Extremal Completions of Triangular Norms Known on a Subregion of the Unit Interval

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Abstract. The strongest and the weakest t-norms that coincide with the given t-norm on a subregion of the unit interval are discussed. The question whether such a t-norm can be obtained as a limit of the sequence of continuous t-norms that coincide with the original t-norm on the given subregion is investigated.

Keywords: t-norm · Ordinal sum · Additive generator

1 Introduction

The (left-continuous) t-norms and their dual t-conorms are special aggregation functions and they have an indispensable role in many domains [2,4,5,10,11]. In real-world applications it can happen that only a part of the aggregation function is known, either it is observed from the input-output relationships in the training data or implied by the requirements of the modelled problem. With additional requirement that the aggregation function has to be a t-norm we are looking for the weakest and the strongest t-norms which coincide with the original t-norm which is known only on the interval $[a, b]^2 \subsetneq [0, 1]^2$. In the case when a and b are idempotent elements this problem was studied in [8] (see also [4]), in a broader context of aggregation functions. We would like to extend these results also for the case when a or b (or both) is not an idempotent element. In [9] we have studied continuous t-norms which coincide with the original t-norm known on $[a, b]^2$. In this contribution we will focus on extremal t-norms and we will be also interested whether these t-norms can be approximated, i.e., obtained as a limit of a sequence of continuous t-norms T_i that coincide with the t-norm T_1 on $[a, b]^2$. Therefore we will focus on such t-norms T_1 which are continuous on $[a, b]^2$.

After recalling several basic notions and results in Sect. 2, we will focus on the case when there is no non-trivial idempotent point of T_1 in $[a, b]^2$ (Sect. 3). We will study several special cases: when $a = b$, when $a = 0$, when $b = 1$ and a general case when $0 < a < b < 1$. In Sect. 4 we will discuss the case when there is a non-trivial idempotent point of T_1 in $[a, b]^2$. We give our conclusions in Sect. 5.

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2 Basic Notions and Results

Let us recall several useful definitions and results on t-norms (see [1,7]).

Definition 1. *(i) A binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ is a* t-norm *if it is commutative, associative, non-decreasing in both variables and* 1 *is its neutral element.*
(ii) A binary function $C\colon [0,1]^2 \longrightarrow [0,1]$ is a t-conorm *if it is commutative, associative, non-decreasing in both variables and* 0 *is its neutral element.*
(iii) A binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ is a t-subnorm *if it is commutative, associative, non-decreasing in both variables and $S(x,y) \leq \min(x,y)$ for all $(x,y) \in [0,1]^2$.*

Thus every t-norm is also a t-subnorm. Due to the associativity, n-ary form of any t-norm (t-conorm) is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm C by the equation

$$C(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa. Therefore all results that we obtain for t-norms can be immediately obtained also for t-conorms.

A t-norm T is called *Archimedean* if for all $x,y \in \,]0,1[$ there exists an $n \in \mathbb{N}$ such that $x_T^{(n)} < y$, where $x_T^{(n)} = T(\underbrace{x, T(x,\ldots)}_{n\text{-times}})$.

A continuous t-norm is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T is either strict, i.e., strictly increasing on $]0,1]^2$, or nilpotent, i.e., there exists $(x,y) \in \,]0,1]^2$ such that $T(x,y) = 0$.

For every t-norm it holds

$$T_{\mathbf{D}}(x,y) \leq T(x,y) \leq T_{\mathbf{M}},$$

where $T_{\mathbf{D}}$ is the drastic product t-norm given by $T_{\mathbf{D}}(x,y) = 0$ if $\max(x,y) < 1$ and $T_{\mathbf{D}}(x,y) = \min(x,y)$ otherwise, and $T_{\mathbf{M}}$ is the minimum t-norm given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ for all $(x,y) \in [0,1]^2$.

Proposition 1. *Let $t\colon [0,1] \longrightarrow [0,\infty]$ be a continuous strictly decreasing function such that $t(1) = 0$. Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

is a continuous t-norm. The function t is called an additive generator *of T.*

Note that every continuous Archimedean t-norm possesses a continuous additive generator. Non-continuous t-norms can be additively generated by non-continuous additive generators.

Definition 2. *(i) Let* $t\colon [0,1] \longrightarrow [0,\infty]$ *be a non-increasing function. Then the function* $t^{(-1)}\colon [0,\infty] \longrightarrow [0,1]$ *given by*

$$t^{(-1)}(x) = \sup\{y \in [0,1] \mid t(y) > x\}$$

is called the pseudo-inverse *of* t.

