Chapter 2
Joseph Fourier

Joseph Fourier (1768–1830)

2.1 Introduction

Fourier series are trigonometric series used to represent a function, and they are widely used throughout pure and applied mathematics. Fourier was not the first to use them, but his name is rightly attached to them because he was the first to use them in the study of heat diffusion, to display their use in the solution of a partial differential equation, and to argue successfully for their generality.

There had been a long 18th-century debate about trigonometric series in connection with solutions to the wave equation and the shape of a vibrating string. On the one hand it seemed reasonable that a string could have any continuous initial shape—that was Euler's view—on the other hand the equation could only be solved by functions to which the calculus applied (we would say that the solutions had to

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be twice differentiable) and this made them what d'Alembert called analytic. Convergence questions were not central to this debate, which was left unresolved in a number of ways.

Fourier proposed to reopen the debate by boldly asserting that any solution to the heat equation, which he was the first to derive, could be written as an infinite sum of sines and cosines for the simple reason that any function could be written that way. This is a dramatic claim, and it was still more so in his day, because the consensus was that however broadly a function might be defined all the functions that arise in practice are finite sums of familiar ones: polynomials, sines, cosines, exponentials and logarithms, nth roots, and the like. They could also be infinite power series, and indeed infinite trigonometric series, but nonetheless they had the usual sorts of properties, such as smoothly varying graphs. No-one said so in so many words, but it is clear that the expectation was there, and Fourier in particular simply assumed that every function is continuous, as is clear from his account of the coefficients of a Fourier series in his (1822, §423).

One of the dramas introduced by Fourier's series was that they readily flout all these expectations. As we shall see, at various stages in the 19th century they provided fresh, and disturbing, examples of just what functions could do. Contrary to what Fourier himself believed, if Cauchy's work began the exploration of what rigorous mathematics can do, Fourier series can indicate just what theory is up against.

2.2 Fourier's Career

Joseph Fourier was born in Auxerre, France in 1768. He was orphaned at the age of 9 and placed in the town's military school where he learned mathematics and a sense of civic responsibility. He was nearly guillotined at the height of the Terror in 1794, but the sentence was withdrawn and Fourier was able to go to the École Normale. In 1795 he was appointed an assistant lecturer at the École Polytechnique, working under Lagrange and Monge, and in 1798 Monge, a prominent supporter of Napoleon, selected Fourier to go on the French expedition to Egypt. After the British defeated them there, Fourier returned to France in 1801, hoping to resume his work at the École Polytechnique, but Napoleon had been impressed by his organisational talents and sent him instead to be the prefect of Governor of the Department of Isère.[1] He was so successful here that Napoleon made him a Baron in 1808, and in 1809 he finished his contribution to the *Description d'Egypt*, a massive account and glorification of ancient Egypt based on the surveys that French engineers had made of Egyptian pyramids and other remains.[2]

[1] An administrative region of France that extended from Grenoble to the French border.

[2] This period is the start of the celebration of ancient Egypt in the modern world, from Cleopatra's and other needles to fanciful statements about ancient wisdom, secret knowledge, and so forth, none of which can be held against Fourier. See Buchwald and Feingold (2012).

The eventual defeat of Napoleon was the lowest point of Fourier's life, but in 1816 he obtained a position as Director of the Bureau of Statistics for the Department of the Seine, a position which left him good time for research. His political enemies now in power delayed his appointment to the reformed Academy of Sciences for a year but he eventually rose to become the permanent secretary of the Academy in 1822 and to be elected to the Académie Française in 1827. He died in 1830 as the result of complications from an illness caught in Egypt.

2.3 Fourier and Series of Sines and Cosines

The book *Théorie analytique de la chaleur*, in which Fourier presented his ideas, was written work in several stages. He submitted a version to the Paris Academy of Sciences in 1807, but although Laplace, Lacroix and Monge were in favour of publishing it, Lagrange blocked publication, apparently because its treatment of trigonometric series differed markedly from the way he, Lagrange, had stipulated in the 1750s. Another chance came in 1810, when the Academy of Sciences announced a prize competition on heat diffusion. Fourier submitted a revised memoir, which won, but was criticised for a lack of rigour and generality. Fourier thought the criticism unfair, but revised it again, and the resulting book came out in 1822 (after Lagrange's death and when Fourier's standing was rising in the Academy).

