

Chapter 2

General Facts on Dissipative Systems

In this chapter we deal with the qualitative theory pertinent to (infinite-dimensional) dissipative systems. Our presentation is based mainly on some new criteria for asymptotic compactness which rely on a certain weak form of quasi-stability. We also emphasize a role of gradient systems for the existence of global attractors. A similar approach was discussed earlier in CHUESHOV/LASIECKA [56, 58] in a short form without many details. For other possible approaches to the topic we refer to the monographs BABIN/VISHIK [9], CHUESHOV [39], HALE [116], HENRY [123], LADYZHENSKAYA [142], ROBINSON [195], SELL/YOU [206], TEMAM [216] and the surveys BABIN [7] and RAUGEL [188].

Our main focus is on questions such as the existence of global attractors and their structure. We present ideas and methods which are applicable to the systems generated by nonlinear partial differential equations. We discuss these applications in Chapters 4–6 in detail for several PDE classes. In the current chapter we illustrate general results on long-time dynamics by means of finite-dimensional ODE examples only. Questions related to dimensions and smoothness of attractors for infinite-dimensional systems are considered in Chapter 3.

2.1 Dissipative dynamical systems

The main topic of this book is that of dissipative dynamical systems. As already mentioned in the Introduction, from a physical point of view, dissipative systems are characterized by relocation and dissipation of energy. This means that the energy of higher modes is dissipated and relocated to low modes. The interaction of these two mechanisms can lead to the appearance of complicated limit regimes and structures in the system that are stable in a suitable sense.

We start with a description of several concepts which present both dissipation and relocation on a formal level.

Definition 2.1.1. Let S_t be an evolution operator on a complete metric space X and (X, S_t) be the corresponding dynamical system.

- A closed set $B \subset X$ is said to be *absorbing* for S_t if for any bounded set $D \subset X$ there exists $t_0(D)$ such that $S_t D \subset B$ for all $t \geq t_0(D)$.
- S_t is said to be *(bounded) dissipative* if it possesses a bounded absorbing set B . If the phase space X of a dissipative evolution operator S_t is a Banach space, then the radius of a ball containing an absorbing set is called a *radius of dissipativity* of S_t .
- S_t is said to be *point dissipative* if there exists a bounded set $B_0 \subset X$ such that for any $x \in X$ there is $t_0(x)$ such that $S_t x \in B_0$ for all $t \geq t_0(x)$.

We apply the same terminology to the corresponding dynamical system (X, S_t) . ■

The following criterion of dissipativity covers many cases which are important from an applications point of view.

Theorem 2.1.2 (Criterion of dissipativity). *Let (X, S_t) be a continuous dynamical system in some Banach space X . Assume that*

- *there exists a continuous function $U(x)$ on X possessing the properties*

$$\phi_1(\|x\|) \leq U(x) \leq \phi_2(\|x\|), \quad \forall x \in X, \quad (2.1.1)$$

where ϕ_i are continuous functions on \mathbb{R}_+ such that $\phi_i(r) \rightarrow +\infty$ as $r \rightarrow +\infty$;

- *there exist a derivative $\frac{d}{dt}U(S_t y)$ for every $t > 0$ and $y \in X$, a positive function¹ $\alpha(r)$ on \mathbb{R}_+ , and a positive number ϱ such that*

$$\frac{d}{dt}U(S_t y) \leq -\alpha(\|y\|) \quad \text{provided} \quad \|S_t y\| > \varrho. \quad (2.1.2)$$

Then the dynamical system (X, S_t) is dissipative with an absorbing set of the form

$$B_* = \{x : \|x\| \leq R_*\}, \quad (2.1.3)$$

where the constant R_ depends on the functions ϕ_1 and ϕ_2 and the constant ϱ only.*

Proof. The argument involves some kind of “barrier method”; see, e.g., REISSING/SANSONE/CONTI [189] for a discussion in the ODE case.

Let us choose $R_0 > \varrho$ such that $\phi_1(r) > 0$ for all $r \geq R_0$. Let

$$L = \sup\{\phi_2(r) : r \leq 1 + R_0\}.$$

¹This function $\alpha(r)$ may tend to zero as $r \rightarrow +\infty$.

We show that the ball B in (2.1.3) is absorbing provided $R_* \geq R_0 + 1$ is chosen such that $\phi_1(r) > L$ for $r \geq R_*$. This choice is definitely possible and R_* can be taken dependent on ϕ_1 , ϕ_2 , and ϱ only.

Our argument consists of two steps.

Step 1. First we show that

$$\|S_t y\| \leq R_* \quad \text{for all } t \geq 0 \text{ and } \|y\| \leq R_0. \quad (2.1.4)$$

Indeed, if this is not true, then for some $y \in X$ such that $\|y\| \leq R_0$ there exists a time $\tilde{t} > 0$ possessing the property $\|S_{\tilde{t}} y\| > R_*$. By the continuity of $S_t y$ this implies that there exists $0 < t' < \tilde{t}$ such that $\|S_{t'} y\| = 1 + R_0 > \varrho$. Let

$$t_0 = \sup\{\tau < \tilde{t} : \|S_\tau y\| = 1 + R_0\}. \quad (2.1.5)$$

It is clear that $\|S_{t_0} y\| = 1 + R_0 > \varrho$. Therefore, equation (2.1.2) implies that

$$\phi_1(\|S_t y\|) \leq U(S_t y) \leq U(S_{t_0} y) \leq L \quad \text{for } t \in [t_0, t_1],$$

where

$$t_1 = \sup\{t : \|S_\tau y\| \geq \varrho \text{ for all } t_0 \leq \tau \leq t\}.$$

This means that $\|S_t y\| \leq R_*$ for all $t \in [t_0, t_1]$. Since $\|S_{\tilde{t}} y\| > R_*$ we have that $t_0 < t_1 < \tilde{t}$. Moreover, it is clear that $\|S_{t_1} y\| = \varrho$. Thus, there exists $t_2 \in (t_1, \tilde{t})$ (hence $t_2 > t_0$) such that $\|S_{t_2} y\| = 1 + R_0$. This contradicts the definition of t_0 in (2.1.5) and thus (2.1.4) is proved.

Step 2. Let us assume now that B is an arbitrary bounded set in X that lies outside the closed ball with the radius R_0 . Then equation (2.1.2) implies that

$$U(S_t y) \leq U(y) - \alpha(\|y\|)t \leq L_B - \alpha_B t \quad \text{for } t \in [0, \tilde{t}], \quad y \in B, \quad (2.1.6)$$

where $\tilde{t} = \sup\{t : \|S_\tau y\| \geq \varrho \text{ for all } 0 \leq \tau \leq t\}$ and

$$L_B = \sup\{U(x) : x \in B\}, \quad \alpha_B = \inf\{\alpha(x) : x \in B\}.$$

We can assume that $L_B > L$. If $\tilde{t} \leq t_B \equiv (L_B - L)/\alpha_B$, then, since $\|S_{\tilde{t}} y\| = \varrho$, by (2.1.4) we have that $\|S_t y\| \leq R_*$ for all $t \geq t_B$. If $\tilde{t} > t_B$, then by (2.1.6) and (2.1.1)

$$\phi_1(\|S_t y\|) \leq U(S_t y) \leq L \quad \text{for } t \in [t_B, \tilde{t}]$$

and hence $\|S_t y\| \leq R_*$ for $t \in [t_B, \tilde{t}]$. Since $\|S_{\tilde{t}} y\| = \varrho$, by (2.1.4) we have that $\|S_t y\| \leq R_*$ for all $t \geq \tilde{t}$. Consequently, the set B_* given by (2.1.3) is absorbing.

□

We conclude this section with several exercises which illustrate the notion of dissipativity.

Exercise 2.1.3. Show that the hypothesis (2.1.2) in Theorem 2.1.2 can be replaced by the requirement

$$\frac{d}{dt}U(S_t y) + \phi_3(\|S_t y\|) \leq \beta, \quad (2.1.7)$$

where $\phi_3(r)$ is a continuous function such that

$$\liminf_{r \rightarrow \infty} \phi_3(r) > \beta$$

and β is a positive constant. In particular, (2.1.7) is true with $\phi_3(r) = \alpha\phi_1(r)$ if we assume that

$$\frac{d}{dt}U(S_t y) + \alpha U(S_t y) \leq \beta, \quad (2.1.8)$$

where α and β are positive constants. ■

Exercise 2.1.4. Let (2.1.8) be in force. Solving the inequality in (2.1.8), show that the set

$$\{x \in X : U(x) \leq R\}$$

is a forward invariant absorbing set provided $R > \beta/\alpha$. ■

Exercise 2.1.5. Show that the dynamical system generated in \mathbb{R} by the differential equation $\dot{x} + f(x) = 0$ (see Section 1.7 in Chapter 1) is dissipative, provided the function $f(x)$ possesses the additional property: $xf(x) \geq \delta x^2 - c$, where $\delta > 0$ and c are constants. Hint: Take $U(x) = x^2$ and use the result of Exercise 2.1.3. Find an upper estimate for the minimal radius of dissipativity. ■

Exercise 2.1.6. Let (X, S_t) be a dissipative system and let B_0 be a bounded absorbing set. Show that there exists $t_* \geq 0$ such that the set $B_* = \cup\{S_t B_0 : t \geq t_*\}$ is a bounded forward invariant absorbing set for (X, S_t) . ■

Exercise 2.1.7. Consider a discrete dynamical system (\mathbb{R}, f) , where f is a continuous function on \mathbb{R} . Show that the system is dissipative, provided there exist $\rho > 0$ and $0 < \alpha < 1$ and such that $|f(x)| \leq \alpha|x|$ when $|x| \geq \rho$. ■

Exercise 2.1.8 (Duffing equation). Consider a dynamical system in \mathbb{R}^2 generated (see Remark 1.8.19 and Example 1.8.18) by the Duffing equation

$$\ddot{x} + \gamma\dot{x} + x^3 - ax = b,$$

where a and b are real numbers and $\gamma > 0$. Using the properties of the function

$$U(x, \dot{x}) = \dot{x}^2 + \frac{1}{2}x^4 + ax^2 + \nu[2x\dot{x} + \gamma x^2],$$

where $\nu > 0$ is small enough, show that this dynamical system is dissipative. ■

Exercise 2.1.9 (Lorenz system). Consider the Lorenz system arising as a three-mode Galerkin approximation in the problem of convection in a thin layer of liquid:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y, \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy. \end{cases}$$

Here σ , r , and b are positive numbers. Prove the dissipativity of the dynamical system generated by these equations in \mathbb{R}^3 . Hint: Consider the function

$$V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2$$

on the trajectories of the system. ■

Exercise 2.1.10 (Krasovskii equation). Consider the system generated by the equation

$$\ddot{x} + k(x^2 + \dot{x}^2)\dot{x} + x = 0,$$

in \mathbb{R}^2 , see Exercise 1.8.22. Show that this system is dissipative under the conditions

$$k \in L_\infty(\mathbb{R}_+) \cap Lip_{loc}(\mathbb{R}_+), \quad \liminf_{r \rightarrow +\infty} k(r) > 0.$$

Hint: Consider the function $V(x, \dot{x}) = x^2 + \dot{x}^2 + \nu x\dot{x}$ with $\nu > 0$ small enough on the trajectories. ■

2.2 Asymptotic compactness and smoothness

To study the long-time dynamics of infinite-dimensional systems we also need some properties of asymptotic compactness. There are several ways to formulate these properties depending on the structure of the corresponding model (see, e.g., the monographs BABIN/VISHIK [9], HALE [116], LADYZHENSKAYA [142], TEMAM [216] and the references therein). Below we mainly concentrate on the approaches suggested by LADYZHENSKAYA [142] and HALE [116]. We also refer to HARAUX [118], where some concepts of asymptotic compactness were used for the first time.

2.2.1 Basic definitions and facts

We start with several notions of compactness of an evolution operator.

Definition 2.2.1. Let S_t be an evolution operator on a complete metric space X .

- S_t is said to be *compact* if it possesses a compact absorbing set.
- S_t is said to be *conditionally compact* if for any bounded set D such that $S_t D \subset D$ for $t > 0$ there exist $t_D > 0$ and a compact set K in the closure \overline{D} of D , such that $S_t D \subset K$ for all $t \geq t_D$.
- S_t is said to be *asymptotically compact* if the following Ladyzhenskaya condition (see LADYZHENSKAYA [142] and the references therein) holds: for any bounded set B in X such that the tail $\gamma^\tau(B) := \cup_{t \geq \tau} S_t B$ is bounded for some $\tau \geq 0$ we have that any sequence of the form $\{S_{t_n} x_n\}$ with $x_n \in B$ and $t_n \rightarrow \infty$ is relatively compact.
- An evolution operator S_t is said to be *asymptotically smooth* if the following Hale condition (see, e.g., HALE [116]) is valid: for every bounded set D such that $S_t D \subset D$ for $t > 0$ there exists a compact set K in the closure \overline{D} of D , such that $S_t D$ converges uniformly to K in the sense that

$$\lim_{t \rightarrow +\infty} d_X\{S_t D \mid K\} = 0, \quad \text{where } d_X\{A \mid B\} = \sup_{x \in A} \text{dist}_X(x, B). \quad (2.2.1)$$

We apply the same terminology in the case of dynamical systems. Below we also use the notation $S_t D \rightrightarrows K$ as $t \rightarrow \infty$ in the case when (2.2.1) holds. ■

Exercise 2.2.2. If S_t is a compact evolution operator, then S_t is conditionally compact. The latter property implies that S_t is asymptotically compact and asymptotically smooth. ■

Exercise 2.2.3. Show that any dissipative conditionally compact system is compact. Hint: By Exercise 2.1.6 there exists a bounded forward invariant absorbing set. ■

The proposition below shows that asymptotic smoothness is equivalent to asymptotic compactness.

Proposition 2.2.4. *An evolution operator S_t in some metric space X is asymptotically compact if and only if it is asymptotically smooth.*

Proof. We start with the following key lemma, which is also important in further considerations.

Lemma 2.2.5. *Let an evolution operator S_t be asymptotically compact on X and D be a bounded set. Assume the tail $\gamma^\tau(D)$ is bounded for some $\tau \geq 0$. Then the ω -limit set² $\omega(D)$ is a nonempty compact strictly invariant set such that $S_t D \rightrightarrows \omega(D)$ as $t \rightarrow \infty$.*

²We recall that the notions of a tail and an ω -limit set were introduced in Chapter 1.

Proof. It follows from asymptotic compactness that for any $t_n \rightarrow \infty$ and $x_n \in D$ the sequence $\{S_{t_n}x_n\}$ is relatively compact. Therefore, there exist a subsequence $\{n_m\}$ and an element $z \in X$ such that $S_{t_{n_m}}x_{n_m} \rightarrow z$ as $m \rightarrow \infty$. By Proposition 1.3.3,

$$\omega(D) = \left\{ z \in X : z = \lim_{t \rightarrow \infty} S_t x_n \text{ for some } t_n \rightarrow +\infty, x_n \in D \right\}. \quad (2.2.2)$$

Hence $\omega(D)$ contains the element z , at least, and thus $\omega(D)$ is not empty.

To prove compactness of $\omega(D)$ we note that by (2.2.2) for any sequence $\{z_n\}$ in $\omega(D)$ there exist $t_n \rightarrow \infty$ and $x_n \in D$ such that $\text{dist}_X(S_{t_n}x_n, z_n) \leq 1/n$. By asymptotic compactness there exist a subsequence $\{n_m\}$ and an element \hat{z} such that $S_{t_{n_m}}x_{n_m} \rightarrow \hat{z} \in \omega(D)$ as $m \rightarrow \infty$. Thus, we have that $z_{n_m} \rightarrow \hat{z}$. This means that $\omega(D)$ is relatively compact. In the same way, if $z_n \rightarrow \bar{z}$ as $n \rightarrow \infty$, then $\bar{z} = \hat{z} \in \omega(D)$, i.e., $\omega(D)$ is closed.

