Chapter 2
Symmetric Functions

This chapter provides a lean but solid introduction to symmetric functions. All of the theory needed for our later chapters is carefully introduced while simultaneously giving the reader a firm hook on which to hang future studies. Our method is novel in that we emphasize the combinatorics of transition matrices and most of our proofs are combinatorial.

The subject is vast and an attempt to create an encyclopedic account would distract from our focus of using symmetric functions to solve enumeration problems. Therefore we have made heartbreaking choices on what topics to include or not include in this chapter, although we admit to succumbing a few interesting digressions which are not strictly needed in our development.

2.1 Standard Bases for Symmetric Functions

Let $x_1, x_2, \ldots$ be an infinite collection of indeterminates and, just as introduced in Section 1.3, let $B\mathbb{Q}[[x_1, x_2, \ldots]]$ be the subring of $\mathbb{Q}[[x_1, x_2, \ldots]]$ containing those monomials with bounded degree. Given a permutation $\sigma = \sigma_1 \ldots \sigma_N \in S_N$ and $P(x_1, x_2, \ldots) \in B\mathbf{Q}[[x_1, x_2, \ldots]]$, we define

$$\sigma P(x_1, x_2, \ldots x_N, x_{N+1}, x_{N+2}, \ldots) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots x_{\sigma_N}, x_{N+1}, x_{N+2}, \ldots).$$

We say that $P(x_1, x_2, \ldots)$ is a symmetric function if for all $N \geq 1$ and all $\sigma \in S_N$,

$$\sigma P(x_1, x_2, \ldots) = P(x_1, x_2, \ldots).$$

Thus $P(x_1, x_2, \ldots)$ is a symmetric function if it is invariant under all finite permutations of the variables $x_1, x_2, \ldots$.

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We define $\Lambda(x_1,x_2,\dots)$ to be the set of all symmetric functions in $B\mathbf{Q}[[x_1,x_2,\dots]]$. Since the sum and the product of any two symmetric functions are again symmetric functions, it follows that $\Lambda(x_1,x_2,\dots)$ is a ring. Further, we let

$$\Lambda_n(x_1,x_2,\dots) = \Lambda(x_1,x_2,\dots) \cap B\mathbf{Q}_n[[x_1,x_2,\dots]]$$

and we will refer to $\Lambda_n(x_1,x_2,\dots)$ as the vector space of symmetric functions of degree n. Our definitions ensure that we can write any symmetric function $P(x_1,x_2,\dots)$ in the form

$$P(x_1,x_2,\dots) = \sum_{n=0}^{N} P_n(x_1,x_2,\dots),$$

where $P_n(x_1,x_2,\dots) \in \Lambda_n[[x_1,x_2,\dots]]$ for all n by breaking $P(x_1,x_2,\dots)$ into its degree n components. In symbols, this means

$$\Lambda(x_1,x_2,\dots) = \bigoplus_{n=0}^{\infty} \Lambda_n(x_1,x_2,\dots).$$

By taking $x_i = 0$ for all $i \geq N+1$, the ring of symmetric functions $\Lambda(x_1,x_2,\dots)$ specializes to the polynomial ring $\Lambda(x_1,\dots,x_N)$. In this situation, an element $f \in \Lambda(x_1,\dots,x_N)$ is called a symmetric polynomial in the variables $x_1,\dots,x_N$ with coefficients in $\mathbb{Q}$. This means that for all permutations $\sigma = \sigma_1 \cdots \sigma_N \in S_N$,

$$f(x_1,\dots,x_N) = f(x_{\sigma_1},\dots,x_{\sigma_N}).$$

For example, one symmetric polynomial in the variables x_1,x_2, and x_3 is

$$2x_1 + 2x_2 + 2x_3 - x_1x_2 - x_1x_3 - x_2x_3 + 4x_1x_2x_3.$$

There are six standard bases for Λ_n: the monomial symmetric functions, the elementary symmetric functions, homogeneous symmetric functions, power symmetric functions, the Schur symmetric functions, and the forgotten symmetric functions. The main objective of this book is to exploit the relationships between these bases in order to solve counting problems.

The Monomial Symmetric Functions

If $\gamma = (\gamma_1,\dots,\gamma_N)$ is a weak composition of n, then we let $\lambda(\gamma)$ the partition found by sorting γ in weakly decreasing order. For example, if $\gamma = (2,0,3,1,0,1,0,0,4)$, then $\lambda(\gamma) = (4,3,2,1,1)$.

Let $\lambda = (\lambda_1,\dots,\lambda_\ell)$ be an integer partition of n. The monomial symmetric function $m_\lambda = m_\lambda(x_1,x_2,\dots)$ is defined to be

$$m_\lambda = \sum_{\gamma \in WC_n, \lambda(\gamma)=\lambda} x^\gamma.$$

Put differently, m_λ is the sum of all the monomials whose exponents can be rearranged to give the partition λ. For example, the monomial symmetric polynomial $m_{(2,1)}(x_1,x_2,x_3)$ is

$$m_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2^1 + x_1^2x_3^1 + x_1^1x_2^2 + x_2^2x_3^1 + x_1^1x_3^2 + x_2^1x_3^2.$$

Theorem 2.1. *The set $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

Proof. If α and β are two weak compositions of n such that $\lambda(\alpha) = \lambda(\beta) = \lambda$, then α and β are rearrangements of one another. Thus the coefficients of x^α and x^β in any given symmetric function $P(x_1,x_2,\ldots)$ are the same. This implies that we can write $P(x_1,x_2,\ldots)$ in the form

$$P(x_1,x_2,\ldots) = \sum_{\lambda \vdash n} c_\lambda m_\lambda$$

for constants c_λ, implying that $\{m_\lambda : \lambda \vdash n\}$ spans $\Lambda(x_1,x_2,\ldots)$.

Since m_λ and m_μ have no monomials in common if $\lambda \neq \mu$, the set $\{m_\lambda : \lambda \vdash n\}$ is an independent set, thereby showing the theorem true. □

Theorem 2.1 tells us that the dimension of $\Lambda_n(x_1,x_2,\ldots)$ is $p(n)$, the number of partitions of n.

The Elementary, Homogeneous, and Power Symmetric Functions

The n^{th} elementary symmetric function e_n is defined using a generating function. Let $E(z)$ denote the generating function for the sequence $e_0, e_1, e_2, \ldots$. Define e_n by

$$E(z) = \sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty}(1+x_i z) = (1+x_1 z)(1+x_2 z)\cdots.$$

For example, if $0 = x_4 = x_5 = \cdots$, the generating function $E(z)$ becomes

$$(1+x_1z)(1+x_2z)(1+x_3z) \\ = 1 + (x_1+x_2+x_3)z + (x_1x_2+x_1x_3+x_2x_3)z^2 + x_1x_2x_3z^3$$

and so the first few elementary symmetric polynomials in three variables are $e_0 = 1$, $e_1 = x_1+x_2+x_3$, $e_2 = x_1x_2+x_1x_3+x_2x_3$, and $e_3 = x_1x_2x_3$. In general, we can employ similar logic as found in the proof of Theorem 1.8 to conclude each variable x_i can appear at most once in a given monomial in e_n. In other words, the elementary symmetric function e_n is the sum of all square-free monomials of degree n—this means that each monomial in e_n is not divisible by x_i^2 for any x_i. The symmetric function e_n is also equal to $m_{(1^n)}$.

The elementary symmetric function e_n can be expressed as a sum of column strict tableaux of shape 1^n. We have

$$e_n = \sum_{T \in CS_{(1^n)}} w(T).$$

For example, the terms in the symmetric polynomial

$$e_3(x_1,x_2,x_3,x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

are the weights of the following column strict tableaux of shape 1^3 which are filled with integers no larger than 4:

$$\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline\end{array}$$

For any integer partition $\lambda = (\lambda_1,\ldots,\lambda_\ell) \vdash n$, we define $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The fundamental theorem of symmetric functions says that the set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n; this is our forthcoming Theorem 2.17.

The n^{th} homogeneous symmetric function h_n is defined in a similar manner as e_n. Letting $H(z)$ denote the generating function for h_n, we define

$$H(z) = \sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} \frac{1}{1-x_i z}.$$

For example, if we take $0 = x_4 = x_5 = \cdots$, then $H(z)$ becomes

$$\begin{aligned}
&\left(\frac{1}{1-x_1z}\right)\left(\frac{1}{1-x_2z}\right)\left(\frac{1}{1-x_3z}\right) \\
&\qquad = (1+x_1z+x_1^2z^2+\cdots)(1+x_2z+x_2^2z^2+\cdots)(1+x_3z+x_3^2z^2+\cdots) \\
&\qquad = 1+(x_1+x_2+x_3)z^1+(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3)z^2+\cdots
\end{aligned}$$

and so the first few homogeneous symmetric polynomials in three variables are $h_0 = 1$, $h_1 = x_1+x_2+x_3$, and $h_2 = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$. In general, by writing each term in the infinite product as a geometric series and expanding, we see that h_n contains all possible degree n monomials, each with leading coefficient 1. In other words,

$$h_n = \sum_{\lambda \vdash n} m_\lambda.$$

The homogeneous symmetric function h_n can be expressed as a sum of column strict tableaux of shape n; we have

$$h_n = \sum_{T \in CS_{(n)}} w(T).$$

For example, the terms in the symmetric polynomial

$$h_2(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

are the weights of the following column strict tableaux of shape 2 which are filled with integers no larger than 3:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$$

For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. The set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_n; this is our Corollary 2.20.

The n^{th} power symmetric function p_n is defined to be

$$p_n(x_1,x_2,x_3,\ldots) = x_1^n + x_2^n + x_3^n + \cdots$$

and so $p_n = m_{(n)}$. The power symmetric function p_n can be expressed as a weighted sum of tableaux if we require that every integer in a tableau of shape (n) be the same. Just like the elementary and homogeneous symmetric functions, we define $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$ for any integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$. We will show that $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ_n in Corollary 2.24.

The Schur Symmetric Functions

The most important basis for Λ_n with respect to its relationship to other areas of mathematics is the Schur symmetric functions—they are crucial in understanding the representation theory of the symmetric group. Given an integer partition $\lambda \vdash n$, we define the Schur symmetric function s_λ by

$$s_\lambda = \sum_{T \in CS_\lambda} w(T).$$

For example, all possible column strict tableaux of shape $(2,1)$ which are filled with integers less than or equal to 3 are

$$\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

and so

$$s_{(2,1)}(x_1,x_2,x_3) = 2x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

From this definition it may not be clear that the Schur symmetric function is even a symmetric function, much less a basis for Λ_n.

Theorem 2.2. *For any $\lambda \vdash n$, the Schur symmetric function s_λ is an element of Λ_n.*

Proof. Every element in S_n is a product of adjacent transpositions, that is, every element in S_n is the product of permutations of the form $(i\; i+1)$, so it is enough to show that $s_\lambda(x_1, x_2, \ldots)$ is unchanged under the action of switching x_i and x_{i+1} for all positive integers i. This means that we need to show that for every column strict tableau with k occurrences of i and j occurrences of $i+1$, there is a corresponding column strict tableau of the same shape with j occurrences of i and k occurrences of $i+1$.

Take $T \in CS_\lambda$. The appearances of i in relationship to the appearances of $i+1$ in T must look something like the appearances of the 3s and 4s below:

3	3	4	4	4	4								
			3	3	3	4	4	4					
							3	3	3	3	3	3	3	4	4

Each row in T may have a sequence of is followed by a sequence of $(i+1)$s. Rows are aligned so that an i cannot appear atop another i in the row below.

Suppose a given row of T contains a sequence of k is followed by j $(i+1)$s such that none of these is or $(i+1)$s are immediately above or below cells containing an i or $i+1$. Change this row of T so that it now contains j is followed by k $(i+1)$s. Make this change for every row of T to create the column strict tableau T'. For example, changing the 3s and 4s in the cells displayed above produces:

3	4	4	4	4	4										
			3	3	3	3	4	4							
							3	3	3	3	4	4	4	4	4

The new column strict tableau has the same shape as before with the number of is and $(i+1)$s switched, as needed. □

We end this section by using Vandermonde determinants to provide an alternative definition of the Schur symmetric function in the case where there are finitely many variables $x_1, \ldots, x_N$.

Given an integer partition $\lambda \vdash n$ where $n \le N$, write λ as $(\lambda_1, \ldots, \lambda_N)$ so that λ has N parts (the last parts of λ can be 0). We define $\Delta_\lambda(x_1, \ldots, x_N)$ to be the determinant of the matrix with i, j entry equal to $x_i^{\lambda_j+N-j}$. For example,

$$\Delta_{(3,1,0,0)}(x_1,x_2,x_3,x_4) = \begin{vmatrix} x_1^{3+3} & x_1^{1+2} & x_1^{0+1} & x_1^{0+0} \\ x_2^{3+3} & x_2^{1+2} & x_2^{0+1} & x_2^{0+0} \\ x_3^{3+3} & x_3^{1+2} & x_3^{0+1} & x_3^{0+0} \\ x_4^{3+3} & x_4^{1+2} & x_4^{0+1} & x_4^{0+0} \end{vmatrix} = \begin{vmatrix} x_1^6 & x_1^3 & x_1 & 1 \\ x_2^6 & x_2^3 & x_2 & 1 \\ x_3^6 & x_3^3 & x_3 & 1 \\ x_4^6 & x_4^3 & x_4 & 1 \end{vmatrix}.$$

Using the expansion of the determinant of a matrix as a signed sum over the symmetric group S_n, we have

$$\Delta_\lambda(x_1,\ldots,x_N) = \sum_{\sigma=\sigma_1\cdots\sigma_N\in S_N} \operatorname{sign}(\sigma) x_{\sigma_1}^{\lambda_1+N-1} x_{\sigma_2}^{\lambda_2+N-2} \cdots x_{\sigma_N}^{\lambda_N+0},$$

where $\operatorname{sign}(\sigma)$ is the sign of the permutation σ. This implies that Δ_λ is a polynomial in $x_1,\ldots,x_N$. Theorem 2.3 says that the polynomial $\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N)$, known as the Vandermonde determinant, factors nicely.

Theorem 2.3. *We have* $\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N) = \prod_{1\le i<j\le N} (x_i - x_j)$.

Proof. Switching x_i and x_j interchanges two rows in the determinant

$$\Delta_{(0,\ldots,0)} = \begin{vmatrix} x_1^{N-1} & x_1^{N-2} & \cdots & 1 \\ x_2^{N-1} & x_2^{N-2} & \cdots & 1 \\ & & \vdots & \\ x_N^{N-1} & x_N^{N-2} & \cdots & 1 \end{vmatrix},$$

which has the net effect of changing the sign of $\Delta_{(0,\ldots,0)}$ by a factor of -1. This implies that $\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N)$ is divisible by $(x_i - x_j)$ for every $i < j$.

More generally, this shows $\prod_{1\le i<j\le N}(x_i - x_j)$ divides $\Delta_{(0,\ldots,0)}$. Since these two polynomials are sums of monomials of degree $(N-1)+(N-2)+\cdots+0$, they must be equal up to some constant factor. By considering the main diagonal of the determinant, the coefficient of $x_1^{N-1}x_2^{N-2}\cdots x_N^0$ is 1 in both the determinant and the product, so this constant factor is 1. $\square$

Slight modifications of the proof of Theorem 2.3 show that $\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N)$ divides $\Delta_\lambda(x_1,\ldots,x_N)$ for all $\lambda \vdash n$. Furthermore, since switching x_i and x_j changes $\Delta_\lambda(x_1,\ldots,x_N)$ by a factor of -1, $\Delta_\lambda(x_1,\ldots,x_N)/\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N)$ is a symmetric polynomial. For example,

$$\frac{\Delta_{(2,1,0)}(x_1,x_2,x_3)}{\Delta_{(0,0,0)}(x_1,x_2,x_3)} = \frac{\begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix}}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}$$

which, when expanded and simplified, is equal to

$$2x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Theorem 2.4 explains why this calculation gives the Schur symmetric polynomial $s_{(2,1)}(x_1,x_2,x_3)$.

