Preface

This fifth volume of the series *Lévy Matters* consists of three chapters, each devoted to an important aspect of Lévy processes and their applications. They all concern distributions of certain functionals of Lévy processes, which appear naturally in different settings.

Historically, processes with independent increments have been considered by Paul Lévy to reveal the fine structure of infinitely divisible distributions; the paradigm being that a probability measure, say μ, is infinitely divisible if and only if there is a Lévy process $\{X_t\}$ such that X_1 has the law μ. In turn, the notion of infinite divisibility for probability measures arises naturally in the context of limit theorems for sums of triangular arrays. Well-known special cases of infinitely divisible laws include stable distributions, which describe the weak limits of certain properly rescaled random walks with heavy-tailed step distributions, and more generally self-decomposable distributions, which in turn arise similarly for sums of independent variables. The so-called Generalized Gamma Convolutions, or distributions in the Thorin class, lie somewhat in between the two former. The **first chapter** of this volume, by Makoto Maejima, surveys representations of the main sub-classes of infinitely divisible distributions in terms of mappings of certain Lévy processes via stochastic integration

$$\{X_t\} \mapsto \int_I f(t)dX_t,$$

where f is some specific deterministic function over some interval I. An important motivation for studying such mappings stems from free probability, and more specifically from the role of free cumulants in this area. The study of the compositions and the iterations of such mappings, and of their limits, then sheds light on the nested structure of those subclasses. A great variety of classical and not-yet classical examples of infinitely divisible distributions are then analyzed from this perspective. Overall, this chapter can be seen as a companion to the contribution by K. Sato "Fractional integrals and extensions of self-decomposability," which appeared in the first volume of this Series, and in which relations between many nested subclasses of infinitely divisible laws are discussed.

Reflecting a path at barriers is a fundamental concept for stochastic processes, both in theory and in applications to modeling physical phenomena. In the setting of one-dimensional Lévy processes, reflection at a single fixed barrier (usually 0) lies at the core of fluctuation theory and its connections with the Wiener-Hopf factorization. One-sided reflected Lévy processes can be used as basic models for stochastic storage processes; they arise in a variety of applications including queuing, dams, insurance, and data communication, to name just the main ones, and there is already a vast mathematical literature on this classical area. The **second chapter** of this volume, by Lars Nørvang Andersen, Søren Asmussen, Peter W. Glynn and Mats Pihlsgård, concerns real Lévy processes reflected at two barriers. Two-sided reflection can be used for modeling systems with a finite capacity, which is of course a crucial hypothesis to fit many real-life situations. Roughly speaking, as the two-sided reflected process V stays in a compact interval, it is positive recurrent. Thus, V possesses a stationary distribution, π, and the ergodic theorem applies. Theoretically, this should enable one to answer natural questions about the long-run behavior of the system; unfortunately in practice and except in some special situations that we shall discuss later on, the invariant measure π is hard to determine, even numerically. A most important quantity for the applications is the overflow, or the loss occurring at the upper barrier, which, for instance, in communication models, represents the number of bits, which are lost when the buffer is full. Explicit asymptotics of the average loss are obtained when the upper barrier goes to infinity, both in the discrete time (i.e., for random walks) and continuous time frameworks; different regimes occur depending on whether the tail distribution of a typical increment is light or heavy. Whereas in discrete time, the construction of a reflected chain raises no difficulty, the continuous time setting is somewhat less intuitive and requires a formulation *à la* Skorokhod. Therefore, stochastic calculus and martingale techniques, in particular using optional sampling for the Wald martingale and the Kella-Whitt martingale, provide fundamental tools for studying quantities related to two-sided reflected Lévy processes. As we mentioned previously, there are also important situations where handy expressions for the stationary distribution π can be obtained. Typically, this is the case whenever the two-sided exit problem can be solved, that is the probability that the Lévy process crosses the upper-barrier before the lower-barrier can be expressed explicitly as a function of its starting point. A first important situation when this occurs is when the jumps of X have a phase-type distribution, a situation that is amply discussed in this chapter. Another well-known case is when the Lévy process X is spectrally negative then a solution of the two-sided exit problem can be given in terms of the so-called scale function. Readers wishing to learn more about scale functions are invited to consult the contribution by Alexey Kuznetsov, Andreas Kyprianou, and Victor Rivero in the second volume of this series.

If now processes are killed rather than reflected when they cross the boundary, probably the most natural and important questions that one can ask concern the lifetime. Typically, for a one-dimensional process, say $\{Y_t\}_{t\geq 0}$, and when the domain is a semi-infinite interval (x, ∞), one is thus interested in the passage time $T_x := \inf\{t \geq 0 : Y_t > x\}$. The one-dimensional distributions of the process $\{T_x\}_{x>0}$ are characterized by those of the running supremum process $\{\sup_{0\leq s\leq t} Y_s\}_{t\geq 0}$ through the basic identity

$$\mathbb{P}(T_x \leq t) = \mathbb{P}\left(\sup_{0\leq s\leq t} Y_s > x\right)$$

that holds whenever x is not the location of a local maximum of Y. In general, these quantities are hard to compute explicitly, and merely determining the tail behavior of the survival probability $\mathbb{P}(T_x > t)$ is already highly challenging. In particular, one is interested in deciding whether the latter decays polynomially in t, i.e. if there is an exponent $\theta > 0$ with $\mathbb{P}(T_x > t) = t^{-\theta+o(1)}$ as $t \to \infty$. In that case, θ (which may depend on x) is called the persistence exponent. This is a fundamental issue in applications, notably for a number of models in physics, and the **third chapter** of this volume, by Frank Aurzada and Thomas Simon, is devoted to this question. Whereas for general Markov processes, this question is usually addressed by considering the eigenvalues and eigenfunctions of the infinitesimal generator of the killed process, for random walks or for Lévy processes, the deep formulas of fluctuation theory are the key to many classical results in this area. The problem is of course much harder for non-Markovian processes, and the main part of this chapter concerns the situation when the process Y is given by either the partial sum of a random walk or the integral of a Lévy process. Many very recent advances and developments are discussed in this setting.

We are confident that you will enjoy reading these new contributions to the theory of Lévy type processes and their applications, as much as those which already appeared in the preceding volumes of *Lévy Matters*.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
Zurich, Switzerland — Jean Bertoin
Paris, France — Jean Jacod
Munich, Germany — Claudia Küppelberg
June 2015

Springer

http://www.springer.com/978-3-319-23137-2

Lévy Matters V
Functionals of Lévy Processes
Andersen, L.N.; Asmussen, S.; Aurzada, F.; Glynn, P.W.;
Maejima, M.; Pihlsgård, M.; Simon, Th.
2015, XVI, 224 p. 8 illus., 7 illus. in color., Softcover
ISBN: 978-3-319-23137-2