Now we prove invariance of $\omega(D)$. Let $z \in \omega(D)$ and $z = \lim_{n \rightarrow \infty} S_{t_n}x_n$. Then $S_t z = \lim_{n \rightarrow \infty} S_{t+t_n}x_n$. Thus, due to (2.2.2) $\omega(D)$ is forward invariant. To prove backward invariance we consider the sequence $\{S_{t_n-t}x_n\}$ for some fixed $t > 0$ and n such that $t_n > t$. By asymptotic compactness this sequence is relatively compact. Thus, there exist a sequence $\{n_m\}$ and an element $v \in \omega(D)$ such that $y_m \equiv S_{t_{n_m}-t}x_{n_m} \rightarrow v$. We also have that $S_t y_m \rightarrow z$. Thus $z = S_t v$ and hence $S_t \omega(D) \supset \omega(D)$, i.e., $\omega(D)$ is backward invariant.

Assume that $S_t D \not\Rightarrow \omega(D)$ is not true. Then there exist $\delta > 0$ and sequences $t_n \rightarrow \infty$ and $x_n \in D$ such that $\text{dist}_X(S_{t_n}x_n, \omega(D)) \geq \delta$ for all n . As above, $\{S_{t_n}x_n\}$ is relatively compact. Therefore, $S_{t_{n_m}}x_{n_m} \rightarrow z \in \omega(D)$ for some subsequence $\{n_m\}$. This contradicts the relation $\text{dist}_X(S_{t_n}x_n, \omega(D)) \geq \delta$. \square

Now we return to the proof of Proposition 2.2.4.

Let S_t be asymptotically compact and $B \subset X$ be an invariant bounded set. By Lemma 2.2.5, $\omega(B)$ is a compact set which attracts B . Thus, the Hale condition (see Definition 2.2.1) holds.

Let S_t be asymptotically smooth and $B \subset X$ be a bounded set such that the tail $\gamma^\tau(B) = \cup_{t \geq \tau} S_t B$ is bounded for some $\tau \geq 0$. Since $B_* \equiv \gamma^\tau(B)$ is forward invariant, by the Hale condition $S_t B_*$ converges uniformly to a compact set K . Thus $S_{t_n}x_n \rightarrow K$ for any sequences $x_n \in B$ and $t_n \rightarrow \infty$. Hence $\{S_{t_n}x_n\}$ is relatively compact. \square

The following exercise provides some sufficient conditions of asymptotic compactness of semiflows. They were established and applied by many authors (see, e.g., TEMAM [216, Chapter 1] and also LADYZHENSKAYA [142] and RAUGEL [188]).

Exercise 2.2.6. An evolution operator S_t in some metric space X is asymptotically compact provided one of the following conditions is valid:

- (A) There exists a compact set K such that $S_t B \Rightarrow K$ as $t \rightarrow \infty$ for every bounded set B in X .

- (B) For any bounded set B there exists a compact set K_B such that $S_t B \rightrightarrows K_B$ as $t \rightarrow \infty$.
- (C) X is a Banach space and there exists a decomposition $S_t = S_t^{(1)} + S_t^{(2)}$, where $S_t^{(1)}$ is uniformly compact for large t ; that is, for any bounded set B there exists $t_0 = t_0(B)$ such that the set $\gamma^{(1)}(B; t_0) := \bigcup_{\tau \geq t_0} S_\tau^{(1)} B$ is relatively compact in X and $S_t^{(2)}$ is uniformly stable in the sense that

$$r_B(t) = \sup \left\{ \|S_t^{(2)} x\|_X : x \in B \right\} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (2.2.3)$$

Hint: (C) implies (B) with $K_B = \text{Closure}_X \{ \gamma^{(1)}(B; t_0) \}$. Statement (B) applied to a bounded sequence $B = \{x_n\}$ yields the convergence of $S_{t_n} x_n$ to a compact set as $t_n \rightarrow \infty$. ■

2.2.2 Kuratowski's measure of noncompactness

To obtain effective criteria of asymptotic compactness, it is convenient to use Kuratowski's α -measure of noncompactness (see, e.g., AKHMEROV ET AL. [2] and the references therein). The latter is defined by the formula

$$\alpha(B) = \inf \{ d : B \text{ has a finite cover by open sets of diameter } < d \}$$

on bounded sets of a complete metric space X . We recall that the diameter of the set is defined by the relation $\text{diam } B = \sup \{ \text{dist}(x, y) : x, y \in B \}$.

Some elementary properties of the α -measure are collected in the following exercises.

Exercise 2.2.7. Let X be a complete metric space.

- (A) Show that in the definition of α -measure we can consider arbitrary coverings, i.e.,

$$\alpha(B) = \inf \{ d : B \text{ has finite cover by (arbitrary) sets of diameter } < d \}.$$

This implies that $\alpha(B) \leq \text{diam } B$.

- (B) Show that if $K_1 \subset K_2$, then $\alpha(K_1) \leq \alpha(K_2)$ (monotonicity).
 (C) Show that $\alpha(K) = \alpha(\overline{K})$, where \overline{K} is the closure of K .
 (D) Show that $\alpha(A \cup B) \leq \max \{ \alpha(A), \alpha(B) \}$ (semi-additivity).
 (E) Show that $\alpha(K) = 0$ if and only if the closure \overline{K} of K is compact.
 (F) Show that the set B is bounded if and only if $\alpha(B) < \infty$. ■

Exercise 2.2.8. Let X be a Banach space. Show that

- (A) $\alpha(\lambda B) = |\lambda| \alpha(B)$ for any $\lambda \in \mathbb{R}$, where $\lambda B = \{ \lambda x : x \in B \}$ (homogeneity).

(B) $\alpha(y + B) = \alpha(B)$ for any $y \in X$, where $y + B = \{y + x : x \in B\}$ (invariance under translations). ■

It is known (see, e.g., AKHMEROV ET AL. [2]) that $\alpha(B_R(y)) = 2R$ for every ball $B_R(y) = \{x \in X : \|x - y\|_X < R\}$ in an infinite-dimensional Banach space X . The following propositions are also important (see HALE [116] and SELL/YOU [206]) in the study of asymptotic smoothness of evolution operators.

Proposition 2.2.9. *Let A and B be bounded sets in a Banach space X . Then*

$$\alpha(A + B) \leq \alpha(A) + \alpha(B), \quad (2.2.4)$$

where $A + B = \{x + y : x \in A, y \in B\}$

Proof. Take arbitrary $\varepsilon > 0$. Let $\{\mathcal{O}_i^A\}$ and $\{\mathcal{O}_j^B\}$ be coverings of A and B with diameters less than $\alpha(A) + \varepsilon$ and $\alpha(B) + \varepsilon$. Then $\{\mathcal{O}_i^A + \mathcal{O}_j^B\}$ is a covering for $A + B$. It is clear that

$$\text{diam}\{\mathcal{O}_i^A + \mathcal{O}_j^B\} \leq \text{diam}\{\mathcal{O}_i^A\} + \text{diam}\{\mathcal{O}_j^B\} \leq \alpha(A) + \alpha(B) + 2\varepsilon.$$

This implies (2.2.4). □

Proposition 2.2.10. *Let X be a complete metric space and $U_1 \supset U_2 \supset U_3 \dots$ be nonempty closed sets in X . If $\alpha(U_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\cap_{n \geq 1} U_n$ is nonempty and compact.*

Proof. For each n take $u_n \in U_n$ and consider the sequence $\{u_n\}$. For every $\varepsilon > 0$ we can find N such that $\alpha(U_N) < \varepsilon$. Thus,

$$K \equiv \{u_n : n = 1, \dots\} \subset \{u_n : n = 1, \dots, N-1\} \cup U_N.$$

Hence by Exercise 2.2.7(B,D), $\alpha(K) \leq \alpha(U_N) < \varepsilon$ for every $\varepsilon > 0$. Therefore, $\alpha(K) = 0$ and thus \bar{K} is compact. Thus, there exist $u \in X$ and a subsequence $\{u_m\}$ such that

$$u_{n_m} \in U_{n_m} \text{ and } u_{n_m} \rightarrow u \text{ as } m \rightarrow \infty.$$

Since U_n is closed for every n , we have that $u \in U_n$ for every n , i.e., $U = \cap_{n \geq 1} U_n$ is not empty. It is clear that U is closed. Since $\alpha(U) \leq \alpha(U_n)$ for $n = 1, 2, \dots$, we have $\alpha(U) = 0$ and thus by Exercise 2.2.7(E) U is compact. □

Exercise 2.2.11. Prove the continuous analog of Proposition 2.2.10: if $\alpha(U_t) \rightarrow 0$ as $t \rightarrow \infty$ for some decreasing family $\{U_t\}$ of nonempty closed sets, then $\cap_{t \geq 0} U_t$ is nonempty and compact. ■

Exercise 2.2.12. Let $B_1 \supset B_2 \supset \dots$ be a sequence of closed sets in a complete metric space X . Assume that $\text{diam } B_n \rightarrow 0$ as $n \rightarrow \infty$. Show that there exists a unique element $x \in X$ such that $x \in B_n$ for all n (in the case when $\{B_k\}$ are balls

in a Banach space, this fact is known as the principle of nested balls). Hint: Apply Proposition 2.2.10 and also the observation made in Exercise 2.2.7(A). ■

The following assertion allows us to reformulate the asymptotic smoothness/compactness in the terms of Kuratowski's α -measure.

Proposition 2.2.13. *An evolution operator S_t is asymptotically smooth if and only if for any bounded forward invariant set B we have that $\alpha(S_t B) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. Let B be a bounded forward invariant set for S_t .

If S_t is an asymptotically smooth evolution operator, then there exists a compact set K_B such that $S_t B \rightrightarrows K_B$ as $t \rightarrow \infty$. By the compactness of K_B , for any $\varepsilon > 0$ there exists a finite set $\{x_k : k = 1, \dots, N_\varepsilon\}$ in K_B such that

$$K_B \subset \bigcup_{k=1}^{N_\varepsilon} \mathcal{B}_k, \text{ where } \mathcal{B}_k = \{x \in X : \text{dist}_X(x_k, x) < \varepsilon\}.$$

Since $S_t B \rightrightarrows K_B$, there exists $t_\varepsilon > 0$ such that $S_t B \subset \bigcup_{k=1}^{N_\varepsilon} \mathcal{B}_k$ for all $t \geq t_\varepsilon$. Thus $\alpha(S_t B) < 2\varepsilon$ for all $t \geq t_\varepsilon$. This implies that $\alpha(S_t B) \rightarrow 0$ as $t \rightarrow \infty$.

Assume now that $\alpha(S_t B) \rightarrow 0$ as $t \rightarrow \infty$. Then we can apply the result of Exercise 2.2.11 to a family of the sets $U_t = \overline{S_t B}$ and conclude that

$$\omega(B) = \bigcap_{t>0} \overline{S_t B} \text{ is a nonempty compact set.}$$

Thus, it is sufficient to show that $S_t B \rightrightarrows \omega(B)$. If this is not true, then there exist $\delta > 0$ and sequences $t_n \rightarrow \infty$ and $x_n \in B$ such that $\text{dist}_X(S_{t_n} x_n, \omega(B)) \geq \delta$ for all n . One can see that for any $t > 0$ there exists N_t such that

$$\{S_{t_n} x_n : n = 1, 2, \dots\} \subset \{S_{t_n} x_n : n = 1, 2, \dots, N_t\} \bigcup \overline{S_t B}.$$

Thus $\alpha(\{S_{t_n} x_n : n = 1, 2, \dots\}) \leq \alpha(S_t B)$, which implies that $\alpha(\{S_{t_n} x_n\}) = 0$. Hence $\{S_{t_n} x_n\}$ is relatively compact. Therefore, $S_{t_{n_m}} x_{n_m} \rightarrow z \in \omega(B)$ for some subsequence $\{n_m\}$. This contradicts the relation $\text{dist}_X(S_{t_n} x_n, \omega(B)) \geq \delta$. □

Using Proposition 2.2.13 we can prove the next assertion, which is a slight modification of the statement proved earlier in HALE [116, Lemma 3.2.3] by another method.

Proposition 2.2.14. *Let S_t be an evolution operator in a Banach space X . Assume that for each $t > 0$ there exists a decomposition $S_t = S_t^{(1)} + S_t^{(2)}$, where $S_t^{(2)}$ is a mapping in X satisfying (2.2.3) and $S_t^{(1)}$ is compact in the sense that for each $t > 0$ the set $S_t^{(1)} B$ is a relatively compact set in X for every $t > 0$ large enough and for every bounded forward invariant set B in X . Then S_t is asymptotically smooth.*

We note that this proposition improves the statement of Exercise 2.2.6(C), because we do not assume compactness of $\gamma^{(1)}(B; t_0)$ here. The size of $S_t^{(1)}B$ may be unbounded as $t \rightarrow +\infty$.

Proof. For any bounded forward invariant set B we have that $S_t B \subset S_t^{(1)}B + S_t^{(2)}B$. Therefore, Proposition 2.2.9 (see also Exercise 2.2.7) yields

$$\alpha(S_t B) \leq \alpha(S_t^{(1)}B) + \alpha(S_t^{(2)}B) \leq \alpha(S_t^{(2)}B) \leq \text{diam}\{S_t^{(2)}B\} \leq 2 \sup_{y \in B} \|S_t^{(2)}y\|$$

for all t large enough. Thus by (2.2.3), $\alpha(S_t B) \rightarrow 0$ as $t \rightarrow \infty$. Hence by Proposition 2.2.13, S_t is asymptotically smooth. \square

Keeping in mind Proposition 2.2.13, it is convenient to introduce the following notion (see HALE [116]).

Definition 2.2.15. A (nonlinear) operator V on a complete metric space X is said to be an α -contraction if there exists $0 \leq \kappa < 1$ such that $\alpha(VB) \leq \kappa \alpha(B)$. \blacksquare

The following simple result connects this notion with dynamics.

Exercise 2.2.16. A dynamical system (X, S_t) is asymptotically smooth if there exists $t_* > 0$ such that S_{t_*} is an α -contraction. Hint: For every forward invariant set D we have that $S_t D \subset S_{nt_*} D$, where n is the integer part of t/t_* . \blacksquare

For more discussion of the α -measure from the point of view of dynamical systems we refer to HALE [116] and the references therein; see also SELL/YOU [206, Lemma 22.2].

2.2.3 Criteria of asymptotic compactness via weak quasi-stability

We conclude this section with several assertions that give convenient criteria for asymptotic smoothness/compactness of evolution operators and dynamical systems. These criteria generalize the corresponding statements known due to KHANMAMEDOV [134], MA/WANG/ZHONG [156] and CERON/LOPES [28]. A posteriori they can be treated as some weak forms of quasi-stability discussed in Chapter 3. Roughly speaking, this *weak* quasi-stability means that the difference of two trajectories can be made small for large moments of time modulo some functional which demonstrates some (rather weak) compactness behavior (see, e.g., (2.2.5) below).

We start with the criterion which relies on the idea presented in KHANMAMEDOV [134] and provides more flexibility with respect to more standard methods (see, e.g., the discussion in CHUESHOV/LASIECKA [56, 58] and also the references cited therein).

Theorem 2.2.17. *Let S_t be an evolution operator on a complete metric space X . Assume that for any bounded forward invariant set B in X and for any $\epsilon > 0$ there exists $T \equiv T(\epsilon, B)$ such that*

$$\text{dist}(S_T y_1, S_T y_2) \leq \epsilon + \Psi_{\epsilon, B, T}(y_1, y_2), \quad y_i \in B, \quad (2.2.5)$$

where $\Psi_{\epsilon, B, T}(y_1, y_2)$ is a functional defined on $B \times B$ such that

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \Psi_{\epsilon, B, T}(y_n, y_m) = 0 \quad \text{for every sequence } \{y_n\} \subset B. \quad (2.2.6)$$

Then S_t is an asymptotically smooth evolution operator.

The result stated in Theorem 2.2.17 is an abstract version of Theorem 2 in KHANMAMEDOV [134] and can be derived from the arguments given in KHANMAMEDOV [134]. Our proof is shorter and can be easily derived from the following assertion.³

Proposition 2.2.18. *Let S_t be an evolution operator on a complete metric space X . Assume that for any bounded positively invariant set B in X and for any $\epsilon > 0$ there exists $T \equiv T(\epsilon, B)$ such that*

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \text{dist}(S_T y_n, S_T y_m) \leq \epsilon \quad \text{for every sequence } \{y_n\} \subset B. \quad (2.2.7)$$

Then S_t is an asymptotically smooth evolution operator.