(ii) A strictly decreasing function $t\colon [0,1] \longrightarrow [0,\infty]$, $t(1) = 0$, *is called an* additive generator *of a t-norm* $T\colon [0,1]^2 \longrightarrow [0,1]$ *if*

$$T(x,y) = t^{(-1)}(t(x) + t(y)))$$

for all $(x,y) \in [0,1]^2$.

Further we recall a construction of t-norms via the ordinal sum. The basic stones for construction of t-norms via the ordinal sum (see [3]) are t-subnorms (see [6]).

Proposition 2. *Let* $(]a_k, b_k[)_{k\in K}$ *be a disjoint system of open subintervals of* $[0,1]$, *where* K *is a finite or countably infinite index set. Let* $(S_k)_{k\in K}$ *be a system of left-continuous t-subnorms such that if* $b_{k_0} = 1$ *for some* $k_0 \in K$ *then* S_{k_0} *is a t-norm, and if* $b_{k_1} = a_{k_2}$ *for some* $k_1, k_2 \in K$ *then either* S_{k_2} *has no zero divisors or* S_{k_1} *is a t-norm. Then the ordinal sum* $T = (\langle a_k, b_k, S_k\rangle \mid k \in K)$ *given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in \,]a_k, b_k]^2, \\ \min(x,y) & \text{else} \end{cases}$$

is a left-continuous t-norm.

Recall that each continuous t-norm can be expressed as an ordinal sum of continuous Archimedean t-norms. In the following sections $\{\varepsilon_i\}_{i\in\mathbb{N}}$ will always be a sequence of small enough $\varepsilon_i > 0$ which converges to 0.

3 Extremal Extensions of t-norms Without a Non-trivial Idempotent Element in $[a,b]$

Let the t-norm T_1 be known only on $[a,b]^2$ and let it be continuous on $[a,b]^2$. In this section we will suppose that there is no non-trivial idempotent point of T_1 in $[a,b]$, i.e., that $T_1(x,x) < x$ for all $x \in [a,b] \setminus \{0,1\}$. We will suppose several subcases: when $a = b$, when $a = 0$, when $b = 1$ and a general case when $0 < a < b < 1$.

3.1 Case When $a = b$

In this case T_1 is known only in one point (a,a). Since all t-norms coincide on the boundary of the unit square if $a \in \{0,1\}$ then the strongest t-norm that coincides with T_1 in (a,a) is the minimum t-norm and the weakest is the drastic product t-norm. Note that the drastic product can be obtained as a limit of the sequence of continuous t-norms (see [7]). Similarly, since the minimum t-norm is continuous it can be obtained as a limit of the sequence of continuous t-norms.

Suppose $a \in \,]0,1[$. Then since a is not an idempotent element we have $T_1(a,a) = q$ for some $q \in [0,a[$. Due to the monotonicity the strongest t-norm T_2 that coincides with T_1 in (a,a) is given by

$$T_2(x,y) = \begin{cases} q & \text{if } (x,y) \in [q,a]^2 \,, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

We see that T_2 is an ordinal sum on the zero t-subnorm Z on $[q,a]$, where Z is given by $Z(x,y) = 0$ for all $(x,y) \in [0,1]^2$. The t-norm T_2 can be obtained as a limit of the sequence of continuous t-norms T_i that coincide with T_1 in (a,a). Let $\{V_i\}_{i \in \mathbb{N}}$ be the sequence of t-norms that converges to the drastic product t-norm (see [7]). Then we define a sequence of ordinal sum t-norms $T_i = (\langle q, a + \varepsilon_i, V_i \rangle)$. Then $\{T_i\}_{i \in \mathbb{N}}$ converges to T_2.

The weakest t-norm T_3 that coincides with T_1 in (a,a) is given by

$$T_3(x,y) = \begin{cases} 0 & \text{if } \min(x,y) < a, \max(x,y) < 1, \\ \min(x,y) & \text{if } \max(x,y) = 1, \\ q & \text{otherwise.} \end{cases}$$

Next we would like to know whether T_3 can be obtained as a limit of the sequence of continuous t-norms that coincide with T_1 in (a,a). This sequence can be obtained as a sequence of continuous Archimedean t-norms T_i with additive generators t_i with $t_i(q) = 2 \cdot t_i(a)$ which converges to a function $t \colon [0,1] \longrightarrow [0,\infty]$ which is linear on $[0,q^-]$, on $[q^+,a^-]$ and on $[a^+,1^-]$ with $t(0) = 2.5$, $t(q^-) = 2.1$, $t(q) = 2$, $t(q^+) = 1.7$, $t(a^-) = 1.6$, $t(a) = t(a^+) = 1$, $t(1^-) = 0.9$ and $t(1) = 0$. Then $\{T_i\}_{i \in \mathbb{N}}$ converges to T_3.

