_{ij,k}^{[s,t]} - \tilde{P}_{ij,k}^{[s+h,t]} = \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,s+1+h]} x_r^{(s)} \big(\tilde{P}_{jr,q}^{[s,s+1+h]} x_m^{(s)} - \tilde{P}_{jr,q} x_m^{(s)}$$
$$+\tilde{P}_{jr,q} x_m^{(s)} - \tilde{P}_{jr,q} x_m^{(s+h)}\big) \tilde{P}_{lq,r}^{[s+1+h,t]}$$
$$+ \sum_{m,l,r,q} \tilde{P}_{jr,q} x_m^{(s+h)} \big(\tilde{P}_{im,l}^{[s,s+1+h]} x_r^{(s)} - \tilde{P}_{im,l} x_r^{(s)}$$
$$+\tilde{P}_{im,l} x_r^{(s)} - \tilde{P}_{im,l} x_r^{(s+h)}\big) \tilde{P}_{lq,r}^{[s+1+h,t]}.$$

Simplifying the last one we find

$$\tilde{P}_{ij,k}^{[s,t]} - \tilde{P}_{ij,k}^{[s+h,t]} = \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,s+1+h]} x_m^{(s)} x_r^{(s)} \big(\tilde{P}_{jr,q}^{[s,s+1+h]} - \tilde{P}_{jr,q}\big) \tilde{P}_{lq,r}^{[s+1+h,t]}$$
$$+ \sum_{m,l,r,q} \tilde{P}_{im,l}^{[s,s+1+h]} \tilde{P}_{jr,q} x_r^{(s)} \big(x_m^{(s)} - x_m^{(s+h)}\big) \tilde{P}_{lq,r}^{[s+1+h,t]}$$
$$+ \sum_{m,l,r,q} \tilde{P}_{jr,q} x_m^{(s+h)} x_r^{(s)} \big(\tilde{P}_{im,l}^{[s,s+1+h]} - \tilde{P}_{im,l}\big) \tilde{P}_{lq,r}^{[s+1+h,t]}$$
$$+ \sum_{m,l,r,q} \tilde{P}_{jr,q} \tilde{P}_{im,l} x_m^{(s+h)} \big(x_r^{(s)} - x_r^{(s+h)}\big) \tilde{P}_{lq,r}^{[s+1+h,t]},$$

where dividing both sides of this equality by h and passing to the limit as $h \to 0$, we obtain the second system of partial differential equations

$$\frac{\partial \tilde{P}_{ij,k}^{[s,t]}}{\partial s} = \sum_{m,l,r,q} \big(\tilde{P}_{im,l} \tilde{P}_{jr,q} (x_m^{(s)} x_r^{(s)})'$$
$$- \tilde{P}_{im,l} \tilde{a}_{jr,q}(s+1) + \tilde{P}_{jr,q} \tilde{a}_{im,l}(s+1)) x_m^{(s)} x_r^{(s)}\big) \tilde{P}_{lq,k}^{[s+1+h,t]}, \tag{4.10}$$

where $(x_m^{(s)} x_r^{(s)})' = \frac{d(x_m^{(s)} x_r^{(s)})}{ds}$ and $i, j, k = 1, \cdots, n$.

Theorem 4.1.2 *Let $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ be a q.s.p. of type (B). Then it satisfies the partial differential equations* (4.9) *and* (4.10)*.*

Note that all the derived differential equations are equations with delaying argument [40].

Example 4.1.1 Let $E = \{1, 2\}$, $x^{(0)} = (x, 1 - x)$, be an initial distribution on E and $a_{11,1} = (x - 1)\ln 2$, $a_{12,1} = [(2x - 1)/2]\ln 2$, $a_{22,1} = x \ln 2$. Since $a_{ij,2} = -a_{ij,1}$, and due to (4.7), we obtain the following system of differential equations:

$$\dot{x}_1^{(t)} = (x - x_1^{(t-1)}) \ln 2,$$
$$\dot{x}_2^{(t)} = -\dot{x}_1^{t}. \tag{4.11}$$

From $x_1^{(0)} = x$ we have that $\dot{x}_1^{(t)}|_{t=1} = 0$, i.e., $x_1^{(1)} = x$, therefore $x_1^{(t)} = x$, $x_2^{(t)} = 1 - x$. In this case systems (4.3) and (4.5) have the following form:

$$\frac{\partial P_{ij,1}^{[s,t]}}{\partial t} = \ln\sqrt{2}\big(x - P_{ij,1}^{[s,t-1]}\big),$$
$$\frac{\partial P_{ij,2}^{[s,t]}}{\partial t} = \ln\sqrt{2}\big(1 - x - P_{ij,2}^{[s,t-1]}\big), \tag{4.12}$$

and

$$\frac{\partial P_{11,k}^{[s,t]}}{\partial s} = -(x-1)\ln 2\big[x \cdot P_{11,k}^{[s+1,t]} + (1-2x)P_{12,k}^{[s+1,t]} + (1-x)P_{22,k}^{[s+1,t]}\big],$$
$$\frac{\partial P_{12,k}^{[s,t]}}{\partial s} = -\frac{(2x-1)}{2}\ln 2\big[x \cdot P_{11,k}^{[s+1,t]} + (1-2x)P_{12,k}^{[s+1,t]} + (1-x)P_{22,k}^{[s+1,t]}\big],$$
$$\frac{\partial P_{22,k}^{[s,t]}}{\partial s} = -x\ln 2\big[x \cdot P_{11,k}^{[s+1,t]} + (1-2x)P_{12,k}^{[s+1,t]} + (1-x)P_{22,k}^{[s+1,t]}\big]. \tag{4.13}$$

The process defined in Chap. 3, Sect. 3.2 (see Example 4.2), with $\varepsilon = 0$, is a solution of the systems (4.12) and (4.13).

