Specht, et al: Euclidean Geometry

Exercises and Answers: Contents

Introduction to Exercises and Answers

This file contains answers (solutions) to exercises in *Euclidean Geometry and its Subgeometries* by Specht, Jones, Calkins, and Rhoads (Birkhäuser, 2015).

All references to chapters, theorems, definitions, remarks, and figures refer to this work. Most exercises are in the form of statements, so that "answers" are actually proofs, and are labeled as such.

Here we state all the exercises for each chapter, but give solutions only to those that are starred; most of these are needed for the development of the book. Thus, a solution is given for Exercise I.13* but not for Exercise I.14.

We encourage you to read each exercise, make a sketch if it helps you to visualize it, and get in mind how to prove it, even if you don't actually put the details together. You can justify skipping an occasional exercise only if you are *quite sure* you could construct the proof if you had to, and feel it is a waste of your time to supply all the details. But beware that supplying all the details may look deceptively simple when you give a theorem a cursory glance.

It is possible that you may create new solutions for exercises that are more elegant than the ones we have given. The authors will appreciate receiving any such improvements, as well as corrections to errors you may find in the proofs given.

Chapter 1: Exercises and Answers for Preliminaries and Incidence Geometry (I)

Exercise I.1* If $\mathcal{L}$ and $\mathcal{M}$ are distinct lines and if $\mathcal{L} \cap \mathcal{M} \neq \emptyset$, then $\mathcal{L} \cap \mathcal{M}$ is a singleton.

Exercise I.1 Proof. Assume $\mathcal{L} \cap \mathcal{M}$ has two distinct members A and B; then each of the points A or B belongs to $\mathcal{L}$ and to $\mathcal{M}$. By Axiom I.1 $\mathcal{L} = \mathcal{M}$, contradicting the given fact that $\mathcal{L}$ and $\mathcal{M}$ are distinct. Hence our assumption that $A \neq B$ is false, and since $\mathcal{L} \cap \mathcal{M} \neq \emptyset$, $\mathcal{L} \cap \mathcal{M}$ is a singleton. $\square$

Exercise I.2* (A) If A and B are distinct points, and if C and D are distinct points on $\overleftrightarrow{AB}$, then $\overleftrightarrow{CD} = \overleftrightarrow{AB}$.

(B) If A, B, and C are noncollinear points, and if D, E, and F are noncollinear points on $\overleftrightarrow{ABC}$, then $\overleftrightarrow{DEF} = \overleftrightarrow{ABC}$.

Exercise I.2 Proof. (A) Since C and D are distinct points, by Axiom I.1, $\overleftrightarrow{CD} = \overleftrightarrow{AB}$.

(B) Since D, E, and F are noncollinear points on $\overleftrightarrow{ABC}$, by Axiom I.2, $\overleftrightarrow{DEF} = \overleftrightarrow{ABC}$. $\square$

Exercise I.3* If $\mathcal{L}$ and $\mathcal{M}$ are lines and $\mathcal{L} \subseteq \mathcal{M}$, then $\mathcal{L} = \mathcal{M}$.

Exercise I.3 Proof. By Axiom I.5(A) there exist distinct points A and B on $\mathcal{L}$. Since $\mathcal{L} \subseteq \mathcal{M}$, A and B are on $\mathcal{M}$. By Axiom I.1 $\mathcal{L} = \mathcal{M}$. $\square$

Exercise I.4* Let A and B be two distinct points, and let D, E, and F be three noncollinear points. If $\overleftrightarrow{AB}$ contains only one of the points D, E, and F, then each of the lines $\overleftrightarrow{DE}$, $\overleftrightarrow{EF}$, and $\overleftrightarrow{DF}$ intersects $\overleftrightarrow{AB}$ in at most one point.

Exercise I.4 Proof. We may choose our notation so that $\overleftrightarrow{AB}$ contains the point D. Then if $\overleftrightarrow{DE}$ (or $\overleftrightarrow{DF}$) intersects $\overleftrightarrow{AB}$ in two points, by Exercise I.2(A) $\overleftrightarrow{DE} = \overleftrightarrow{AB}$, (or $\overleftrightarrow{DF} = \overleftrightarrow{AB}$) and both D and E (D and F) are members of $\overleftrightarrow{AB}$, contradicting the assumption that this line contains only one of the points D, E, or F. Since E and F do not belong to $\overleftrightarrow{AB}$, $\overleftrightarrow{EF} \neq \overleftrightarrow{AB}$ and by Exercise I.1, if these intersect they intersect in a singleton. $\square$

Exercise I.5* If $\mathcal{E}$ is a plane, $\mathcal{L}$ is a line such that $\mathcal{E} \cap \mathcal{L} \neq \emptyset$, and $\mathcal{L}$ is not contained in $\mathcal{E}$, then $\mathcal{E} \cap \mathcal{L}$ is a singleton.