Theorem 2.4. *For any $\lambda \vdash n \leq N$, we have* $\dfrac{\Delta_\lambda(x_1,\ldots,x_N)}{\Delta_{(0,\ldots,0)}(x_1,\ldots,x_N)} = s_\lambda(x_1,\ldots,x_N)$.

Proof. Expanding the determinant $\Delta_\lambda(x_1,\ldots,x_N)$ as a sum over permutations in S_N and using Theorem 2.3, the identity in the statement of the theorem is the same as

$$\prod_{1\leq i<j\leq N} \frac{1}{x_i - x_j} \sum_{\sigma=\sigma_1\cdots\sigma_N \in S_N} \mathrm{sign}(\sigma) x_{\sigma_1}^{\lambda_1+N-1} x_{\sigma_2}^{\lambda_2+N-2} \cdots x_{\sigma_N}^{\lambda_N+0} = \sum_{T\in CS_\lambda} w(T).$$

Multiply both sides of this equation by $x_1^N x_2^{N-1}\cdots x_N^1$. With this term, we factor out the first term in each of the parentheses in $\prod_{i<j} 1/(x_i - x_j)$ and use $x_1^{N-1}x_2^{N-2}\cdots x_N^0$ to turn this product into $\prod_{i<j} 1/(1-x_j/x_i)$. The remaining $x_1\cdots x_N$ is used to increase each exponent in the sum on the left by 1. Our equation becomes

$$\prod_{1\leq i<j\leq N} \frac{1}{1-\frac{x_i}{x_j}} \sum_{\sigma\in S_N} \mathrm{sign}(\sigma) x_{\sigma_1}^{\lambda_1+N} x_{\sigma_2}^{\lambda_2+N-1} \cdots x_{\sigma_N}^{\lambda_N+1} = x_1^N x_2^{N-1}\cdots x_N^1 \sum_{T\in CS_\lambda} w(T). \tag{2.1}$$

We will prove this formulation of the identity with a sign reversing involution.

Looking at the left-hand side of this equality, we begin by constructing combinatorial objects in the following manner:

1. Affix an additional $N-j+1$ cells to the left of the j^{th} row of the Young diagram of λ, counting rows from bottom to top.
2. Select a permutation $\sigma = \sigma_1\cdots\sigma_N \in S_N$ and write σ vertically to the right of the Young diagram, reading bottom to top.
3. Starting from the bottom, place the integer σ_i into each cell in row i of our picture.

For example, if $\lambda = (3,1,0,0)$, the choice of $\sigma = 2\ 1\ 4\ 3$ gives

			3				3
		4	4				4
	1	1	1	1			1
2	2	2	2	2	2	2	2

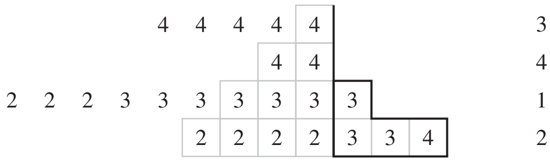
by following steps 1, 2, and 3. These three steps account for the sum on the left-hand side of (2.1). To account for the product

$$\prod_{i<j} \frac{1}{1-\frac{x_j}{x_i}} = \prod_{i<j}\left(1 + \left(\frac{x_j}{x_i}\right) + \left(\frac{x_j}{x_i}\right)^2 + \left(\frac{x_j}{x_i}\right)^3 + \cdots\right),$$

we finish creating our combinatorial in step 4:

4. In each row i, change any number of σ_is to an integer larger than σ_i. If every σ_i is changed, select any number of integers larger than σ_i to write down to the left of row i. Arrange the integers in each row so as to form a nondecreasing sequence.

For example, we can choose to change the above object into the one below:



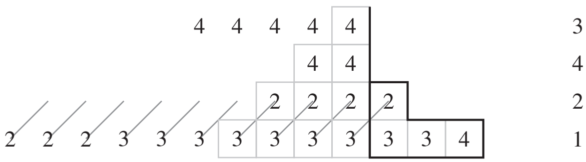
We define the sign of such an object T to be $\mathrm{sign}(\sigma)$ and we define the weight of the object to be

$$\left(\frac{x_1^{\text{the number of 1's in } T}}{x_1^{\text{the number of integers not in a cell in row } \sigma_1}}\right)\cdots\left(\frac{x_n^{\text{the number of } n\text{'s in } T}}{x_n^{\text{the number of integers not in a cell in row } \sigma_n}}\right).$$

For instance, the sign of the object displayed above is $\mathrm{sign}(2\ 1\ 4\ 3) = +1$ and the weight is $x_1^{-6}x_2^7x_3^5x_4^9$. By construction, the signed, weighted sum over all possible objects T is equal to the left-hand side of (2.1).

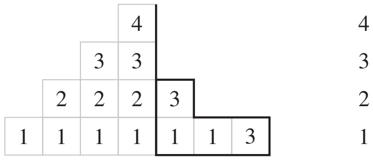
Let T be an object under consideration. We now describe how to create a new object $\varphi(T)$ with the same weight as T but with opposite sign. Starting from the most north cell in the most east column, scan the columns of T from top to bottom, moving right to left, looking for the first violation of column strictness. In the sample object displayed above, this violation occurs at the place where a 3 appears above another 3.

If T has no violations of column strictness, define $\varphi(T) = T$. Otherwise, let c be this first violating cell—this means the integer in c is not greater than the integer in the cell immediately below c. Define $\varphi(T)$ to be T with c and every cell in the same row and to the left of c switched with the cell kitty-corner to its south west. Additionally, if c is in the i^{th} row, switch the positions of σ_i and σ_{i-1} in σ. Below we show the image of the object T displayed above together with added diagonal lines to help the reader more readily identify how cells have been changed:



Since integers increase within rows, the integer in c is switched with a cell containing an integer no greater than the integer in c. Therefore the first violating cell in $\varphi(T)$ must be in the same position as the first violating cell in T, that is, φ is an involution. Introducing the transposition (σ_i, σ_{i+1}) changes the sign of σ by a factor of -1 and, since the integers both inside and outside of the cells in T and $\varphi(T)$ are the same, the weights of T and $\varphi(T)$ are also the same. In conclusion, φ is an involution which is weight preserving and, unless T is a fixed point, sign reversing.

The fixed points under the involution φ must look something like below:



There can be no violations of column strictness and so the column immediately to the left of the Young diagram of shape λ must contain the integers $1,\dots,n$ reading bottom to top. Therefore every fixed point must have σ equal to $1\ 2\ \cdots\ n$, every integer to the left of the Young diagram in row i containing i, and there cannot be any integers appearing outside of a cell.

These fixed points correspond to column strict tableaux of shape λ with an additional weight of $x_1^N x_2^{N-1} \cdots x_N^1$ coming from the cells to the left of the tableau; in other words, we have found the right-hand side of equation (2.1). $\square$

2.2 Relationships Between Bases for Symmetric Functions

Our first relationship between symmetric functions, Theorem 2.5, follows immediately from our definitions of e_n and h_n. However, although simple, we will reap an incredible amount of information about generating functions for permutation statistics from Theorem 2.5.

Theorem 2.5. *The generating functions $E(z)$ and $H(z)$ for the elementary and homogeneous symmetric polynomials satisfy $H(z) = 1/E(-z)$.*

Proof. By definition, $H(z) = \prod_{i=1}^{\infty} \frac{1}{1-x_i z} = \frac{1}{\prod_{i=1}^{\infty}(1+x_i(-z))} = 1/E(-z)$. $\square$

Rewriting Theorem 2.5 as $1 = H(z)E(-z)$, we find

$$1 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} (-1)^i e_i h_{n-i} \right) z^n.$$

Comparing coefficients of z^n shows that $\sum_{i=0}^{n}(-1)^i e_i h_{n-i}$ is equal to 0 for all $n \geq 1$. In following the philosophy of providing simple combinatorial proofs whenever reasonable, we will prove this fact with a sign reversing involution on pairs of column strict tableaux.

Proof (A second proof of Theorem 2.5). Consider ordered pairs (S,T) where S is a column strict tableau of shape 1^i and T a column strict tableau of shape $(n-i)$ for some $i \leq n$. Define the sign of (S,T) to be $(-1)^i$ and the weight to be $w(S)w(T)$. Then the signed, weighted sum over all possible pairs (S,T) is equal to $\sum_{i=0}^{n}(-1)^i e_i h_{n-i}$.

If the topmost integer in S is not smaller than the rightmost in T, move this integer from S to T. Otherwise, move the rightmost integer in T to the top of S. An example:

$$\begin{array}{|c|}\hline 6\\\hline 4\\\hline 2\\\hline 1\\\hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline 1&1&4&4\\\hline\end{array}\;\longleftrightarrow\;\begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array}\;,\;\begin{array}{|c|c|c|c|c|}\hline 1&1&4&4&6\\\hline\end{array}$$

This process is the desired sign reversing involution. □

The next theorem nicely illustrates a common theme in our work: after a theorem is proved combinatorially (that is, proved with a bijection or a sign reversing involution), we can usually modify the proof to arrive at new, related results.

Theorem 2.6. *For $k \geq 1$ and $n \geq 1$, $\sum_{i=0}^{n-1}(-1)^i e_i s_{(1^k,n-i)} = (-1)^{n-1}e_{n+k}$.*

Proof. The left hand side of this equation is the signed, weighted sum over all pairs of the form (S,T) where S is a column strict tableau of shape 1^i, T is a column strict tableau of shape $(1^k, n-i)$ with $n-i \geq 1$, the sign is $(-1)^i$, and the weight is $w(S)w(T)$. Apply the same sign reversing and weight preserving involution as in the second proof of Theorem 2.5: if the topmost integer in S is not smaller than the rightmost in T, move this integer from S to T. Otherwise, undo this operation.

Since we require that $n-i \geq 1$, there are fixed points which cannot be changed by this involution. Such a fixed point (S,T) must have the topmost integer in S smaller than the single element on the bottom row of T. For example, one fixed point when $k=3$ and $n=5$ is

$$\begin{array}{|c|}\hline 5\\\hline 4\\\hline 3\\\hline 1\\\hline\end{array}\;,\;\begin{array}{|c|}\hline 9\\\hline 8\\\hline 7\\\hline 6\\\hline\end{array}$$

Since the sign of such a fixed point is $(-1)^{n-1}$, the weighted sum over all fixed points (S,T) corresponds to $(-1)^{n-1}e_{n+k}$; this can be seen by affixing T atop S. □

Corollary 2.7. *For $k \geq 1$,*

$$\sum_{n=1}^{\infty} s_{(1^k,n)} z^{n+k} = \frac{\sum_{n=k+1}^{\infty}(-1)^{n-k-1}e_n z^n}{E(-z)}.$$

Proof. Multiplying both sides of this equation by $E(-z) = \sum_{n=0}^{\infty}(-1)^n e_n z^n$ and expanding, we find this statement:

$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1}(-1)^i e_i s_{(1^k,n-i)}\right) z^{n+k} = \sum_{n=k+1}^{\infty}(-1)^{n-k-1}e_n z^n.$$

The result follows by replacing the inner summand on the left-hand side with $(-1)^{n-k-1}e_{n+k}$ as allowed by Theorem 2.6 and reindexing. □

Simple bijections and involutions can give relationships between the elementary, homogeneous, and power symmetric functions, as we show in Theorems 2.8 and 2.9. These two theorems are commonly attributed to Isaac Newton or Albert Girard.

Theorem 2.8. *For* $n \geq 1$, $\sum_{i=0}^{n-1} h_i p_{n-i} = nh_n$.

Proof. The right-hand side corresponds to the weighted sum over all column strict tableaux of shape n where one of the n cells is shaded. Define a bijection on such objects in this way: if a cell c containing i is marked, remove c and all of the cells right of c that also contain i to create two-column strict tableau. This process is depicted below:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 2 & 4 & 5 \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 5 \\ \hline \end{array}\,, \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array}$$

The result is a pair (S,T) where S is a column strict tableau of shape i and T is a column strict tableau of shape $n-i$ where every cell in T contains the same integer. The weighted sum over all such pairs corresponds to the left-hand side of the equation. □

Theorem 2.9. *For* $n \geq 1$, $\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} ne_n$.

Proof. The left-hand side corresponds to the signed and weighted sum over all pairs of the form (S,T) where S is a column strict tableau of shape 1^i, T is a column strict tableau of shape $n-i$ where every cell in T contains the same integer, the sign of (S,T) is $(-1)^i$, and the weight is $w(S)w(T)$.

Define a weight preserving and sign reversing involution on such pairs (S,T) in the following way. If the integer in T also appears in S, move that integer from S to T. If the integer in T does not appear in S and T contains more than one cell, then move one cell from T to S. Otherwise, declare (S,T) to be a fixed point. This operation is displayed below:

$$\begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}\,, \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline 1 \\ \hline \end{array}\,, \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 2 \\ \hline \end{array}$$

The fixed points (S,T) under this operation have sign $(-1)^{n-1}$, have only one cell in T, and the integer in that cell does not appear in S. If we place the single cell in T into S and shade it gray, these fixed points correspond to $(-1)^{n-1}ne_n$ since there are ne_n ways to form a column strict tableau of shape 1^n with one cell shaded gray. □

Corollary 2.10. *We have*

$$\sum_{n=1}^{\infty} p_n z^n = \frac{\sum_{n=1}^{\infty} (-1)^{n-1} ne_n z^n}{E(-z)}.$$

Proof. Multiplying both sides of this equation by $E(-z) = \sum_{n=0}^{\infty}(-1)^n e_n z^n$ and expanding, we find this statement:

$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1}(-1)^i e_i p_{n-i}\right) z^n = \sum_{n=1}^{\infty}(-1)^{n-1} n e_n z^n.$$

This follows immediately by replacing the inner summand on the left-hand side with $(-1)^{n-1}e_n$ as allowed by Theorem 2.9. □

Theorem 2.11. *For $n \geq 1$, $\sum_{\lambda \vdash n} n!p_\lambda/z_\lambda = n!h_n$.*

Proof. Write a permutation $\sigma \in S_n$ above the cells of a standard tableau of shape n. The weighted sum over all possible objects is equal to $n!h_n$.

Suppose that the largest integer inside a cell in an object T is i. Locate the largest integer in σ atop an i in T, say σ_j. Cut the σ_j cell and all cells to the right off of T, creating two objects. Repeat this procedure on the remaining portion of T until there are no more cuts to be made. For example, if the object T is shown below,

$$\begin{array}{cccccccccccc} 8 & 1 & 5 & 2 & 6 & 10 & 11 & 4 & 12 & 3 & 7 & 9 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

then we would change T into

$$\begin{array}{cccc} 8 & 1 & 5 & 2 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{c} 6 \\ \boxed{2} \end{array} \quad \begin{array}{c} 10 \\ \boxed{2} \end{array} \quad \begin{array}{c} 11 \\ \boxed{2} \end{array} \quad \begin{array}{c} 4 \\ \boxed{3} \end{array} \quad \begin{array}{cccc} 12 & 3 & 7 & 9 \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Even if many components which result from these cuts were rearranged, the process could be reversed in order to reconstruct T.