Example 4.1.2 Let $E = \{1, 2\}$, $x^{(0)} = (x, 1-x)$, be an initial distribution and $a_{11,1}(t) = \ln 2 \ln x \cdot 2^{t+1}$, $a_{12,1} = a_{21,1} = a_{22,1} = 0$. Since $a_{ij,2} = -a_{ij,1}$, for the distribution $x^{(t)}$ we obtain the following system of equations:

$$\dot{x}_1^{(t)} = \ln 2 \ln x \cdot 2^{t+1}\big(x_1^{(t-1)}\big)^2,$$
$$\dot{x}_2^{(t)} = -\ln 2 \ln x \cdot 2^{t+1}\big(1 - x_2^{(t-1)}\big)^2.$$

It has the following solution

$$x_1^{(t)} = x^{2^t}, \quad x_2^{(t)} = 1 - x^{2^t}$$

for $x_1^{(0)} = x$ and $x_2^{(0)} = 1 - x$. In this case the systems (4.3) and (4.5) have the following form:

$$\frac{\partial P_{ij,1}^{[s,t]}}{\partial t} = \ln 2 \ln x x^{2^{t-1}} P_{ij,1}^{[s,t-1]},$$
$$\frac{\partial P_{ij,2}^{[s,t]}}{\partial t} = -\ln 2 \ln x x^{2^{t-1}} \left(1 - P_{ij,2}^{[s,t-1]}\right), \tag{4.14}$$

and

$$\frac{\partial P_{11,k}^{[s,t]}}{\partial s} = -\ln 2 \ln x 2^{s+1} \Bigg(x^{2^{s+1}} \left(P_{11,k}^{[s+1,t]} - 2P_{12,k}^{[s+1,t]} + P_{22,k}^{[s+1,t]}\right),$$
$$-P_{12,k}^{[s+1,t]} - P_{22,k}^{[s+1,t]} \Bigg),$$
$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = 0 \qquad \text{in all other cases.} \tag{4.15}$$

The process defined in Chap. 3, Sect. 3.2 (see Example 3.2.4) is a solution of the systems (4.14) and (4.15).

4.2 Quadratic Processes with a Continuous Set of States

First let us give an example of a quadratic process with continuous set E.

Example 4.2.1 Let $(E, \Im)$ be a measurable space and m_0 be an initial measure on $(E, \Im)$. A transition function

$$P(s, x, y, t, A) = \frac{1}{2^{t-s-1}} \left(\frac{\delta_x(A) + \delta_y(A)}{2} + (2^{t-s-1} - 1) m_0(A) \right),$$

defined at $t - s \geq 1$ for $x, y \in E$ and $A \in \Im$, satisfies all conditions I, II, II, IV_A, IV_B. Moreover, it determines a homogeneous quadratic stochastic process of both types A and B.

Now let us produce differential equations for q.s.p.s with continuous E. As before, we first consider processes of type (A). Let $\{(E, \Im), P(s, x, y, t, A), m_0\}$ be a q.s.p. of type (A). As shown above, according to condition IV_A) we have for $t > s+2$

$$P(s, x, y, t + h, A) - P(s, x, y, t, A)$$
$$= \int_E \int_E P(s, x, y, t - 1, du)[P(t - 1, u, v, t + h, A) - P(t - 1, u, v, t, A)] m_{t-1}(dv).$$

Let

$$c(t, u, v, A) = \lim_{h \to 0+} \frac{P(t-1, u, v, t+h, A) - P(0, u, v, 1, A)}{h}$$

provided the limit exists. Then from the previous equality we get the first integro-differential equation

$$\frac{\partial P(s, x, y, t, A)}{\partial t} = \int_E \int_E P(s, x, y, t-1, du) c(t-1, u, v, A) m_{t-1}(dv). \tag{4.16}$$

Similarly as in (4.6) and (4.7) one can produce the following second integro-differential equation

$$\frac{\partial P(s, x, y, t, A)}{\partial s} = -\int_E \int_E c(s+1, x, y, du) P(s+1, u, v, t, A) m_{s+1}(dv). \tag{4.17}$$

Using the same argument, for a q.s.p. $\{(E, \Im), \tilde{P}(s, x, y, t, A), m_0\}$ of type (B), we obtain similar kinds of equations

$$\begin{aligned} \frac{\partial \tilde{P}(s, x, y, t, A)}{\partial t} = & \int_E \int_E \int_E \int_E \tilde{c}(t, u, w, A) \tilde{P}(s, x, z, t-1, du) \\ & \times \tilde{P}(s, y, v, t-1, dw) m_s(dz) m_s(dv) \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} \frac{\partial \tilde{P}(s, x, y, t, A)}{\partial s} = & -\int_E \int_E \tilde{P}(x, u, dv) \tilde{P}(y, w, dz) (m_s(du) m_s(dw))' \\ & - (\tilde{c}(s+1, y, w, dz) \tilde{P}(x, u, dv) + \tilde{c}(s+1, x, u, dv) \tilde{P}(y, w, dz)) \\ & \times \tilde{P}(s+1, v, z, t, A) m_s(dv) m_s(dz), \end{aligned} \tag{4.19}$$

where $\tilde{c}$ is defined similarly as c for $\tilde{P}(s, x, y, t, A)$, and $\tilde{P}(x, y, A) \equiv \tilde{P}(0, x, y, 1, A)$.

Let us consider the following case: $E = \mathbb{R}$ and $A_z = (-\infty, z], z \in \mathbb{R}$. Assume that $F(s, x, y, t, z) = P(s, x, y, t, A_z)$ and $\tilde{F}(s, x, y, t, z) = \tilde{P}(s, x, y, t, z)$. It is evident that F and $\tilde{F}$ are the distribution functions for q.s.p.s of type (A) and type (B), respectively. If the functions F and $\tilde{F}$ are absolutely continuous with respect to the variable z, then there are nonnegative functions f and $\tilde{f}$ such that

$$F(s, x, y, t, z) = \int_z^\infty f(s, x, y, t, u) du$$

and

$$\tilde{F}(s,x,y,t,z) = \int_z^{\infty} \tilde{f}(s,x,y,t,u)du,$$

where du is the usual Lebesgue measure on $\mathbb{R}$.