Exercise I.5 Proof. Since $\mathcal{E} \cap \mathcal{L} \neq \emptyset$, there exists a point A belonging to $\mathcal{E}$ and to $\mathcal{L}$. Assume there exists a point B distinct from A which belongs to $\mathcal{E}$ and to $\mathcal{L}$. By Axiom I.1 $\mathcal{L} = \overleftrightarrow{AB}$ and by Axiom I.3 $\mathcal{L} \subseteq \mathcal{E}$. This contradicts the given fact that $\mathcal{L}$ is not contained in $\mathcal{E}$. Hence $\mathcal{E} \cap \mathcal{L}$ is a singleton. $\square$

Exercise I.6 Let $\mathcal{D}$ and $\mathcal{E}$ be distinct planes such that $\mathcal{D} \cap \mathcal{E} \neq \emptyset$, so that (by Theorem I.4) $\mathcal{D} \cap \mathcal{E}$ is a line $\mathcal{L}$; let P be a point on $\mathcal{D}$ but not on $\mathcal{L}$; and let Q be a point on $\mathcal{E}$ but not on $\mathcal{L}$. Then $\overleftrightarrow{PQ}$ and $\mathcal{L}$ are not coplanar.

Exercise I.7* Given a line $\mathcal{L}$ and a point A not on $\mathcal{L}$, there exists one and only one plane $\mathcal{E}$ such that $A \in \mathcal{E}$ and $\mathcal{L} \subseteq \mathcal{E}$.

Exercise I.7 Proof. By Axiom I.5 there are two distinct points on $\mathcal{L}$, and by assumption these are not collinear with A. By Axiom I.2 there exists exactly one plane $\mathcal{E}$ containing all three points. By Axiom I.3 $\mathcal{L} \subseteq \mathcal{E}$. $\square$

Exercise I.8* Let A, B, C, and D be noncoplanar points. Then each of the triples $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$ is noncollinear.

Exercise I.8 Proof. If one of the triples is collinear, all of its points belong to a line $\mathcal{L}$. By Exercise I.7 there exists a unique plane $\mathcal{E}$ such that $\mathcal{L}$ and the one point among A, B, C, and D which is not in the triple are in that plane, hence the four points are coplanar. $\square$

Exercise I.9 There exist four distinct planes such that no point is common to all of them.

Exercise I.10 Every plane $\mathcal{E}$ contains at least three lines $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ such that $\mathcal{L} \cap \mathcal{M} \cap \mathcal{N} = \emptyset$.

Exercise I.11 Every plane contains (at least) three distinct lines.

Exercise I.12 Space contains (at least) six distinct lines.

Exercise I.13* If $\mathcal{L}$ is a line contained in a plane $\mathcal{E}$, then there exists a point A belonging to $\mathcal{E}$ but not belonging to $\mathcal{L}$.

Exercise I.13 Proof. Axiom I.5 says that there exist at least three noncollinear points on $\mathcal{E}$. Therefore not all of these points can be in any line $\mathcal{L}$; let A be that point from among these three that is not on this line. $\square$

Exercise I.14 If P is a point in a plane $\mathcal{E}$, then there is a line $\mathcal{L}$ such that $P \in \mathcal{L}$ and $\mathcal{L} \subseteq \mathcal{E}$.

Exercise I.15 If a plane $\mathcal{E}$ has (exactly) three points, then each line contained in $\mathcal{E}$ has (exactly) two points.

Exercise I.16 If a plane $\mathcal{E}$ has exactly four points, and if all of the lines contained in $\mathcal{E}$ have the same number of points, then each line contained in $\mathcal{E}$ has (exactly) two points.

Exercise I.17 If each line in space has at least three points, then:

(1) Each point of a plane is a member of at least three lines of the plane;

(2) Each plane has at least seven points;

(3) Each plane contains at least seven lines.