The integers on the top of the object created by cutting T can be considered a permutation of n written in cyclic notation. If the cycle type of this permutation is the integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, then each part λ_i corresponds to a column strict tableau of shape (λ_i) where every integer is the same. Since Theorem 1.10 gives that the number of permutations with cycle type λ is $n!/z_\lambda$, these objects are counted by $\sum_{\lambda \vdash n} n!p_\lambda/z_\lambda$. □

Theorem 2.12. *For $n \geq 1$, $\sum_{\lambda \vdash n}(-1)^{n-\ell(\lambda)} n!p_\lambda/z_\lambda = n!e_n$.*

Proof. Take a permutation $\sigma \in S_n$ written in cyclic notation and, underneath each cycle of length λ_i, write a column strict tableau of shape (λ_i) where each cell contains the same integer. If σ has cycle type λ, then we define the sign of such an object to be $(-1)^{n-\ell(\lambda)}$. The signed, weighed sum over all such objects is equal to the left-hand side of the equation.

Momentarily ignoring the sign of an object T, apply the inverse to the bijection found in the proof of Theorem 2.11. We will now define a sign reversing involution in order to cancel any terms with a sign of -1.

If no two integers appearing in the cells of T are the same, define T as a fixed point. Otherwise, scan the cells of T from left to right looking for the first occurrence of two consecutive cells containing the same integer, say i. When this happens, find the largest two integers in the permutation σ which appear above an i and switch them. As an example, our involution pairs these objects:

$$\begin{array}{ccccccccc} 5 & 1 & 2 & 3 & 6 & 4 & 8 & 9 & 7 \\ \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{5} \end{array} \longleftrightarrow \begin{array}{ccccccccc} 2 & 1 & 5 & 3 & 6 & 4 & 8 & 9 & 7 \\ \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{5} \end{array}$$

This is a sign reversing involution because we have introduced exactly one transposition into the permutation σ. Fixed points correspond to a permutation atop a column strict tableau of shape (n) where no two cells contain the same integer. These fixed points, which have sign $(-1)^{n-n} = 1$, naturally correspond to $n!e_n$. □

We end this section by showing that some of these relationships between symmetric functions can be rephrased in terms of matrix determinants.

Theorem 2.13. *For all $n \geq 1$,*

$$e_n = \begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ 1 & h_1 & h_2 & \cdots & h_{n-1} \\ 0 & 1 & h_1 & \cdots & h_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & h_1 \end{vmatrix}.$$

Proof. The assertion is true when $n = 1$ because $e_1 = h_1$. We proceed by induction.

Removing the i^{th} row and last column of the $n \times n$ determinant leaves a determinant of the form

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix}.$$

where A is an $(i-1) \times (i-1)$ matrix of the same form as the original $n \times n$ matrix, B is an $(i-1) \times (n-i)$ matrix, 0 is the $(i-1) \times (n-i)$ zero matrix, and C is an $(n-i) \times (n-i)$ upper triangular matrix with 1s along the diagonal. By the induction hypothesis, the determinant of this matrix is e_{i-1}.

Expanding the determinant of the original $n \times n$ matrix along the last column, we find

$$\sum_{i=0}^{n-1} (-1)^{n+i-1} h_{n-i} e_i = e_n + (-1)^{n-1} \sum_{i=0}^{n} (-1)^i e_i h_{n-i}$$

which, by theorem 2.5, is equal to e_n. □

Theorem 2.14. *For all $n \geq 1$,*

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & \cdots & 0 \\ 3e_3 & e_2 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{vmatrix}.$$

Proof. The assertion is true when $n = 1$ because $p_1 = e_1$. We proceed by induction.

Removing the i^{th} column and last row of the $n \times n$ determinant leaves a determinant of the form

$$\begin{vmatrix} A & 0 \\ B & C \end{vmatrix}.$$

where A is an $(i-1) \times (i-1)$ matrix of the same form as the original $n \times n$ matrix, 0 is the $(i-1) \times (n-i)$ zero matrix, B is an $(n-i) \times (i-1)$ matrix, and C is an $(n-i) \times (n-i)$ lower triangular matrix with 1s along the diagonal. By the induction hypothesis, the determinant of this matrix is p_{i-1}.

Expanding the determinant of the original $n \times n$ matrix along the last row, we find

$$(-1)^{n-1} n e_n p_0 - \sum_{i=1}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n - \sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} + p_n$$

which, by Theorem 2.9, is equal to p_n. □

2.3 Transition Matrices

Let $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ be two bases for Λ_n. There is a $p(n) \times p(n)$ change of basis matrix A with entries indexed by partitions λ and μ such that

$$a_\mu = \sum_{\lambda \vdash n} A_{\lambda,\mu} b_\lambda, \tag{2.2}$$

where $A_{\lambda,\mu}$ is the λ,μ entry of A. The matrix A is the a-to-b transition matrix.

Let $\lambda^{(1)}, \ldots, \lambda^{(p(n))}$ be the integer partitions of n listed in the reverse lexicographic order, say, and take $f \in \Lambda_n$. Since $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ are bases, there are constants $c_{\lambda^{(1)}}, \ldots, c_{\lambda^{(p(n))}}$ and $d_{\lambda^{(1)}}, \ldots, d_{\lambda^{(p(n))}}$ such that

$$\begin{aligned} f &= c_{\lambda^{(1)}} a_{\lambda^{(1)}} + \cdots + c_{\lambda^{(p(n))}} a_{\lambda^{(p(n))}} \\ &= d_{\lambda^{(1)}} b_{\lambda^{(1)}} + \cdots + d_{\lambda^{(p(n))}} b_{\lambda^{(p(n))}}. \end{aligned}$$

Using standard matrix notation, equation (2.2) is equivalent to

$$\begin{bmatrix} A_{\lambda^{(1)},\lambda^{(1)}} & \cdots & A_{\lambda^{(1)},\lambda^{(p(n))}} \\ \vdots & \ddots & \vdots \\ A_{\lambda^{(p(n))},\lambda^{(1)}} & \cdots & A_{\lambda^{(p(n))},\lambda^{(p(n))}} \end{bmatrix} \begin{bmatrix} c_{\lambda^{(1)}} \\ \vdots \\ c_{\lambda^{(p(n))}} \end{bmatrix} = \begin{bmatrix} d_{\lambda^{(1)}} \\ \vdots \\ d_{\lambda^{(p(n))}} \end{bmatrix}.$$

Thus multiplying by the a-to-b transition matrix A allows us to take a symmetric function f expressed in terms of the a basis and write f in terms of the b basis.

This section is devoted to providing combinatorial interpretations for the entries of the transition matrices between various symmetric functions.

We do not have to use infinitely many variables to find such transition matrices. To see this, notice that for the monomial symmetric function $m_\lambda(x_1,x_2,\dots,x_N)$ to be nonzero, it must be the case that $N \geq n$—otherwise there may not be enough variables to create a monomial with the needed exponents. Thus $\{m_\lambda(x_1,\dots,x_N) : \lambda \vdash n\}$ is a basis for $\Lambda_n(x_1,\dots,x_N)$.

This means that the a-to-m transition matrix A is the same in $\Lambda_n(x_1,\dots,x_N)$ as it is in $\Lambda_n(x_1,x_2,\dots)$. If the b-to-m transition matrix is B, then it follows that the a-to-b transition matrix is $B^{-1}A$. Since A and B are the same in $\Lambda_n(x_1,\dots,x_N)$ and $\Lambda_n(x_1,x_2,\dots)$, the a-to-b transition matrix is also the same in $\Lambda_n(x_1,\dots,x_N)$ and $\Lambda_n(x_1,x_2,\dots)$. Thus when we are studying the transition matrices between bases of $\Lambda_n(x_1,x_2,\dots)$, it is enough to only consider symmetric polynomials in N variables $x_1,x_2,\dots,x_N$ for some $N \geq n$.

The s-to-m Transition Matrix

If we let $K_{\lambda,\mu}$ equal the number of column strict tableau of shape λ and content μ, then definition of the Schur symmetric function says that the coefficient of m_λ in s_μ is $K_{\mu,\lambda}$. This coefficient is called a Kostka number.

The Kostka matrix is the square matrix indexed by integer partitions of n written in reverse lexicographic order with λ,μ entry equal to $K_{\mu,\lambda}$. For example, the Kostka matrix with rows indexed by λ (the content) and columns indexed by μ (the shape) when $n=4$ is

$$\begin{array}{c}
\\ (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4)
\end{array}
\begin{array}{c}
\begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\
\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 3 & 1
\end{array}\right]
\end{array}.$$

The $(2,1^2),(3,1)$ entry is 2 because there are two-column strict tableau of shape $(3,1)$ and type $(2,1^2)$:

$$\begin{array}{|c|cc}
\hline
3 & \multicolumn{1}{|c}{} & \\
\hline
1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|cc}
\hline
2 & \multicolumn{1}{|c}{} & \\
\hline
1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}$$

The Kostka matrix is the s-to-m transition matrix, or the change of basis matrix, which turns a linear combination of Schur functions into a linear combination of monomial symmetric functions by matrix multiplication. In other words, the coefficient of m_λ in $a_1 s_{(4)} + a_2 s_{(3,1)} + a_3 s_{(2^2)} + a_4 s_{(2,1^2)} + a_5 s_{(1^4)}$ can be found by performing the matrix multiplication

$$\begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&0 \\ 1&2&1&1&0 \\ 1&3&2&3&1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

Theorem 2.15. *The set $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

Proof. If $\lambda < \mu$ in the reverse lexicographic order, then the first part in which λ and μ disagree is larger in λ than in μ. In this case there are no column strict Young tableau of shape μ and type λ. Further, $K_{\lambda,\lambda} = 1$ for all $\lambda \vdash n$. This tells us that the Kostka matrix is invertible because it is lower triangular with ones along the diagonal. Since $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n, so is $\{s_\lambda : \lambda \vdash n\}$. $\square$

The e-to-m Transition Matrix

Given integer partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_k)$, let $\mathbf{Z}_2 M_{\lambda,\mu}$ be the number of $\ell \times k$ matrices with entries either 0 or 1 such that the sum of the i^{th} row is λ_i and the sum of the j^{th} column is μ_j. For example, if $\lambda = (3,2,1)$ and $\mu = (2,2,2)$, then one possible matrix is

$$\begin{bmatrix} 1&1&1 \\ 0&1&1 \\ 1&0&0 \end{bmatrix}$$

because the row sums are λ and the column sums are μ.

Theorem 2.16. *The coefficient of m_λ in e_μ is $\mathbf{Z}_2 M_{\lambda,\mu}$.*

Proof. Given $\lambda \vdash n$, we will count the number of ways we can form the monomial $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ by multiplying out $e_\mu = e_{\mu_1} \cdots e_{\mu_\ell}$ by organizing our work into a table where rows are indexed by $x_1, \ldots, x_k$ and columns are indexed by $e_{\mu_1}, \ldots, e_{\mu_\ell}$. Place a 1 in the x_i row and e_{μ_j} column entry of the table if the monomial selected from e_{μ_j} to contribute to a final product contains x_i and place a 0 in the table otherwise.

For example, when $\lambda = (3^2, 2, 1^2)$ and $\mu = (3^2, 2^2)$, one possible table is

$$\begin{array}{c c} & \begin{array}{cccc} e_3 & e_3 & e_2 & e_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{bmatrix} 1&1&0&1 \\ 1&1&1&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix} \end{array}.$$

This table corresponds to the terms in each parenthesis in

$$e_{(3^2,2^2)}(x_1,x_2,\dots) = (x_1x_2x_3 + x_1x_2x_4 + \cdots)^2(x_1x_2 + x_1x_3 + \cdots)^2$$

which are selected to form the monomial $x_1^3x_2^3x_3^2x_4^1x_5^1$.

The number of ways to form such a table is the coefficient of m_λ in e_μ. Each table is an element in $\mathbf{Z}_2M_{\lambda,\mu}$ and so the theorem is proved. □

Theorem 2.16 gives a combinatorial interpretation for the entries in the e-to-m transition matrix. This matrix in the case $n = 4$ is shown below:

$$\begin{array}{c} \\ (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \end{array} \begin{array}{c} \begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 2 & 5 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix} \end{array}.$$

This is a symmetric matrix because the number of matrices with row sum λ and column sum μ is the same as the number of matrices with column sum λ and row sum μ by transposition.

The next theorem is known as the fundamental theorem of symmetric functions.

Theorem 2.17. *The set* $\{e_\lambda : \lambda \vdash n\}$ *is a basis for* Λ_n.

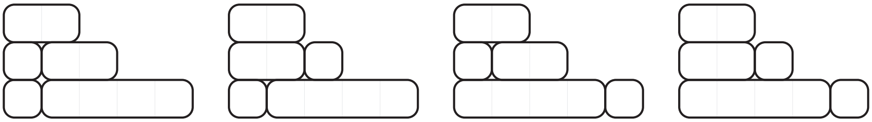
Proof. The only possible 0-1 matrix with row sum λ and column sum λ' is the matrix with the upside-down Young diagram of λ displayed in 1s in the matrix. For instance, the only 0-1 matrix with row sum $(4,2,1)$ and column sum $(3,2,1,1)$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The same argument will show that if $\lambda' < \mu'$, then there are no possible 0-1 matrices with row sum μ and column sum λ' because there are not enough parts in μ to account for the first part of λ. Therefore a reordering of the rows and columns of the e-to-m transition matrix results in a triangular matrix with 1s along the diagonal. This transition matrix is therefore invertible, implying that $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n. □

The h-to-e and e-to-h transition matrices

Let $B_{\lambda,\mu}$ be the set of all possible Young diagrams of μ where the rows of μ are partitioned into "bricks" of lengths giving the integer partition λ. The four $T \in B_{\lambda,\mu}$ when $\lambda = (4,2,2,1,1)$ and $\mu = (5,3,2)$ are here:



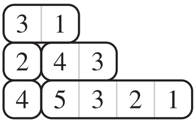
These elements in $B_{\lambda,\mu}$ are called brick tabloids of content λ and shape μ.

Theorem 2.18. *The coefficient of e_λ in h_μ is $(-1)^{n-\ell(\lambda)}\left|B_{\lambda,\mu}\right|$. In other words,*

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \left|B_{\lambda,\mu}\right| e_\lambda .$$

Proof. The right-hand side of this identity can be interpreted combinatorially. Use the summand and the $\left|B_{\lambda,\mu}\right|$ term to select a brick tabloid of content λ and shape μ for some $\lambda \vdash n$. Using the e_λ term, fill each brick with a decreasing sequence of distinct positive integers. Define the weight of such a brick tabloid to be the usual weight of a tableau. Finally, define the sign of such an object to be $(-1)^{n-\ell(\lambda)}$ (this power is the total number of cells in brick tabloid plus the number of bricks in the tabloid). The signed sum over all such combinatorial objects is equal to the right-hand side of the identity in the statement of the theorem.

For example, one such combinatorial object created in this way is shown below:

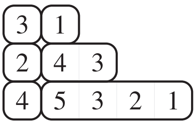


The weight of this object is $x_1^2 x_2^2 x_3^3 x_4^2 x_5$ and the sign is $(-1)^{10-5}$.

Let $\mathcal{B}$ be the set of combinatorial objects created in this way. We now define a sign reversing weight preserving involution φ on $\mathcal{B}$. Starting in the top row and scanning the bricks in $B \in \mathcal{B}$ from left to right, locate the first time if there is either a brick of length ≥ 2 or there is a brick of length 1 followed by another brick in the same row such that the integer labels between the two consecutive bricks decrease.

If there is a brick of length ≥ 2, then let $\varphi(B)$ be the object found by chopping the first cell off the brick of length ≥ 2, thereby creating two bricks. If there is a brick of length 1 followed by another brick in the same row such that the integer labels between the two consecutive bricks decrease, then let $\varphi(B)$ be the object found by combining the bricks. If neither situation is found after scanning all the rows of B, let $\varphi(B) = B$.