3.2 Case when $a = 0$

Now we will focus on the interval $[0,b]$. Let the t-norm T_1 be known and continuous on $[0,b]^2$. Then we will use the following result.

Lemma 1. *Let $S \colon [0,b]^2 \longrightarrow [0,b]$ be a t-subnorm for some $b \in [0,1]$. We define the binary operation $T^* \colon [0,1]^2 \longrightarrow [0,1]$ by*

$$T^*(x,y) = \begin{cases} S(x,y) & \textit{if } \max(x,y) \le b, \\ \min(x,y) & \textit{otherwise.} \end{cases}$$

Then T^* is a t-norm.

The t-norm T^* from the previous result is an ordinal sum of a t-subnorm S_1 on $[0,b]$, where S_1 on $[0,1]^2$ is linearly isomorphic with S on $[0,b]^2$. Then we get the following.

Proposition 3. *Let* $T_1\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm. Then the strongest t-norm* T_2 *which coincides with* T_1 *on* $[0,b]^2$ *is given by*

$$T_2(x,y) = \begin{cases} T_1(x,y) & \text{if } \max(x,y) \le b, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Now the question is whether T_2 can be obtained as a limit of a sequence of continuous t-norms T_i which coincide with T_1 on $[0,b]^2$. This is possible only if T_1 fulfills the conditions necessary for existence of a continuous t-norm which coincides with T_1 on $[0,b]^2$ (for more details see [9]). In such a case every continuous t-norm that coincides with T_1 on $[0,b]^2$ is Archimedean on $[0,b]^2$ and thus it possess an additive generator s on $[0,b]$. Then we construct a sequence $\{T_i\}_{i\in\mathbb{N}}$ of t-norms where $T_i = (\langle 0, b+\varepsilon_i, V_i\rangle)$, where V_i is a continuous, Archimedean t-norm and additive generator s_i of T_i on $[0, b+\varepsilon_i]$ satisfies $s_i(x) = s(x)$ for all $x \in [0,b]$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_2.

For the weakest extension we will use the following result.

Lemma 2. *Let* $S\colon\ [0,b]^2 \longrightarrow [0,b]$ *be a t-subnorm for some* $b \in [0,1]$. *We define the binary operation* $T_*\colon [0,1]^2 \longrightarrow [0,1]$ *by*

$$T_*(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ S(\min(x,b), \min(y,b)) & \text{otherwise.} \end{cases}$$

Then T_* *is a t-norm.*

We get the following.

Proposition 4. *Let* $T_1\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm. Then the weakest t-norm* T_3 *which coincides with* T_1 *on* $[0,b]^2$ *is given by*

$$T_3(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ T_1(\min(x,b), \min(y,b)) & \text{otherwise.} \end{cases}$$

Similarly as above, if T_1 fulfills the conditions necessary for existence of a continuous t-norm which coincides with T_1 on $[0,b]^2$ then T_1 has an additive generator s on $[0,b]$. We will construct a sequence of continuous, Archimedean t-norms $\{T_i\}_{i\in\mathbb{N}}$, with additive generators t_i such that $t_i(x) = s(x)$ for all $x \in [0,b]$, t_i is linear on $[b, 1-\varepsilon_i]$ and on $[1-\varepsilon_i, 1]$ and $t_i(1-\varepsilon_i) = s(b) - \varepsilon_i$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_3.

3.3 Case When $b = 1$

Now we will focus on the interval $[a,1]$. Here we recall a result from [9].

Lemma 3. *Let* $T_1\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm continuous on* $[a,1]^2$, $0 < a < 1$. *Then for any t-norm* T_2 *such that* T_1 *coincides with* T_2 *on* $[a,1]^2$, *we have* $T_2(x,y) = T_2(y,x) = T_1(x,y) = T_1(y,x)$ *for all* $(x,y) \in A$, *where*

$$A = \{(x,y),(y,x) \in [0,1]^2 \mid \text{ there exists a } z \in [0,1], T_1(z,a) = x, T_1(z,y) \ge a\}.$$

From the previous lemma we see that T_2 is on A uniquely given by values of T_1 on $[a,1]^2$. Moreover, since T_1 is continuous on $[a,1]^2$ for every $x \in [T_1(a,a), a]$ there exists a $z \in [a,1]$ such that $x = T_1(a,z) = T_2(a,z)$. By continuity again, considering the fact that $T_1(a,z) \le a$, $T_1(z,1) = z$, we see that there exists a $p \in [a,1]$ such that $T_1(z,p) = a$. Then $T_1(z,y) \ge a$ for all $y \ge p$. Thus the set A is a symmetric connected set which contains all points from $[0,1]^2$ greater than the points from the lower border of A, where the lower border of A is the set $B = \{(x,y),(y,x) \in [0,1]^2 \mid \text{ there exists a } z \in [0,1], T_1(z,a) = x, T_1(z,y) = a\}$. Note that if $(x,y) \in B$ then $T_2(x,y) = T_1(a,a)$ (see [9]). Therefore we get the following.