Exercise I.18 In this exercise we will use the symbolism "$\mathcal{P} \parallel \mathcal{Q}$" to indicate that two planes $\mathcal{P}$ and $\mathcal{Q}$ do not intersect. Consider what can happen if the restrictions of $\mathcal{P}_1 \cap \mathcal{P}_2$, $\mathcal{P}_1 \cap \mathcal{P}_3$, and $\mathcal{P}_2 \cap \mathcal{P}_3$ being nonempty are removed in Theorem I.9. Sketch at least four possibilities ($\mathcal{P}_1 \parallel \mathcal{P}_2 \parallel \mathcal{P}_3$, $\mathcal{P}_1 = \mathcal{P}_2 \parallel \mathcal{P}_3$, $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3$, $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, but $\mathcal{P}_1 \cap \mathcal{P}_3 = \mathcal{L}_2$ and $\mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{L}_1$) and determine if these can be proved within incidence geometry.

Exercise I.19 Count the number of lines in the 8-point model. Compare this with $T_n = \frac{n(n+1)}{2}$, triangular numbers, for $n = 7$. Compare it also with ${}_nC_r = \frac{n!}{r!(n-r)!}$, the number of combinations of n items taken r at a time, where $n = 8$ and $r = 2$.

Exercise I.20 Count the number of planes in the 8-point model. Compare this with ${}_nC_r$ for $n = 8$ and $r = 3$. Note the reduction by a factor of four due to the fact that each plane has four points. Can you form a similar argument with $r = 4$?

Exercise I.21 Consider a 4-point model with the four points configured like the vertices of a tetrahedron. Label these points A, B, C, and D. Specify six lines and four planes and verify that this model satisfies the axioms and theorem of incidence geometry. Compare this with Exercises I.7, I.10, I.12, and I.13. How does Theorem I.9 apply in this geometry?

Chapter 2: Exercises and Answers for Affine Geometry: Incidence with Parallelism (IP)

Exercise IP.1* If $\mathcal{L}$ and $\mathcal{M}$ are parallel lines, then there is exactly one plane containing both of them.

Exercise IP.1 Proof. If $\mathcal{L} \parallel \mathcal{M}$, by Definition IP.0 there exists a plane $\mathcal{E}$ containing both $\mathcal{L}$ and $\mathcal{M}$ such that $\mathcal{L} \cap \mathcal{M} = \emptyset$. If there is a second plane $\mathcal{F}$ containing both $\mathcal{L}$ and $\mathcal{M}$, let A and B both belong to $\mathcal{L}$ and $C \in \mathcal{M}$. Then A, B, and C are not collinear and by Axiom I.2, $\mathcal{F} = \mathcal{E}$. □

Exercise IP.2* Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be distinct lines contained in a single plane.

(A) If $\mathcal{L} \parallel \mathcal{M}$ and $\mathcal{M} \parallel \mathcal{N}$, then $\mathcal{L} \parallel \mathcal{N}$.

(B) If $\mathcal{L}$ intersects $\mathcal{M}$, then $\mathcal{N}$ must intersect $\mathcal{L}$ or $\mathcal{M}$, possibly both.

Exercise IP.2 Proof. (A) This follows immediately from Theorem IP.6. However, since all the lines are in a single plane, a simpler proof may be constructed as follows: if $\mathcal{L} \not\parallel \mathcal{N}$ then $\mathcal{L} \cap \mathcal{N} \neq \emptyset$; let $\mathcal{L} \cap \mathcal{N} = \{A\}$; then since $\mathcal{L}$ and $\mathcal{N}$ are both parallel to $\mathcal{M}$, there are two lines through A parallel to $\mathcal{M}$, violating the parallel postulate PS.

(B) If $\mathcal{N}$ does not intersect either $\mathcal{L}$ or $\mathcal{M}$, then it is parallel to both, and thus, by part (A), $\mathcal{L}$ is parallel to $\mathcal{M}$, contradicting the assumption that $\mathcal{L}$ and $\mathcal{M}$ intersect. □

Exercise IP.3* Let $\mathbb{E}$ be a pencil of lines on the plane $\mathcal{P}$. If $\mathcal{L}$ and $\mathcal{M}$ are distinct members of $\mathbb{E}$ which intersect at the point O, then the members of $\mathbb{E}$ are concurrent at O.