For example, the image of the combinatorial object shown above is here:



Fixed points under this involution must have every brick of length 1 and the integer labels within each row must weakly increase. The sign of such an object is $(-1)^{n-n} = 1$ and the weights give rise to exactly the homogeneous symmetric function h_μ. This proves the desired identity. $\square$

Theorem 2.18 gives a combinatorial interpretation for the entries of the h-to-e transition matrix. This matrix in the case $n=4$ is shown below; the entries are $(-1)^{n-\ell(\lambda)}\left|B_{\lambda,\mu}\right|$ with λ (the content) indexing the rows and μ (the shape) indexing the columns:

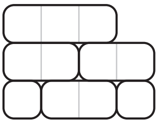
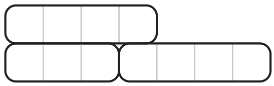
$$\begin{array}{c@{\;}c}
 & \begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\
\begin{array}{c} (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \end{array} &
\left[\begin{array}{ccccc}
-1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
-3 & -2 & -2 & -1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{array}\right].
\end{array}$$

Theorem 2.19. *The h-to-e transition matrix is its own inverse.*

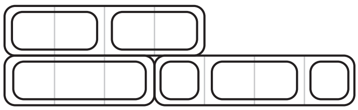
Proof. Writing down the matrix multiplication explicitly, we wish to show that

$$\sum_{\alpha\vdash n}(-1)^{n-\ell(\lambda)+n-\ell(\alpha)}\left|B_{\lambda,\alpha}\right|\left|B_{\alpha,\mu}\right| = \begin{cases} 1 & \text{if } \lambda=\mu, \\ 0 & \text{if } \lambda\neq\mu. \end{cases} \tag{2.3}$$

Given an element $T_1\in B_{\lambda,\alpha}$ and $T_2\in B_{\alpha,\mu}$, form a "double brick tabloid" by placing the bricks in each row of T_1 into the corresponding brick in T_2. For example, if T_1 and T_2 are the brick tabloids shown below

then we would combine T_1 and T_2 to create the double brick tabloid shown here:



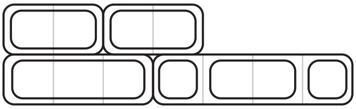
Given a double brick tabloid created from T_1 and T_2, call the bricks in the rows of T_1 "big bricks" and call the bricks inside the big bricks "little bricks." If we define the sign of a double brick tabloid to be $(-1)^{\text{the number of big and little bricks}}$, then the signed, weighted sum of all possible double brick tabloids of shape μ filled with little bricks of content λ is equal to the left-hand side of (2.3).

To find the right-hand side of (2.3), we will define a sign reversing involution. Scan the double brick tabloid from top to bottom and then from left to right looking for the first time there are either

1. two consecutive big bricks within a row, or
2. two little bricks inside of one big brick.

The double brick tabloid is a fixed point if there are no instances of either situation 1 or situation 2. Otherwise, if we encounter 1 first, then combine the two big bricks into one. If we encounter 2 first, then split the violating big brick b into two big bricks immediately after the first little brick in b. These are inverse operations which reverse the sign of the double brick tabloid.

For example, the double brick tabloid shown earlier in this proof would be changed to this double brick tabloid:



Each row in a fixed point must contain exactly one big brick containing exactly one little brick. Therefore only one fixed point of positive sign exists exactly when $\lambda = \mu$, which is the right-hand side of (2.3). $\square$

Corollary 2.20. *The set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

Proof. The h-to-e transition matrix is invertible and the elementary symmetric functions are a basis, so the homogeneous symmetric functions are also a basis. $\square$

Corollary 2.21. *The coefficient of h_λ in e_μ is $(-1)^{n-\ell(\lambda)} \left|B_{\lambda,\mu}\right|$. In other words,*

$$e_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \left|B_{\lambda,\mu}\right| h_\lambda .$$

The p-to-e and p-to-m Transition Matrices

Small modifications to brick tabloids can help us describe the elements in both the p-to-e and the p-to-m transition matrices. Define the weight of $T \in B_{\lambda,\mu}$, denoted $w(T)$, to be the product of the lengths of the bricks ending each row in T and let

$$w(B_{\lambda,\mu}) = \sum_{T \in B_{\lambda,\mu}} w(T).$$

For example, the weights of the four brick tabloids displayed after the proof of Theorem 2.17 on page 51 are 16, 8, 4, and 2, showing that $w(B_{(4,2^2,1^2),(5,3,2)}) = 30$. Theorem 2.22 below tells us that the λ,μ entry of the p-to-e transition matrix is equal to $(-1)^{n-\ell(\lambda)} w(B_{\lambda,\mu})$.

Theorem 2.22. *The coefficient of e_λ in p_μ is $(-1)^{n-\ell(\lambda)} w(B_{\lambda,\mu})$.*

Proof. Let $c_{\lambda,\mu}$ be the coefficient of e_λ in p_μ. We will show the following facts:

1. $c_{(n),(n)} = (-1)^{n-1} n$.
2. If $\lambda \vdash n$ has more than one part, then $c_{\lambda,(n)} = \sum_{i=1}^{n-1} (-1)^{i-1} c_{\lambda \setminus i,(n-i)}$ where $\lambda \setminus i$ denotes the integer partition λ with one part of size i removed with $c_{\lambda \setminus i,\mu} = 0$ if λ does not have a part of size i.

3. If $\alpha+\beta$ denotes the partition created by the multiset union of α and β where $\alpha \vdash \mu_1$ and $\beta \vdash n-\mu_1$, then
$$c_{\lambda,\mu} = \sum_{\alpha+\beta=\lambda} c_{\alpha,(\mu_i)} c_{\beta,\mu\backslash\mu_i}.$$

After proving these identities true, we will show that the integers $(-1)^{n-\ell(\lambda)}w(B_{\lambda,\mu})$ satisfy the same identities, thereby proving the theorem since both integers satisfy the same recursion and initial conditions.

Theorem 2.9 tells us that $(-1)^{n-1}ne_n = \sum_{i=0}^{n-1}(-1)^i e_i p_{n-i}$. Rewriting this,

$$\begin{aligned} p_n &= (-1)^{n-1}ne_n + \sum_{i=1}^{n-1}(-1)^{i-1}e_i p_{n-i} \\ &= (-1)^{n-1}ne_n + \sum_{i=1}^{n-1}(-1)^{i-1}e_i \left(\sum_{\alpha\vdash n-i} c_{\alpha,(n-i)} e_\alpha\right) \\ &= (-1)^{n-1}ne_n + \sum_{\lambda\vdash n}\left(\sum_{i=1}^{n-1}(-1)^{i-1}c_{\lambda\backslash i,(n-i)}\right) e_\lambda \end{aligned}$$

where in the last line we have combined the e_i and the e_α terms to create e_λ where λ is an integer partition with more than one part. Looking at the coefficients of e_n and e_λ in this expression verifies items 1 and 2. As for the third item,

$$\begin{aligned} \sum_{\lambda\vdash n} c_{\lambda,\mu}e_\lambda = p_\mu &= p_{\mu_1}p_{\mu\backslash\mu_1} \\ &= \left(\sum_{\alpha\vdash n} c_{\alpha,(\mu_1)}e_\alpha\right)\left(\sum_{\beta\vdash n} c_{\beta,\mu\backslash\mu_1}e_\beta\right) \\ &= \sum_{\alpha+\beta=\lambda} c_{\alpha,(\mu_1)}c_{\beta,\mu\backslash\mu_1}e_\lambda. \end{aligned}$$

Comparing coefficients of e_λ on the extremes gives item 3.

Now we show that $(-1)^{n-\ell(\lambda)}w(B_{\lambda,\mu})$ satisfies the same recursions. Item 1 follows since $(-1)^{n-\ell((n))}w(B_{(n),(n)}) = (-1)^{n-1}n$.

Item 2 follows by sorting the bricks appearing in the one row of (n) by the length of the first brick. Suppose that $\lambda \neq (n)$ and i is a part of λ. Then there are $w(B_{\lambda\backslash i,(n-i)})$ ways to create weighted brick tabloid of shape (n) after starting with a brick of length i. Therefore we have

$$\begin{aligned} (-1)^{n-\ell(\lambda)}w(B_{\lambda,(n)}) &= (-1)^{n-\ell(\lambda)}\sum_{i=1}^{n-1} w(B_{\lambda\backslash i,(n-i)}) \\ &= \sum_{i=1}^{n-1}(-1)^{i-1}\left((-1)^{(n-i)-(\ell(\lambda)-1)}w(B_{\lambda\backslash i,(n-i)})\right), \end{aligned}$$

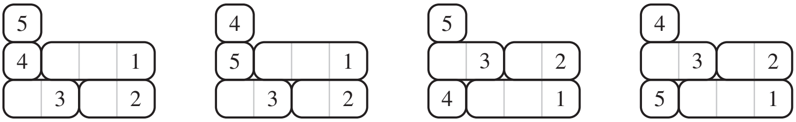
which verifies item 2.

Item 3 follows by sorting the bricks according to the bricks appearing in the top row. The number of weighted brick tabloids with bricks in the lengths appearing in α in the top row is equal to $w(B_{\alpha,(\mu_1)})w(B_{\beta,\mu\backslash\mu_1})$ where β is the integer partition for which $\alpha+\beta=\lambda$. Therefore $(-1)^{n-\ell(\lambda)}w(B_{\lambda,\mu})$ is equal to

$$\sum_{\alpha+\beta=\lambda}\left((-1)^{\mu_1-\ell(\alpha)}w(B_{\alpha,(\mu_1)})\right)\left((-1)^{(n-\mu_1)-\ell(\beta)}w(B_{\beta,\mu\backslash\mu_1})\right),$$

which verifies item 3 and completes the proof. □

An ordered brick tabloid of content $\mu=(\mu_1,\ldots,\mu_\ell)$ and shape λ is a brick tabloid in $B_{\mu,\lambda}$ such that the bricks of length $\mu_1,\ldots,\mu_\ell$ are labeled with $1,\ldots,\ell$ such that brick labels decrease within rows. For example, all four possible ordered brick tabloids of content $(3,2^2,1^2)$ and shape $(4^2,1)$ are shown below:



Let $OB_{\mu,\lambda}$ be the number of ordered brick tabloids of content μ and shape λ.

Theorem 2.23. *The coefficient of m_λ in p_μ is $OB_{\mu,\lambda}$.*

Proof. The number of ordered brick tabloids of content μ and shape λ corresponds directly to the number of times the monomial $x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ appears in the expansion of the product

$$p_\mu = p_{\mu_1}\cdots p_{\mu_\ell} = \left(x_1^{\mu_1}+x_2^{\mu_1}+\cdots\right)\cdots\left(x_1^{\mu_\ell}+x_2^{\mu_\ell}+\cdots\right).$$

Specifically, if row λ_i in an ordered brick tabloid contains bricks labeled $\mu_{i_1},\ldots,\mu_{i_k}$, then this ordered brick tabloid corresponds to selecting the x_i term from each of $p_{\mu_1},\ldots,p_{\mu_{i_k}}$ to contribute to the final monomial. □

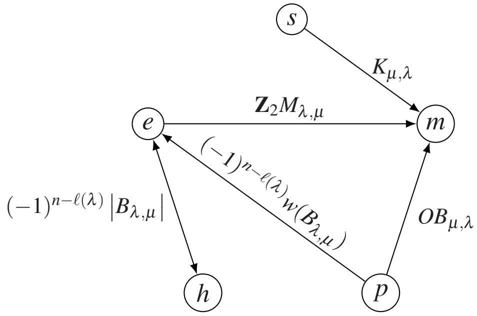
Theorem 2.23 tells us the coefficient in the p-to-m transition matrix is $OB_{\mu,\lambda}$. For clarity and for reference in section 6, we display this matrix in the case $n=4$ with rows indexed by λ (the shape) and columns indexed by μ (the content):

$$\begin{array}{c}
 \\ (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4)
\end{array}
\begin{array}{c}
\begin{array}{ccccc} (4) & (3,1) & (2^2) & (2,1^2) & (1^4) \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 2 & 4 \\
0 & 0 & 2 & 2 & 6 \\
0 & 0 & 0 & 2 & 12 \\
0 & 0 & 0 & 0 & 24
\end{array}\right]
\end{array}.$$

Corollary 2.24. *The set $\{p_\lambda : \lambda\vdash n\}$ is a basis for Λ_n.*

Proof. The p-to-m transition matrix is invertible since it is upper triangular with nonzero diagonal entries. □

At this point we have a number of combinatorial descriptions for the entries of the transition matrices between standard bases of the ring of symmetric functions. We have recorded what we have done so far by labeling the edge on the directed graph below with the λ, μ entry of the corresponding transition matrix:



There are combinatorial interpretations for the other transition matrices we have not included in this section; many are developed in the exercises. For reference, we have drawn a more complete diagram which includes all of the transition matrices introduced in this text in Appendix A.

This graph is connected if edge directions are ignored, so we can now combine transition matrices by matrix inversion or multiplication to turn any one basis into another. For instance, to find the m-to-h transition matrix, we multiply the inverse of the e-to-m matrix and the e-to-h matrix; in the case of $n = 4$ this is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -3 & -2 & -2 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 2 & 5 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -4 & -2 & 4 & -1 \\ -4 & 7 & 2 & -7 & 2 \\ -2 & 2 & 3 & -4 & 1 \\ 4 & -7 & -4 & 10 & -3 \\ -1 & 2 & 1 & -3 & 1 \end{bmatrix}.$$

2.4 A Scalar Product

This section defines a scalar product on Λ_n. This scalar product has a relationship to some of the results in Chapter 5. Although not discussed in this book, the scalar product is also closely related to an inner product in the representation theory of the symmetric group, see [104] for more details on that connection.

We define a scalar product on Λ_n by declaring that $\{p_\lambda/\sqrt{z_\lambda} : \lambda \vdash n\}$ is an orthonormal basis. In other words, we define our scalar product so that

$$\left\langle \frac{p_\lambda}{\sqrt{z_\lambda}}, \frac{p_\mu}{\sqrt{z_\mu}} \right\rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$

for all $\lambda, \mu \vdash n$ and then extend the definition by linearity.

We say that two bases $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ of Λ_n are dual bases if

$$\langle a_\lambda, b_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$

for all $\lambda, \mu \vdash n$. This means that the basis $\{p_\lambda/\sqrt{z_\lambda} : \lambda \vdash n\}$ is dual with itself. The next theorem provides a useful characterization of dual bases in Λ_n.

Theorem 2.25. *Bases $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ of Λ_n are dual if and only if*

$$\sum_{\lambda \vdash n} a_\lambda(X) b_\lambda(Y) = \sum_{\lambda \vdash n} \frac{p_\lambda(X) p_\lambda(Y)}{z_\lambda}$$

where $X = (x_1, x_2, \dots)$ and $Y = (y_1, y_2, \dots)$.