Then one can rewrite the fundamental equations IV_A and IV_B, and integro-differential equations (4.16)–(4.19) with respect to the density functions f and $\tilde{f}$, respectively. Namely, the equations IV_A and IV_B are reduced to

IV'_A

$$f(s,x,y,t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,x,y,\tau,u)f(\tau,u,v,t,z)dum_{\tau}(dv),$$

IV'_B

$$\tilde{f}(s,x,y,t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(s,x,u,\tau,v)\tilde{f}(s,y,w,\tau,h)$$
$$\times\tilde{f}(\tau,v,h,t,z)dvdhm_s(du)m_s(dw),$$

respectively. Similarly the Eqs. (4.16)–(4.19) can be reduced to

$$\frac{\partial f(s,x,y,t,z)}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t,u,v,z)f(s,x,y,t-1,u)dum_{t-1}(dv), \tag{4.20}$$

$$\frac{\partial f(s,x,y,t,z)}{\partial s} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(s+1,x,y,u)f(s+1,u,v,t,z)dum_{s+1}(dv), \tag{4.21}$$

$$\frac{\partial \tilde{f}(s,x,y,t,z)}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{a}(t,v,h,z)\tilde{f}(s,x,u,t-1,v)$$
$$\times\tilde{f}(s,y,w,t-1,h)dvdhm_s(du)m_s(dw), \tag{4.22}$$

$$\frac{\partial \tilde{f}(s,x,y,t,A)}{\partial s} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(x,u,v)\tilde{f}(y,w,h)(m_s(du)m_s(dw))'\right.$$
$$-(\tilde{a}(s+1,y,w,h)\tilde{f}(x,u,v)$$
$$\left.+\tilde{a}(s+1,x,u,v)\tilde{f}(y,w,h))m_s(du)m_s(dw)\right)$$
$$\times\tilde{f}(s+1,v,h,t,z)dvdh, \tag{4.23}$$

where $f(x, y, z) \equiv f(0, x, y, 1, z), \tilde{f}(x, y, z) \equiv \tilde{f}(0, x, y, 1, z)$ and

$$a(t, x, y, z) = \lim_{h \to 0} \frac{f(t-1, x, y, t+h, z) - f(x, y, z)}{h},$$

$$\tilde{a}(t, x, y, z) = \lim_{h \to 0} \frac{\tilde{f}(t-1, x, y, t+h, z) - \tilde{f}(x, y, z)}{h},$$

respectively.

If $m_\tau(dv) = r_\tau(v)dv$, then the fundamental equations IV'_A and IV'_B can be rewritten as follows

$I\tilde{V}_A$

$$f(s, x, y, t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, x, y, \tau, u) f(\tau, u, v, t, z) r_\tau(v) du dv,$$

$I\tilde{V}_B$

$$\tilde{f}(s, x, y, t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(s, x, u, \tau, v) \tilde{f}(s, y, w, \tau, h)$$
$$\times \tilde{f}(\tau, v, h, t, z) r_s(u) r_s(w) du dw dh dv,$$

respectively.

Example 4.2.2 Let us consider a family of functions

$$f(s, x, y, t, z) = \frac{\exp(-\frac{(z-x-y)^2}{2^{t+1}-2^{s+2}+1})}{\sqrt{(2^{t+1} - 2^{s+2} + 1)\pi}}$$

with

$$r_t(v) = \frac{\exp(-\frac{v^2}{2^{t+1}-1})}{\sqrt{(2^{t+1} - 1)\pi}}.$$

One can establish that such a family determines a quadratic stochastic process of type (A).

Example 4.2.3 A family of functions

$$f(s, x, y, t, z) = \frac{\exp(-\frac{(z-x)^2}{t-s}) + \exp(-\frac{(z-y)^2}{t-s})}{2^{t-s}\sqrt{(t-s)\pi}} + \frac{2^{t-s-1} - 1}{2^{t-s-1}} \cdot \frac{\exp(-\frac{z^2}{t+1})}{\sqrt{(t+1)\pi}}$$

with

$$r_t(v) = \frac{1}{\sqrt{(t+1)\pi}} \exp(-\frac{v^2}{t+1})$$

determines a quadratic stochastic process of type (A).

Example 4.2.4 A family of functions

$$f(s,x,y,t,z) = \frac{2^{t-s-1} \exp(-\frac{4^{t-s-1}}{2^{2(t-s)-1}-1}) \cdot (z - \frac{x+y}{2^{t-s}})^2}{\sqrt{(2^{2(t-s)-1}-1)\pi}}$$

with

$$r_t(v) = \frac{\exp(-\frac{v^2}{2})}{\sqrt{2\pi}}$$

determines a quadratic stochastic process of both types A and B.

Under some conditions on the density functions f and $\tilde{f}$, one can reduce the derived integro-differential equations (4.20)–(4.23) to some partial differential equations. Indeed, let us consider (4.20). First define

$$\Delta(s,x,y,t,z,h) = \frac{f(s,x,y,t+h,z) - f(s,x,y,t,z)}{h}. \tag{4.24}$$

Then from $\mathrm{IV}_{A'}$ we get the following equality:

$$\Delta(s,x,y,t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,x,y,t-1,u)\Delta(t-1,u,v,t,z)m_{t-1}(dv)du. \tag{4.25}$$