Proof. By Proposition 2.2.13 it is sufficient to prove that

$$\lim_{t \rightarrow \infty} \alpha(S_t B) = 0,$$

where $\alpha(B)$ is Kuratowski's α -measure of noncompactness.

Because $S_{t_1} B \subset S_{t_2} B$ for $t_1 > t_2$, the function $\alpha(t) \equiv \alpha(S_t B)$ is non-increasing. Therefore, it is sufficient to prove that for any $\epsilon > 0$ there exists $T > 0$ such that $\alpha(S_T B) \leq \epsilon$. If this is not true, then there is $\epsilon_0 > 0$ such that $\alpha(S_T B) \geq 5\epsilon_0$ for all $T > 0$. For this ϵ_0 we choose T_0 such that (2.2.7) holds. The relation $\alpha(S_{T_0} B) \geq 5\epsilon_0$ implies that there exists an infinite sequence $\{y_n\}_{n=1}^\infty$ such that

$$\text{dist}(S_{T_0} y_n, S_{T_0} y_m) \geq 2\epsilon_0 \quad \text{for all } n \neq m, \quad n, m = 1, 2, \dots \quad (2.2.8)$$

If such a sequence does not exist, then we can use the following construction: take arbitrary $y_1 \in B$ and choose $y_2 \in B$ such that $\text{dist}(S_{T_0} y_1, S_{T_0} y_2) \geq 2\epsilon_0$. Then we take $y_3 \in B$ such that $\text{dist}(S_{T_0} y_3, S_{T_0} y_i) \geq 2\epsilon_0$ for $i = 1, 2$, and so on. If this procedure stops, we obtain a finite $2\epsilon_0$ -net for $S_{T_0} B$. This means that $\alpha(S_{T_0} B) \leq 4\epsilon_0$ and contradicts the relation $\alpha(S_{T_0} B) \geq 5\epsilon_0$. Thus (2.2.8) holds true. This contradicts (2.2.7). \square

³In many cases we can use Proposition 2.2.18 directly. Theorem 2.2.17 is formulated mainly due to priority and historical reasons.

Proposition 2.2.18 can also be used to obtain the following criterion.

Proposition 2.2.19. *Let S_t be an evolution operator on a reflexive Banach space X . Assume that for any bounded forward invariant set B in X and any $\varepsilon > 0$ there exist $T > 0$ and a compact operator K such that*

$$\|(I - K)S_T y\| \leq \varepsilon, \quad \forall y \in B. \quad (2.2.9)$$

Then the evolution operator S_t is asymptotically smooth.

This proposition was proved in MA/WANG/ZHONG [156] for the case when K is a finite-dimensional projector. Now the relation in (2.2.9) with a projector is known as the “flattening” property (see the discussion in CARVALHO/LANGA/ROBINSON [26] and KLOEDEN/RASMUSSEN [135]).

Proof. By (2.2.9) we have that

$$\begin{aligned} \|S_T y_1 - S_T y_2\| &\leq \|(I - K)S_T y_1\| + \|(I - K)S_T y_2\| + \|K(S_T y_1 - S_T y_2)\| \\ &\leq 2\varepsilon + \|K(S_T y_1 - S_T y_2)\|, \quad \forall y_1, y_2 \in B. \end{aligned}$$

Let $\{y_n\} \subset B$. Since $\{S_T y_n\} \subset B$ is a bounded sequence, there exists a weakly convergent subsequence $\{S_T y_{n_k}\}$. By the compactness of K , we have that

$$\lim_{k, m \rightarrow \infty} \|K(S_T y_{n_k} - S_T y_{n_m})\| = 0$$

which implies that

$$\liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \|K(S_T y_n - S_T y_m)\| = 0 \quad \text{for every sequence } \{y_n\} \subset B.$$

Thus, we can apply Proposition 2.2.18. □

The following exercise presents another asymptotic smoothness criterion in reflexive Banach spaces.

Exercise 2.2.20. Let S_t be an evolution operator on a Hilbert space. Assume that S_t is weakly continuous for every $t > 0$; i.e., the condition $x_n \rightarrow x$ weakly in X implies that $S_t x_n \rightarrow S_t x$ weakly. Show that the evolution operator S_t is asymptotically smooth provided that for any bounded forward invariant set B and for any $\varepsilon > 0$ there exists $T \equiv T(\varepsilon, B)$ such that

$$\limsup_{n \rightarrow \infty} \|S_T y_n\| \leq \|S_T y\| + \varepsilon \quad (2.2.10)$$

for every sequence $\{y_n\} \subset B$ such that $y_n \rightarrow y$ weakly. Hint: Prove first that

$$\limsup_{n \rightarrow \infty} \|S_T y_n - S_T y\| \leq \varepsilon,$$

then apply Proposition 2.2.18. ■

The following assertion is a generalization of the results presented in HALE [116] and CERON/LOPES [28] (see also CHUESHOV/LASIECKA [56, 58] where this fact is established by a different method).

Theorem 2.2.21. *Let S_t be an evolution operator on a complete metric space X . Assume that for any bounded forward invariant set B in X there exist $T > 0$, a continuous nondecreasing function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and a pseudometric ϱ_B^T on the set B such that*

- (i) $g(0) = 0$; $g(s) < s$, $s > 0$.
- (ii) The pseudometric ϱ_B^T is precompact (with respect to the topology of X) in the sense that any sequence $\{x_n\} \subset B$ has a subsequence $\{x_{n_k}\}$ which is Cauchy with respect to ϱ_B^T .
- (iii) The following estimate holds for every $y_1, y_2 \in B$:

$$\text{dist}_X(S_T y_1, S_T y_2) \leq g(\text{dist}_X(y_1, y_2)) + \varrho_B^T(y_1, y_2). \quad (2.2.11)$$

Then the evolution operator S_t is asymptotically smooth.

Remark 2.2.22. The difference between pseudometrics and metrics is that a pseudometric can be degenerate. In our case this means that the property $\varrho_B^T(y_1, y_2) = 0$ does not imply $y_1 = y_2$. We also know that instead of (2.2.11) one may also assume that

$$\text{dist}_X(S_T y_1, S_T y_2) \leq g(\text{dist}_X(y_1, y_2) + \varrho_B^T(y_1, y_2)),$$

(pseudometric inside g); see some details in [56, Chapter 2]. ■

Proof. We use Proposition 2.2.18.

Let B be a bounded forward invariant set in X with diameter L . One can see that for any $\varepsilon > 0$ we can choose N such that $g^N(L) \leq \varepsilon$, where g^N denotes the composition $g \circ \dots \circ g$. Iterating (2.2.11) we have that

$$\begin{aligned} \text{dist}_X(S_T^N y_1, S_T^N y_2) &\leq g(\text{dist}_X(S_T^{N-1} y_1, S_T^{N-1} y_2)) + \varrho_B^T(S_T^{N-1} y_1, S_T^{N-1} y_2) \\ &\leq g(g(\dots g(g(L)) + \varrho_B^T(y_1, y_2))) \\ &\quad + \varrho_B^T(S_T y_1, S_T y_2) \dots + \varrho_B^T(S_T^{N-1} y_1, S_T^{N-1} y_2). \end{aligned}$$

The right-hand side of the relation above is a continuous function of L and the expressions of the form

$$\varrho_B^T(S_T^m y_1, S_T^m y_2), \quad m = 1, \dots, N-1.$$

Since the pseudometric ϱ_B^T is precompact, any sequence $\{x_n\} \subset B$ has a subsequence $\{\hat{x}_{n_k}\}$ such that

$$\lim_{p,q \rightarrow \infty} \varrho_B^T(S_T^m \hat{x}_{n_p}, S_T^m \hat{x}_{n_q}) = 0, \quad \forall m = 1, \dots, N-1.$$

This implies that

$$\liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \text{dist}_X(S_T^N x_n, S_T^N x_k) \leq g^N(L) \leq \varepsilon.$$

By Proposition 2.2.18 this implies that S_t is asymptotically smooth. \square

Theorem 2.2.21 implies the following result which was proved earlier in the paper of CERON/LOPES [28].

Proposition 2.2.23. *Let (X, S_t) be a dynamical system in a Banach space X . Assume that for any bounded forward invariant set B in X there exist functions $C_B(t) \geq 0$ and $K_B(t) \geq 0$ such that $\lim_{t \rightarrow \infty} K_B(t) = 0$, a time $t_0 = t_0(B)$, and a precompact pseudometric ϱ on X such that*

$$\|S_t y_1 - S_t y_2\| \leq K_B(t) \cdot \|y_1 - y_2\| + C_B(t) \cdot \varrho(y_1, y_2), \quad t \geq t_0, \quad (2.2.12)$$

for every $y_1, y_2 \in B$. Then (X, S_t) is an asymptotically smooth dynamical system.

Proof. We apply Theorem 2.2.21 with $g(s) = K_B(T) \cdot s$, where T is chosen such that $K_B(T) < 1$. \square

2.3 Global attractors

The main objects arising in the analysis of the long-time behavior of dissipative dynamical systems are attractors. Their study makes it possible to answer a number of fundamental questions on the properties of limit regimes that can arise in the system. There are several general approaches and methods that allow us to study attractors for a large class of dynamical systems generated by nonlinear partial differential equations (see, e.g., BABIN/VISHIK [9], CHUESHOV [39], HALE [116], LADYZHENSKAYA [142], TEMAM [216] and the references listed therein). In this section we present the main general tools which are usually involved in the theory of infinite-dimensional dissipative systems.

2.3.1 Existence and basic properties

Several definitions of an attractor are available (see, e.g., the discussion in CHUESHOV [39, Section 1.3]). From the point of view of infinite-dimensional systems, the most convenient concept is a global attractor.⁴

Definition 2.3.1 (Global attractor). Let S_t be an evolution operator on a complete metric space X . A bounded closed set $\mathfrak{A} \subset X$ is said to be a *global attractor* for S_t if

- (i) \mathfrak{A} is an invariant set; that is, $S_t\mathfrak{A} = \mathfrak{A}$ for $t \geq 0$.
- (ii) \mathfrak{A} is uniformly attracting; that is, for all bounded set $D \subset X$

$$\lim_{t \rightarrow +\infty} d_X\{S_t D \mid \mathfrak{A}\} = 0 \quad \text{for every bounded set } D \subset X, \quad (2.3.1)$$

where $d_X\{A \mid B\} = \sup_{x \in A} \text{dist}_X(x, B)$ is the Hausdorff semidistance. ■

In many sources (see, e.g., BABIN [7], CHEPYZHOV [31], HALE [116], TEMAM [216]) the definition of a global attractor requires this to be a compact set. We do not assume this property because, hypothetically, situations when a global attractor is not compact are possible for systems with degenerate damping mechanisms. See, e.g., Section 5.3.3 in Chapter 5.

Exercise 2.3.2. Show that if a global attractor exists, then it is unique. ■

Exercise 2.3.3. Show that any backward invariant bounded set belongs to the global attractor. In particular, every stationary point lies in the attractor. ■

Exercise 2.3.4. Show that

- (A) A full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ belongs to the global attractor if and only if γ is a bounded set.
- (B) For any x from the attractor \mathfrak{A} there exists a full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ such that $u(0) = x$ and $\gamma \subset \mathfrak{A}$. Hint: The strict invariance property of the attractor implies that there exists a sequence $\{x_{-n} : n = 1, 2, \dots\} \subset \mathfrak{A}$ such that $S_1 x_{-n} = x_{-(n-1)}$ for all $n = 1, 2, \dots$ with $x_0 = x$.

Thus, the global attractor can be described as a set of all bounded full trajectories. ■

The main result on the existence of global attractors is the following assertion.

⁴ Below we use the Fraktur (Gothic) “A” for notation of global attractors because the Latin version of this letter is overloaded, especially in Chapters 4–6.

Theorem 2.3.5. *Let (X, S_t) be a dissipative asymptotically compact dynamical system on a complete metric space X . Then S_t possesses a unique compact global attractor \mathfrak{A} such that*

$$\mathfrak{A} = \omega(B_0) = \bigcap_{t>0} \overline{\bigcup_{\tau \geq t} S_\tau B_0} \quad (2.3.2)$$

for every bounded absorbing set B_0 and

$$\lim_{t \rightarrow +\infty} (d_X\{S_t B_0 \mid \mathfrak{A}\} + d_X\{\mathfrak{A} \mid S_t B_0\}) = 0, \quad (2.3.3)$$

where as above $d_X\{A \mid B\} = \sup_{x \in A} \text{dist}_X(x, B)$. Moreover, if there exists a connected absorbing bounded set,⁵ then \mathfrak{A} is connected.

Property (2.3.3) states that \mathfrak{A} attracts bounded absorbing sets in the Hausdorff metric which is defined by the formula

$$\text{dist}_H\{A \mid B\} = d_X\{A \mid B\} + d_X\{B \mid A\}$$

for all bounded sets A and B . The convergence in the Hausdorff metric means that for any $\varepsilon > 0$ and for any absorbing set B there exists $t_\varepsilon > 0$ such that $S_t B \subset \mathcal{O}_\varepsilon(A)$ and $A \subset \mathcal{O}_\varepsilon(S_t B)$ for all $t \geq t_\varepsilon$. Here $\mathcal{O}_\varepsilon(D)$ denotes the ε -vicinity of the set D .

We note that in finite-dimensional systems for the existence of a global attractor we need the dissipativity property only. This observation implies that the Duffing (Exercise 2.1.8) and Lorenz (Exercise 2.1.9) systems possess global attractors.

Exercise 2.3.6. Show that the 1D system generated by the equation $\dot{x} + x^3 - x = 0$ on \mathbb{R} possesses a global attractor \mathfrak{A} and \mathfrak{A} is the interval $[-1, 1]$. Hint: See Exercise 2.1.5 for dissipativity; also, make use of the fact that the attractor is a connected set containing the rest points $x = \pm 1$. ■

Further applications of Theorem 2.3.5 will be presented later.

Proof of Theorem 2.3.5. Since S_t is dissipative, there exists a bounded absorbing set B_0 . This implies that for every bounded set D the tail γ'_D lies in B_0 for all $t \geq t_D$. Therefore, using the asymptotic compactness of (X, S_t) , by Lemma 2.2.5 we conclude that $\omega(B_0)$ is a nonempty compact strictly invariant set such that (2.3.1) holds. Thus, the formula in (2.3.2) gives a global attractor.

To prove (2.3.3) we need to show that

$$\lim_{t \rightarrow +\infty} \sup \{d_X(x, S_t B_0) : x \in \mathfrak{A}\} = 0.$$

⁵We can assume instead that X is a connected space in the sense that every two points from X can be connected by a continuous path.

This follows from the fact that $\mathfrak{A} \subset B_0$, which implies that

$$\mathfrak{A} = S_t \mathfrak{A} \subset S_t B_0 \text{ for all } t > 0.$$

To prove connectedness we use (2.3.3) and the contradiction argument.

Let B_0 be connected. Assume that \mathfrak{A} is not connected, i.e., $\mathfrak{A} = K \cup K_*$, where K and K_* are two nonempty disjoint compact sets such that $\text{dist}(K, K_*) = 3\delta > 0$. By (2.3.3) we have that

$$S_t B_0 \subset \mathcal{O}_\delta(\mathfrak{A}) \equiv \{x \in X : \text{dist}_X(x, \mathfrak{A}) < \delta\} \quad (2.3.4)$$

for all t large enough. Obviously $S_t B_0$ is connected for each t . Thus, by (2.3.4) we have that $S_t B_0 \subset \mathcal{O}_\delta(\tilde{K})$, where \tilde{K} is either K or K_* , say $\tilde{K} = K$. Using (2.3.3) again we have that

$$K_* \subset \mathfrak{A} \subset \mathcal{O}_\delta(S_t B_0) \subset \mathcal{O}_{2\delta}(\tilde{K})$$

for all t large enough. This is impossible because $\text{dist}(K, K_*) = 3\delta > 0$. \square

It is clear that if an evolution operator possesses a compact global attractor, then it is dissipative and asymptotically compact. Thus, Theorem 2.3.5 implies that a dynamical system (X, S_t) has a compact global attractor *if and only if* it is dissipative and asymptotically compact (or asymptotically smooth).