Proof. Let A be the a-to-$p_\lambda/\sqrt{z_\lambda}$ transition matrix and let B be the b-to-$p_\lambda/\sqrt{z_\lambda}$ transition matrix. This means

$$a_\lambda = \sum_{\alpha \vdash n} A_{\alpha,\lambda} \frac{p_\alpha}{\sqrt{z_\alpha}} \qquad \text{and} \qquad b_\mu = \sum_{\beta \vdash n} B_{\beta,\mu} \frac{p_\beta}{\sqrt{z_\beta}}.$$

Then we have

$$\begin{aligned} \langle a_\lambda, b_\mu \rangle &= \left\langle \sum_{\alpha \vdash n} A_{\alpha,\lambda} \frac{p_\alpha}{\sqrt{z_\alpha}}, \sum_{\beta \vdash n} B_{\beta,\mu} \frac{p_\beta}{\sqrt{z_\beta}} \right\rangle \\ &= \sum_{\alpha,\beta \vdash n} A_{\alpha,\lambda} B_{\beta,\mu} \left\langle \frac{p_\alpha}{\sqrt{z_\alpha}}, \frac{p_\beta}{\sqrt{z_\beta}} \right\rangle \\ &= \sum_{\alpha \vdash n} A_{\alpha,\lambda} B_{\alpha,\mu}. \end{aligned}$$

This last sum is the μ, λ entry in the matrix multiplication $B^{\mathrm{T}}A$. Therefore the bases $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ of Λ_n are dual if and only if $B^{\mathrm{T}}A = I$.

On the other hand, we have

$$\begin{aligned} \sum_{\lambda \vdash n} a_\lambda(X) b_\lambda(Y) &= \sum_{\lambda \vdash n} \left(\sum_{\alpha \vdash n} A_{\alpha,\lambda} \frac{p_\alpha(X)}{\sqrt{z_\alpha}} \right) \left(\sum_{\beta \vdash n} B_{\beta,\lambda} \frac{p_\beta(Y)}{\sqrt{z_\beta}} \right) \\ &= \sum_{\alpha,\beta,\lambda \vdash n} A_{\alpha,\lambda} B_{\beta,\lambda} \frac{p_\alpha(X)}{\sqrt{z_\alpha}} \frac{p_\beta(Y)}{\sqrt{z_\beta}} \end{aligned}$$

The coefficient of $p_\alpha(X) p_\beta(Y)/\sqrt{z_\alpha z_\beta}$ in this last line is

$$\sum_{\lambda \vdash n} A_{\alpha,\lambda} B_{\beta,\lambda},$$

which is the β, α entry in the matrix multiplication BA^{T}. Therefore identity in the statement of the theorem is true if and only if $BA^{\mathrm{T}} = I$.

The theorem follows since $B^{\mathrm{T}}A = I$ if and only if $BA^{\mathrm{T}} = I$. $\square$

The next theorem gives an alternative expression for the sums in Theorem 2.25.

Theorem 2.26. *We have* $\displaystyle\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} = \sum_{\lambda} \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda}$.

Proof. Starting with the left-hand side of the identity, we have

$$\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} = \exp\left(\ln \prod_{i,j\geq 1} \frac{1}{1-x_iy_j}\right) = \exp\left(\sum_{i,j\geq 1} \ln \frac{1}{1-x_iy_j}\right).$$

Using $\ln 1/(1-x) = \sum_{k\geq 1} x^k/k$ and $\exp x = \sum_{m\geq 0} x^m/m!$, the above expression is

$$\exp\left(\sum_{i,j,k\geq 1} \frac{x_i^k y_j^k}{k}\right) = \exp\left(\sum_{k\geq 1} \frac{p_k(X)p_k(y)}{k}\right) = \sum_{m\geq 0}\left(\sum_{k\geq 1} \frac{p_k(X)p_k(Y)}{k}\right)^m \frac{1}{m!}.$$

Let $\cdot|_{2n}$ denote the degree $2n$ terms for $n \geq 1$ in a sum or product. Applying degree $2n$ extraction on both sides of our string of inequalities gives

$$\begin{aligned}\prod_{i,j\geq 1} \frac{1}{1-x_iy_j}\Bigg|_{2n} &= \sum_{m\geq 0}\left(\sum_{k\geq 1} \frac{p_k(X)p_k(Y)}{k}\right)^m \frac{1}{m!}\Bigg|_{2n} \\ &= \sum_{m=1}^{n}\left(\sum_{k=1}^{n} \frac{p_k(X)p_k(Y)}{k}\right)^m \frac{1}{m!}\Bigg|_{2n}\end{aligned}$$

where we are able to truncate the infinite sums since the tail end of the series cannot contribute to a degree $2n$ term. Using the multinomial theorem $(x_1 + \cdots + x_n)^m = \sum_{a_1+\cdots+a_n=m} \binom{m}{a_1,\ldots,a_n} x_1^{a_1}\cdots x_n^{a_n}$, this expression is equal to

$$\sum_{m=1}^{n} \frac{1}{m!} \sum_{a_1+\cdots+a_n=m} \frac{m!}{a_1!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{p_k(X)p_k(Y)}{k}\right)^{a_k}\Bigg|_{2n}.$$

The degree of the terms in $\prod_{i=1}^{n} (p_k(X)p_k(Y)/k)^{a_k}$ are $2(a_1 + 2a_2 + \cdots + na_n)$. Furthermore, if $\lambda = (1^{a_1}\cdots n^{a_n})$ is a partition of n with m parts, then $a_1 + \cdots + a_n = m$, $a_1 + 2a_2 + \cdots + na_n = n$, and $p_\lambda(X)p_\lambda(Y)/z_\lambda = \prod_{k=1}^{n}(p_k(X)p_k(Y))^{a_k}/(k^{a_i}a_i!)$. We now have

$$\prod_{i,j\geq 1} \frac{1}{1-x_iy_j}\Bigg|_{2n} = \sum_{m=1}^{n} \sum_{\substack{a_1+\cdots+a_n=m \\ a_1+2a_2+\cdots+na_n=n}} \prod_{k=1}^{n} \frac{p_k(X)^{a_k}p_k(Y)^{a_k}}{k^{a_k}a_k!} = \sum_{\lambda\vdash n} \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda}.$$

The theorem follows by summing this identity over all nonnegative integers n. $\square$

Theorem 2.27. *The homogeneous symmetric functions* $\{h_\lambda : \lambda \vdash n\}$ *and the monomial symmetric functions* $\{m_\lambda : \lambda \vdash n\}$ *are dual bases in* Λ_n.

Proof. The definition of the homogeneous symmetric function says

$$\prod_{i\geq 1}\frac{1}{1-x_iy_j}=\sum_{n\geq 0}h_n(X)y_j^n$$

for any $j\geq 1$, and so

$$\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}=\prod_{j=1}^{\infty}\sum_{n\geq 0}h_n(X)y_j^n.$$

The left-hand side of this identity is symmetric in the variables $y_1,y_2,\ldots$. If we take the coefficient of $m_\lambda(Y)$ on the right-hand side of the equation, the coefficient is $h_\lambda(X)$. This proves

$$\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}=\sum_{\lambda}h_\lambda(X)m_\lambda(Y),$$

which, by Theorems 2.25 and 2.26, is enough to prove the theorem. □

Theorem 5.6, found in Chapter 5, will provide a combinatorial proof that the Schur symmetric functions $\{s_\lambda:\lambda\vdash n\}$ are an orthonormal basis for Λ_n.

Theorem 2.28. *Let $\{a_\lambda:\lambda\vdash n\}$ and $\{b_\lambda:\lambda\vdash n\}$ be one pair of dual bases in Λ_n and let $\{a'_\lambda:\lambda\vdash n\}$ and $\{b'_\lambda:\lambda\vdash n\}$ be second pair of dual bases. If A is the a-to-a' transition matrix and B is the b-to-b' transition matrix, then $A=\left(B^{-1}\right)^{\mathrm{T}}$.*

Proof. Since A is the a-to-a' transition matrix, $a_\mu=\sum_{\lambda\vdash n}A_{\lambda,\mu}a'_\lambda$. The b'-to-b transition matrix is B^{-1}, and so $b'_\lambda=\sum_{\mu\vdash n}B^{-1}_{\mu,\lambda}b_\mu$ where $B^{-1}_{\mu,\lambda}$ is the μ,λ entry of B^{-1}. We now have

$$A_{\lambda,\mu}=\left\langle\sum_{\lambda\vdash n}A_{\lambda,\mu}a'_\lambda,b'_\lambda\right\rangle=\langle a_\mu,b'_\lambda\rangle=\left\langle a_\mu,\sum_{\mu\vdash n}B^{-1}_{\mu,\lambda}b_\mu\right\rangle=B^{-1}_{\mu,\lambda},$$

which is the same as $A=\left(B^{-1}\right)^{\mathrm{T}}$. □

At this point we know the dual basis for the monomial symmetric functions (the homogeneous), the homogeneous symmetric functions (the monomials), the power symmetric functions (the power symmetric functions, divided by a factor of z_λ), and the Schur symmetric functions (the Schur symmetric functions). But what is dual to the elementary symmetric functions? We define the forgotten symmetric functions $\{f_\lambda:\lambda\vdash n\}\subseteq\Lambda_n$ to be dual to the basis $\{e_\lambda:\lambda\vdash n\}$.

The forgotten symmetric functions can be found using Theorem 2.28. The homogeneous and the monomial symmetric functions are dual, and, by Theorem 2.18, the h-to-e transition matrix has μ,λ entry $(-1)^{n-\ell(\lambda)}\left|B_{\lambda,\mu}\right|$. By Theorem 2.28, the f-to-m transition matrix has μ,λ entry $(-1)^{n-\ell(\mu)}\left|B_{\mu,\lambda}\right|$. Put differently,

$$f_\mu=\sum_{\lambda\vdash n}(-1)^{n-\ell(\mu)}\left|B_{\mu,\lambda}\right|m_\lambda.$$

Thus the forgotten symmetric functions can be expanded into monomials by counting brick tabloids.

2.5 The ω Transformation

In this section we define a ring homomorphism ω on Λ. This function will expand our understanding of fundamental relationships between standard bases for Λ and will allow us to explain why the transition matrices between certain bases in Λ_n are the same as previously described transition matrices.

Since the elementary symmetric functions $\{e_\lambda : \lambda \vdash n\}$ are basis for Λ_n for all n, the functions $e_0, e_1, \ldots$ are algebraically independent and generate Λ. This means every element of Λ can be uniquely expressed as a polynomial in the functions $e_1, \ldots, e_N$ for some N.

This means we can define a ring homomorphism ω on Λ by defining $\omega(e_n)$ for each $n \geq 1$ and then extending ω by linearity. Defining various ring homomorphisms on Λ can reveal many combinatorial identities; this is one of our major themes.

For this section we will take ω to be the ring homomorphism defined by setting $\omega(e_n) = h_n$ for all $n \geq 1$. It follows that $\omega(e_\lambda) = h_\lambda$ for all $\lambda \vdash n$.

Theorem 2.29. *The function ω is an involution.*

Proof. Using Theorem 2.18 to expand h_n in terms of the elementary symmetric functions, we have

$$\begin{aligned}
\omega(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \omega(e_\lambda) \\
&= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| h_\lambda .
\end{aligned}$$

Corollary 2.21 says this sum is equal to e_n. Therefore $\omega^2(e_n) = \omega(h_n) = e_n$, showing that ω is an involution. □

Theorem 2.30. *For all $n \geq 1$, $\omega(p_n) = (-1)^{n-1} p_n$.*

Proof. We show this by induction on n, with the case $n = 1$ being true since $p_1 = e_1 = h_1$ and thus $\omega(p_1) = \omega(e_1) = h_1 = (-1)^{1-1} p_1$.

Assume by induction that $\omega(p_k) = (-1)^{k-1} p_k$ for $k \leq n$. By Theorem 2.8,

$$p_n = nh_n - \sum_{i=1}^{n-1} h_i p_{n-i}.$$

Applying ω to both sides and using the induction hypothesis, we find

$$\omega(p_n) = ne_n - \sum_{i=1}^{n-1} (-1)^{n-i-1} e_i p_{n-i},$$

which, by Theorem 2.9, we can conclude is equal to $(-1)^{n-1} p_n$. □

Theorem 2.30 implies that $\omega(p_\lambda) = (-1)^{n-\ell(\lambda)} p_\lambda$ for all $\lambda \vdash n$.

Theorem 2.31. *For any symmetric functions $f,g \in \Lambda_n$, $\langle \omega(f), \omega(g)\rangle = \langle f,g\rangle$.*

Proof. For any $\lambda,\mu \vdash n$,

$$\left\langle \omega\left(\frac{p_\lambda}{\sqrt{z_\lambda}}\right), \omega\left(\frac{p_\mu}{\sqrt{z_\mu}}\right)\right\rangle = \left\langle (-1)^{n-\ell(\lambda)}\frac{p_\lambda}{\sqrt{z_\lambda}}, (-1)^{n-\ell(\mu)}\frac{p_\mu}{\sqrt{z_\mu}}\right\rangle$$
$$= \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$
$$= \left\langle \frac{p_\lambda}{\sqrt{z_\lambda}}, \frac{p_\mu}{\sqrt{z_\mu}}\right\rangle.$$

It is enough to prove the theorem true for a basis, like we just did for the basis $\{p_\lambda/\sqrt{z_\lambda} : \lambda \vdash n\}$. □

Theorem 2.31 allows us to find the image of the monomial symmetric functions under the ring homomorphism ω. For any $\lambda,\mu \vdash n$, we have

$$\langle e_\lambda, \omega(m_\mu)\rangle = \langle \omega(e_\lambda), m_\mu\rangle = \langle h_\lambda, m_\mu\rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

This says that the bases $\{e_\lambda : \lambda \vdash n\}$ and $\{\omega(m_\lambda) : \lambda \vdash n\}$ are dual. Since the forgotten symmetric functions are the functions which are dual to the elementary symmetric functions, it must be the case that $\omega(m_\lambda) = f_\lambda$.

At this point we know the values of ω on the elementary, homogeneous, power, monomial, and forgotten bases. What about the Schur symmetric functions? We will use Theorem 2.32 to prove that $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the conjugate partition to λ. The identities in Theorem 2.32 are known as the Jacobi–Trudi identities and are of interest in their own right. The proof we have chosen to include is due to Ira Gessel and Xavier Viennot.

Theorem 2.32. *Let $\lambda \vdash n$ be an integer partition with ℓ parts $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell$ written in nondecreasing order. Then*

$$s_\lambda = \det\left(h_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell} \qquad \textit{and} \qquad s_{\lambda'} = \det\left(e_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell}$$

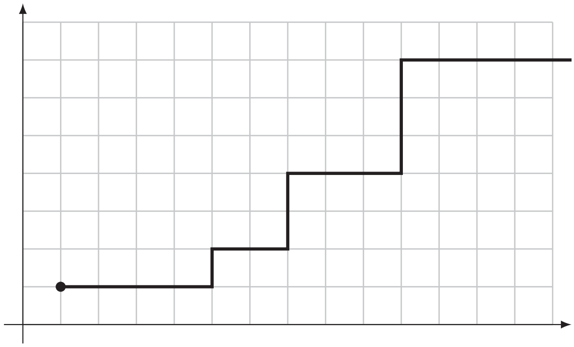
where we set $h_k = 0$ and $e_k = 0$ if $k < 0$.

Proof. We first prove the identity involving the homogeneous symmetric functions.

Each homogeneous symmetric function h_{λ_i+i-j} is the weighted sum over all column strict tableaux of shape $(\lambda_i + i - j)$. By interpreting the integers appearing in the column strict tableaux as the x-coordinates of the north steps, each choice of such a column strict tableaux corresponds to a weighted path p in the plane which starts at $(1, j)$, makes unit steps either north or east, and ends in an infinite number of east steps at height $\lambda_i + i$.