Assume that the function $f(s,x,y,t-1,u)$ has partial derivatives up to third order with respect to the argument u, and consider its Taylor expansion in a neighborhood of the point z:

$$f(s,x,y,t-1,u) = f(s,x,y,t-1,z) + \frac{\partial f(s,x,y,t-1,z)}{\partial z}(u-z)$$
$$+ \frac{\partial^2 f(s,x,y,t-1,z)}{\partial z^2} \cdot \frac{(u-z)^2}{2} + \theta \frac{(u-z)^3}{6}$$

and substitute this expansion into (4.25). Then one finds

$$
\begin{aligned}
&\Delta(s,x,y,t,z)\\
&=f(s,x,y,t-1,z)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(t-1,u,v,t,z,h)m_{t-1}(dv)du\\
&\quad+\frac{\partial f(s,x,y,t-1,z)}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(t-1,u,v,t,z,h)(u-z)m_{t-1}(dv)du\\
&\quad+\frac{\partial^2 f(s,x,y,t-1,z)}{\partial z^2}\int_R\int_R\frac{\Delta(t-1,u,v,t,z,h)}{2}(u-z)^2m_{t-1}(dv)du\\
&\quad+\theta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta(t-1,u,v,t,z,h)}{6}(u-z)^3m_{t-1}(dv)du. \qquad (4.26)
\end{aligned}
$$

Assume that the following limits exist:

$$\lim_{h\to 0+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(t-1,u,v,t,z,h)m_{t-1}(dv)du=\bar{N}(t,z);$$

$$\lim_{h\to 0+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(t-1,u,v,t,z,h)(u-z)m_{t-1}(dv)du=\bar{A}(t,z);$$

$$\lim_{h\to 0+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Delta(t-1,u,v,t,z,h)}{2}(u-z)^2m_{t-1}(dv)du=\bar{B}^2(t,z);$$

$$\lim_{h\to 0+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(t-1,u,v,t,z,h)|u-z|^3m_{t-1}(dv)du=0.$$

Then passing to the limit in (4.26) when $h\to 0$, we obtain the following partial differential equation with delaying argument

$$
\begin{aligned}
\frac{\partial f(s,x,y,t,z)}{\partial t}&=\bar{N}(t,z)f(s,x,y,t-1,z)+\bar{A}(t,z)\frac{\partial f(s,x,y,t-1,z)}{\partial z}\\
&\quad+\bar{B}^2(t,z)\frac{\partial^2 f(s,x,y,t,z)}{\partial z^2}. \qquad (4.27)
\end{aligned}
$$

Now let us elaborate on the integro-differential equation (4.21). Define

$$\tilde{\Delta}(s,x,y,t,z,h)=f(s,x,y,t,z)-f(s+h,x,y,t,z). \qquad (4.28)$$

Then again using $\mathrm{IV}_{A'}$ one finds

$$\tilde{\Delta}(s,x,y,t,z,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\Delta}(s,x,y,s+1+h,u,h) \times f(s+1+h,u,v,t,z)m_{s+1+h}(dv)du. \tag{4.29}$$

Assuming that the function $f(s+1+h,u,v,t,z)$ has partial derivatives up to third order, we expand it into a Taylor series in a neighborhood of the point (x,y):

$$\begin{aligned}
&f(s+1+h,u,v,t,z)\\
&\quad = f(s+1+h,x,y,t,z) + \frac{\partial f(s+1+h,x,y,t,z)}{\partial x}(u-x)\\
&\qquad + \frac{\partial f(s+1+h,x,y,t,z)}{\partial y}(v-y)\\
&\qquad + \frac{1}{2}\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x^2}(u-x)^2\\
&\qquad + \frac{1}{2}\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial y^2}(v-y)^2\\
&\qquad + \frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x\partial y}(u-x)(v-y)\\
&\qquad + \frac{1}{6}\frac{\partial^3 f(s+1+h,x+\theta_3(u-x),y+\theta_3(v-y),t,z)}{\partial x^3}(u-x)^3\\
&\qquad - \frac{1}{2}\frac{\partial^3 f}{\partial x^2\partial y}(u-x)^2(v-y) + \frac{1}{2}\frac{\partial^3 f}{\partial x\partial y^2}(u-x)(v-y)^2\\
&\qquad + \frac{1}{6}\frac{\partial^3 f}{\partial y^3}(v-y)^3.
\end{aligned}$$

By substituting this expansion into (4.29) let us evaluate the integrals from each summand. Then one finds

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{\Delta}(s,x,y,s+1+h,u,h)f(s+1+h,x,y,t,z)m_{s+1+h}(dv)du = 0;$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{\Delta}(s,x,y,s+1+h,u,h)f(s+1+h,x,y,t,z)\frac{\partial f}{\partial x}(u-x)m_{s+1+h}(dv)du\\
&\quad = \frac{\partial f(s+1+h,x,y,t,z)}{\partial x}\cdot\int_{-\infty}^{\infty}\tilde{\Delta}(s,x,y,s+1+h,u,h)(u-x)du.
\end{aligned}$$

Let

$$a(s,x,y,h) = \int_{-\infty}^{\infty} \tilde{\Delta}(s,x,y,s+1+h,u,h)(u-x)du$$

and assume that

$$\int_{-\infty}^{\infty} \tilde{\Delta}(s,x,y,s+1+h,u,h)\frac{\partial f}{\partial y}(v-y)m_{s+1+h}(dv)du$$
$$= \frac{\partial f(s+1+h,x,y,t,z)}{\partial x}\int_{-\infty}^{\infty} \tilde{\Delta}(s,x,y,s+1+h,u,h)du\int_{-\infty}^{\infty}(v-y)m_{s+1+h}(dv)$$
$$= 0.$$

Now consider the second moments

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\Delta}(s,x,y,s+1+h,u,h)\frac{1}{2}\frac{\partial f(s+1+h,x,y,t,z)}{\partial x}(u-x)^2 m_{s+1+h}(dv)du$$
$$= \frac{1}{2}\cdot\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x^2}\int_{-\infty}^{\infty} \tilde{\Delta}(s,x,y,s+1+h,u,h)(u-x)^2 du.$$

Put

$$b^2(s,x,y,h) = \int_{-\infty}^{\infty}[f(s,x,y,s+1+h,u) - f(s+h,x,y,s+1+h,u)](u-x)^2du.$$