Exercise IP.3 Proof. By Definition IP.0(D) O belongs to every member of $\mathbb{E}$, so the members of $\mathbb{E}$ are concurrent at O. □

Exercise IP.4* Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be distinct lines in a plane $\mathcal{E}$ such that $\mathcal{L} \parallel \mathcal{M}$. Then if $\mathcal{L} \cap \mathcal{N} \neq \emptyset$, $\mathcal{M} \cap \mathcal{N} \neq \emptyset$.

Exercise IP.4 Proof. If $\mathcal{L}$ and $\mathcal{N}$ intersect, and neither intersects $\mathcal{M}$, then both $\mathcal{L}$ and $\mathcal{N}$ are lines through a point, both parallel to $\mathcal{M}$. By Axiom PS, this is impossible, so that one of these lines must intersect $\mathcal{M}$. By assumption $\mathcal{L} \parallel \mathcal{M}$ so $\mathcal{N}$ intersects $\mathcal{M}$. □

Exercise IP.5* Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, and $\mathcal{M}_2$ be lines on the plane $\mathcal{P}$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a point, $\mathcal{L}_1 \parallel \mathcal{M}_1$, and $\mathcal{L}_2 \parallel \mathcal{M}_2$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect at a point.

Exercise IP.5 Proof. Assume $\mathcal{M}_1$ and $\mathcal{M}_2$ are parallel, then by Exercise IP.2(B) $\mathcal{L}_1 \parallel \mathcal{L}_2$. This contradicts the given fact that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a point. Hence our assumption is false and so $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect at a point. □

Exercise IP.6* Let $\mathcal{E}$ and $\mathcal{F}$ be planes such that $\mathcal{E} \parallel \mathcal{F}$, and let $\mathcal{L}$ be a line in $\mathcal{E}$. Then $\mathcal{L} \parallel \mathcal{F}$.

Exercise IP.6 Proof. By Definition IP.0(C), $\mathcal{E} \parallel \mathcal{F}$ means that $\mathcal{E} \cap \mathcal{F} = \emptyset$. Since $\mathcal{L} \subseteq \mathcal{E}$, $\mathcal{L} \cap \mathcal{E} = \emptyset$, and by Definition IP.0(B) $\mathcal{L} \parallel \mathcal{E}$. □

Exercise IP.7* Let $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ be planes such that $\mathcal{E} \parallel \mathcal{F}$, $\mathcal{E} \cap \mathcal{G} \neq \emptyset$, and $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Then $\mathcal{E} \cap \mathcal{G}$ is a line $\mathcal{L}$, $\mathcal{F} \cap \mathcal{G}$ is a line $\mathcal{M}$, and $\mathcal{L} \parallel \mathcal{M}$. See Figure 2.2 in Chapter 2 (Exercises), reproduced here.

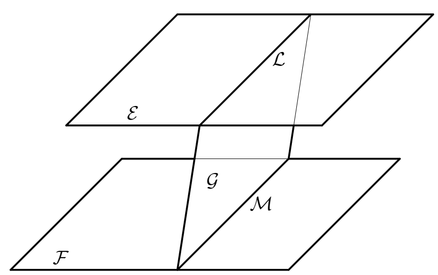

Figure 2.2 for Exercise IP.7.

Exercise IP.7 Proof. By Theorem I.4 $\mathcal{E} \cap \mathcal{G}$ is a line $\mathcal{L}$ and $\mathcal{F} \cap \mathcal{G}$ is a line $\mathcal{M}$. Since lines $\mathcal{L}$ and $\mathcal{M}$ are subsets of $\mathcal{G}$, they are coplanar. Since $\mathcal{L} \subseteq \mathcal{E}$, $\mathcal{M} \subseteq \mathcal{F}$, and $\mathcal{E} \cap \mathcal{F} = \emptyset$, $\mathcal{L} \cap \mathcal{M} \neq \emptyset$. By Definition IP.0(A), $\mathcal{L} \parallel \mathcal{M}$. □

Exercise IP.8 If $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are distinct planes such that $\mathcal{E} \parallel \mathcal{F}$ and $\mathcal{F} \parallel \mathcal{G}$, then $\mathcal{E} \parallel \mathcal{G}$.