For example, suppose that $\lambda = (4,4,4,4)$, $i = 3$, and $j = 1$. Looking at $h_{\lambda_i+i-j} = h_{4+3-1} = h_6$, the column strict tableau on the left is interpreted as the lattice path on the right in the figure below:

5	7	7	10	10	10



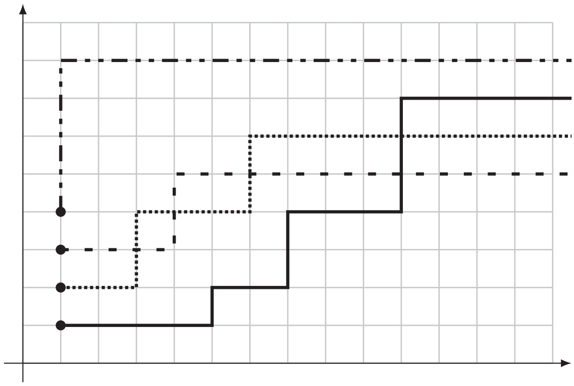
The column strict tableau on the left corresponds to the path on the right because the path has north steps at x-coordinates $5,7,7,10,10$, and 10. From such a path it is easy to find i (since the maximum height is $\lambda_i + i$) and j (since the starting point is at height j).

Let $\mathcal{P}_{j,\lambda_i+i}$ be the set of such lattice paths which begin at $(1, j)$ and end with an infinite sequence of east steps at height $\lambda_i + i$. If we define the weight of $p \in \mathcal{P}_{j,\lambda_i+i}$ to be the weight of the corresponding column strict tableau, then h_{λ_i+i-j} is the weighted sum over all $p \in \mathcal{P}_{j,\lambda_i+i}$.

Expanding the determinant as a signed sum over permutations $\sigma \in S_n$, we have

$$\det\left(h_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) h_{\lambda_1+1-\sigma(1)} \cdots h_{\lambda_\ell+\ell-\sigma(\ell)}.$$

The terms in this sum can be considered collections of paths $(p_1,\ldots,p_\ell)$ where $p_i \in \mathcal{P}_{\sigma(i),\lambda_i+i}$ for $i = 1,\ldots,\ell$. The ordered ℓ-tuple of lattice paths $(p_1,\ldots,p_\ell)$ will be called a lattice path family. For example, if $\lambda = (4,4,4,4)$ and $\sigma = 3\ 2\ 1\ 4$, one such lattice path family is represented below:



If we define the weight of the lattice path family $(p_1,\ldots,p_\ell)$ to be the product of the weights of the paths $p_1,\ldots,p_\ell$ and if we define the sign of the family to be the sign of the underlying permutation σ (which can be deduced from the lattice path family since the integer $\sigma(i)$ can be found for each i), then by construction, $\det\left(h_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell}$ is the weighted, signed sum over all lattice path families.

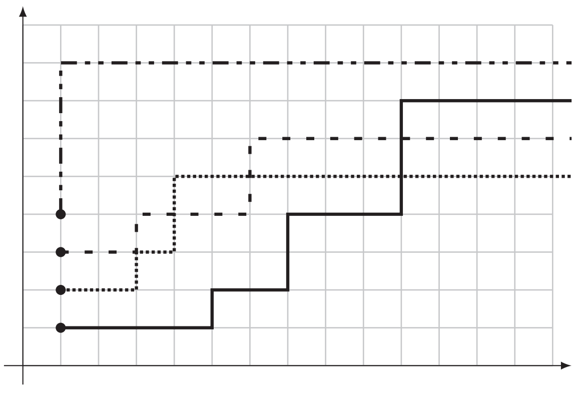
Because we ordered the parts of λ in nondecreasing order, the maximum heights of the paths in a lattice path family $(p_1,\ldots,p_\ell)$, namely $\lambda_1+1,\ldots\lambda_\ell+\ell$, are distinct. Moreover, the ith highest path reading bottom to top on the right side of a lattice path family is the path p_i. This path must appear as the $\sigma(i)$th path reading bottom to top on the left. This means that the permutation σ can be found easily: the ith highest path on the right ends up as the $\sigma(i)$th highest path on the right.

To prove the identity involving the homogeneous symmetric functions in the statement of the theorem, we will describe a weight preserving, sign reversing involution on lattice path families which will leave fixed points corresponding to column strict tableau of shape λ.

The involution φ is as follows. If there is no place in the lattice path family $(p_1,\ldots,p_\ell)$ where two paths intersect, define the lattice path family to be fixed under the involution φ. Otherwise, find the most south and then most west coordinate where two paths intersect. Exactly two paths must intersect here, for if three lattice paths intersect at the same point, then two of these paths must have intersected at a more south or more west coordinate, contradicting our choice of intersection. Furthermore, by our choice of intersection, the paths involved must begin at consecutive coordinates.

Suppose this intersection involves the paths p_i and p_{i+1}. Define φ to be the lattice path family found by switching the tail ends of p_i and p_{i+1} after this point of intersection and leaving all other paths alone.

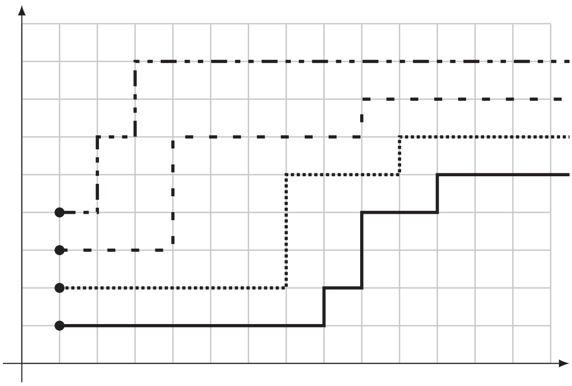
For example, considering the lattice path family displayed earlier in the proof, the first point of intersection is at the $(3,-3)$ coordinate and involves the second and the third paths. Applying the involution φ to this lattice path family gives the picture below:



The function φ is weight preserving and is an involution because $(p_1,\ldots,p_\ell)$ and $\varphi(p_1,\ldots,p_\ell)$ have the same set of north steps and any coordinates of intersection remain unchanged; in particular, the most south and most west coordinate of intersection is preserved. Furthermore, since we have switched the ending positions of exactly two paths, the permutation σ is changed by one transposition, changing the sign of σ by -1.

Thus $\det\left(h_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell}$ is equal the weighted sum over all lattice path families where no two paths intersect. Since each path p_i lies below the path p_{i+1} for all i, the underlying permutation in such a nonintersecting lattice path family must be the identity permutation, which has sign $+1$.

Each nonintersecting lattice path family $(p_1,\ldots,p_\ell)$ naturally corresponds to a column strict tableaux of shape λ. Starting from the top path and working downwards, fill the rows in a tableau of shape λ working bottom up with the x-coordinates of the north steps in each path. For example, below we display one nonintersecting lattice path family together with the corresponding tableau:



8	9	9	11
7	7	7	10
4	4	4	9
2	2	3	3

Since each path in the nonintersecting lattice path family moves north and east only, each row of the tableau is weakly increasing. Furthermore, by construction, the kth column in the tableau is strictly increasing since the kth north step in path p must appear higher than the kth north step in any path below p.

Since the Schur symmetric function is the weighted sum over all column strict tableaux of shape λ, at this point we have proved the identity

$$s_\lambda = \det\left(h_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell}.$$

The proof for the elementary symmetric function determinant is the same as that for the homogeneous symmetric functions with a few small modifications. We outline the main points but leave some of the finer details for the reader to verify.

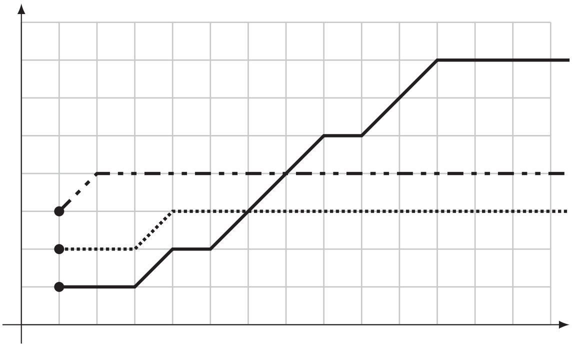
The key difference is that each elementary symmetric function e_{λ_i+i-j} is the sum over column strict tableaux of shape $1^{(\lambda_i+i-j)}$, meaning that, unlike the homogeneous symmetric function, we cannot have repeated integers in a tableau. Thus the corresponding lattice paths cannot have two consecutive north steps—every north step must be immediately followed by an east step.

To adjust for this difference, we will associate each column strict tableau coming from e_{λ_i+i-j} with a lattice path p in the plane which starts at $(1, j)$, takes steps of the form $(1,0)$ or $(1,1)$, and ends in an infinite number of $(1,0)$ steps at height $\lambda_i + i$. We create the path p such that if the integer k appears in the column strict tableaux, then p has a diagonal step beginning at x-coordinate k.

Therefore the determinant

$$\det\left(e_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell} = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)e_{\lambda_1+1-\sigma(1)}\cdots e_{\lambda_\ell+\ell-\sigma(\ell)}$$

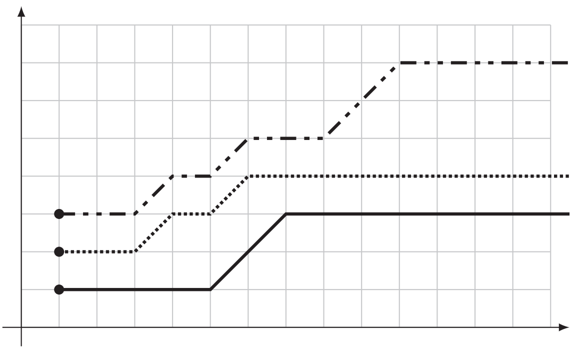
can be interpreted as a signed, weighted sum over lattice path families with east and diagonal steps instead of east and north steps. For example, one such lattice path family when $\lambda = (2,2,4)$ is shown below:



The weight of this lattice path family shown above is $x_1x_3^2x_5x_6x_7x_9x_{10}$ (since the diagonal steps begin at x-coordinates $1,3,3,5,6,7,9$, and 10) and the underlying permutation is $\sigma = 2\ 3\ 1$ with sign $+1$.

We can now apply the same involution as described for the homogeneous symmetric functions; find the most south and most west coordinate where two paths intersect and switch their tails.

The positions of the diagonal steps in a fixed point can be used to fill the rows of a tableau of shape λ in the same way as we did for the homogeneous symmetric functions. Fixed points naturally correspond to tableau with strictly increasing rows and weakly increasing columns. For example, the fixed point shows on the left corresponds to the tableau on the right:



5	6		
3	5		
3	5	8	9

This proves

$$s_{\lambda'} = \det\left(e_{\lambda_i+i-j}\right)_{i,j=1,\ldots,\ell}$$

since conjugating these tableaux which correspond to fixed points gives the necessary column strict tableaux. □

Corollary 2.33. *For all $\lambda \vdash n$, $\omega(s_\lambda) = s_{\lambda'}$.*

Proof. Apply ω to the first identity in 2.32 to find the second identity. □

Exercises

2.1. Show that $s_{(1^k,n)} = \sum_{i=0}^{k} (-1)^{k-i} e_i h_{n+k-i}$.

2.2. Show that $p_r = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)}$ for $r \geq 1$.

2.3. Show $\sum_{n=1}^{\infty} p_n z^n = zH'(z)/H(z)$ where $H'(z)$ is the derivative of $H(z) = \sum_{n=0}^{\infty} h_n z^n$.

2.4. Prove that the coefficient of h_λ in e_μ is $(-1)^{n-\ell(\lambda)} \left|B_{\lambda,\mu}\right|$ using an involution similar to that found in the proof of Theorem 2.18.

2.5. Show that for integer partitions $\lambda, \mu \vdash n$,

$$\sum_{\alpha \vdash n} (-1)^{\ell(\lambda)+\ell(\alpha)} OB_{\lambda,\alpha} w(B_{\alpha,\mu}) = \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ z_\lambda & \text{if } \lambda = \mu. \end{cases}$$

2.6. Using Exercise 2.5, show that the coefficient of p_λ in e_μ is $(-1)^{n-\ell(\lambda)} OB_{\lambda,\mu}/z_\lambda$ and the coefficient of p_λ in m_μ is $(-1)^{\ell(\lambda)+\ell(\mu)} w(B_{\mu,\lambda})/z_\lambda$.

2.7. Define an alternating polynomial f in the variables $x_1, \ldots, x_n$ to be a polynomial with coefficients in $\mathbb{Q}$ such that

$$f(x_1, \ldots, x_n) = \operatorname{sign}(\sigma) f(x_{\sigma_1}, \ldots, x_{\sigma_n})$$

for all permutations $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. For example, one alternating polynomial in the variables x_1, x_2, and x_3 is $x_1 x_2 x_3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. Show that any alternating polynomial must be divisible by the Vandermonde determinant $\Delta_{(0,\ldots,0)}$ and therefore must have minimum degree $\binom{n}{2}$.

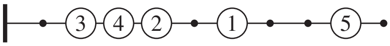
2.8. Show that Δ_λ is an alternating polynomial (see Exercise 2.7). Further, show that if a monomial $x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n+n-n}$ is a term in an alternating polynomial f, then f must have all terms present in Δ_λ. These two facts imply that $\{\Delta_\lambda : \lambda \vdash k\}$ is a basis for the set of alternating polynomials of degree $k + \binom{n}{2}$.

2.9. Let RCS_λ denote the set of reverse column strict tableaux, that is, all tableaux where the integer labeling weakly decreases in rows and strictly decreases in columns. For example, here are all elements in $RCS_{(2,1)}$ that are filled with integers ≤ 3:

$$\begin{array}{|c|c}\hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\quad \begin{array}{|c|c}\hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ .$$

Show that $s_\lambda = \sum_{RCS_\lambda} w(T)$ for any $\lambda \vdash n$.

2.10. A labeling of the mathematical abacus a is a filling of the k beads in a with a permutation in S_k. Below we display a labeled abacus of length 10 with 5 beads filled with the permutation $3\ 4\ 2\ 1\ 5 \in S_5$:



Let $b_1,\ldots,b_k$ be the beads in a labeled abacus a when read left to right, let $\text{label}(b_i)$ be the integer in bead b_i, and let $\text{position}(b_i)$ be the position of bead b_i. We define the weight of a to be $x_{\text{label}(b_1)}^{\text{position}(b_1)} \cdots x_{\text{label}(b_k)}^{\text{position}(b_k)}$ and we define the sign of a to be the sign of the permutation $\text{label}(b_k)\cdots\text{label}(b_1)$. For instance, the weight of the labeled abacus shown above is $x_3^2 x_4^3 x_2^4 x_1^6 x_5^9$ and the sign is $\text{sign}(5\ 1\ 2\ 4\ 3) = -1$.

Let λ be the integer partition corresponding to the abacus a. Show that

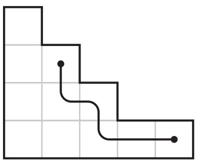
$$x_1 \cdots x_k \Delta_\lambda(x_1,\ldots,x_k) = \sum \text{sign}(\ell)\text{weight}(\ell)$$

where the sum runs over all possible labelings ℓ of the abacus a.

2.11. Let $\lambda \vdash n$. Using Exercise 2.10, show that $e_j \Delta_\alpha = \sum \Delta_\lambda$ where the sum runs over the integer partitions $\lambda \vdash (n+j)$ found by adding 1 to j distinct parts of α.

2.12. Using Exercise 2.11 to expand $e_\mu \Delta_{(0,\ldots,0)}$ into a sum of terms of the form Δ_λ, show that the λ,μ entry of the e-to-s transition matrix is equal to $K_{\lambda',\mu}$.