Furthermore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(s,x,y,s+1+h,u) - f(s+h,x,y,s+1+h,u)]$$
$$\times\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x\partial y}(u-x)(v-y)m_{s+1+h}(dv)du$$
$$= \frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x\partial y}\times\int_{-\infty}^{\infty}[f(s,x,y,s+1+h,u)$$
$$-f(s+h,x,y,s+1+h,u)](u-x)du\cdot\int_{-\infty}^{\infty}(v-y)m_{s+1+h}(dv).$$

Let

$$d(s+1,y,h) = \int_{-\infty}^{\infty}(v-y)m_{s+1+h}(dv).$$

It is evident that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(s,x,y,s+1+h,u)-f(s+h,x,y,s+1+h,u)]$$
$$\times\frac{1}{2}\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial y^2}(v-y)^2 m_{s+1+h}(dv)du=0.$$

Assume that the other integrals tend to zero when $h\to 0$. Then we get

$$\frac{f(s,x,y,t,z)-f(s+h,x,y,t,z)}{h}$$
$$=\frac{a(s,x,y,h)}{h}\cdot\frac{\partial f(s+1+h,x,y,t,z)}{\partial x}$$
$$+\frac{b^2(s,x,y,h)}{h}\cdot\frac{1}{2}\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x^2}$$
$$+\frac{a(s,x,y,h)}{h}\cdot d(s+1,y,h)\cdot\frac{\partial^2 f(s+1+h,x,y,t,z)}{\partial x\partial y}. \tag{4.30}$$

Letting

$$A(s,x,y)=\lim_{h\to 0}\frac{a(s,x,y,h)}{h},$$
$$B^2(s,x,y)=\lim_{h\to 0}\frac{b^2(s,x,y,h)}{h},$$
$$D(s+1,y)=\lim_{h\to 0}d(s+1,y,h),$$

provided that these limits exist, equality (4.30) is transformed into the following differential equation:

$$\frac{\partial f(s,x,y,t,z)}{\partial s}=-A(s,x,y)\frac{\partial f(s+1,x,y,t,z)}{\partial x}-B^2(s,x,y)\frac{\partial^2 f(s+1,x,y,t,z)}{\partial x^2}$$
$$-A(s,x,y)D(s+1,y)\frac{\partial^2 f(s+1,x,y,t,z)}{\partial x\partial y}. \tag{4.31}$$

Thus the integro-differential equations (4.20) and (4.21) with delaying argument (with respect to t and s respectively) have reduced to differential equations (4.27) and (4.31), respectively. The latter are also equations with delaying argument. In the next chapter, these equations will be reduced to well-known differential equations that are not equations with delaying argument. The integro-differential equations (4.22) and (4.23) will also be reduced to differential equations using another approach.

4.3 Averaging of Quadratic Stochastic Processes

In this chapter we are going to consider continuous analogues of Theorems 3.3.1 and 3.3.2. Namely, we consider relations between quadratic and Markovian processes.

Theorem 4.3.1 *Let* $\{(E, \Im), P(s, x, y, t, A), m_0\}$ *be a q.s.p. Then the function*

$$H(s, x, t, A) = \int_E P(s, x, y, t, A) m_s(dy) \tag{4.32}$$

is the transition function for some Markovian process with initial distribution m_0.

Proof We consider two cases with respect to the type of the q.s.p. Assume that the q.s.p. is of type (A). Then one gets

$$\begin{aligned}
H(s, x, t, A) &= \int_E P(s, x, y, t, A) m_s(dy) \\
&= \int_E \left(\int_E \int_E P(s, x, y, \tau, du) P(\tau, u, v, t, A) m_\tau(dv) m_s(dy) \right) \\
&= \int \left(\int_E P(s, x, y, \tau, du) m_s(dy) \right) \left(\int_E P(\tau, u, v, t, A) m_\tau(dv) \right) \\
&= \int_E H(s, x, \tau, du) H(\tau, u, t, A).
\end{aligned}$$

Similarly, for a type (B) process, we obtain

$$\begin{aligned}
\tilde{H}(s, x, t, A) &= \int_E \tilde{P}(s, x, y, t, A) m_s(dy) \\
&= \int_E \left(\int_E \int_E \int_E \int_E \tilde{P}(s, x, z, \tau, du) \tilde{P}(s, y, u, \tau, dw) \right. \\
&\qquad \left. \times \tilde{P}(\tau, u, w, t, A) m_s(dv) \right) m_s(dy)
\end{aligned}$$

$$= \int \int_E \int_E \tilde{P}(s,x,z,\tau,du) m_s(dz) \int_E \int_E \tilde{P}(s,y,v,\tau,dw)$$

$$\times \tilde{P}(\tau,u,w,t,A) m_s(dv)$$

$$= \int_E \int_E \tilde{P}(s,x,z,du) m_s(dz) \int_E \tilde{P}(\tau,u,w,t,A) m_\tau(dw)$$

$$= \int_E \tilde{H}(s,x,\tau,du) \tilde{H}(\tau,u,t,A).$$

Hence, the theorem is proved.

Remark 4.3.1 Note that a process generated by transition probabilities $\{H(s,x,t,A)\}$, in general, forms a non-homogenous Markov process. Thus, starting from a quadratic process one can construct a non-homogenous Markov process.

Remark 4.3.2 This theorem allows us to simplify the obtained system of differential and integro-differential equations. We demonstrate this below.