Heat Diffusion

Fourier was interested in finding the temperature at every point of a solid body, perhaps as a function of time, when the shape of the body, its physical properties, and the temperature on some or all of its boundary is given. He made no assumptions about the nature of heat and concentrated on how it flowed.

He considered that any solid body could be regarded as made of infinitesimal cubes, and argued, on the basis of some observational evidence, that the amount of heat that passes from the hotter part of the body to an adjacent colder part in an instant of time is proportional to the duration of the instant, the infinitesimal temperature difference between opposite faces of the cubes and a certain function of the distance between the particles that depends on the nature of the body. So each body determines some constants that characterise how heat flows in them, such as its conductivity and its specific heat. In what follows all these physical matters will be consumed in the single letter K.

He considered what happens as the heat flows through one of these infinitesimal cubes, where temperature v is a function of x, y, z and t, the time. What enters the face with sides dx and dy is $Kdxdy\frac{\partial v}{\partial z}$ evaluated at that face.[3] What leaves the opposite face is $Kdxdy\frac{\partial v}{\partial z}$ evaluated at that face. This amount Fourier evaluated by saying that the faces are a distance dz apart, so he replaced z by $z + dz$, and the difference in the amount of heat between what enters and what leaves is given by

[3] I have modernised Fourier's 'd' notation by writing ∂v where he wrote dv.

$$Kdydxd\left(\frac{\partial v}{\partial z}\right) = Kdxdydz\frac{\partial^2 v}{\partial z^2}.$$

Should the temperature be in a steady state the sum of these quantities taken over the three pairs of opposite faces of a cube is zero and the resulting equation is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0. \tag{2.1}$$

The more important situation is when the temperature is changing. Fourier now argued that the amount of heat leaving a cube in the z direction is once again $Kdxdydz\frac{\partial^2 v}{\partial z^2}$, but now the sum over the pairs of opposite faces equals the rate of change of temperature, which is given by $\frac{dv}{dt}$. The result (see *Théorie* §128, p. 102) is the heat equation:

$$\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right). \tag{2.2}$$

In each case, solutions of this partial differential equation are required that satisfy the given boundary conditions, and Fourier confined his attention to bodies with simple shapes, such as a cuboid, or one equivalent to this by a suitable coordinate transformation.

For example, in (§166, p. 133) Fourier considered a semi-infinite strip of a given width, π in suitable units. He supposed that the temperature at the base is kept constant at 1 in some units and that the temperature of the infinite sides is kept at at 0, and looked for the corresponding steady state distribution of temperature. Let y measure the height above the base and x the horizontal distance of a point from the mid-line of the strip (I have relabelled his coordinates) so the differential equation, which now involves only two variables, is

$$K\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{2.3}$$

Fourier looked for a solution of the form

$$v(x, y) = f(x)g(y),$$

which leads to the equation $g''(y)/g(y) = -f''(x)/f(x)$ in which both sides must be constant, say m, so the solutions are of the form

$$f(x) = \cos(mx), \quad g(y) = e^{-my} \tag{2.4}$$

The temperature in the bar surely does not become infinite, so the exponential term must decrease and so m must be positive. Also, m must be odd so that the solution vanishes for $x = \pm\pi$ for all y, as required.

It was clear to Fourier that a sum of solutions of this form is also a solution and he proceeded at once to consider infinite sums, solutions of the form, as he wrote (§169, p. 135),

$$ae^{-y}\cos x + be^{-3y}\cos 3x + ce^{-5y}\cos 5x + de^{-7y}\cos 7x + etc. \tag{2.5}$$

subject to the boundary condition at the base that

$$1 = a\cos x + b\cos 3x + c\cos 5x + d\cos 7x + etc. \tag{2.6}$$

The arbitrary constants had now to be determined. Fourier first gave a marvellous argument that involved him in solving the infinitely many equations he could obtain for his infinitely many unknowns by differentiating equation (2.6) arbitrarily often (see §§171–176 of the *Théorie*). Only then did he give the simpler and more general way that has become standard, and start to claim (§220, see §A.1) that every function can be written as one of these series. By this he meant that every function is equal to its corresponding series, and that there is a simple rule for writing down the coefficients of the series.