2.13. Let ν be a rim hook (see Exercise 1.3). The sign of ν, denoted $\text{sign}(\nu)$, is defined to be $(-1)^{\text{the number of rows spanned by } \nu - 1}$. For instance, the sign of the rim hook of length 6 pictured below is $(-1)^{3-1} = +1$:

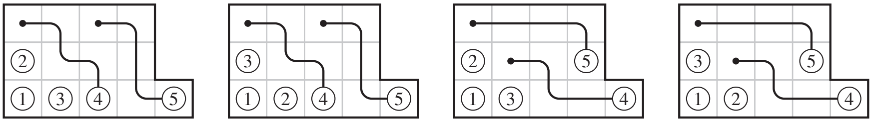


Using Exercise 2.10, show that

$$p_j \Delta_\alpha = \sum \text{sign}(\nu)\Delta_\lambda$$

where the sum runs over the integer partitions λ which can be found by adding a rim hook ν of length j to α.

2.14. A rim hook tableau of shape λ and content $\mu = (\mu_1, \ldots, \mu_\ell)$ is a filling of the cells of the Young diagram of λ with rim hooks of lengths $\mu_1, \ldots, \mu_\ell$ (see Exercise 1.3) labeled with $1, \ldots, \ell$ such that the removal of the last i rim hooks leaves the Young diagram of a smaller integer partition for all i. For example, below we display all possible rim hook tableau of shape $(5,4,4)$ and content $(5,5,1,1,1)$:



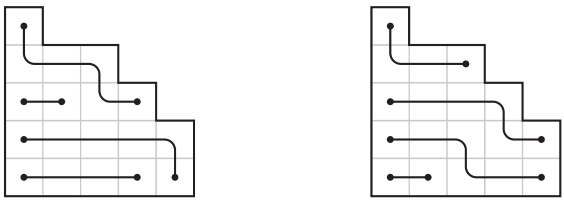
The sign of a rim hook tableau T is defined to be the product of the signs of the rim hooks in T (see Exercise 2.13). The four rim hook tableaux pictured above all have sign $+1$. We define

$$\chi_\mu^\lambda = \sum_{\substack{\text{rim hook tableaux } T \text{ with} \\ \text{shape } \lambda \text{ and content } \mu}} \operatorname{sign}(T).$$

Using Exercise 2.13 to expand $p_\mu \Delta_{(0,\ldots,0)}$ into a sum of terms of the form Δ_λ, show that the λ, μ entry of the p-to-s transition matrix is χ_μ^λ.

2.15. A special rim hook tabloid of shape λ and content μ is a rim hook tableau of shape λ and content μ (see Exercise 2.14) such that the labels on the rim hooks are erased and every rim hook contains at least one cell in the first column of the Young diagram of λ. The change of nomenclature from "tableau" to "tabloid" indicates that the rim hooks within a special rim hook tabloid are unordered.

For example, here are the only two possible special rim hook tabloids of content $(6,6,4,2)$ and shape $(5,5,4,3,1)$:



Let $K_{\mu,\lambda}^{-1}$ to be the integer defined by

$$K_{\mu,\lambda}^{-1} = \sum_{\substack{\text{special rim hook tabloids } T \\ \text{of shape } \lambda \text{ and content } \mu}} \operatorname{sign}(T)$$

where $\operatorname{sign}(T)$ is defined in Exercise 2.14. The goal of this exercise is to show that the inverse to the Kostka matrix has λ, μ entry equal $K_{\mu,\lambda}^{-1}$, that is, we want to show

$$\sum_{\alpha \vdash n} K_{\mu,\alpha} K_{\alpha,\lambda}^{-1} = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda. \end{cases} \tag{2.4}$$

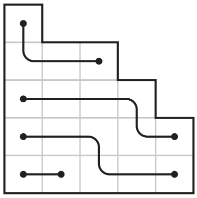
for all λ, μ. This can be done by considering pairs (C,S) where C is a column strict tableau and S is a special rim hook tabloid such that:

1. The special rim hook tabloid S of shape λ and content α is chosen first. Let a_i be the length of the special rim hook which begins in row i of S reading bottom to top. This number might be 0.
2. The column strict tableau C has shape μ and contains a_1 1s, a_2 2s, etc. By Theorem 2.2, the number of ways to form C is independent of this specification of number of 1's, 2's, etc.
3. The sign of (C,S) is equal to $\mathrm{sign}(S)$.

For example, one pair when $\mu = (7,6,5)$ and $\lambda = (5,5,4,3,1)$ is

3	3	5	5	5		
2	2	3	3	3	5	
1	1	2	2	2	2	3

,



.

There is a unique way to switch the tail ends of two consecutive special rim hooks in S. Use this "tail-switching" idea to define a sign reversing involution on such pairs (C,S) in order to verify (2.4).

2.16. Given integer partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_k)$, let $NM_{\lambda,\mu}$ be the number of $\ell \times k$ matrices with nonnegative integer entries such that the sum of the i^{th} row is λ_i and the sum of the j^{th} column is μ_j. Show that the coefficient of m_λ in h_μ is $NM_{\lambda,\mu}$.

2.17. Show that

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ p_2 & p_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & n-1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 \end{vmatrix}.$$

Using the ω transformation, find a similar relationship involving the homogeneous and power symmetric functions.

Solutions

2.1 Looking at the right-hand side of the identity, we can consider ordered pairs (S,T) where S is a column strict tableau of shape 1^i for $0 \le i \le k$ and T is a column strict tableau of shape $(n+k-i)$. The sign is $(-1)^{k-i}$. We now apply a sign reversing involution similar to that found in the second proof of Theorem 2.5.

If the bottommost integer in S is not larger than the leftmost integer in T, then move this integer from S to T. Otherwise, if S has height smaller than k and if the bottommost integer in S is larger than the leftmost integer in T, then move this integer from T to S.

Fixed points under this sign reversing involution must have S with height k and the bottommost integer in S is larger than the leftmost integer in T, like the picture below if $k = 4$:

$$\begin{array}{|c|}\hline 6\\\hline 4\\\hline 3\\\hline 2\\\hline\end{array}\;,\quad \begin{array}{|c|c|c|c|}\hline 1&1&4&4\\\hline\end{array}$$

These fixed points, which have sign $+1$, correspond to tableaux of shape $(1^k, n)$ by gluing S atop T.

2.2 Define the sign of a column strict tableau T of shape $(r-k, 1^k)$ to be $(-1)^k$. Consider the following sign reversing involution: locate the largest integer m appearing in T. If m appears in the first column of T and this first column has more than one cell, then move m to the right of the bottom row of T. If m appears in bottom row of T and m is larger than the largest cell in the first column of T, then move m to the first column.

Fixed points under this sign reversing involution must have the largest integer m appearing in both the bottom row of T and the first column of T. Furthermore, the first column of T must have only one cell. It follows that T must contain exactly one row, and that every cell in that row contains the same integer m. These fixed points correspond to p_r, as desired.

2.3 Using Theorem 2.8,

$$H(z)\sum_{n=1}^{\infty} p_n z^n = \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} h_i p_{n-i}\right) z^n = \sum_{n=1}^{\infty} n h_n z^n = z\sum_{n=0}^{\infty} n h_n z^{n-1} = zH'(z).$$

2.4 The proof is similar to the proof of Theorem 2.18 except that we allow weakly decreasing sequences in the bricks instead of strictly decreasing sequences. In particular, the desired identity is

$$e_\mu = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} \left|B_{\lambda,\mu}\right| h_\lambda. \tag{2.5}$$

The right-hand side of (2.5) can be interpreted combinatorially. Use the summand and the $\left|B_{\lambda,\mu}\right|$ term to select a brick tabloid of content λ and shape μ for some $\lambda \vdash n$. Using the h_λ term, fill each brick with a weakly decreasing sequence of positive integers. Define the weight and sign in the same way as in the proof of theorem 2.18. The signed sum over all such combinatorial objects is equal to the right-hand side of (2.5).

Define an involution φ by starting in the top row and scanning the bricks from left to right, locating the first time there is either a brick of length ≥ 2 or there is a brick of length 1 followed by another brick in the same row such that the integer labels between the two consecutive bricks weakly decrease.

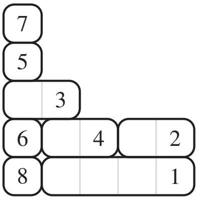
If there is a brick of length ≥ 2, change the object by chopping the first cell off the brick of length ≥ 2, thereby creating two bricks. If there is a brick of length 1 followed by another brick in the same row such that the integer labels between the two consecutive bricks weakly decrease, then change the object by combining the bricks. Do nothing if neither situation is found.

Fixed points must consist of only bricks of length 1 (and thus must have sign $+1$) and must have strictly increasing sequences of integers within each row, corresponding directly with e_μ, as desired.

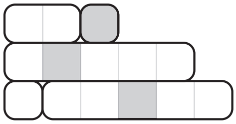
2.5 Construct a set of combinatorial objects by following these steps:

1. Select an ordered brick tabloid of content λ and shape α for some $\alpha \vdash n$.
2. Select a brick tabloid of content α and shape μ. Select one cell in the last brick in each row and shade it gray. This shading accounts for the weight in a weighted brick tabloid.
3. Combine the brick tabloids selected in step 1 and step 2 by placing the bricks in each row of the ordered brick tabloid into the corresponding brick in the weighted brick tabloid.

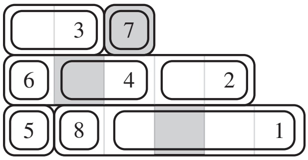
For example, if the brick tabloids



and



are selected in steps 1 and 2, then combining them in step 3 would create



.

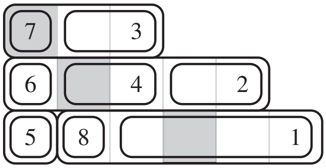
Let $\mathcal{T}$ be the set of all objects created by following steps 1, 2, and 3. Call a smaller brick appearing inside of another brick a "little brick" and the larger bricks "big bricks." Define the sign of $T \in \mathcal{T}$ to be $(-1)^{\text{the number of big and little bricks in } T}$. By construction, the signed, weighted sum of $T \in \mathcal{T}$ is the sum in the statement of this exercise.

Define a sign reversing involution by examining the last big brick in each row of T, starting from top to bottom, looking for either

1. a last big brick which contains more than one little brick or

2. a row with more than one big brick such that the last big brick contains only one little brick.

If case 1 is found, break the big brick into smaller big bricks by moving the little brick containing the shaded cell into its own big brick at the end of the row. If case 2 is found, combine the two big bricks into one big brick, sorting the little bricks so that the little brick labels decrease within the big brick. For example, the image of the $T \in \mathcal{T}$ displayed earlier is shown below:



Fixed points must have exactly one big brick in each row, and that big brick must contain exactly one little brick. These fixed points, which all have sign $+1$, occur exactly when $\lambda = \mu$. If $\lambda = 1^{m_1}2^{m_2}\cdots$, then the total number of fixed points is $z_\lambda = 1^{m_1}2^{m_2}\cdots m_1!m_2!\cdots$ since $1^{m_1}2^{m_2}\cdots$ accounts for the placement of the shaded cell in each row and $m_1!m_2!\cdots$ accounts for the ways to rearrange the labels on little bricks of the same length.

2.6 Exercise 2.5 gives the λ, μ entry in the matrix multiplication

$$\|(-1)^{n-\ell(\lambda)}OB_{\lambda,\mu}\|_{\lambda,\mu\vdash n}\|(-1)^{n-\ell(\lambda)}w(B_{\lambda,\mu})\|_{\lambda,\mu\vdash n}.$$

The product of these two matrices is the diagonal matrix with λ^{th} diagonal entry equal to z_λ, which is nearly the identity matrix. From this we can say two things:

1. the inverse to $\|(-1)^{n-\ell(\lambda)}w(B_{\lambda,\mu})\|_{\lambda,\mu\vdash n}$ is $\|(-1)^{n-\ell(\lambda)}OB_{\lambda,\mu}/z_\lambda\|_{\lambda,\mu\vdash n}$ and
2. the inverse to $\|OB_{\lambda,\mu}\|_{\lambda,\mu\vdash n}$ is $\|(-1)^{\ell(\lambda)+\ell(\mu)}w(B_{\mu,\lambda})/z_\lambda\|_{\lambda,\mu\vdash n}$.

Stated differently,

1. the e-to-p transition matrix, which is the inverse to the p-to-e transition matrix given in Theorem 2.22, has λ,μ entry $(-1)^{n-\ell(\lambda)}OB_{\lambda,\mu}/z_\lambda$ and
2. the m-to-p transition matrix, which is the inverse to the p-to-m transition matrix given in Theorem 2.23, has λ,μ entry $(-1)^{\ell(\lambda)+\ell(\mu)}w(B_{\mu,\lambda})/z_\lambda$, as desired.

2.7 Considering the transposition $(i\ \ j)$, an alternating polynomial f must satisfy

$$f(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = -f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N)$$

for all integers i, j. Therefore f must be divisible by $(x_i - x_j)$ for all i, j. This means that f must be divisible by the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$ and that the degree of f must be at least $(N-1)+\cdots+0 = \binom{N}{2}$.

2.8 The determinant of a matrix is changed by a factor of -1 when two rows are interchanged. Since Δ_λ is the determinant

$$\begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n+0} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n+0} \\ \vdots & \vdots & & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n+0} \end{vmatrix},$$

switching the roles of x_i and x_j changes the sign of Δ_λ by -1. It is therefore an alternating polynomial.

If $x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n+0}$ is a term in an alternating polynomial f, then f must contain all terms of the form $\mathrm{sign}(\sigma)x_{\sigma_1}^{\lambda_1+n-1}x_{\sigma_2}^{\lambda_2+n-2}\cdots x_{\sigma_n}^{\lambda_n+0}$ for σ a permutation of S_n. Therefore f contains

$$\Delta_\lambda(x_1,\ldots,x_n) = \sum_{\sigma=\sigma_1\cdots\sigma_n\in S_n} \mathrm{sign}(\sigma)x_{\sigma_1}^{\lambda_1+n-1}x_{\sigma_2}^{\lambda_2+n-2}\cdots x_{\sigma_n}^{\lambda_n+0},$$

as desired.

2.9 We will describe a weight preserving function which turns any $T \in CS_\lambda$ into a $T' \in RCS_\lambda$.

Take $T \in CS_\lambda$. If a 1 appears in the same column as a 2 in T, switch their positions. Then if a sequence of 1s appears in the same row as a sequence of 2s in T, switch the appearances of these sequences. Now every 1 appears above or to the right of every 2. Repeat this procedure with the 1s and 3s in T, then the 1s and 4s, and so on, until every 1 is appears above or to the right of every larger integer in T.

Inductively repeat this process with 2, moving all appearances of 2 above or to the right of all larger integers. Then repeat this process with 3, 4, and so on. The result is the desired reverse column strict tableau T'.