Proposition 5. *Let $T_1\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm. Then the strongest t-norm T_2 which coincides with T_1 on $[a,1]^2$ is given by*

$$T_2(x,y) = \begin{cases} T_1(a,T_1(z,y)) & \textit{if } (x,y) \in A, x = T_1(a,z), \\ T_1(a,T_1(z,x)) & \textit{if } (x,y) \in A, y = T_1(a,z), \\ T_1(a,a) & \textit{if } (x,y) \in [T_1(a,a),1]^2 \setminus A, \\ \min(x,y) & \textit{otherwise.} \end{cases}$$

Here $A = \{(x,y),(y,x) \in [0,1]^2 \mid$ there exists a $z \in [0,1], T_1(a,z) = x, T_1(z,y) \ge a\}$.

Since T_2 is continuous it is easy to see that T_2 can be obtained as a limit of a sequence of continuous t-norms that coincide with T_1 on $[a,1]^2$.

Example 1. Assume that T is the product t-norm and $a = \frac{1}{2}$. Then the strongest t-norm that coincide with T on $[a,1]^2$ is given by

$$T^*(x,y) = \begin{cases} x \cdot y & \text{if } x \cdot y \ge \frac{1}{4}, \\ \frac{1}{4} & \text{if } (x,y) \in \left[\frac{1}{4},1\right]^2, x \cdot y < \frac{1}{4} \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Proposition 6. *Let $T_1\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm. Then the weakest t-norm T_3 which coincides with T_1 on $[a,1]^2$ is given by*

$$T_3(x,y) = \begin{cases} T_1(a,T_1(z,y)) & \textit{if } (x,y) \in A, x = T_1(a,z), \\ T_1(a,T_1(z,x)) & \textit{if } (x,y) \in A, y = T_1(a,z), \\ \min(x,y) & \textit{if } \max(x,y) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Here again $A = \{(x,y),(y,x) \in [0,1]^2 \mid$ there exists a $z \in [0,1], T_1(a,z) = x, T_1(z,y) \ge a\}$.

Since the strongest t-norm T_2 which coincides with T_1 on $[a,1]^2$ is continuous and Archimedean on $[T_1(a,a),1]^2$ it has an additive generator s on $[T_1(a,a),1]$.

We will define a sequence $\{T_i\}_{i\in\mathbb{N}}$ of continuous, Archimedean t-norms which are generated by respective additive generators t_i such that $t_i(x) = s(x)$ for all $x \in [T_1(a,a), 1]$ and t_i is linear on $[0, T_1(a,a)]$ and $t_i(0) = s(T_1(a,a)) + \varepsilon_i$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_3.

Example 2. Let us again assume that T is the product t-norm and $a = \frac{1}{2}$. Then the weakest t-norm that coincide with T on $[a,1]^2$ is given by

$$T_*(x,y) = \begin{cases} x \cdot y & \text{if } x \cdot y \geq \frac{1}{4}, \\ \min(x,y) & \text{if } \max(x,y) = 1, \min(x,y) < \frac{1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Case When $0 < a < b < 1$

Let T_1 be known and continuous on $[a,b]^2$. First we will focus on the strongest and the weakest extensions of T_1 to $[0,b]^2$. Since each t-norm on $[0,b]^2$ is linearly isomorphic with some t-subnorm S on $[0,1]^2$ we will use the following result.

Lemma 4. *Let $S_1\colon [0,1]^2 \longrightarrow [0,1]$ be a t-subnorm. Then for any t-subnorm S_2 such that S_1 coincides with S_2 on $[a,1]^2$, we have $S_2(x,y) = S_2(y,x) = S_1(x,y) = S_1(y,x)$ for all $(x,y) \in \overline{A}$, where*

$$\overline{A} = \{(x,y),(y,x) \in [0,1]^2 \mid \text{ there exists } z,q \in [a,1], S_1(z,q) = x, S_1(z,y) \geq a\}.$$

Suppose that $T_1(a,b) = a$. Then $a = T_1(a, T_1(\underbrace{b,\ldots,b}_{n\text{-times}}))$ and since T_1 is continuous on $[a,b]^2$ and has no idempotents in $[a,b]$ there exists an $n \in \mathbb{N}$ such that $T_1(\underbrace{b,\ldots,b}_{n\text{-times}}) = u < a$. Then, however, $a = T_1(a,u) \leq u < a$ what is a contradiction. Thus we have always $T_1(a,b) < a$. We have now two possibilities: either $T_1(b,b) \geq a$, or $T_1(b,b) < a$.