Exercise 2.3.7. Show that under the hypotheses and the notation of Theorem 2.3.5 we have that

$$\mathfrak{A} = \bigcap_{n \geq N} S_{nT} B_0 \text{ for every } N \in \mathbb{Z}_+ \text{ and } T > 0. \quad (2.3.5)$$

Hint: $\mathfrak{A} \subset B_0$ and thus $\mathfrak{A} = S_{nT} \mathfrak{A} \subset S_{nT} B_0$ for every $n \in \mathbb{Z}_+$ and $T > 0$. \blacksquare

Exercise 2.3.8. Let a system (X, S_t) be dissipative and $V = S_{t_*}$ be an α -contraction for some $t_* > 0$ (see Definition 2.2.15). Then (X, S_t) possesses a compact global attractor which can be written in the form (2.3.5). Hint: See Exercise 2.2.16. \blacksquare

In some cases it is convenient to use the condition of point dissipativity instead of (bounded) dissipativity. The following assertion can be found in HALE [116] and RAUGEL [188]; see also CARVALHO/LANGA/ROBINSON [26].

Theorem 2.3.9. *An evolution semigroup S_t on some complete metric space X possesses a compact global attractor if and only if*

- (i) S_t is point dissipative;
- (ii) for every bounded set B there exists $\tau > 0$ such that the tail $\gamma_B^\tau = \bigcup_{t \geq \tau} S_t B$ is bounded;
- (iii) S_t is asymptotically smooth.

Proof. Due to Theorem 2.3.5 it is sufficient to prove that under the conditions above the system (X, S_t) is (bounded) dissipative. To show this, we use the same idea as in RAUGEL [188].

We first establish the following “locally compact” dissipativity property. Namely, we show that there exists a bounded forward invariant set B_* possessing the property: for every compact set K

$$\exists \epsilon = \epsilon_K > 0, \exists t_K \geq 0 : S_t \mathcal{O}_\epsilon(K) \subset B_* \text{ for all } t \geq t_K, \quad (2.3.6)$$

where $\mathcal{O}_\epsilon(K)$ is the ϵ -neighborhood of K . Indeed, since S_t is point dissipative, there exists a bounded set B_0 such that

$$\forall x_0 \in X, \exists t_{x_0} \geq 0 : S_t x_0 \in B_0 \text{ for all } t \geq t_{x_0}.$$

We can assume that B_0 is open. In this case, by the continuity of $S_{t_{x_0}}$ there is $\epsilon = \epsilon_{x_0} > 0$ such that

$$S_{t_{x_0}} \mathcal{O}_{\epsilon_{x_0}}(x_0) \subset B_0.$$

Let τ_0 be such that $B_* \equiv \gamma_{B_0}^{\tau_0}$ is bounded. In this case,

$$S_{\tau+t_{x_0}} \mathcal{O}_{\epsilon_{x_0}}(x_0) \subset \gamma_{B_0}^{\tau_0} = B_* \text{ for all } \tau \geq \tau_0.$$

If K is a compact set, then we can find a finite set $\{x_i\}$ in K such that

$$K \subset \mathcal{U} \equiv \cup \mathcal{O}_{\epsilon_{x_i}}(x_i).$$

It is clear that

$$S_t \mathcal{U} = \cup S_t \mathcal{O}_{\epsilon_{x_i}}(x_i) \subset B_* \text{ for all } t \geq \tau_0 + \max_i t_{x_i}.$$

Since \mathcal{U} is open, we can find $\epsilon = \epsilon_K > 0$ such that $\mathcal{O}_\epsilon(K) \subset \mathcal{U}$. Thus (2.3.6) is established.

To conclude the proof we note that for every bounded set B there exists $\tau = \tau_B$ such that γ_B^τ is bounded and forward invariant. Thus, by asymptotic smoothness, there is a compact set K such that

$$\forall \epsilon > 0, \exists t_\epsilon \geq 0 : S_t [\gamma_B^\tau] \subset \mathcal{O}_\epsilon(K) \text{ for all } t \geq t_\epsilon.$$

Hence the locally compact dissipativity property in (2.3.6) implies the desired conclusion. \square

The study of the structure of the global attractors is an important problem from the point of view of applications. There are no universal approaches to this problem. It is well known that even in finite-dimensional cases an attractor can possess an extremely complicated structure. However, some sets that belong to the attractor can

be easily pointed out. For example, every stationary point and every bounded full trajectory belong to the global attractor (see Exercises 2.3.3 and 2.3.4). The global attractor also contains unstable motions which can be introduced by the following definition (see, e.g., BABIN/VISHIK [9], CHUESHOV [39], TEMAM [216]).

Definition 2.3.10. Let \mathcal{N} be the set of stationary points of a dynamical system (X, S_t) :

$$\mathcal{N} = \{v \in X : S_t v = v \text{ for all } t \geq 0\}.$$

We define the *unstable manifold* $\mathcal{M}^u(\mathcal{N})$ emanating from the set \mathcal{N} as a set of all $y \in X$ such that there exists a full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ with the properties

$$u(0) = y \text{ and } \lim_{t \rightarrow -\infty} \text{dist}_X(u(t), \mathcal{N}) = 0. \quad (2.3.7)$$

■

Exercise 2.3.11. Show that $\mathcal{M}^u(\mathcal{N})$ is a (strictly) invariant set. ■

The following assertion can be found in BABIN/VISHIK [9], CHUESHOV [39], or TEMAM [216], for instance.

Proposition 2.3.12. Let \mathcal{N} be the set of stationary points of a dynamical system (X, S_t) possessing a global attractor \mathfrak{A} . Then $\mathcal{M}^u(\mathcal{N}) \subset \mathfrak{A}$.

Proof. Let $y \in \mathcal{M}^u(\mathcal{N})$ and $\gamma = \{u(t) : t \in \mathbb{R}\}$ be the trajectory possessing property (2.3.7). Then there exists $s \leq 0$ such that the set

$$\gamma_s \equiv \{u(t) : -\infty < t \leq s\} \subset \{z : \text{dist}(z, \mathcal{N}) \leq 1\}$$

Thus γ_s is bounded. It is also clear that γ_s is backward invariant, i.e., $\gamma_s \subset S_t \gamma_s$ for every $t > 0$. Therefore, the result of Exercise 2.3.3 implies that $\gamma_s \subset \mathfrak{A}$. Since $y \in S_{-s} \gamma_s$, this implies the desired conclusion. □

In some cases (see Section 2.4 below) it is possible to show that the unstable manifold coincides with the attractor; that is, $\mathcal{M}^u(\mathcal{N}) = \mathfrak{A}$.

To exclude unstable motions from consideration, it is convenient to use the concept of a global *minimal* attractor (see LADYZHENSKAYA[142]). This concept is also useful for a description of the long-time behavior of individual trajectories.

Definition 2.3.13 (Global minimal attractor). Let S_t be an evolution operator on a complete metric space X . A bounded closed set $\mathfrak{A}_{\min} \subset X$ is said to be a *global minimal attractor* for S_t if the following properties hold.

- (i) \mathfrak{A}_{\min} is a positively invariant set; that is, $S_t \mathfrak{A}_{\min} \subseteq \mathfrak{A}_{\min}$ for $t \geq 0$;
- (ii) \mathfrak{A}_{\min} attracts every point x from X ; that is,

$$\lim_{t \rightarrow +\infty} \text{dist}_X(S_t x, \mathfrak{A}_{\min}) = 0 \text{ for any } x \in X;$$

- (iii) \mathfrak{A}_{\min} is minimal; that is, \mathfrak{A}_{\min} has no proper closed subsets possessing (i) and (ii). ■

One can prove the following assertion.

Theorem 2.3.14. *Let S_t be an evolution operator on a complete metric space X . Assume that S_t is point dissipative (see Definition 2.1.1). If any semitrajectory γ_v^+ is Lagrange stable (see Definition 1.4.1), then S_t possesses a (unique) global minimal attractor \mathfrak{A}_{\min} . Moreover, the attractor \mathfrak{A}_{\min} has the representation*

$$\mathfrak{A}_{\min} = \overline{\bigcup \{\omega(x) : x \in X\}}. \quad (2.3.8)$$

Proof. Since any positive semitrajectory is Lagrange stable, then by Theorem 1.4.5 and Proposition 1.4.6 each ω -limit set $\omega(x)$ is a strictly invariant compact set which attracts $S_t x$. Thus,

$$A_{\min} = \bigcup \{\omega(x) : x \in X\} \quad (2.3.9)$$

is a strictly invariant set attracting all semitrajectories. Due to point dissipativity this set A_{\min} is bounded. Now using the continuity of S_t one can see that the closure $\overline{A_{\min}}$ of A_{\min} is forward⁶ invariant, and thus $\mathfrak{A}_{\min} \equiv \overline{A_{\min}}$ is a minimal global attractor. □

Exercise 2.3.15. Assume that a system (X, S_t) possesses a compact global minimal attractor \mathfrak{A}_{\min} . Show that in this case any semitrajectory γ_v^+ is Lagrange stable, and thus \mathfrak{A}_{\min} has form (2.3.8). ■

The following assertion (see DE [83]) shows how global and global minimal attractors are related.

Theorem 2.3.16. *Assume that an evolution operator S_t on a complete metric space X possesses a compact global attractor \mathfrak{A} . Then there exists a global minimal attractor \mathfrak{A}_{\min} which is a compact subset of \mathfrak{A} and has the form (2.3.8). Moreover, \mathfrak{A}_{\min} is strictly invariant and*

$$\mathfrak{A} = \omega(\mathcal{O}_\delta(\mathfrak{A}_{\min})) = \bigcap_{t>0} \overline{\bigcup_{\tau \geq t} S_\tau(\mathcal{O}_\delta(\mathfrak{A}_{\min}))} \text{ for every } \delta > 0, \quad (2.3.10)$$

where $\mathcal{O}_\delta(D)$ denotes the δ -neighborhood of the set D . Thus, any small neighborhood of \mathfrak{A}_{\min} “generates” the global attractor \mathfrak{A} .

Proof. It is clear that we can apply Theorem 2.3.14 and show that \mathfrak{A}_{\min} given by (2.3.8) is a global minimal attractor. Since $\omega(x) \subset \mathfrak{A}$ for every x , we have that $\mathfrak{A}_{\min} \subset \mathfrak{A}$ and thus it is compact.

The set A_{\min} given by (2.3.9) is strictly invariant. Therefore, we can apply Exercise 1.2.1(F) to show that $\mathfrak{A}_{\min} = \overline{A_{\min}}$ is strictly invariant.

⁶ In general we cannot guarantee the strict invariance of this closure, see Exercise 1.2.1(F).

To prove (2.3.10) we note that for every $\delta > 0$ the set $B_0 = \mathcal{O}_\delta(\mathfrak{A}_{\min})$ is point-absorbing, i.e.,

$$\forall x \in X, \exists t_x > 0 : S_t x \in B_0, \forall t \geq t_x.$$

Thus, we can apply the same argument as in the proof of Theorem 2.3.9 and show that there exist $\varepsilon_0 > 0$ and $t_0 > 0$ such that

$$S_{t_0} \mathcal{O}_{\varepsilon_0}(\mathfrak{A}) \subset \gamma_{B_0}^{t_*} = \bigcup_{\tau \geq t_*} S_\tau B_0$$

for some $t_* \geq 0$. Therefore,

$$\mathfrak{A} = S_{t+t_0} \mathfrak{A} \subset S_t (S_{t_0} \mathcal{O}_{\varepsilon_0}(\mathfrak{A})) \subset \gamma_{B_0}^{t+t_*} \subset \gamma_{B_0}^t \text{ for every } t \geq 0.$$

Thus,

$$\mathfrak{A} \subset \bigcap_{t > 0} \overline{\gamma_{B_0}^t} \equiv \omega(B_0).$$

This completes the proof of Theorem 2.3.16. \square

For some further discussions of properties of global minimal attractors we refer to LADYZHENSKAYA [142] and DE [83].

2.3.2 Weak global attractor

The most restrictive assumption guaranteeing the existence of a global attractor is asymptotic compactness of the corresponding dynamical system (see Theorem 2.3.5). However, in some cases it is possible to get rid of this requirement. For this we need the notion of a global *weak* attractor.

Definition 2.3.17 (Global weak attractor). Let S_t be an evolution operator in a reflexive Banach space X . A bounded weakly closed set \mathfrak{A} in X is called a *global weak attractor* if (i) it is invariant ($S_t \mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$) and (ii) it is uniformly attracting in the weak topology: for any weak vicinity \mathcal{O} of the set \mathfrak{A} and for every bounded set $B \subset X$ there exists $t_* = t(\mathcal{O}, B) > 0$ such that $S_t B \subset \mathcal{O}$ for all $t \geq t_*$. \blacksquare

It is clear that if a global attractor exists and is weakly closed, then it is also weak. Thus, in the finite-dimensional case they are the same.

Theorem 2.3.18. *Let S_t be an evolution semigroup on a separable reflexive Banach space X . Assume that S_t is weakly closed; i.e., for every $t > 0$ the weak convergence properties $x_n \rightarrow x$ and $S_t x_n \rightarrow y$ imply that $y = S_t x$. If this semigroup S_t is dissipative, then it possesses a weak global attractor.*

The argument for the proof relies on weak compactness of bounded sets in a separable reflexive Banach space. For details we refer to BABIN/VISHIK [9] or CHUESHOV [39]. Here we give an alternative argument by showing that the situation can be reduced to Theorem 2.3.5.

Proof. Let D be an absorbing bounded set for S_t . We can suppose that D is weakly closed. Let $t_* \geq 0$ be such that $S_t D \subset D$ for all $t \geq t_*$ and

$$D_* = \gamma_D^{t_*} \equiv \bigcup_{t \geq t_*} S_t D.$$

One can see that D_* is a bounded forward invariant set. Since S_t is weakly closed, the weak closure D_*^w of D_* possesses the same properties. Moreover (see DUNFORD/SCHWARTZ [88, Chapter 5, Section 5]), this set D_*^w endowed with weak topology is a *compact* metric space with respect to the distance

$$\varrho(f, g) = \sum_{n=1}^{\infty} \frac{|l_n(f - g)|}{1 + |l_n(f - g)|}, \quad f, g \in D_*^w,$$

where $\{l_n\}$ is a complete set of functionals on X . Thus, the evolution operator S_t is automatically (asymptotically) compact, and we can apply Theorem 2.3.5. \square

We note that sometimes it is also convenient to use not only strong or weak convergences but also other topologies in the definition of global attractors. We refer to BABIN/VISHIK [9] (see also the recent survey BABIN [7]) for the theory of attractors involving two phase spaces with different topologies. We also refer to CHESKIDOV/FOIAS [33], CHESKIDOV [32], FOIAS/ROSA/TEMAM [106] and the literature cited there for some development of the theory of weak attractors with application to 3D hydrodynamics.

2.3.3 Stability properties and reduction principle

In order to describe the stability properties of attractors, we need the following notions.

Definition 2.3.19 (Lyapunov stability of invariant sets). A forward invariant set M is said to be *stable* (in the Lyapunov sense) if for any vicinity \mathcal{O} of the closure \bar{M} of M there exists an open set \mathcal{O}' such that $\bar{M} \subset \mathcal{O}' \subset \mathcal{O}$ and $S_t \mathcal{O}' \subset \mathcal{O}$ for all $t \geq 0$. The set M is *asymptotically stable* iff it is stable and $S_t x \rightarrow M$ as $t \rightarrow \infty$ for every $x \in \mathcal{O}'$. This set is *uniformly asymptotically stable* if it is stable and

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathcal{O}'} \text{dist}_X(S_t x, M) = 0.$$

■

We note that when $M = \gamma_v^+ = \{S_t v : t \geq 0\}$ is a semitrajectory, the stability of M as an invariant set follows from its stability as a trajectory (see Definition 1.6.1). However, as we can see in the following exercise, the inverse statement is not true.