4.3.1 The Set E Is Finite

In this subsection we consider the case when E is a finite set. Let $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ and $(E, \tilde{P}_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ be q.s.p.s of types A and B, respectively. As before, we assume that q.s.p.s are homogeneous per unit time. Then the corresponding Markov processes are

$$H_{ij}^{[s,t]} = \sum_{k=1}^{n} P_{ik,j}^{[s,t]} x_k^{(s)}, \tag{4.33}$$

(respectively

$$\tilde{H}_{ij}^{[s,t]} = \sum_{k=1}^{n} \tilde{P}_{ik,j}^{[s,t]} x_k^{(s)}).$$

Remark 4.3.3 Note that with any q.s.p, one can connect a Markovian chain with the same initial distribution. It is necessary to note that although the function $P_{ij,k}^{[t,t+h]}$ is not defined for $0 \leq h < 1$, the quantity $H_{ij}^{[t,t+h]}$ is defined by means of a Markovian property but it cannot be represented in the form $\sum_{k=1}^{n} \tilde{P}_{ik,j}^{[t,t+h]} x_k^{(t)}$.

From Theorems 3.3.1 and 3.3.2 one has

$$P_{ij,k}^{[s,t]} = \sum_{m=1}^{n} P_{ij,m}^{[s,\tau]} H_{mk}^{[\tau,t]}, \tag{4.34}$$

(respectively

$$\tilde{P}_{ij,k}^{[s,t]} = \sum_{m,l=1}^{n} \tilde{H}_{im}^{[s,\tau]} \tilde{H}_{jl}^{[s,\tau]} \tilde{P}_{ml,k}^{[\tau,t]}), \tag{4.35}$$

where $\tau - s \geq 0$, and $t - \tau \geq 1$ and

$$x_k^{(\tau)} = \sum_{i=1}^{n} H_{ik}^{[0,\tau]} x_i^{(0)}. \tag{4.36}$$

Following [121] we assume

$$\lim_{h \to 0+} \frac{H_{ij}^{[t,t+h]} - H_{ij}^{[t,t]}}{h} = A_{ij}(t) \tag{4.37}$$

and

$$\lim_{h \to 0+} \frac{\tilde{H}_{ij}^{[t,t+h]} - \tilde{H}_{ij}^{[t,t]}}{h} = \tilde{A}_{ij}(t). \tag{4.38}$$

Proposition 4.3.2 *Let $(E, P_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ and $(E, \tilde{P}_{ij,k}^{[s,t]}, \mathbf{x}^{(0)})$ be q.s.p.s of types A and B, respectively. Then the following equalities hold:*

$$a_{ij,k}(t) = \sum_{m=1}^{n} P_{ij,m} A_{mk}(t) \tag{4.39}$$

and

$$\tilde{a}_{ij,k}(t) = \sum_{l=1}^{n} (\tilde{A}_{il}(t) \tilde{P}_{lj,k} + \tilde{A}_{jl}(t) \tilde{P}_{il,k}). \tag{4.40}$$

Proof Let us first consider the case when the q.s.p. has type (A). Then according to (4.34), one finds

$$P_{ij,k}^{[t,t+1+h]} = \sum_{m=1}^{n} P_{ij,m} H_{mk}^{[t+1,t+1+h]}$$

$$P_{ij,k}^{[t,t+1]} = \sum_{m=1}^{n} P_{ij,m} H_{mk}^{[t+1,t+1]}.$$

Therefore, we obtain

$$\begin{aligned} a_{ij,k}(t+1) &= \lim_{h\to 0+} \frac{P_{ij,k}^{[t,t+1+h]} - P_{ij,k}^{[t,t+1]}}{h} \\ &\times \lim_{h\to 0+} \frac{\sum_{m=1}^{n} P_{ij,m}\left(H_{mk}^{[t+1,t+1+h]} - H_{mk}^{[t+1,t+1]}\right)}{h} \\ &= \sum_{m=1}^{n} P_{ij,m} A_{mk}(t+1). \end{aligned}$$

Now let us turn to a q.s.p. of type (B). By putting

$$\tilde{H}_{il}^{[t,t]} = \begin{cases} 1 \text{ if } i = l \\ 0 \text{ if } i \neq l \end{cases} \tag{4.41}$$

from (4.35) one finds

$$\begin{aligned} \frac{\tilde{P}_{ij,k}^{[t,t+1+h]} - \tilde{P}_{ij,k}^{[t,t+1]}}{h} &= \sum_{l,m=1}^{n} \tilde{H}_{im}^{[t,t+h]} \frac{\tilde{H}_{jl}^{[t,t+h]} - \tilde{H}_{jl}^{[t,t]}}{h} \cdot \tilde{P}_{lm,k} \\ &+ \sum_{l,m=1}^{n} \tilde{H}_{jl}^{[t,t]} \frac{\tilde{H}_{im}^{[t,t+h]} - \tilde{H}_{im}^{[t,t]}}{h} \cdot \tilde{P}_{lm,k}. \end{aligned}$$

Therefore, passing to the limit as $h \to 0$ and taking into account (4.41), we obtain the equality (4.40).

Remark 4.3.4 In [121] it was proved that the continuity of $H_{ij}^{[s,t]}$ with respect to s and t is sufficient for the existence of limits (4.37) and (4.38). From (4.39) and (4.40) it follows that if $P_{ij,k}^{[s,t]}$ and $\tilde{P}_{ij,k}^{[s,t]}$ are continuous with respect to s and t, then the limits (4.2) and (4.9) exist.

The equalities (4.39) and (4.40) allow us to simplify the system of equations produced in Sect. 4.1. Let us consider the first system of equations (4.3). According to (4.39) we have

$$\begin{aligned}\sum_{m,l=1}^{n} a_{ml,k}(t)x_l^{(t-1)}P_{ij,m}^{[s,t-1]} &= \sum_{m,l,r=1}^{n} p_{ml,r}A_{r,k}(t)x_l^{(t-1)}P_{ij,m}^{[s,t-1]} \\ &= \sum_{r=1}^{n}\left(\sum_{m,l=1}^{n} P_{ij,m}^{[s,t-1]}P_{ml,r}x_l^{(t-1)}\right)A_{rk}(t) \\ &= \sum_{r=1}^{n} P_{ij,r}^{[s,t]}A_{r,k}(t).\end{aligned}$$

Hence, (4.3) is reduced to

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial t} = \sum_{l=1}^{n} A_{lk}(t)P_{ij,l}^{[s,t]}. \tag{4.42}$$

This system (4.42) is similar to Kolmogorov's direct differential equations for Markov chains [121]:

$$\frac{\partial q_{ik}(s,t)}{\partial t} = \sum_{l=1}^{n} A_{lk}(t)q_{il}(s,t).$$

Hence, the system of differential equations with delaying argument (4.3) is reduced to the well-known system of equations (4.42).