First suppose $T_1(b,b) \geq a$. Then there exists an $r \in [a,b]$ such that $T_1(r,r) = a$. Since $T_1(a,b) < a$ also $T_1(a,r) < a$. We then have the following.

Lemma 5. *Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm such that T is continuous on $[a,b]^2$ for $0 < a < b < 1$, and $T(a,b) < a$, $T(r,r) = a$ for some $r \in [a,b]$. Then the values of T on $A_1 \cup A_2$ are determined by the values of T on $[a,b]^2$, where $A_1 = \{(x,y),(y,x) \in [0,b]^2 \mid$ there exists $z \in [a,r], T_1(r,z) = x, T_1(z,y) \geq T_1(a,r)\}$, $A_2 = \{(x,y),(y,x) \in [0,b]^2 \mid$ there exists $z \in [a,r], T_1(a,z) = x, T_1(z,y) \geq a\}$.*

If we combine this with Lemma 1 we see that the strongest t-norm T_2 which coincides with T_1 on $[a,b]^2$, if $T(a,b) < a$ and $T(b,b) \geq a$ is given by

$$T_2(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,1]^2 \setminus [T_1(a,a),b]^2, \\ T_1(x,y) & \text{if } (x,y) \in [a,b]^2, \\ T_1(a,w) & \text{if } T_1(r,z) = x, T_1(z,y) = T_1(r,w), z,w \in [a,r], \\ T_1(a,w) & \text{if } T_1(r,z) = y, T_1(z,x) = T_1(r,w), z,w \in [a,r], \\ T_1(r,T_1(z,y)) & \text{if } T_1(r,z) = x, z \in [a,b], T_1(z,y) > a, \\ T_1(r,T_1(z,x)) & \text{if } T_1(r,z) = y, z \in [a,b], T_1(z,x) > a, \\ T_1(a,T_1(z,y)) & \text{if } T_1(a,z) = x, z \in [a,r], T_1(z,y) \geq a, \\ T_1(a,T_1(z,x)) & \text{if } T_1(a,z) = y, z \in [a,r], T_1(z,x) \geq a, \\ T_1(a,a) & \text{otherwise.} \end{cases}$$

The t-norm T_2 is an ordinal sum t-norm and $T_1(a,a)$ is its idempotent point. If there exists a continuous t-norm that coincides with T_1 on $[a,b]^2$ then such a t-norm has an additive generator s on $[T_1(a,a),b]$. We construct a sequence of t-norms T_i which are ordinal sums of one continuous, Archimedean summand on $[T_1(a,a), b+\varepsilon_i]$ and T_i is generated on $[T_1(a,a), b+\varepsilon_i]$ by an additive generator t_i such that $t_i(x) = s(x)$ for all $x \in [T_1(a,a), b]$ and t_i is linear on $[b, b+\varepsilon_i]$ with $t_i(b+\varepsilon_i) = 0$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_2.

Similarly, using Lemma 2 we see that the weakest t-norm T_3 which coincides with T_1 on $[a,b]^2$, if $T(a,b) < a$ and $T(b,b) \geq a$ is given by

$$T_3(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ T_2(\min(x,b), \min(y,b)) & \text{if } (x,y) \in [T_1(a,p),1]^2 \setminus [T_1(a,p),b]^2, \\ T_2(x,y) & \text{if } (x,y) \in A_1 \cup A_2 \cup [a,b]^2, \\ 0 & \text{otherwise,} \end{cases}$$

where p is the smallest point from $[a,b]$ such that $T_1(b,p) = a$ and $A_1 = \{(x,y),(y,x) \in [0,b]^2 \mid \text{ there exists } z \in [a,r], T_1(r,z) = x, T_1(z,y) \geq T_1(a,r)\}$, and $A_2 = \{(x,y),(y,x) \in [0,b]^2 \mid \text{ there exists } z \in [a,r], T_1(a,z) = x, T_1(z,y) \geq a\}$.

If there exists a continuous t-norm that coincides with T_1 on $[a,b]^2$ then such a t-norm has an additive generator s on $[T_1(a,a),b]$. We construct a sequence of t-norms T_i with respective additive generators t_i such that $t_i(x) = s(x)$ for all $x \in [T_1(a,a),b]$ and t_i is linear on $[0, T_1(a,a)]$, $[b, 1-\varepsilon_i]$ and $[1-\varepsilon_i, 1]$ with $t_i(0) = s(T_1(a,a)) + \varepsilon_i$, $t_i(1-\varepsilon_i) = s(b) - \varepsilon_i$, $t_i(1) = 0$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_3.