Exercise 2.3.20. Show that any (nontrivial) trajectory in the system described in Exercise 1.8.10 is stable as an invariant set but unstable as a trajectory. ■

To distinguish these two types of (Lyapunov) stability, the stability of a trajectory as an invariant set is often called the *orbital stability* of the trajectory.

The following stability property of compact global attractors is important in many situations (see, e.g., BABIN/VISHIK [9] or CHUESHOV [39]).

Theorem 2.3.21. *Let (X, S_t) be a dynamical system in a complete metric space X possessing a compact global attractor \mathfrak{A} . Assume that there exists a bounded vicinity \mathcal{U} of \mathfrak{A} such that the mapping $(t; x) \mapsto S_t x$ is continuous on $\mathbb{R}_+ \times \mathcal{U}$. Then \mathfrak{A} is uniformly asymptotically stable.*

Proof. Let \mathcal{O} be a vicinity of \mathfrak{A} . Then there exists $T > 0$ such that $S_t \mathcal{U} \subset \mathcal{O}$ for all $t \geq T$. Now we show that there exists a vicinity \mathcal{O}^* of the attractor such that $S_t \mathcal{O}^* \subset \mathcal{O}$ for all $t \in [0, T]$. If this is not true, then there exist sequences $\{u_n\} \subset X$ and $\{t_n\} \subset [0, T]$ such that $\text{dist}\{u_n, \mathfrak{A}\} \rightarrow 0$ and $S_{t_n} u_n \notin \mathcal{O}$. Since \mathfrak{A} is compact, we can choose a subsequence $\{n_k\}$ such that $u_{n_k} \rightarrow u \in \mathfrak{A}$ and $t_{n_k} \rightarrow t \in [0, T]$. Therefore, the continuity property of the function $(t; x) \mapsto S_t x$ gives us that $S_{t_{n_k}} u_{n_k} \rightarrow S_t u \in \mathfrak{A}$. This contradicts the equation $S_{t_n} u_n \notin \mathcal{O}$. Thus, there exists an $\mathcal{O}^* \supset \mathfrak{A}$ such that $S_t \mathcal{O}^* \subset \mathcal{O}$ for $t \in [0, T]$. This implies that $S_t(\mathcal{O}^* \cap \mathcal{U}) \subset \mathcal{O}$ for all $t \in \mathbb{R}_+$. Therefore, the attractor \mathfrak{A} is stable. Thus, by the global attraction property the attractor \mathfrak{A} is uniformly asymptotically stable. □

In certain situations the following *reduction principle* enables us to significantly decrease the number of degrees of freedom in the problem. This is important in the study of infinite-dimensional systems.

Theorem 2.3.22 (Reduction principle). *Let (X, S_t) be a dissipative dynamical system in a complete metric space X . Assume that there exists a positively invariant locally compact⁷ closed set M possessing the property of uniform attraction:*

$$\lim_{t \rightarrow +\infty} \sup_{x \in D} \text{dist}_X(S_t x, M) = 0 \text{ for every bounded set } D. \quad (2.3.11)$$

If \mathfrak{A} is a global attractor of the restriction (M, S_t) of the system (X, S_t) on M , then \mathfrak{A} is also a global attractor for (X, S_t) .

Proof. We use the same method as in CHUESHOV [39, Chapter 1].

It is sufficient to verify that

$$\lim_{t \rightarrow +\infty} \sup_{x \in D} \text{dist}_X(S_t x, \mathfrak{A}) = 0 \quad (2.3.12)$$

⁷In the sense that every bounded subset of that set is relatively compact.

for any bounded set D in X . Assume that there exists a bounded closed set $B \subset X$ such that (2.3.12) does not hold. Then there exist sequences $\{y_n\} \subset B$ and $\{t_n : t_n \rightarrow +\infty\}$ such that

$$\text{dist}_X(S_{t_n}y_n, \mathfrak{A}) \geq \delta \text{ for some } \delta > 0. \quad (2.3.13)$$

Let B_0 be a bounded absorbing set for (X, S_t) . We choose a time t_* such that

$$\sup_{x \in M \cap B_0} \text{dist}_X(S_{t_*}x, \mathfrak{A}) \leq \delta/2 \quad (2.3.14)$$

This choice is possible because \mathfrak{A} is a global attractor for (M, S_t) . Equation (2.3.11) implies that

$$\text{dist}_X(S_{t_n - t_*}y_n, M) \rightarrow 0, \quad n \rightarrow +\infty.$$

The dissipativity of (X, S_t) gives us that $S_{t_n - t_*}y_n \in B_0$ for all n large enough. Therefore, local compactness of the set M guarantees the existence of an element $z \in M \cap B_0$ and a subsequence $\{n_k\}$ such that $z = \lim_{k \rightarrow \infty} S_{t_{n_k} - t_*}y_{n_k}$. This implies that $S_{t_{n_k}}y_{n_k} \rightarrow S_{t_*}z$. Therefore, equation (2.3.13) gives us that $\text{dist}_X(S_{t_*}z, \mathfrak{A}) \geq \delta$, which contradicts (2.3.14). This completes the proof of Theorem 2.3.22. \square

Example 2.3.23. Consider the following system of ODEs:

$$\begin{cases} \dot{y} + y^3 - \lambda y = yz^2, & t > 0, & y|_{t=0} = y_0, \\ \dot{z} + z(1 + y^2) = 0, & t > 0, & z|_{t=0} = z_0, \end{cases} \quad (2.3.15)$$

where $\lambda \in \mathbb{R}$. One can see that for any initial data the problem in (2.3.15) has a unique solution on some semi-interval $[0, t_*)$, where $t_* \leq \infty$ depends on $(y_0; z_0)$. If we multiply the first equation by $y(t)$ and the second equation by $z(t)$, then after taking the sum we obtain

$$\frac{1}{2} \frac{d}{dt} [y^2 + z^2] + y^4 - \lambda y^2 + z^2 = 0, \quad 0 < t < t_*.$$

This implies that the function $V(y, z) = y^2 + z^2$ possesses the property

$$\frac{d}{dt} V(y(t), z(t)) + 2V(y(t), z(t)) \leq \frac{(1 + \lambda)^2}{2}, \quad 0 < t < t_*.$$

Therefore,

$$V(y(t), z(t)) \leq V(y_0, z_0)e^{-2t} + \frac{(1 + \lambda)^2}{4}(1 - e^{-2t}), \quad 0 < t < t_*.$$

This implies that any solution to problem (2.3.15) can be extended to the whole semi-axis \mathbb{R}_+ and the dynamical system (\mathbb{R}^2, S_t) generated by (2.3.15) is dissipative. Obviously, the set $M = \{(y; 0) : y \in \mathbb{R}\}$ is positively invariant. Moreover, the second equation in (2.3.15) yields that

$$\frac{1}{2} \frac{d}{dt} [z(t)]^2 + [z(t)]^2 \leq 0, \quad t > 0.$$

on the solutions. Hence, $|z(t)| \leq |z_0|e^{-t}$ for all $t > 0$. Thus, the set M exponentially attracts all bounded sets from \mathbb{R}^2 . Consequently, Theorem 2.3.22 yields that the global attractor of the dynamical system (M, S_t) is also the attractor of the system (\mathbb{R}^2, S_t) .

On the set M , equations (2.3.15) are reduced to the problem

$$\dot{y} + y^3 - \lambda y = 0, \quad t > 0, \quad y|_{t=0} = y_0. \quad (2.3.16)$$

Thus, the global attractors of the dynamical systems generated by equations (2.3.15) and (2.3.16) coincide, and the study of the dynamics on the plane is reduced to the investigation of properties of a certain one-dimensional dynamical system. ■

Exercise 2.3.24. Using the same idea as in Exercise 2.3.6, show that the global attractor \mathfrak{A} of the system (\mathbb{R}, \hat{S}_t) generated by (2.3.16) is the interval $[-\sqrt{\lambda_+}, \sqrt{\lambda_+}]$ in \mathbb{R} , where $\lambda_+ = \max\{0, \lambda\}$. Therefore, by Theorem 2.3.22 the global attractor \mathfrak{A} of the dynamical system (\mathbb{R}^2, S_t) generated by (2.3.15) has the form $\mathfrak{A} = \{(y; z) : -\sqrt{\lambda_+} \leq y \leq \sqrt{\lambda_+}, z = 0\}$. ■

Another example of a model with the reduction possibility is described in the following exercise.

Exercise 2.3.25 (Two-mode plasma equation). This model arises as the lowest mode approximation of some equations arising in plasma physics (see, e.g., CHUE-SHOV/SHCHERBINA [70, 71] and the references therein). We consider the following system of ODEs:

$$\begin{cases} \ddot{y} + \gamma \dot{y} + y^3 - y = |z|^2, & t > 0, & y|_{t=0} = y_0, & \dot{y}|_{t=0} = y_1, \\ i\dot{z} - z(1 + y) + i\delta z = 0, & t > 0, & z|_{t=0} = z_0, \end{cases} \quad (2.3.17)$$

where y is a real and z is a complex unknown function. Assume that γ and δ are positive parameters. Prove the following statements:

- (A) Equations (2.3.17) generate a dynamical system (X, S_t) in $X = \mathbb{R}^2 \times \mathbb{C}$. Hint: One can see that for any initial data the problem in (2.3.17) has a unique solution on some semi-interval $[0, t_*)$, where $t_* \leq \infty$ depends on $(y_0; y_1; z_0)$. If we multiply the second equation by $\bar{z}(t)$ and take the imaginary part, then we get the relation

$$\frac{d}{dt}|z(t)|^2 + 2\delta|z(t)|^2 = 0$$

on the existence time interval. This allows us to apply the non-explosion criterion (see Theorem A.1.2).

- (B) The subspace $X_0 = \{U = (y_0; y_1; 0) : (y_0; y_1) \in \mathbb{R}^2\}$ is (strictly) invariant, and the restriction $(X_0; S_t)$ of (X, S_t) on X_0 is generated by the Duffing equation

$$\ddot{y} + \gamma\dot{y} + y^3 - y = 0, \quad t > 0, \quad y|_{t=0} = y_0, \quad \dot{y}|_{t=0} = y_1. \quad (2.3.18)$$

- (C) The subspace $X_0 = \{U = (y_0; y_1; 0) : (y_0; y_1) \in \mathbb{R}^2\}$ is an exponentially attracting set for S_t .
- (D) Show that system (2.3.17) is dissipative. Hint: Make use of the same Lyapunov function as in Exercise 2.1.8 for the first equation and the fact that $|z(t)| \leq |z_0| \exp\{-\delta t\}$.
- (E) Using the reduction principle, describe the global attractor for the system (X, S_t) .

■

We can also formulate a reduction principle with exponential convergence properties (see FABRIE ET AL. [94] and also CHUESHOV [39, Lemma 1.9.6]).

Theorem 2.3.26. *Let (X, S_t) be a dissipative dynamical system in a complete metric space X . In addition to the hypotheses of Theorem 2.3.22, we assume:*

- *There is an absorbing set B_0 and constants $K, \alpha > 0$ such that*

$$\text{dist}_X(S_t x, S_t y) \leq L e^{\alpha t} \text{dist}_X(x, y) \quad \text{for any } x, y \in B_0. \quad (2.3.19)$$

- *The convergence in (2.3.11) holds with exponential rate, i.e., there exist $K, \gamma > 0$ such that*

$$\sup_{x \in B_0} \text{dist}_X(S_t x, M) \leq K e^{-\gamma t}, \quad t > 0. \quad (2.3.20)$$

- *The attractor \mathfrak{A} is exponential in M , i.e., for any bounded set D in M there exist positive constants K_D and γ_D and time t_D such that*

$$\sup_{x \in D} \text{dist}_X(S_t x, \mathfrak{A}) \leq K_D e^{-\gamma_D t}, \quad t \geq t_D. \quad (2.3.21)$$

Then \mathfrak{A} is an exponential attractor for (X, S_t) , i.e., for any bounded set B in X there exist positive constants K_B and γ_D and time t_B such that

$$\sup_{x \in B} \text{dist}_X(S_t x, \mathfrak{A}) \leq K_B e^{-\gamma_B t}, \quad t \geq t_B. \quad (2.3.22)$$

Proof. We first prove the following lemma.

Lemma 2.3.27 (Fabrie et al. [94]). *Let (X, S_t) be a dynamical system in a complete metric space X . Assume that there exist $L, \alpha > 0$ such that*

$$\text{dist}_X(S_t x, S_t y) \leq L e^{\alpha t} \text{dist}_X(x, y) \quad \text{for any } x, y \in X. \quad (2.3.23)$$

Let $M_i, i = 0, 1, 2$, be subsets in X such that $S_t M_i$ converges to M_{i+1} with exponential speed, $i = 0, 1$. This means that

$$\sup_{x \in M_0} \text{dist}_X(S_t x, M_1) \leq K_1 e^{-\gamma_1 t} \quad \text{and} \quad \sup_{x \in M_1} \text{dist}_X(S_t x, M_2) \leq K_2 e^{-\gamma_2 t} \quad (2.3.24)$$

for some positive constants K_i and γ_i . Then $S_t M_0 \rightarrow M_2$ exponentially, i.e.,

$$\sup_{x \in M_0} \text{dist}_X(S_t x, M_2) \leq (L K_1 + K_2) e^{-\gamma t} \quad \text{with} \quad \gamma = \frac{\gamma_1 \gamma_2}{\alpha + \gamma_1 + \gamma_2}. \quad (2.3.25)$$

Proof. Let $x \in M_0$ and $z \in M_2$. Then

$$\text{dist}_X(S_t x, z) \leq \text{dist}_X(S_{\kappa t} S_{(1-\kappa)t} x, S_{\kappa t} w) + \text{dist}_X(S_{\kappa t} w, z)$$

for any $0 \leq \kappa \leq 1$ and $w \in M_1$. By (2.3.23),

$$\text{dist}_X(S_t x, z) \leq L e^{\alpha \kappa t} \text{dist}_X(S_{(1-\kappa)t} x, w) + \text{dist}_X(S_{\kappa t} w, z).$$

Therefore,

$$\text{dist}_X(S_t x, M_2) \leq L e^{\alpha \kappa t} \text{dist}_X(S_{(1-\kappa)t} x, w) + \sup_{y \in M_1} \text{dist}_X(S_{\kappa t} y, M_2)$$

for any $0 \leq \kappa \leq 1$ and $w \in M_1$. This implies that

$$\sup_{x \in M_0} \text{dist}_X(S_t x, M_2) \leq L K_1 e^{[\alpha \kappa - \gamma_1 (1-\kappa)]t} + K_2 e^{-\gamma_2 \kappa t}$$

for any $0 \leq \kappa \leq 1$. Taking $\kappa = \gamma_1 (\alpha + \gamma_1 + \gamma_2)^{-1}$ we obtain (2.3.25). \square

To conclude the proof of Theorem 2.3.26, we apply Lemma 2.3.27 with

$$X = \hat{B} \equiv \overline{\bigcup_{t \geq \hat{t}} S_t B_0} = M_0, \quad M_1 = M \cap \hat{B}, \quad M_2 = \mathfrak{A},$$

where \hat{t} is chosen such that $\hat{B} \subset B_0$ is a forward invariant absorbing set. \square

Exercise 2.3.28. Show that any solution to the 1D ODE in (2.3.16) with $\lambda > 0$ and with initial data $|y_0| \geq \lambda$ has the form

$$y(t) = \sqrt{\lambda} y_0 \left[y_0^2 - (y_0^2 - \lambda) e^{-2\lambda t} \right]^{-1/2}, \quad t \geq 0.$$

Using this formula, prove that the attractor $\hat{\mathfrak{A}}$ (see Exercise 2.3.24) for the system generated by (2.3.16) is exponential. Then apply Theorem 2.3.26 to show that the global attractor \mathfrak{A} of the dynamical system (\mathbb{R}^2, S_t) generated by (2.3.15) is also exponential. ■

Exercise 2.3.29. Show that the attractor \mathfrak{A} for (2.3.15) for $\lambda = 0$ is not exponential. Hint: Use the formula from Exercise 1.7.5. ■

2.3.4 Stability of attractors with respect to parameters

We next deal with the stability of attractors with respect to perturbations of a dynamical system. For this we consider a family of dynamical systems (X, S_t^λ) with the same phase space X and with evolution operators S_t^λ depending on a parameter λ from a complete metric space Λ .