Now let us consider the system of differential equations (4.7). From (4.39) we get

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij,k}(t)x_i^{(t-1)}x_j^{(t-1)} &= \sum_{i,j=}^{n}\left(\sum_{l=1}^{n} P_{ij,l}A_{lk}(t)x_i^{(t-1)}x_j^{(t-1)}\right) \\ &= \sum_{l=1}^{n} A_{lk}(t)\left(\sum_{i,j=1}^{n} P_{ij,l}x_i^{(t-1)}x_j^{(t-1)}\right) \\ &= \sum_{l=1}^{n} A_{mk}(t)x_m^{(t)}.\end{aligned}$$

Then (4.7) can be rewritten as

$$\dot{x}_k^{(t)} = \sum_{m=1}^{n} A_{mk}(t)x_m^{(t)}. \tag{4.43}$$

Similarly, the reverse equation (4.5), according to (4.39), can be rewritten as follows

$$\frac{\partial P_{ij,k}^{[s,t]}}{\partial s} = -\sum_{m,l,r=1}^{n} P_{ij,r}A_{rm}(s+1)x_l^{(s+1)}P_{mj,k}^{[s+1,t]} \tag{4.44}$$

i.e., in this case the Eq. (4.5) cannot be reduced to an ordinary differential equation.

Now let us consider the system of differential equations (4.9) and (4.10) produced for a q.s.p. of type (B). According to (4.40), the Eq. (4.9) is reduced to the following one:

$$\frac{\partial \tilde{P}_{ij,k}^{[s,t]}}{\partial t} = \sum_{m,l,r,q,u,v=1}^{n} (\tilde{A}_{qv}(t-1)\tilde{P}_{lv,k} + \tilde{A}_{lu}(t-1)\tilde{P}_{uq,k})x_m^{(s)}x_r^{(s)}\tilde{P}_{im,l}^{[s,t-1]}\tilde{P}_{jr,q}^{[s,t-1]}. \tag{4.45}$$

As in the previous case, the delaying of the argument is preserved.

We are going to reproduce the reverse equations (4.10) by means of (4.35). Namely, from (4.35) we obtain

$$\frac{\tilde{P}_{ij,k}^{[s,t]} - \tilde{P}_{ij,k}^{[s+h,t]}}{h} = \sum_{l,m=1}^{n} \tilde{H}_{jm}^{[s,s+h]}\frac{\tilde{H}_{il}^{[s,s+h]} - \tilde{H}_{il}^{[}s,s]}{h}\cdot \tilde{P}_{lm,k}^{[s+h,t]}$$
$$+\sum_{l,m=1}^{n} \tilde{H}_{il}^{[s,s]}\frac{\tilde{H}_{jm}^{[s,s+h]} - \tilde{H}_{jm}^{[s,s]}}{h}\cdot \tilde{P}_{lm,k}^{[s+h,t]}.$$

Therefore, passing to limit as $h \to 0$ and taking into account (4.41), one finds

$$\frac{\partial \tilde{P}_{ij,k}^{[s,t]}}{\partial s} = -\sum_{l=1}^{n}(\tilde{A}_{jl}(s)\tilde{P}_{il,k}^{[s,t]} + \tilde{A}_{il}(s)\tilde{P}_{lj,k}^{[s,t]}). \tag{4.46}$$

Hence, the reverse equations are reduced to ordinary differential equations, which differ from the reverse Kolmogorov's equation for Markov chains [121] only by a number of summands.

So, as in [121], one can establish the existence and the uniqueness of the solutions for given initial conditions.

Consequently, for quadratic stochastic processes of type (A), the direct system of differential equations are similar to Kolmogorov's direct system for Markov chains [124], and in the case of processes of type (B) the reverse system of differential equations are similar to Kolmogorov's reverse system for Markov chains [121].

Definition 4.3.1 If a quadratic stochastic process satisfies both the fundamental equations IV_A and IV_B, then it is called *simple*.

Examples 4.1 and 4.2 in Chap. 3 are simple quadratic processes.

The above results can be interpreted in the following way.

Theorem 4.3.3 *The analytic theory of simple quadratic processes coincides with Kolmogorov's analytic theory of Markov chains.*

Similarly, one can consider the case when E is a countable set, but here we shall omit it.

4.3.2 The Set E Is a Continuum

Now we consider the case when the set E is a continuum. Let f and $\tilde{f}$ be the density functions. Then according to Theorem 4.3.1 the functions

$$g(s,x,t,z) = \int_{-\infty}^{\infty} f(s,x,y,t,z) m_s(dy) \tag{4.47}$$

and

$$\tilde{g}(s,x,t,z) = \int_{-\infty}^{\infty} \tilde{f}(s,x,y,t,z) m_s(dy) \tag{4.48}$$

are the density functions for some Markov process. Then (4.47) and (4.48) can be rewritten as follows:

$$f(s,x,y,t,z) = \int_{-\infty}^{\infty} f(s,x,y,\tau,u) g(\tau,u,t,z) du \tag{4.49}$$

and

$$\tilde{f}(s,x,y,t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}(s,x,\tau,v)\tilde{g}(s,y,\tau,h)\tilde{f}(\tau,v,h,t,z) dv dh \tag{4.50}$$

where $\tau - s \geq 0, t-\tau \geq 1$.