Finally, we will suppose that $T_1(b,b) < a$.

Then using Lemma 2 we see that the weakest t-norm T_3 which coincides with T_1 on $[a,b]^2$, if $T(a,b) < a$ and $T(b,b) < a$ is given by

$$T_3(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ T_1(\min(x,b), \min(y,b)) & \text{if } (x,y) \in [a,1]^2, \\ 0 & \text{otherwise.} \end{cases}$$

If there exists a continuous t-norm that coincides with T_1 on $[a,b]^2$ then this t-norm has an additive generator s on $[T_1(a,a),b]$. Then its values on $[T_1(a,a),T_1(b,b)]\cup[a,b]$ determines T_1 on $[a,b]^2$. However, in [9] we have shown that values of such a generator on $[T_1(a,a),T_1(b,b)]\cup[a,b]$ are not uniquely determined by T_1 on $[a,b]^2$. More precisely, we can obtain a whole class of such additive generators with $s(a)=1$ which are dependent on a parameter $s(b)=w$, where $1>w>\frac{1}{2}$. We will select such an additive generator for which $3s(b)>2s(a)$. Then we will define a sequence of t-norms T_i generated by respective additive generators t_i, where $t_i(x)=s(x)$ for $x\in[T_1(a,a),T_1(b,b)]\cup[a,b]$ and t_i is linear on $[0,T_1(a,a)]$, on $[T_1(b,b),a-\varepsilon_i]$, on $[a-\varepsilon_i,a]$, on $[b,1-\varepsilon_i]$, and on $[1-\varepsilon_i,1]$, with $t_i(0)=s(T_1(a,a))+\varepsilon_i$, $t_i(a-\varepsilon_i)=s(T_1(b,b))-\varepsilon_i$, $t_i(1-\varepsilon_i)=s(b)-\varepsilon_i$, $t_i(1)=0$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_3.

In the case of the strongest extension the situation is more complicated. Let $T\colon[0,1]^2\longrightarrow[0,1]$ be a t-norm that coincides with T_1 on $[a,b]^2$. Then it is clear that for all $(x,y)\in[T_1(a,a),a]^2$ we have $T(x,y)\le T_1(a,a)$. Further, $T(x,y)\le T_1(a,y)$ for all $x\le a\le y$. If $T_1(a,b)=T_1(a,a)$ then the strongest t-norm that coincides with T_1 on $[a,b]^2$ is given by

$$T_2(x,y)=\begin{cases}\min(x,y) & \text{if } (x,y)\in[0,1]^2\setminus[T_1(a,a),b]^2,\\ T_1(\max(a,x),\max(a,y)) & \text{otherwise.}\end{cases}$$

If T_1 on $[a,b]^2$ can be extended to a continuous t-norm then this t-norm has an additive generator s on $[T_1(a,a),b]$ and values of s on $[T_1(a,a),T_1(b,b)]\cup[a,b]$ determine the values of T_1 on $[a,b]^2$. Since $T_1(a,b)=T_1(a,a)$ we see that $T_1(a,a)$ is an idempotent element of such a t-norm and $s(a)+s(b)\ge 2\cdot s(a)$. Thus if we define a sequence of t-norms T_i which are ordinal sums of one continuous, Archimedean summand on $[T_1(a,a),b+\varepsilon_i]$, where T_i is on $[T_1(a,a),b+\varepsilon_i]$ generated by respective additive generators t_i, where $t_i(x)=s(x)$ for $x\in[T_1(a,a),T_1(b,b)]\cup[a,b]$ and t_i is linear on $[T_1(b,b),a]$ and on $[b,b+\varepsilon_i]$, with $t_i(b+\varepsilon_i)=0$, then $\{T_i\}_{i\in\mathbb{N}}$ converges to T_2.