We start with the following simple assertion (see, e.g., ROBINSON [195, Theorem 10.16]). We also refer to BABIN/VISHIK [9] and HALE [115] for similar results on semicontinuity.

Proposition 2.3.30. *Let X be a complete metric space and S_t^λ be a family of evolution semigroups on X possessing global attractors \mathfrak{A}^λ for $\lambda \in \Lambda$. Assume that*

- *the attractors \mathfrak{A}^λ are uniformly bounded, i.e., there exists a bounded set B_0 such that $\mathfrak{A}^\lambda \subset B_0$;*
- *there exists $t_0 \geq 0$ such that $S_t^\lambda x \rightarrow S_t^{\lambda_0} x$ as $\lambda \rightarrow \lambda_0$ for each $t \geq t_0$ uniformly with respect to $x \in B_0$, i.e.,*

$$\sup_{x \in B_0} \text{dist}_X(S_t^\lambda x, S_t^{\lambda_0} x) \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0. \quad (2.3.26)$$

Then the family $\{\mathfrak{A}^\lambda\}$ of attractors is upper semicontinuous at the point λ_0 , i.e.,

$$d_X \{\mathfrak{A}^\lambda \mid \mathfrak{A}^{\lambda_0}\} \equiv \sup \{\text{dist}_X(x, \mathfrak{A}^{\lambda_0}) : x \in \mathfrak{A}^\lambda\} \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0.$$

Proof. Given $\varepsilon > 0$ there exists $t > t_0$ such that $S_t^{\lambda_0} B_0 \subset \mathcal{O}_\varepsilon(\mathfrak{A}^{\lambda_0})$. We also have that

$$\text{dist}_X(S_t^\lambda x, S_t^{\lambda_0} B_0) \leq \sup_{y \in B_0} \text{dist}_X(S_t^\lambda y, S_t^{\lambda_0} y), \quad \forall x \in B_0.$$

Thus,

$$\exists \delta > 0 : S_t^\lambda B_0 \subset \mathcal{O}_{2\varepsilon}(\mathfrak{A}^{\lambda_0}) \quad \text{as soon as } \text{dist}_\Lambda(\lambda, \lambda_0) < \delta.$$

Consequently,

$$\forall \varepsilon > 0 : \mathfrak{A}^\lambda = S_t^\lambda \mathfrak{A}^\lambda \subset S_t^\lambda B_0 \subset \mathcal{O}_{2\varepsilon}(\mathfrak{A}^{\lambda_0}) \text{ when } \text{dist}_\Lambda(\lambda, \lambda_0) < \delta.$$

This implies the conclusion. \square

We illustrate the statements of this proposition and the next theorem in Exercises 2.3.34 and 2.3.35 below.

The following assertion, which was proved by KAPITANSKY/KOSTIN [130] (see also BABIN/VISHIK [9] and CHUESHOV [39]), assumes a much weaker hypothesis concerning convergence of semigroups. This can be critical in singularly perturbed evolutions; see examples in KAPITANSKY/KOSTIN [130]. However, in contrast with the previous assertion, some uniform compactness property of the attractors is assumed.

Theorem 2.3.31 (Kapitansky-Kostin [130]). *Assume that a dynamical system (X, S_t^λ) in a complete metric space X possesses a compact global attractor \mathfrak{A}^λ for every $\lambda \in \Lambda$. Assume that the following conditions hold.*

- (i) *There exists⁸ a compact $K \subset X$ such that $\mathfrak{A}^\lambda \subset K$.*
- (ii) *If $\lambda_k \rightarrow \lambda_0$, $x_k \rightarrow x_0$, and $x_k \in \mathfrak{A}^{\lambda_k}$, then*

$$S_\tau^{\lambda_k} x_k \rightarrow S_\tau^{\lambda_0} x_0 \text{ for some } \tau > 0. \quad (2.3.27)$$

Then the family $\{\mathfrak{A}^\lambda\}$ of attractors is upper semicontinuous at the point λ_0 ; that is,

$$d_X \{\mathfrak{A}^\lambda \mid \mathfrak{A}^{\lambda_0}\} = \sup \{ \text{dist}_X(x, \mathfrak{A}^{\lambda_0}) : x \in \mathfrak{A}^\lambda \} \rightarrow 0 \text{ as } \lambda \rightarrow \lambda_0. \quad (2.3.28)$$

Moreover, if (2.3.27) holds for every $\tau > 0$, then the upper limit $\mathfrak{A}(\lambda_0, \Lambda)$ of the attractors \mathfrak{A}^λ at λ_0 defined by the formula

$$\mathfrak{A}(\lambda_0, \Lambda) = \bigcap_{\delta > 0} \overline{\bigcup \{ \mathfrak{A}^\lambda : \lambda \in \Lambda, 0 < \text{dist}(\lambda, \lambda_0) < \delta \}} \quad (2.3.29)$$

is a nonempty compact strictly invariant set lying in the attractor \mathfrak{A}^{λ_0} and possessing the property

$$d_X \{ \mathfrak{A}^\lambda \mid \mathfrak{A}(\lambda_0, \Lambda) \} \rightarrow 0 \text{ as } \lambda \rightarrow \lambda_0. \quad (2.3.30)$$

Proof. Assume that equation (2.3.28) does not hold. Then there exists a sequence $\lambda_m \rightarrow \lambda_0$ such that $d_X \{ \mathfrak{A}^{\lambda_m} \mid \mathfrak{A}^{\lambda_0} \} \geq 2\delta$ for all $m = 1, 2, \dots$ and for some $\delta > 0$.

⁸This property can be relaxed, see Exercise 2.3.33 below.

Thus we can find a sequence $x_m \in \mathfrak{A}^{\lambda_m}$ such that $\text{dist}_X(x_m, \mathfrak{A}^{\lambda_0}) \geq \delta$. But this sequence $\{x_m\}$ lies in the compact K . Therefore, without loss of generality we can assume that $x_m \rightarrow x_0$ for some $x_0 \in K$ such that $x_0 \notin \mathfrak{A}^{\lambda_0}$. We show now that this fact leads to a contradiction.

Let $\gamma_m = \{u_m(t) : t \in \mathbb{R}\} \subset \mathfrak{A}^{\lambda_m}$ be a full trajectory of the dynamical system $(X, S_t^{\lambda_m})$ passing through the element x_m ($u_m(0) = x_m$). Using the standard diagonal process, one can see that there exist a subsequence $\{m_n\}$ and a sequence of elements $u_N \in K$ such that

$$\lim_{n \rightarrow \infty} u_{m_n}(-N\tau) = u_N \quad \text{for all } N = 0, 1, \dots,$$

with $u_0 = x_0$, where τ is the same as in (2.3.27). The condition (ii) also implies that

$$u_{N-L} = \lim_{n \rightarrow \infty} u_{m_n}(-(N-L)\tau) = \lim_{n \rightarrow \infty} S_{L\tau}^{\lambda_{m_n}} u_{m_n}(-N\tau) = S_{L\tau}^{\lambda_0} u_N$$

for all $N = 1, 2, \dots$ and $L = 1, \dots, N$. Therefore, the function

$$u(t) = \begin{cases} S_t^{\lambda_0} u_0, & \text{for } t > 0; \\ S_{t+N\tau}^{\lambda_0} u_N, & \text{for } -\tau N \leq t < -\tau(N-1), \quad N = 1, 2, \dots, \end{cases}$$

gives a full trajectory γ of $(X, S_t^{\lambda_0})$ passing through x_0 . It is obvious that this trajectory is bounded. Therefore, by Exercise 2.3.4(A), $\gamma \subset \mathfrak{A}_0^{\lambda}$. This contradicts the relation $x_0 \notin \mathfrak{A}^{\lambda_0}$ and thus completes the proof of (2.3.28).

To prove the assertion concerning the set $\mathfrak{A}(\lambda_0, A)$ given by (2.3.29), we first note that, by the assumption in (i),

$$\mathfrak{A}^\delta(\lambda_0, A) = \overline{\bigcup \{\mathfrak{A}^\lambda : \lambda \in A, 0 < \text{dist}(\lambda, \lambda_0) < \delta\}}$$

is a compact set for each $\delta > 0$ and $\mathfrak{A}^\delta(\lambda_0, A) \supset \mathfrak{A}^{\delta'}(\lambda_0, A)$ for every $\delta \geq \delta'$. Thus

$$\mathfrak{A}(\lambda_0, A) = \bigcap_{\delta > 0} \mathfrak{A}^\delta(\lambda_0, A)$$

is a nonempty compact set. By (2.3.28) we have that $\mathfrak{A}(\lambda_0, A) \subseteq \mathfrak{A}^{\lambda_0}$. The (strict) invariance of $\mathfrak{A}(\lambda_0, A)$ follows from the obvious relation

$$x \in \mathfrak{A}(\lambda_0, A) \quad \text{if and only if} \quad \left\{ \exists \lambda_n \rightarrow \lambda_0, \exists x_n \in \mathfrak{A}^{\lambda_n} : x = \lim_{n \rightarrow \infty} x_n \right\}. \quad (2.3.31)$$

By (2.3.27) with $\tau > 0$ arbitrary we obtain that

$$S_t^{\lambda_0} x = \lim_{n \rightarrow \infty} S_t^{\lambda_n} x_n, \quad \forall t \geq 0.$$

Since $S_t^{\lambda_n} x_n \in \mathfrak{A}^{\lambda_n}$, the criterion in (2.3.31) implies that $S_t^{\lambda_0} x \in \mathfrak{A}^{\lambda_0}$ for every $t \geq 0$, i.e., $S_t^{\lambda_0} \mathfrak{A}(\lambda_0, \Lambda) \subset \mathfrak{A}(\lambda_0, \Lambda)$. To prove the backward invariance of $\mathfrak{A}(\lambda_0, \Lambda)$ we note that by invariance of the attractors \mathfrak{A}^{λ_n} there exists a sequence $y_n \in \mathfrak{A}^{\lambda_n}$ such that $x_n = S_t^{\lambda_n} y_n$. Due to the assumption in (i) and the criterion in (2.3.31), we can choose a subsequence $\{n_k\}$ such that $y_{n_k} \rightarrow y \in \mathfrak{A}(\lambda_0, \Lambda)$. Thus,

$$x = \lim_{k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} S_t^{\lambda_{n_k}} y_{n_k} = S_t^{\lambda_0} y,$$

which implies that $S_t^{\lambda_0} \mathfrak{A}(\lambda_0, \Lambda) \supset \mathfrak{A}(\lambda_0, \Lambda)$. Relation (2.3.30) follows from (2.3.31). This completes the proof of Theorem 2.3.31. \square

In the following two exercises we suggest that the reader make sure that condition (i) in Theorem 2.3.31 concerning uniform compactness can be relaxed.

Exercise 2.3.32. Let $\{B_n\}$ be a sequence of bounded sets in a complete metric space X . Assume that there exists a compact set K such that

$$d_X \{B_n \mid K\} = \sup \{\text{dist}_X(x, K) : x \in B_n\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then every sequence $\{x_n\}$ with $x_n \in B_n$ contains a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow z$ as $k \rightarrow \infty$ for some $z \in K$. \blacksquare

Exercise 2.3.33. Using the result of the previous exercise, show that condition (i) in Theorem 2.3.31 can be changed to the following one: there exists a compact set K_{λ_0} such that

$$d_X \{\mathfrak{A}^\lambda \mid K_{\lambda_0}\} \rightarrow 0 \text{ as } \lambda \rightarrow \lambda_0$$

(if condition (i) holds, then this property is definitely true with $K_{\lambda_0} = K$). \blacksquare

The situation with the (full) continuity of attractors \mathfrak{A}^λ with respect to λ is more complicated. In general the family $\{\mathfrak{A}^\lambda\}$ is not lower semicontinuous at the point λ_0 ; that is, the property $d_X \{\mathfrak{A}^{\lambda_0} \mid \mathfrak{A}^{\lambda_k}\} \rightarrow 0$ as $\lambda_k \rightarrow \lambda_0$ does not hold. The corresponding examples (borrowed from BABIN [7] and RAUGEL [188]) are given in the following exercises.

Exercise 2.3.34 (Raugel [188]). We consider a dynamical system generated in \mathbb{R} by the following equation:

$$\dot{x} = (1 - x)(x^2 - \lambda), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}.$$

Prove that for each value of the parameter $\lambda \in [-1, 1]$ this dynamical system possesses a global attractor \mathfrak{A}^λ . Show that

$$\mathfrak{A}^\lambda = \begin{cases} [-\sqrt{\lambda}, 1] & \text{for } \lambda \geq 0; \\ \{1\}, & \text{for } \lambda < 0. \end{cases}$$

Thus $d_X \{\mathfrak{A}^{\lambda_k} \mid \mathfrak{A}^{\lambda_0}\} \rightarrow 0$ as $\lambda_k \rightarrow \lambda_0$ for every $\lambda_0 \in [-1, 1]$ and $d_X \{\mathfrak{A}^0 \mid \mathfrak{A}^{\lambda_k}\} = 1$ as $\lambda_k \rightarrow -0$, which means that \mathfrak{A}^λ is not (fully) continuous at $\lambda = 0$. Moreover,

$$\mathfrak{A}(0, [-1, 0]) = \{1\} \neq \mathfrak{A}^0,$$

where $\mathfrak{A}(0, [-1, 0])$ is the upper limit defined according to (2.3.29). ■

A similar idea is realized in the next exercise.

Exercise 2.3.35 (Babin [7]). Let $(\mathbb{R}, S_t^\lambda)$ be a dynamical system generated by the equation

$$\dot{x} = -x[(|x| - 1)^2 - \lambda], \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}.$$

Prove that for each value of the parameter $\lambda \in \mathbb{R}$ the system $(\mathbb{R}, S_t^\lambda)$ possesses a global attractor \mathfrak{A}_λ and

$$\mathfrak{A}_\lambda = \begin{cases} [-1 - \sqrt{\lambda}, 1 + \sqrt{\lambda}] & \text{for } \lambda \geq 0; \\ \{0\}, & \text{for } \lambda < 0. \end{cases}$$

Thus, \mathfrak{A}_λ is continuous with respect to λ for every $\lambda \neq 0$ and is not lower semicontinuous at $\lambda = 0$. ■

We note that in order to prove lower semicontinuity under the hypotheses of Theorem 2.3.31 some additional assumptions should be imposed (see BABIN/VISHIK [9]). However, the lower semicontinuity property is generic under simple compactness assumptions (see the discussion in the surveys BABIN [7], RAUGEL [188] and also the recent note HOANG/OLSON/ROBINSON [124]). In particular, one can prove the following result (see HOANG/OLSON/ROBINSON [124] for the details).

Theorem 2.3.36 (Full continuity of attractors). *Let (X, S_t^λ) be a collection of dynamical systems on a complete metric space X . We suppose that the set Λ of parameters is also a complete metric space. Assume that the following conditions hold.*

- (i) (X, S_t^λ) possesses a compact global attractor \mathfrak{A}^λ for every $\lambda \in \Lambda$;
- (ii) there exists a compact set $K \subset X$ such that $\mathfrak{A}^\lambda \subset K$ for every $\lambda \in \Lambda$;
- (iii) for each $t > 0$ the function $\lambda \mapsto S_t^\lambda x$ is continuous uniformly for x in compact subsets of X .

Then the family $\{\mathfrak{A}^\lambda\}$ of attractors is continuous in λ with respect to the Hausdorff distance

$$d_H \{A \mid B\} \equiv \sup \{\text{dist}_X(x, B) : x \in A\} + \sup \{\text{dist}_X(x, A) : x \in B\}$$

at every point λ_0 from some residual set. Thus, the full continuity of $\lambda \mapsto \{\mathfrak{A}^\lambda\}$ is a generic property.