He noted (§221) that the integral

$$\int_0^{\pi} \sin jx \sin kx dx = \frac{1}{2}\left(\frac{1}{k-j}\sin(k-j)x - \frac{1}{k+j}\sin(k+j)x\right)\Bigg|_0^{\pi} \tag{2.7}$$

vanishes when $j \neq k$ and is $\pi/2$ when $k = j$, and claimed that the coefficients of a series such as his can be found by integrating the product of the series with $\sin jx$ for each value of $j > 0$. Similar results apply to series of cosines, to series of sines and cosines, and to series obtained when the period is different (as it might be, 2π or 1).

He went on to claim that any function f defined on the interval $[-\pi, \pi]$ can be written as an infinite series of sines and cosines in any of these forms (called the mixed series, the cosine series and the sine series, respectively):

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx.$$

The coefficients of the mixed series are given by the formulae $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx.$$

Examples of these series had already been studied by Euler and Daniel Bernoulli.

He then gave several simple examples of his series, some of which such a function constant on a given interval, or equal to x on a given interval. He showed how to obtain the function $\cos x$ as an infinite series of sines, dealt with cosine series as well as sine series, and

Fourier was very proud of his series for the function $F(x) = \pm\pi/4$:

$$\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots ,$$

as well he might be, once you see what it looks like. Here are three graphs of it: Fig. 2.1 shows the sum of only the first 5 terms in the series, Fig. 2.2 the first 25, and Fig. 2.3 the first 105.

Note that the difference between the sum of the series and the sum of its first 105 terms is certainly less than 1/209 and generally much less.

It is clear that the infinite series represents a function that is $+\pi/4$ on the range $(-\pi/2, \pi/2)$ and that is $-\pi/4$ on the range $(+\pi/2, 3\pi/2)$. Indeed, it represents a function that is $+\pi/4$ on the range $((4n-1)\pi/2, (4n+1)\pi/2)$, that is $-\pi/4$ on the range $((4n+1)\pi, (4n+3)\pi/2)$, and that is zero at the points $x = (4n \pm 1)\pi/2$.

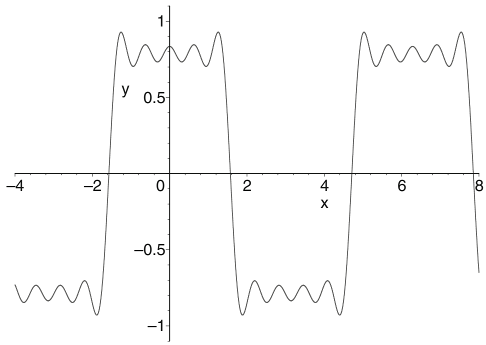

Fig. 2.1 The first 5 terms of a Fourier series

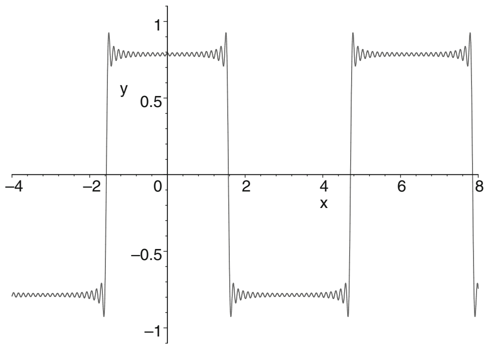

Fig. 2.2 The first 25 terms of a Fourier series

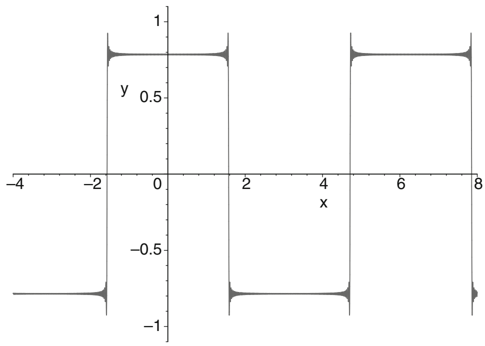

Fig. 2.3 The first 105 terms of a Fourier series

Note that it is plainly not the case that a series of analytic (and therefore in particular continuous) functions is itself continuous. We shall see that Fourier's confident remark (§235) that all these series converge everywhere to the function that they represent, and his proof of the claim in §423 (see Appendix A.1) that the coefficients in a 'Fourier' series can be evaluated as he indicated, were to be the occasion for much significant later work.

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