Further, let $c=T_1(a,b)>T_1(a,a)$. Then $T(c,b)=T(a,T(b,b))\le T_1(a,a)$, and we get the following:

Lemma 6. *Let $T_1\colon[0,1]^2\longrightarrow[0,1]$ be a t-norm which is continuous on $[a,b]^2$, $T_1(b,b)<a$, $T_1(a,b)>T_1(a,a)$. Then each t-norm T which coincides with T_1 on $[a,b]^2$ is smaller than or equal to the binary function $T^*\colon[0,1]^2\longrightarrow[0,1]$ given by*

$$T^*(x,y)=\begin{cases}\min(x,y) & \text{if } (x,y)\in[0,1]^2\setminus[T_1(a,a),b]^2,\\ T_1(x,y) & \text{if } (x,y)\in[a,b]^2,\\ T_1(a,y) & \text{if } y\in[a,b],\, x\in\,]T_1(b,b),a[,\\ T_1(a,x) & \text{if } x\in[a,b],\, y\in\,]T_1(b,b),a[,\\ T_1(a,w) & \text{if } x=T_1(b,q), T_1(y,q)=T_1(b,w), q,w\in[a,b],\\ T_1(a,w) & \text{if } y=T_1(b,q), T_1(x,q)=T_1(b,w), q,w\in[a,b],\\ T_1(a,a) & \text{otherwise.}\end{cases}$$

The binary function T^* is commutative, non-decreasing and has 1 as a neutral element. However, it need not be associative.

The binary function T^* is associative only if the inequalities $T_1(x,y) = T_1(b,u)$, $T_1(u,z) = T_1(b,v)$, $T_1(x,z) = T_1(b,w)$, $T_1(w,y) = T_1(b,c)$, imply $v = c$ for all $x,y,z,u,v,w,c \in [a,b]$. This is satisfied, for example, if there exists a continuous t-norm which coincides with T_1 on $[a,b]^2$, which follows from the existence of an additive generator of such a t-norm on $[T_1(a,a),b]$. Thus in this case T^* is the strongest t-norm which coincides with T_1 on $[a,b]^2$.

Suppose that there exists a continuous t-norm that coincides with T_1 on $[a,b]^2$. Then similarly as above the values of the corresponding additive generator s on $[T_1(a,a), T_1(b,b)] \cup [a,b]$ are not determined uniquely by values of T_1 on $[a,b]^2$, but for $s(a) = 1$ we have $s(b) = w$ for $1 > w > \frac{1}{2}$. We select a sequence of such additive generators s_i, where for the respective parameter $s_i(b) = w_i$ we have $w_i = \frac{1+\varepsilon_i}{2}$. We now define a sequence of t-norms T_i which are ordinal sums of one continuous, Archimedean summand on $[T_1(a,a), b+\varepsilon_i]$, and T_i is on $[T_1(a,a), b+\varepsilon_i]$ generated by respective additive generators t_i such that $t_i(x) = s_i(x)$ for $x \in [T_1(a,a), T_1(b,b)] \cup [a,b]$, and t_i is linear on $[T_1(b,b), a]$ and on $[b, b+\varepsilon_i]$, with $t_i(b+\varepsilon_i) = 0$. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ converges to T_2.

4 Extremal Extensions of t-norms with a Non-trivial Idempotent Element in $[a,b]$

Assume that the t-norm T_1 is known only on $[a,b]^2$ and is continuous on $[a,b]^2$. In this section we will suppose that there is a non-trivial idempotent in $[a,b]$. First suppose that $a = b$, i.e., T_1 is known only in one point (a,a) and $T_1(a,a) = a$. Then it is evident that the strongest t-norm which satisfies $T(a,a) = a$ is the minimum t-norm. The weakest t-norm T_3 with $T_3(a,a) = a$ is given by

$$T_3(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ a & \text{if } (x,y) \in [a,1]^2, \max(x,y) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

However, T_3 cannot be obtained as a limit of a sequence of continuous t-norms T_i with $T_i(a,a) = a$ as for each such a continuous t-norm T_i there is $T_i(x,y) = x$ for all $x \le a \le y$.

From now on we will suppose that $a < b$. Let $q \in [a,b]$ be an idempotent point of T_1, i.e., $T_1(q,q) = q$. Then we have the following.

Lemma 7. *Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm which is continuous on $[a,b]^2$ and let $q \in [a,b]$ be an idempotent point of T_1. Then $T_1(x,y) = x$ for all $x,y \in [a,b]$, $x \le q \le y$.*

From the previous result we see that if any t-norm T coincides with T_1 on $[a,b]^2$ then for all $x \in [a,q]$ we have $T(x,z) = x$ for all $z \in [q,1]$. Recall that for every t-norm T we have $T(x,y) \le \min(x,y)$. Thus we get the following result.