We recall (see, e.g., BOURBAKI [16]) that in the metric space Λ a *residual* set is the complement of a meager set. A subset D of Λ is said to be *meager* (or a first category set in the Baire sense), if it is contained in a countable union of closed nowhere dense subsets of Λ . A set K is said to be *nowhere dense* if its closure contains no open sets. By the Baire categories theorem (see, e.g., BOURBAKI [16]) any residual set is dense. A property \mathcal{P} is said to be *generic* in Λ if \mathcal{P} holds in some residual set of Λ .

In conclusion, we emphasize that the results presented in this subsection deal with stability of attractors with respect to parameters in the Hausdorff (semi)distance and do not consider issues related to uniform stability of individual perturbed trajectories on large time intervals. In this connection we point out the method of finite-dimensional composed trajectories for global tracking of trajectories of a perturbed system which was developed by BABIN/VISHIK [9, Chapters 7 and 8] (see also a short survey in BABIN [7] and the references therein).

2.4 Gradient systems

In this section we consider gradient systems. The main features of these systems are that (i) in the proof of the existence of a global attractor we can avoid a dissipativity property in explicit form, and (ii) the structure of the attractor can be described via unstable manifolds.

2.4.1 Lyapunov function

We start with the following definition.

Definition 2.4.1. Let $Y \subseteq X$ be a forward invariant set of a dynamical system (X, S_t) .

- A continuous functional $\Phi(y)$ defined on Y is said to be a *Lyapunov function* on Y for the dynamical system (X, S_t) if $t \mapsto \Phi(S_t y)$ is a non-increasing function for any $y \in Y$.
- The Lyapunov function $\Phi(y)$ is said to be *strict* on Y if the equation $\Phi(S_t y) = \Phi(y)$ for all $t > 0$ and for some $y \in Y$ implies that $S_t y = y$ for all $t > 0$; that is, y is a stationary point of (X, S_t) .
- The dynamical system (X, S_t) is said to be *gradient* if there exists a strict Lyapunov function for (X, S_t) on the whole phase space X . This Lyapunov function is usually called *global*.

■

The simplest examples of Lyapunov functions are given in the following exercises.

Exercise 2.4.2. Let $F : \mathbb{R}^d \mapsto \mathbb{R}$ be a C^2 function such that $F(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$. Show that the ordinary differential equation

$$\dot{x} = -\nabla F(x), \quad x \in \mathbb{R}^d, \quad t > 0,$$

generates a dynamical system (\mathbb{R}^d, S_t) which possesses a strict Lyapunov function $\Phi(x) = F(x)$ on \mathbb{R}^d . ■

Exercise 2.4.3. Consider the second order in time ordinary differential equation

$$\ddot{y} + \gamma \dot{y} + U'(y) = 0, \quad t > 0, \quad y|_{t=0} = y_0, \quad \dot{y}|_{t=0} = y_1,$$

where $\gamma > 0$ and $U(y)$ is a C^2 function on \mathbb{R} bounded from above. Show that this equation generates a dynamical system (\mathbb{R}^2, S_t) which possesses a strict Lyapunov function

$$\Phi(y, \dot{y}) = \frac{1}{2} \dot{y}^2 + U(y), \quad (y, \dot{y}) \in \mathbb{R}^2.$$

Hint: See Example 1.8.18 and Remark 1.8.19. ■

Example 2.4.4. Using the result of Exercise 2.4.3, one can see that the system generated by the plasma equation in (2.3.17) has a strict Lyapunov function on the attractor \mathfrak{A} . This is true due to the reduction principle, which shows that the dynamics on \mathfrak{A} can be described by the Duffing equation in (2.3.18). We do not know whether the system generated by (2.3.17) possesses a *global* Lyapunov function, i.e., whether it is gradient. The same effect can be seen in the model considered in Example 2.3.23. ■

2.4.2 Geometric structure of the attractor

The following result on the structure of a global attractor is known from many sources, including BABIN/VISHIK [9], CHUESHOV [39], HALE [116], HENRY [123], LADYZHENSKAYA [142], TEMAM [216].

Theorem 2.4.5. *Let a dynamical system (X, S_t) possess a compact global attractor \mathfrak{A} . Assume that there exists a strict Lyapunov function on \mathfrak{A} . Then $\mathfrak{A} = \mathcal{M}^u(\mathcal{N})$, where $\mathcal{M}^u(\mathcal{N})$ denotes the unstable manifold emanating from the set \mathcal{N} of stationary points (see Definition 2.3.10). Moreover, the global attractor \mathfrak{A} consists of full trajectories $\gamma = \{u(t) : t \in \mathbb{R}\}$ such that*

$$\lim_{t \rightarrow -\infty} \text{dist}_X(u(t), \mathcal{N}) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \text{dist}_X(u(t), \mathcal{N}) = 0. \quad (2.4.1)$$

Proof. It is known from Proposition 2.3.12 that $\mathcal{M}^u(\mathcal{N}) \subset \mathfrak{A}$. Thus, we need only prove that $\mathfrak{A} \subset \mathcal{M}^u(\mathcal{N})$.

Let $y \in \mathfrak{A}$. By Exercise 2.3.4(B) there exists a full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ passing through y , $u(0) = y$. Since $\gamma \subset \mathfrak{A}$, the set γ is compact. This implies that the α -limit set

$$\alpha(\gamma) = \bigcap_{\tau < 0} \overline{\bigcup \{u(t) : t \leq \tau\}}$$

of the trajectory γ is a nonempty compact set. One can see that the set $\alpha(\gamma)$ is invariant: $S_t \alpha(\gamma) = \alpha(\gamma)$. This follows from its compactness and the description given in Exercise 1.3.6.

Let us show that the Lyapunov function $\Phi(x)$ is a constant on $\alpha(\gamma)$. Indeed, if $u \in \alpha(\gamma)$, then there exists a sequence $\{t_n\}$ such that $t_n \rightarrow -\infty$ and $u(t_n) \rightarrow u$ as $n \rightarrow \infty$ (see Exercise 1.3.6). Consequently,

$$\Phi(u) = \lim_{n \rightarrow \infty} \Phi(u(t_n)).$$

By the monotonicity of Φ along trajectories, we have

$$\Phi(u) = \sup_{\tau < 0} \Phi(u(\tau)).$$

Therefore, the limit above does not depend on a sequence $\{u_n\}$ and the function $\Phi(u)$ is a constant on $\alpha(\gamma)$. Hence by invariance of $\alpha(\gamma)$ we have that $\Phi(S_t u) = \Phi(u)$ for all $t > 0$ and $u \in \alpha(\gamma)$. This means that $\alpha(\gamma)$ lies in the set \mathcal{N} of stationary points. Now we prove that

$$\lim_{t \rightarrow -\infty} \text{dist}(u(t), \alpha(\gamma)) = 0. \quad (2.4.2)$$

If (2.4.2) is not true, then there exists a sequence $\{t_n \rightarrow -\infty\}$ such that

$$\text{dist}_X(u(t_n), \alpha(\gamma)) \geq \delta > 0 \quad \text{for all } n = 1, 2, \dots \quad (2.4.3)$$

By the compactness of $\overline{\gamma}$ there exist an element $z \in X$ and a subsequence $\{t_{n_m}\}$ such that $u(t_{n_m}) \rightarrow z$ as $m \rightarrow \infty$. Moreover, by Exercise 1.3.6, $z \in \alpha(\gamma)$. This contradicts the property in (2.4.3) and thus (2.4.2) holds.

Since $\alpha(v) \subset \mathcal{N}$, equation (2.4.2) implies the first relation in (2.4.1) and hence $y \in \mathcal{M}^u(\mathcal{N})$ and $\mathfrak{A} = \mathcal{M}^u(\mathcal{N})$.

To prove the second relation in (2.4.1), we use the same idea as above. We consider the ω -limit set

$$\omega(\gamma) = \bigcap_{\tau > 0} \overline{\bigcup \{u(t) : t \geq \tau\}}$$

which is a nonempty compact strictly invariant set. As above, it follows from the monotonicity of Φ and the invariance of $\omega(\gamma)$ that the Lyapunov function $\Phi(x)$ is a constant on $\omega(\gamma)$ and hence $\Phi(S_t u) = \Phi(u)$ for all $t > 0$ and $u \in \omega(\gamma)$. This implies that $\omega(\gamma) \subset \mathcal{N}$. As above, by the contradiction argument,

$$\text{dist}(u(t), \mathcal{N}) \leq \text{dist}(u(t), \omega(\gamma)) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

This completes the proof of Theorem 2.4.5. \square

Remark 2.4.6. It follows from the first equality in (2.4.1) that under the hypotheses of Theorem 2.4.5 the following relation is valid:

$$\sup\{\Phi(u) : u \in \mathfrak{A}\} \leq \sup\{\Phi(u) : u \in \mathcal{N}\}, \quad (2.4.4)$$

where $\Phi(u)$ is the corresponding Lyapunov function. If $\Phi(u)$ topologically dominates the metric of the phase space X , then the inequality in (2.4.4) can be used in order to provide an upper bound for the size of the attractor and an absorbing ball. This method can be applied to obtain uniform (with respect to the parameters of the problem) bounds for the attractor. We refer to Section 5.3 for an application of this idea for some class of second order in time models. \blacksquare

If the system (X, S_t) is gradient; i.e., if a strict Lyapunov function exists on the whole phase space, then the result of Theorem 2.4.5 can be improved (see, e.g., BABIN/VISHIK [9] or CHUESHOV [39]). More precisely, we can describe the long-time behavior of individual trajectories.

Theorem 2.4.7. *Assume that a gradient dynamical system (X, S_t) possesses a compact global attractor \mathfrak{A} . Then*

$$\lim_{t \rightarrow +\infty} \text{dist}_X(S_t x, \mathcal{N}) = 0 \quad \text{for any } x \in X; \quad (2.4.5)$$

that is, any trajectory stabilizes to the set \mathcal{N} of stationary points.⁹ In particular, this means that the global minimal attractor \mathfrak{A}_{\min} coincides with the set of the stationary points, $\mathfrak{A}_{\min} = \mathcal{N}$.

Proof. For every $x \in X$ we consider the ω -limit set $\omega(x) = \bigcap_{\tau > 0} \overline{\{S_t x : t \geq \tau\}}$ and apply the same argument as in the end of the proof of Theorem 2.4.5. \square

Exercise 2.4.8. Show that relation (2.4.5) in the statement of Theorem 2.4.7 remains true if instead of the existence of a compact global attractor we assume that any semitrajectory of the system is Lagrange stable (see Definition 1.4.1). The assertion concerning global minimal attractors remains in force if we assume that the set \mathcal{N} is bounded. \blacksquare

⁹ This property is often referred to as *strong stability* of the set of equilibria.

Assume that $\mathcal{N} = \{z_1, \dots, z_n\}$ is a finite set. In this case $\mathfrak{A} = \bigcup_{i=1}^n \mathcal{M}^u(z_i)$, where $\mathcal{M}^u(z_i)$ is the unstable manifold of the stationary point z_i . That is, $\mathcal{M}^u(z_i)$ consists of all $y \in X$ such that there exists a full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ with the properties $u(0) = y$ and $u(t) \rightarrow z_i$ as $t \rightarrow -\infty$.

Theorems 2.4.5 and 2.4.7 lead us to the following consequences.

Corollary 2.4.9. *Assume that a gradient dynamical system (X, S_t) possesses a compact global attractor \mathfrak{A} and \mathcal{N} is a finite set. Then*

- (i) *The global attractor \mathfrak{A} consists of full trajectories $\gamma = \{u(t) : t \in \mathbb{R}\}$ connecting pairs of stationary points: any $u \in \mathfrak{A}$ belongs to some full trajectory $\gamma \subset \mathfrak{A}$ and for any $\gamma \subset \mathfrak{A}$ there exists a pair $\{z, z^*\} \subset \mathcal{N}$ such that*

$$u(t) \rightarrow z \text{ as } t \rightarrow -\infty \text{ and } u(t) \rightarrow z^* \text{ as } t \rightarrow +\infty.$$

- (ii) *For any $v \in X$ there exists a stationary point z such that $S_t v \rightarrow z$ as $t \rightarrow +\infty$.*

Remark 2.4.10. Assume that the hypotheses of Corollary 2.4.9 hold. Introduce m_0 distinct values $\Phi_1 < \Phi_2 < \dots < \Phi_{m_0}$ of the set $\{\Phi(x) : x \in \mathcal{N}\}$ and let

$$\mathcal{N}^j = \{x \in \mathcal{N} : \Phi(x) = \Phi_j\}, \quad j = 1, \dots, m_0.$$

Then the sets $\mathcal{N}^1, \dots, \mathcal{N}^{m_0}$ provide *Morse decomposition* of the attractor \mathfrak{A} . That is, (i) the subsets \mathcal{N}^j are compact, invariant, and disjoint; and (ii) for any $x \in \mathfrak{A} \setminus \bigcup_j \mathcal{N}^j$ and every full trajectory $\gamma_x \subset \mathfrak{A}$ through x there exist $k > l$ such that $\alpha(\gamma_x) \in \mathcal{N}^k$ and $\omega(\gamma_x) \in \mathcal{N}^l$, where $\alpha(\gamma_x)$ and $\omega(\gamma_x)$ are the α - and ω -limit sets for γ_x (see (1.3.2)).

In the situation considered the set \mathcal{N}^1 is uniformly asymptotically stable (see Definition 2.3.19). Thus, \mathcal{N}^1 is a subattractor of the attractor \mathfrak{A} . We recall that by the definition (see BABIN [7]) any compact strictly invariant uniformly asymptotically stable subset of \mathfrak{A} is called a *subattractor*. If the set \mathcal{N}^1 is not connected (e.g., it consists of isolated equilibria), then we can split \mathcal{N}^1 into several non-intersecting subattractors. This observation motivates (see BABIN [7]) the notion of a *fragmentation number* of the attractor \mathfrak{A} , which is defined as the maximal number of non-intersecting subattractors in \mathfrak{A} . This number characterizes the intrinsic complexity of the attractor. For further discussions we refer to BABIN [7] and the references therein. ■

The following example shows that the strictness of the corresponding Lyapunov function is important in the statements of Theorems 2.4.5 and 2.4.7.

Example 2.4.11 (Non-strict Lyapunov function). We consider the dynamical system (\mathbb{R}^2, S_t) generated by the following equations:

$$\begin{cases} \dot{x}_1 = \mu x_1 - \alpha x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 = \alpha x_1 + \mu x_2 - x_2(x_1^2 + x_2^2) \end{cases} \quad (2.4.6)$$

where $\mu \in \mathbb{R}$ and $\alpha > 0$ are parameters. In the case $\alpha = 1$ this system was considered in Example 1.9.4 as a demonstration for the Andronov-Hopf bifurcation. It was shown that for $\mu \leq 0$ the system has a unique equilibrium $x_* = (0; 0)$. If $\mu > 0$ and $\alpha > 0$ there is also the periodic orbit (the circle $C_{\sqrt{\mu}}$ with center at 0 and the radius $\sqrt{\mu}$). If in the latter case we take $\alpha = 0$, then the circle $C_{\sqrt{\mu}}$ consists of equilibria.

One can see that the function

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)^2 - \mu(x_1^2 + x_2^2)$$

satisfies the equation

$$\frac{dV(x_1, x_2)}{dt} = -2[(x_1^2 + x_2^2) - \mu]^2(x_1^2 + x_2^2) \leq 0$$

on a solution $S_t y_0 = (x_1(t); x_2(t))$. Thus, V is a Lyapunov function. Moreover, we observe the following picture:

- If $\mu \leq 0$ this function is *strict* (and the global attractor consists of a single (zero) equilibrium);
- If $\mu > 0$ and $\alpha = 0$, the function V is still *strict* (the circle $C_{\sqrt{\mu}}$ consists of equilibria) and the global attractor is the disc $D_{\sqrt{\mu}} = \{(x_1^2 + x_2^2) \leq \mu\}$, which can be seen as a collection of trajectories connecting the zero equilibrium and an equilibrium lying on $C_{\sqrt{\mu}}$.
- If $\mu > 0$ and $\alpha > 0$, the function V is *not strict* and the global attractor is the disc $D_{\sqrt{\mu}}$ which contains a nontrivial periodic orbit.