2.10 If $b_1,\ldots,b_k$ are the beads in the abacus a, then the corresponding integer partition λ is equal to $(\mathrm{empty}(b_k),\ldots,\mathrm{empty}(b_1))$ where $\mathrm{empty}(b_i)$ denotes the number of empty places to the left of b_i. We have

$$\begin{aligned} x_1\cdots x_k\Delta_\lambda(x_1,\ldots,x_k) &= \sum_{\sigma=\sigma_1\cdots\sigma_k\in S_k} \mathrm{sign}(\sigma)x_{\sigma_1}^{\lambda_1+k}\cdots x_{\sigma_k}^{\lambda_k+1} \\ &= \sum_{\sigma=\sigma_1\cdots\sigma_k\in S_k} \mathrm{sign}(\sigma)x_{\sigma_1}^{\mathrm{empty}(b_k)+k}\cdots x_{\sigma_k}^{\mathrm{empty}(b_1)+1}. \end{aligned}$$

Since there are exactly $i-1$ beads to the left of bead b_i, we know $\mathrm{position}(b_i) = \mathrm{empty}(b_i)+i$. Therefore our sum is equal to

$$\sum_{\sigma=\sigma_1\cdots\sigma_k\in S_k} \mathrm{sign}(\sigma)x_{\sigma_1}^{\mathrm{position}(b_k)}\cdots x_{\sigma_k}^{\mathrm{position}(b_1)} = \sum_{\text{labeling } \ell \text{ of } a} \mathrm{sign}(\ell)\mathrm{weight}(\ell).$$

2.11 Let a be the mathematical abacus corresponding to the integer partition α. Since e_j is the sum of square-free monomials of degree j, each monomial in the product

$$e_j\Delta_\alpha = e_j \sum_{\ell \text{ is a labeling of } a} \operatorname{sign}(\ell)\operatorname{weight}(\ell)$$

can be associated with an ordered pair of the form $(x_{i_1}\cdots x_{i_j},\ell)$ where $i_1 < \cdots < i_j$ are j distinct positive integers and ℓ is a labeling of a. By defining the sign of such a pair to be $\operatorname{sign}(\ell)$ and the weight to be $x_{i_1}\cdots x_{i_j}\operatorname{weight}(\ell)$, it follows that $e_j\Delta_\alpha$ is the signed, weighted sum over all possible pairs of the form $(x_{i_1}\cdots x_{i_j},\ell)$.

Given $(x_{i_1}\cdots x_{i_j},\ell)$, starting with the leftmost possible bead and working rightward, move the beads with labels given by $i_1,\ldots,i_j$ one space to the right. If we cannot move bead b one space to the right because that space is occupied by another bead b', match $(x_{i_1}\cdots x_{i_j},\ell)$ with the pair found by interchanging the labels on b and b' in both $x_{i_1}\cdots x_{i_j}$ and ℓ. Since the permutations in ℓ and ℓ' differ by a transposition, these two objects have opposite signs. They have the same weight, and so their pairing will cancel them from the sum.

Each time we move one of the k beads to the right, we are increasing one part in the corresponding integer partition by 1. Therefore $e_j\Delta_\alpha$ corresponds to the signed sum over all possible labelings of abaci which correspond to an integer partition $\lambda \vdash (n+j)$ which can be found by adding 1 to j distinct parts of α, as desired.

2.12 Let $\mu = (\mu_1,\ldots,\mu_\ell) \vdash n$. By Exercise 2.11, $e_\mu\Delta_{(0,\ldots,0)} = e_{\mu_1}\cdots e_{\mu_\ell}\Delta_{(0,\ldots,0)}$ is the sum of terms of the form Δ_λ where λ is an integer partition created by adding 1 to μ_1 distinct parts in the integer partition $(0,\ldots,0)$, then adding 1 to μ_2 distinct parts to the result, then adding 1 to μ_3 distinct parts to the result, and so on. Place an i in the cell of the Young diagram for the integer partition λ if the cell was created by adding 1 in step i of this process.

For example, we can create $\Delta_{(6,3,3,2)}$ from $e_{(3,3,3,2,2,1)}\Delta_{(0,\ldots,0)}$ by starting with $(0,0,0,0)$, then successively adding 1 to distinct parts to create $(1,1,1,0)$, $(2,2,1,1)$, $(3,3,2,1)$, $(4,3,3,1)$, $(5,3,3,2)$, and then $(6,3,3,2)$. Recording these steps by placing $1,\ldots,6$ in the cells of the Young diagram gives

2	5				
1	3	4			
1	2	3			
1	2	3	4	5	6

.

The integers in this tableau of shape λ must weakly increase within columns and strictly increase within rows. Therefore the number of terms of the form Δ_λ in the expansion of $e_\mu\Delta_{(0,\ldots,0)}$ is equal to the number of column strict tableau of shape λ' and content μ, namely $K_{\lambda',\mu}$.

We now have that $e_\mu\Delta_{(0,\ldots,0)} = \sum_{\lambda\vdash n} K_{\lambda',\mu}\Delta_\lambda$. Dividing both sides of this equation by $\Delta_{(0,\ldots,0)}$ and using Theorem 2.4, we have $e_\mu = \sum_{\lambda\vdash n} K_{\lambda',\mu}s_\lambda$, as desired.

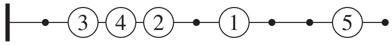
2.13 Let a be the mathematical abacus corresponding to the integer partition α. Since $p_j = x_1^j + x_2^j + \cdots$, each monomial in the product

$$p_j\Delta_\alpha = p_j \sum_{\ell \text{ is a labeling of } a} \operatorname{sign}(\ell)\operatorname{weight}(\ell)$$

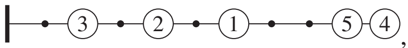
can be associated with an ordered pair of the form (x_i^j, ℓ) where ℓ is a labeling of a. If the sign of such a pair is sign(ℓ) and the weight is x_i^jweight(ℓ), then $p_j\Delta_\alpha$ is the signed, weighted sum over all possible pairs of the form (x_i^j, ℓ).

Given (x_i^j, ℓ), move the bead with label i to the right j spaces. If we cannot do this because that space is occupied by another bead b', match (x_i^j, ℓ) with the pair found by interchanging the labels on b and b' in both x_i^j and ℓ. Since the permutations in ℓ and ℓ' differ by a transposition, these two objects have opposite signs. They have the same weight, and so their pairing will cancel them from the sum.

If we happen to move this bead b over another bead b', then we are multiplying the permutation giving the labels on ℓ by the transposition (label(b) label(b')). For example, if we move the bead with label 4



7 spaces to the right to form

,

then we introduce the transpositions (2 4), (1 4), and (5 4) to the underlying permutation.

Exercise 1.3 tells us that we are adding one rim hook of size j to the corresponding integer partition each time we move a bead b to the right j spaces. The number of beads b passes is one less than the number of rows in the corresponding rim hook ν, and so this move changes the sign by sign(ν). Therefore, by Exercise 1.3, $p_j\Delta_\alpha$ corresponds to the signed sum over all possible labelings of abaci which correspond to an integer partition $\lambda \vdash (n+j)$ which can be found by adding a rim hook ν of length j to α.

2.14 Let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash n$. By Exercise 2.13, $p_\mu \Delta_{(0,\ldots,0)} = p_{\mu_1} \cdots p_{\mu_\ell} \Delta_{(0,\ldots,0)}$ is the sum of terms of the form $\pm\Delta_\lambda$ where λ is an integer partition created by adding a first rim hook μ of size μ_ℓ to $(0, \ldots, 0)$, then adding a rim hook of length $\mu_{\ell-1}$ to the result, then adding a rim hook of length to $\mu_{\ell-2}$ to the result, and so on. The $\pm$ sign on $\pm\Delta_\lambda$ is determined by the product of the rim hooks. Label the order the rim hooks were placed with the numbers $1, \ldots, \ell$ to find a rim hook tableau of shape λ and content μ.

This shows that $p_\mu \Delta_{(0,\ldots,0)} = \sum_{\lambda \vdash n} \chi_\mu^\lambda \Delta_\lambda$. Dividing both sides of this equation by $\Delta_{(0,\ldots,0)}$ and using Theorem 2.4, we have $p_\mu = \sum_{\lambda \vdash n} \chi_\mu^\lambda s_\lambda$, as desired.

2.15 If row i contains only the integer i for each row i in C reading bottom to top, then define (C, S) to be a fixed point of the involution. Otherwise, find the least i such that row i contains an integer larger than i, and let j be the maximum integer in this row.

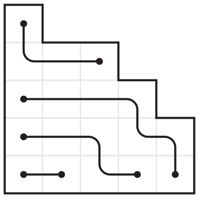
Let ν_j be the special rim hook which begins in row j of S. By switching their tail ends, there is a unique way to change the special rim hooks ν_j and ν_{j-1} to two other special rim hooks which occupy the same cells as ν_j and ν_{j-1}. Change S into the special rim hook tabloid found by making this switch.

Performing this "tail-switching" operation on S changes the lengths of the special rim hooks ν_j and ν_{j-1}, and so to be a valid pair of the form (C,S) we need to change the frequencies of the js and the $(j-1)$s in C accordingly. Make this switch using the involution in the proof of Theorem 2.2.

For example, the image of the pair (C,S) displayed in Exercise 2.15 is shown below:

3	3	5	5	5		
2	2	3	3	3	5	
1	1	2	2	2	3	3

,



.

This "tail-switching" involution changes the sign of S by -1 since it either introduces or removes exactly one extra down step in the rim hook ν_j.

The fixed points under this involution must have row i of C containing exactly i. This means that a special rim hook of length i begins in row i of S for all i. There is exactly one special rim hook tabloid S with this property—the special rim hook tabloid which features a completely flat rim hook in each row. This one fixed point of sign $+1$ only occurs when $\mu = \lambda$, verifying equation (2.4).

2.16 Given $\lambda \vdash n$, we will count the number of ways can we form the monomial $x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ by multiplying out $h_\mu = h_{\mu_1}\cdots h_{\mu_\ell}$ by organizing our work into a table where rows are indexed by $x_1,\ldots,x_k$ and columns are indexed by $h_{\mu_1},\ldots,h_{\mu_\ell}$. Place an m in the x_i row and h_{μ_j} column entry of the table if the monomial selected from h_{μ_j} to contribute to a final product contains x_i^m.

For example, when $\lambda = (3^2,2,1^2)$ and $\mu = (3^2,2^2)$, one possible table is

$$\begin{array}{c c} & \begin{array}{cccc} h_3 & h_3 & h_2 & h_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

This table corresponds to the terms in each parenthesis in

$$h_{(3^2,2^2)}(x_1,x_2,\ldots) = (x_1^3+\cdots+x_1^2x_2+\cdots+x_1x_2x_3+\cdots)^2(x_1^2+\cdots+x_1x_2+\cdots)^2$$

which are selected to form the monomial $x_1^3x_2^3x_3^2x_4^1x_5^1$.

The number of ways to form a such a table is the coefficient of m_λ in h_μ. Each table is an element in $NM_{\lambda,\mu}$ as desired.

2.17 The assertion is true when $n=1$ because $1!e_1 = p_1$. We proceed by induction.

Removing the i^{th} column and last row of the $n\times n$ determinant leaves a determinant of the form

$$\begin{vmatrix} A & 0 \\ B & C \end{vmatrix}$$

where A is an $(i-1)\times(i-1)$ matrix of the same form as the original $n\times n$ matrix, 0 is the $(i-1)\times(n-i)$ zero matrix, B is an $(n-i)\times(i-1)$ matrix, and C is an $(n-i)\times(n-i)$ lower triangular matrix diagonal entries equal to $i,i+1,\ldots,n-1$. By the induction hypothesis, the determinant of this matrix is $(i-1)!e_{i-1}i$ $(i-1)\cdots(n-1)=(n-1)!e_{i-1}$.

Expanding the determinant of the original $n\times n$ matrix along the last row,

$$\sum_{i=1}^{n}(-1)^{n-i}p_{n-(i-1)}(n-1)!e_{i-1}=(-1)^{n-1}(n-1)!\sum_{i=0}^{n-1}(-1)^{i}p_{n-i}e_{i},$$

which, by Theorem 2.9, is equal to $(-1)^{n-1}(n-1)!(-1)^{n-1}ne_n=n!e_n$.

Applying ω to both sides of the identity gives

$$n!h_n=\begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ -p_2 & p_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-2}p_{n-1} & (-1)^{n-3}p_{n-2} & (-1)^{n-4}p_{n-3} & \cdots & -p_1 & n-1 \\ (-1)^{n-1}p_n & (-1)^{n-2}p_{n-1} & (-1)^{n-3}p_{n-2} & \cdots & -p_2 & p_1 \end{vmatrix}.$$

Notes

The theory of symmetric functions has a long history and with many applications to the representation theory of finite groups, special functions, and combinatorics. There are two books on the theory of symmetric functions that we would recommend.

The first is Macdonald's *Symmetric functions and Hall polynomials* [82], which contains a wealth of information not presented here, including several generalizations of the symmetric functions such as the Hall–Littlewood symmetric functions which involve an extra parameter q and Macdonald polynomials which involve two extra parameters q and t. There are many combinatorial applications of both Hall–Littlewood symmetric functions and Macdonald polynomials which are beyond the scope of this book. See, for example, Haglund's book [55].

A second account of the theory of symmetric functions is found in Stanley's *Enumerative Combinatorics, Volume 2* [108]. The latter text contains notes on the history of symmetric functions with numerous references.

There are many approaches of developing the theory of symmetric functions. Our approach has been to give direct combinatorial proofs of identities wherever possible. Moreover, we have made sure that our proofs work over the ring of symmetric functions in infinitely many variables; this will be needed for some of our applications.

The exercises allow us to add 11 edges to the directed graph giving the transition matrices between bases for the ring of symmetric functions featured on page 56. Specifically,

1. Exercise 2.6 says the λ,μ entry of the e-to-p matrix is $(-1)^{n-\ell(\lambda)}\frac{OB_{\lambda,\mu}}{z_\lambda}$.
2. Exercise 2.6 says the λ,μ entry of the m-to-p matrix is $(-1)^{\ell(\lambda)+\ell(\mu)}\frac{w(B_{\mu,\lambda})}{z_\lambda}$.
3. Exercise 2.12 says the λ,μ entry of the e-to-s matrix is $K_{\lambda',\mu}$.
4. Exercise 2.14 says the λ,μ entry of the p-to-s matrix is χ^λ_μ.
5. Exercise 2.15 says the λ,μ entry of the m-to-s matrix is $K^{-1}_{\mu,\lambda}$.
6. Exercise 2.15 says the λ,μ entry of the s-to-e matrix is $K^{-1}_{\lambda,\mu'}$.
7. Exercise 2.16 says the λ,μ entry of the h-to-m matrix is $NM_{\lambda,\mu}$.

The ω transformation implies that the λ,μ entry of the x-to-y transition matrix is the λ,μ entry of the $\omega(x)$-to-$\omega(y)$ transition matrix for all bases x,y. Applying ω to items 1, 3, 6 on the above list as well as Theorem 2.22 gives

8. The λ,μ entry of the h-to-p matrix is $\frac{OB_{\lambda,\mu}}{z_\lambda}$.
9. The λ,μ entry of the h-to-s matrix is $K_{\lambda,\mu}$.
10. The λ,μ entry of the s-to-h matrix is $K^{-1}_{\lambda,\mu}$.
11. The λ,μ entry of the p-to-h matrix is $(-1)^{\ell(\lambda)+\ell(\mu)}w(B_{\lambda,\mu})$.

Theorem 2.28 when applied to the dual bases $\{s_\lambda : \lambda \vdash n\}$ and $\{s_\lambda : \lambda \vdash n\}$ and the dual bases $\{p_\lambda : \lambda \vdash n\}$ and $\{p_\lambda/z_\lambda : \lambda \vdash n\}$ when applied to item 4 on the above list gives

12. The λ,μ entry of the s-to-p transition matrix is $\chi^\mu_\lambda/z_\lambda$.

All of the above information about transition matrices is recorded as a directed graph in Appendix A.

We have not provided a combinatorial interpretation of the entries are the m-to-e and m-to-h transition matrices. Combinatorial interpretations for the entries of these matrices are described in [9], but they do not have straightforward descriptions and are difficult to use in applications, and so we choose to omit them.

In [115] and [116], White gave somewhat lengthy but purely combinatorial proofs that the λ,μ entry of the s-to-p transition matrix is $\chi^\mu_\lambda/z_\lambda$. We shall give a different approach to this result in Chapter 5 where we give a second proof of the so-called Muraghan–Nakyama rule.

The combinatorial interpretations of the entries of the transition matrices for symmetric functions can be found in [33, 73] while the idea of using labeled abaci to prove results about transition matrices is due to Loehr [80, 81].

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