Now using (4.49), (4.50), and the same argument as in the case when E is finite, and following the lines of [121], we then reduce the Eqs. (4.20) and (4.23) to the following:

$$\frac{\partial f(s,x,y,t,z)}{\partial t} = N(t,z) f(s,x,y,t,z) + A(t,z)\frac{\partial f(s,x,y,t,z)}{\partial z} + B^2(t,z)\frac{\partial^2 f(s,x,y,t,z)}{\partial z^2}, \tag{4.51}$$

where

$$N(t,z) = \lim_{h\to 0} \frac{\int_{-\infty}^{\infty} g(t,u,t+h,z)du - 1}{h};$$

$$A(t,z) = \lim_{h\to 0} \frac{\int_{-\infty}^{\infty} g(t,u,t+h,z)(u-z)du}{h};$$

$$B^2(t,z) = \lim_{h\to 0} \frac{\int_{-\infty}^{\infty} g(t,u,t+h,z)(u-z)^2du}{2h},$$

and

$$\frac{\partial\tilde{f}(s,x,y,t,z)}{\partial s} = -\tilde{A}(s,x,z)\frac{\partial\tilde{f}(s,x,y,t,z)}{\partial x} - \tilde{A}(s,y,z)\frac{\partial\tilde{f}(s,x,y,t,z)}{\partial y}$$
$$-B^2(s,x,z)\frac{\partial^2\tilde{f}(s,x,y,t,z)}{\partial x^2} - B^2(s,x,z)\frac{\partial^2\tilde{f}(s,x,y,t,z)}{\partial y^2}, \tag{4.52}$$

where

$$\tilde{A}(t,z) = \lim_{h\to 0} \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}(s-h,x,s,v)\tilde{g}(s-h,y,s,w)(v-z)dvdw}{h};$$

$$\tilde{B}^2(s,x,z) = \lim_{h\to 0} \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{g}(s-h,x,s,v)\tilde{g}(s-h,y,s,v)(v-z)^2dvdw}{2h}.$$

The existence of all above mentioned limits can be proved as in [121]. The existence and uniqueness of solutions for Eqs. (4.51) and (4.52) can be established by the methods of [121] and [42].

4.4 Diffusion Quadratic Processes

Definition 4.4.1 We call a quadratic stochastic process *Wiener* (respectively *diffusion, Poisson*, etc.) if its average is a Wiener (respectively diffusion, Poisson etc.) process.

Let us consider the following process (see Example 4.2.4):

$$f(s,x,y,t,z) = \frac{2^{t-s-1}\exp(-\frac{4^{t-s-1}}{2^{2(t-s)-1}-1}\cdot(z-\frac{x+y}{2^{t-s}})^2)}{\sqrt{(2^{2(t-s)-1}-1)\pi}} \tag{4.53}$$

with

$$r_t(v) = \frac{\exp(-\frac{v^2}{2})}{\sqrt{2\pi}}. \tag{4.54}$$

Proposition 4.4.1 *The quadratic stochastic process generated by* (4.53)–(4.54) *is a diffusion process.*

Proof Let us compute the mean (average) of the process (4.53):

$$\begin{aligned} g(s,x,y,t,z) &= \int_{-\infty}^{\infty} f(s,x,y,t,z) r_s(y) dy \\ &= \int_{-\infty}^{\infty} \frac{2^{t-s-1}\exp(-\frac{4^{t-s-1}}{2^{2(t-s)-1}-1}(z-\frac{x+y}{2^{t-s}})^2)}{\sqrt{(2^{2(t-s)-1}-1)\pi}} \cdot \frac{\exp(-\frac{y^2}{2})}{\sqrt{2\pi}} dy. \end{aligned}$$

We have

$$\int_{-\infty}^{\infty} e^{-r^2x^2} dx = \frac{\sqrt{\pi}}{r} \quad (r > 0),$$

and simple but unwieldy calculations show that

$$g(s,x,t,z) = \frac{1}{\sqrt{2\pi}} \frac{2^{t-s}}{\sqrt{4^{t-s}-1}} \exp\left(-\frac{4^{t-s}}{2(4^{t-s}-1)}\right)\left(z - \frac{x}{2^{t-s}}\right)^2, \tag{4.55}$$

and (4.55) is a density of transition probabilities that defines the diffusion process.

As mentioned above, the process (4.53) is simple. In this case, the corresponding differential equations (4.51) and (4.52) have the following forms:

$$\begin{aligned} \frac{\partial f(s,x,y,t,z)}{\partial t} &= \ln 2\bigg(f(s,x,y,t,z) + z\frac{\partial f(s,x,y,t,z)}{\partial z} \\ &\quad + (1+z)^2 \frac{\partial^2 f(s,x,y,t,z)}{\partial z^2}\bigg), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f(s,x,y,t,z)}{\partial s} &= \ln 2\bigg(x\frac{\partial f(s,x,y,t,z)}{\partial x} + y\frac{\partial f(s,x,y,t,z)}{\partial y} \\ &\quad - \frac{\partial^2 f(s,x,y,t,z)}{\partial x^2} - \frac{\partial^2 f(s,x,y,t,z)}{\partial y^2}\bigg). \end{aligned}$$

4.5 Comments and References

The motivation behind the study of q.s.p.s came from the dynamics of q.s.o.s, where the q.s.p. describes its trajectory (see Chap. 1). A theory of q.s.p.s has been developed in [46, 234–236]. With the exception of the last section, all material in this chapter has essentially been taken from [46, 234, 237]. Note that in [197], the direct and reverse equations have been derived for general q.s.p.s (i.e. the process is not necessarily homogeneous). The results of the last section are taken from [51]. If we consider the Eq. (4.4) and (4.6), then one can ask:

Open problem 4.5.1 *Under what conditions on the coefficients do these equations produce a quadratic stochastic process?*

In [187] we have found some conditions on the coefficients $(a_{ij,k}(t))$ for homogeneous q.s.p.s of type (A) so that the equations produce q.s.p.s.

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