■

The following exercise demonstrates the non-uniqueness of the Lyapunov function V in Example 2.4.11. A similar effect for local Lyapunov functions was observed in Exercises 1.6.5 and 1.7.5(D).

Exercise 2.4.12. Show that

$$W(x_1, x_2) = \frac{1}{3}(x_1^2 + x_2^2)^3 - \frac{\mu}{2}(x_1^2 + x_2^2)$$

is also a Lyapunov function for (2.4.6) which is strict when either $\mu \leq 0$ or $\alpha = 0$.

■

Another example with non-strict Lyapunov function provides the Krasovskii system (see Exercises 1.8.22 and 2.1.10) under the condition that the damping coefficient $k(r)$ is non-negative and has a nonzero root.

To describe additional properties of global attractors for gradient systems, we introduce the following definition.

Definition 2.4.13. Let X be a Banach space. Assume that the evolution operator S_t of a dynamical system (X, S_t) is of class C^1 ; that is, $S_t u$ has a continuous Fréchet derivative¹⁰ with respect to $u \in X$ for each $t > 0$. An equilibrium point z of dynamical system (X, S_t) is said to be *hyperbolic* if the Fréchet derivative $S' \equiv DS_1(z)$ of $S_t z$ at the moment $t = 1$ is a linear operator in X with the spectrum $\sigma(S')$ possessing the property

$$\sigma(S') \cap \{w \in \mathbb{C} : |w| = 1\} = \emptyset.$$

We also define the index $\text{ind}(z)$ (of instability) of the equilibrium z as a dimension of the spectral subspace of the operator S' corresponding to the set $\sigma_+(S') \equiv \{z \in \sigma(S') : |z| > 1\}$. ■

The following assertion is proved in BABIN/VISHIK [9].

Theorem 2.4.14. Assume that a gradient dynamical system (X, S_t) in a Banach space X with a strict Lyapunov function $\Phi(u)$ possesses the following properties.

- (i) It admits a compact global attractor \mathfrak{A} .
- (ii) $S_t \in C^{1+\alpha}$ for some $\alpha > 0$ and there exists a vicinity $\mathcal{O} \supset \mathfrak{A}$ such that

$$\|DS_t(u) - DS_t(v)\|_{X \rightarrow X} \leq C_T \|u - v\|_X^\alpha, \quad u, v \in \mathcal{O}, \quad t \in [0, T].$$

- (iii) $(t, u) \mapsto S_t u$ is continuous over $\mathbb{R}_+ \times \mathfrak{A}$.
- (iv) The operators S_t are injective on \mathfrak{A} for any $t > 0$ and S_t^{-1} are continuous on \mathfrak{A} .
- (v) The Fréchet derivatives $DS_t(u)$ of $S_t u$ at any point $u \in \mathfrak{A}$ have zero kernel.
- (vi) The set $\mathcal{N} = \{z_1, \dots, z_n\}$ of equilibrium points is finite and every point $z_j \in \mathcal{N}$ is hyperbolic.

Let the indexation of equilibrium points be such that

$$\Phi(z_1) \leq \Phi(z_2) \leq \dots \leq \Phi(z_n)$$

and $M_k = \cup_{j=1}^k \mathcal{M}^u(z_j)$, $M_0 = \emptyset$, where $\mathcal{M}^u(z_j)$ is the unstable manifold emanating from z_j . Assume that the function $t \mapsto \Phi(S_t u)$ is strictly decreasing for $u \notin \mathcal{N}$.

Then $\mathfrak{A} = M_n$ and the following properties hold.

- (i) $\mathcal{M}^u(z_i) \cap \mathcal{M}^u(z_j) = \emptyset$ when $i \neq j$.
- (ii) M_k is a compact invariant set.
- (iii) $\partial \mathcal{M}^u(z_i) \equiv \overline{\mathcal{M}^u(z_i)} \setminus \mathcal{M}^u(z_i)$ is an invariant set and $\partial \mathcal{M}^u(z_i) \subset M_{i-1}$.
- (iv) For any compact set $K \subset \mathcal{M}^u(z_i) \setminus \{z_i\}$ we have

$$\lim_{t \rightarrow +\infty} \max\{\text{dist}_X(S_t k, M_{i-1}) : k \in K\} = 0.$$

¹⁰See Section A.5 in the Appendix for the definitions.

- (v) Every set $\mathcal{M}^u(z_i)$ is a C^1 -manifold of finite dimension d_i , this manifold is diffeomorphic to \mathbb{R}^{d_i} , and the embedding $\mathcal{M}^u(z_i) \subset X$ is of class C^1 in a vicinity of any point $v \in \mathcal{M}^u(z_i)$. Moreover, $d_i = \text{ind}(z_i)$.

In many cases it is important to know how fast the trajectories starting from bounded sets converge to global attractors. The result stated below provides conditions sufficient for an exponential rate of stabilization to the attractor along with some additional properties of the attractor (see, e.g., BABIN/VISHIK [9], HALE [116] and also Theorems 4.7 and 4.8 in the survey RAUGEL [188]).

Theorem 2.4.15. *Let (X, S_t) be a dynamical system in a Banach space X . Assume that (i) an evolution operator S_t is C^1 , (ii) the set \mathcal{N} of equilibrium points is finite and all equilibria are hyperbolic, (iii) there exists a function Lyapunov $\Phi(x)$ on X such that $\Phi(S_tx) < \Phi(x)$ for all $x \in X$, $x \notin \mathcal{N}$ and for all $t > 0$, and (iv) there exists a compact global attractor \mathfrak{A} . Then*

- For any $y \in X$ there exists $e \in \mathcal{N}$ such that

$$\|S_ty - e\|_X \leq C_y e^{-\omega t}, \quad t > 0.$$

Moreover,

$$\sup \{\text{dist}(S_ty, \mathfrak{A}) : y \in B\} \leq C_B e^{-\omega t}, \quad t > 0, \quad (2.4.7)$$

for any bounded set B in X . Here C_y , C_B , and ω are positive constants, and ω in (2.4.7) depends on the minimum, over $e \in \mathcal{N}$, of the distance of the spectrum of $D[S_1 e]$ to the unit circle in \mathbb{C} .

- If we assume in addition that (i) S_1 is injective on the attractor and (ii) the linear map $D[S_1 y]$ is injective for every $y \in \mathfrak{A}$, then for each $e \in \mathcal{N}$ the unstable manifold $\mathcal{M}^u(e)$ is an embedded C^1 -submanifold of X of finite dimension $\text{ind}(e)$.

We note that the proof of this result (see BABIN/VISHIK [9] or HALE [116]) relies on geometric consideration of the behavior of trajectories in a vicinity of equilibrium points. The critical assumption for this is that the evolution S_t is C^1 and that equilibria are finite and hyperbolic. The above assumptions allow us to reduce the problem of convergence in the vicinity of equilibria to a linear problem.

We also refer to CARVALHO/LANGA [25] and CARVALHO/LANGA/ROBINSON [26, Chapter 5] for some generalizations of the notion of a gradient system. These generalizations are related to the Morse decomposition of attractors and deal with families of isolated invariant sets rather than with collections of (isolated) equilibria.

Another important issue is persistence of the regular structure of a global attractor under perturbations. On this topic we mention the paper by BABIN/VISHIK [8], which presents some results on the persistence of the gradient structure (i.e., the existence of a strict Lyapunov function) for some classes of PDEs. Recently this question was discussed in great detail in ARAGÃO ET AL. [3] and CARVALHO/LANGA/ROBINSON [26, Chapter 5].

2.4.3 Criteria of existence of global attractors for gradient systems

In this section we prove several assertions on the existence of global attractors which do not assume any dissipativity properties of the system in explicit form.

We start with the following criterion for the existence of a global attractor for gradient systems (see, e.g., RAUGEL [188, Theorem 4.6]), which is useful in many applications.

Theorem 2.4.16. *Let (X, S_t) be an asymptotically smooth gradient system which has the property that for any bounded set $B \subset X$ there exists $\tau > 0$ such that $\gamma_\tau(B) \equiv \bigcup_{t \geq \tau} S_t B$ is bounded. If the set \mathcal{N} of stationary points is bounded, then (X, S_t) has a compact global attractor \mathfrak{A} .*

Remark 2.4.17. By Theorem 2.4.5 the global attractor \mathfrak{A} given by Theorem 2.4.16 coincides with the unstable set $\mathcal{M}_+(\mathcal{N})$ emanating from the set \mathcal{N} of stationary points (see Definition 2.3.10), i.e., $\mathfrak{A} = \mathcal{M}_+(\mathcal{N})$. ■

Proof of Theorem 2.4.16. Let B be a bounded set in X and $B_\tau = \overline{\gamma_\tau(B)}$. We consider the restriction (B_τ, S_t) of the dynamical system (X, S_t) on the (forward invariant) set B_τ . Since B_τ is bounded, (B_τ, S_t) is a dissipative asymptotically smooth dynamical system. By Theorem 2.3.5 this system possesses a compact global attractor \mathfrak{A}_B . By Theorem 2.4.5 $\mathfrak{A}_B = \mathcal{M}_+(\mathcal{N}_B)$, where $\mathcal{N}_B = \mathcal{N} \cap B_\tau$. Under the condition $B \supset \mathcal{N}$ we have that $\mathcal{N} = S_t \mathcal{N} \subset S_t B$ for every $t > 0$. Thus $\mathcal{N} \subset B_\tau$. This implies that $\mathfrak{A}_B = \mathcal{M}_+(\mathcal{N})$ and thus the attractor \mathfrak{A}_B is independent of B when $B \supset \mathcal{N}$. Since $\mathfrak{A}_{B_1} \subset \mathfrak{A}_{B_2}$ for $B_1 \subset B_2$, we have that $\mathfrak{A} := \mathcal{M}_+(\mathcal{N})$ attracts all bounded sets from X . □

Using Theorem 2.4.16 we can obtain the following assertion (see Corollary 2.29 in CHUESHOV/LASIECKA [56]).

Theorem 2.4.18. *Assume that (X, S_t) is a gradient asymptotically smooth dynamical system. Assume its Lyapunov function $\Phi(x)$ is bounded from above on any bounded subset of X and the set $\Phi_R = \{x : \Phi(x) \leq R\}$ is bounded for every R . If the set \mathcal{N} of stationary points of (X, S_t) is bounded, then (X, S_t) possesses a compact global attractor $\mathfrak{A} = \mathcal{M}^u(\mathcal{N})$.*

Proof. Due to Theorem 2.4.16, it is sufficient to show that for any bounded set $B \subset X$ the set $\gamma_+(B) \equiv \bigcup_{t \geq 0} S_t B$ is bounded. To see this, we note that $B \subset \Phi_R$ for some $R > 0$. Since Φ_R is invariant, we have that $\gamma_+(B) \subset \Phi_R$ and thus $\gamma_+(B)$ is bounded. □

Exercise 2.4.19. Show that in the statement of Theorem 2.4.18 the condition concerning the boundedness of the set of stationary points can be changed to the requirement that (X, S_t) is point dissipative (see Definition 2.1.1). ■

Example 2.4.20 (Two-mode fluid-structure model). This model is the lowest Galerkin mode approximation of a system arising in the study of interaction of a

fluid filling a bounded vessel with an elastic wall (see, e.g., CHUESHOV/RYZHKOVA [68] and the references therein). The equations appear as follows:

$$\frac{d}{dt} [z + \gamma \dot{y}] + \alpha z = f, \quad (2.4.8a)$$

$$\frac{d}{dt} [\gamma z + 2\dot{y}] - \lambda y + y^3 = h, \quad (2.4.8b)$$

where $\alpha > 0$, $|\gamma| \leq 1$, and $\lambda, f, h \in \mathbb{R}$ are constants. We endow these equations with initial data

$$z(0) = z_0, \quad y(0) = y_0, \quad \dot{y}(0) = y_1. \quad (2.4.9)$$

One can see that problem (2.4.8) and (2.4.9) has a unique local solution for all initial data $(z_0; y_0; y_1) \in \mathbb{R}^3$. Using the multipliers $z - f/\alpha$ for the first equation and \dot{y} for the second one, we obtain the following energy balance relation:

$$\frac{d}{dt} E(z(t) - f/\alpha, y(t), \dot{y}(t)) + \alpha [z(t) - f/\alpha]^2 = 0 \quad (2.4.10)$$

on the existence interval, where the energy functional E has the form

$$E(z, y, \dot{y}) = \frac{1}{2} z^2 + \gamma z \dot{y} + \dot{y}^2 + \frac{1}{4} y^4 - \frac{\lambda}{2} y^2 - h y.$$

The energy relation in (2.4.10) allows us to use the non-explosion criterion in Theorem A.1.2 and show that problem (2.4.8) and (2.4.9) generates a dynamical system in \mathbb{R}^3 . Moreover, one can see that

$$V(z, y, \dot{y}) = E(z - f/\alpha, y, \dot{y})$$

is a strict global Lyapunov function for this system. Since the set

$$\{(z; y) : \alpha z = f, \quad y^3 - \alpha y = h\}$$

of stationary solutions is finite, by Theorem 2.4.18 the system generated by (2.4.8) and (2.4.9) possesses a global attractor which coincides with the unstable set emanating from the set of equilibria. ■

If a system (X, S_t) is not gradient but possesses a Lyapunov function (which is not strict), we cannot guarantee that $\mathfrak{A} = \mathcal{M}^u(\mathcal{N})$. However, we can prove the following assertion (see also CHUESHOV [39, Theorem 6.2, Chapter 1] and CHUESHOV/LASIECKA [56, Theorem 2.30]).

Theorem 2.4.21. *Let (X, S_t) be an asymptotically smooth dynamical system in some complete metric space X . Assume that there exists a Lyapunov function $\Phi(x)$*

for (X, S_t) on X such that $\Phi(x)$ is bounded from above on any bounded subset of X and the set $\Phi_R = \{x : \Phi(x) \leq R\}$ is bounded for every R . Let \mathcal{B} be the set of elements $x \in X$ such that there exists a full trajectory $\{u(t) : t \in \mathbb{R}\}$ with the properties $u(0) = x$ and $\Phi(u(t)) = \Phi(x)$ for all $t \in \mathbb{R}$. If \mathcal{B} is bounded, then (X, S_t) possesses a compact global attractor and $\mathfrak{A} = \mathcal{M}^u(\mathcal{B})$.

Proof. We choose R_0 such that $\mathcal{B} \subset \Phi_{R_0}$. By Theorem 2.3.5 the dynamical system (Φ_R, S_t) possesses a compact global attractor \mathfrak{A}_R for every R . Let $R \geq R_0$. In this case we have that $\mathcal{B} \subset \Phi_R$. By the same argument as in the proof of Theorem 2.4.5 we can show that for any full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ from the attractor \mathfrak{A}_R we have that $\alpha(\gamma) \subset \mathcal{B}$ and thus $u(t) \rightarrow \mathcal{B}$ as $t \rightarrow -\infty$. This means that $\mathfrak{A}_R \subset \mathcal{M}^u(\mathcal{B})$. Since \mathcal{B} is a bounded strictly invariant set, we have that $\mathcal{B} \subset \mathfrak{A}_R$. This implies that $\mathcal{M}^u(\mathcal{B}) \subset \mathfrak{A}_R$ and thus $\mathfrak{A}_R = \mathcal{M}^u(\mathcal{B})$ for all $R \geq R_0$. Therefore, $\mathfrak{A} := \mathcal{M}^u(\mathcal{B})$ is a global attractor for (X, S_t) . \square

The following two exercises illustrate Theorem 2.4.21.

Exercise 2.4.22. Apply Theorem 2.4.21 to the model in Example 2.4.11 to describe the global attractor in the case when μ and α are positive. \blacksquare

Exercise 2.4.23. Apply Theorem 2.4.21 to describe the structure of the global attractor for the Krasovskii system (see Exercise 2.1.10). \blacksquare

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