Proposition 7. *Let* $T\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm which is continuous on* $[a,b]^2$ *and let* $q_1, q_2 \in [a,b]$ *be respectively the smallest and the biggest idempotent point of* T_1 *in* $[a,b]$. *Then the strongest t-norm* T_2 *which coincides with* T_1 *on* $[a,b]^2$ *is equal to an ordinal sum* $T_2 = (\langle 0, q_1, T_2^1\rangle, \langle q_1, q_2, T_1^2\rangle, \langle q_2, 1, T_2^3\rangle)$. *Here* T_2^1 *is the strongest t-norm that coincides with the t-norm* T_1^1 *on* $\left[\frac{a}{q_1}, 1\right]$, *where* T_1^1 *is linearly isomorphic with* T_1 *on* $[0,q_1]^2$, *and* T_2^3 *is the strongest t-norm that coincides with the t-norm* T_1^3 *on* $\left[0, \frac{b-q_2}{1-q_2}\right]$, *where* T_1^3 *is linearly isomorphic with* T_1 *on* $[q_2,1]^2$. *The t-norm* T_1^2 *is linearly isomorphic with* T_1 *on* $[q_1,q_2]^2$.

Since in the previous result q_1 and q_2 are the smallest and the biggest idempotent point in $[a,b]$ thus in $[a, q_1[$ and in $]q_2, b]$ there is no non-trivial idempotent point of T_1. Therefore T_2^1 and T_2^3 can be determined from the previous section.

We get

$$T_2^3(x,y) = \begin{cases} T_1^3(x,y) & \text{if } (x,y) \in \left[0, \frac{b-q_2}{1-q_2}\right]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and

$$T_2^1(x,y) = \begin{cases} T_1^1(\frac{a}{q_1}, T_1^1(y,z)) & \text{if } (x,y) \in A, x = T_1^1(z, \frac{a}{q_1}), \\ T_1^1(\frac{a}{q_1}, T_1^1(x,z)) & \text{if } (x,y) \in A, y = T_1^1(z, \frac{a}{q_1}), \\ \min(x,y) & \text{if } \min(x,y) \le \frac{T_1(a,a)}{q_1}, \\ \frac{T_1(a,a)}{q_1} & \text{otherwise,} \end{cases}$$

with $A = \{(x,y),(y,x) \in [0,1]^2 \mid \text{there exists } z \in [0,1], x = T_1^1(z, \frac{a}{q_1}), T_1^1(y,z) \ge \frac{a}{q_1}\}$.

As T_2^1 and T_2^2 are continuous and T_2^3 can be obtained as a limit of continuous t-norms that coincide on the corresponding interval, also T_2 can be obtained as a limit of a sequence of continuous t-norms that coincide with T_1 on $[a,b]^2$.

The monotonicity and the results obtained in previous section gives us for the weakest extension the following.

Proposition 8. *Let* $T_1\colon [0,1]^2 \longrightarrow [0,1]$ *be a t-norm which is continuous on* $[a,b]^2$ *and let* $q_1, q_2 \in [a,b]$ *be respectively the smallest and the biggest idempotent point of* T_1 *in* $[a,b]$. *Then the weakest t-norm* T_3 *which coincides with* T_1 *on* $[a,b]^2$ *is given by*

$$T_3(x,y) = \begin{cases} \min(x,y) & \textit{if } \max(x,y) = 1, \\ \min(x,y) & \textit{if } (x,y) \in [q_1,1] \times [T_1(a,a), q_2] \setminus [q_1,q_2]^2, \\ \min(x,y) & \textit{if } (y,x) \in [q_1,1] \times [T_1(a,a), q_2] \setminus [q_1,q_2]^2, \\ T_1(x,y) & \textit{if } (y,x) \in [q_1,q_2]^2, \\ T_1(\min(x,b), \min(y,b)) & \textit{if } (x,y) \in [q_2,1]^2 \\ T_1(a, T_1(z,y)) & \textit{if } (x,y) \in A, x = T_1(a,z), \\ T_1(a, T_1(z,x)) & \textit{if } (x,y) \in A, y = T_1(a,z), \\ 0 & \textit{otherwise,} \end{cases}$$

with $A = \{(x,y),(y,x) \in [0,q_1]^2 \mid$ *there exists* $z \in [0,q_1], x = T_1(a,z), T_1(z,y) \geq a\}$.

If $T_1(a,a) > 0$ then T_3 cannot be obtained as a limit of continuous t-norms that coincide with T_1 on $[a,b]^2$. This is fact that if q_1 is an idempotent element of a continuous t-norm T then $T(x,y) = x$ for all $x \leq q_1 \leq y$. However, we have $T_3(x,y) = 0$ for $x \in [0, T_1(a,a)]$ and $y \in [q_1, 1]$. If $T_1(a,a) = 0$ then T_3 is similarly as T_2 an ordinal sum of three t-norms and the result can be composed from results of the previous section.

5 Conclusion

We have described the strongest and the weakest t-norms that coincide with the given t-norm T_1 on $[a,b]^2$. These results can be applied everywhere when we search for an extremal t-norm that coincides with the given values on some subinterval of the unit interval.

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