Exercise IP.9 If $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ are distinct planes such that $\mathcal{E} \parallel \mathcal{F}$ and $\mathcal{E} \cap \mathcal{G} \neq \emptyset$, then $\mathcal{F} \cap \mathcal{G} \neq \emptyset$.

Exercise IP.10 If $\mathcal{L}$ and $\mathcal{M}$ are noncoplanar lines, then there exist planes $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{E} \parallel \mathcal{F}$ and $\mathcal{L} \subseteq \mathcal{E}$, and $\mathcal{M} \subseteq \mathcal{F}$.

Exercise IP.11 Let $\mathcal{E}$ and $\mathcal{F}$ be parallel planes, and let $\mathcal{L}$ be a line which is parallel to $\mathcal{E}$ and which is not contained in $\mathcal{F}$. Then $\mathcal{L} \cap \mathcal{F} = \emptyset$.

Exercise IP.12 Let $\mathcal{E}$ and $\mathcal{F}$ be parallel planes, and let $\mathcal{L}$ be a line which is not parallel to $\mathcal{E}$ and which is not contained in $\mathcal{E}$. Then $\mathcal{L} \cap \mathcal{F} \neq \emptyset$.

Exercise IP.13 Given a plane $\mathcal{E}$ and a line $\mathcal{L}$ parallel to $\mathcal{E}$, there exists a plane $\mathcal{F}$ containing $\mathcal{L}$ and parallel to $\mathcal{E}$.

Exercise IP.14 Let n be a natural number greater than 1. If there exists a line which has exactly n points, then:

(1) Every line has exactly n points.

(2) For any point P and any plane $\mathcal{E}$ containing P, there are exactly $n+1$ lines through P and contained in $\mathcal{E}$.

(3) For any line $\mathcal{L}$ and any plane $\mathcal{E}$ containing $\mathcal{L}$, there exist exactly $n-1$ lines $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$ such that $\mathcal{L}_k \parallel \mathcal{L}$ for each k in $[1; n-1]$.

(4) Each plane contains $n(n+1)$ lines.

(5) Each plane contains n^2 points.

(6) Given any plane $\mathcal{E}$, there exists exactly $n-1$ planes $\mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$ such that $\mathcal{E}_k \parallel \mathcal{E}$ for each k in $[1; n-1]$.

(7) There are n^3 points in space.

(8) There are $n^2(n^2+n+1)$ lines in space.

(9) There are n^2+n+1 lines through each point.

(10) There are $n+1$ planes containing a given line.

(11) There are n^2+n+1 planes through each point.

(12) There are $n(n^2+n+1)$ planes in space.

Chapter 3: Exercises and Answers for Collineations of an Affine Plane (CAP)

Exercise CAP.1* Let $\mathcal{P}$ be an affine plane and let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be lines on $\mathcal{P}$. If $\mathcal{L} \mathbb{PE} \mathcal{M}$ and $\mathcal{M} \mathbb{PE} \mathcal{N}$, then $\mathcal{L} \mathbb{PE} \mathcal{N}$.

Exercise CAP.1 Proof. There are four cases.
(1) If $\mathcal{L} \parallel \mathcal{M}$ and $\mathcal{M} \parallel \mathcal{N}$, then by Exercise IP.2, $\mathcal{L} \parallel \mathcal{N}$, so $\mathcal{L} \mathbb{PE} \mathcal{N}$.
(2) If $\mathcal{L} = \mathcal{M}$ and $\mathcal{M} \parallel \mathcal{N}$, then $\mathcal{L} \parallel \mathcal{N}$, so $\mathcal{L} \mathbb{PE} \mathcal{N}$.
(3) If $\mathcal{L} \parallel \mathcal{M}$ and $\mathcal{M} = \mathcal{N}$, then $\mathcal{L} \parallel \mathcal{N}$, so $\mathcal{L} \mathbb{PE} \mathcal{N}$.
(4) If $\mathcal{L} = \mathcal{M}$ and $\mathcal{M} = \mathcal{N}$, then $\mathcal{L} = \mathcal{N}$, so $\mathcal{L} \mathbb{PE} \mathcal{N}$. □

Exercise CAP.2* Let $\mathcal{P}$ be any plane where the incidence axioms hold, φ be a collineation of $\mathcal{P}$, and A, B, and C be points on $\mathcal{P}$.

(A) If A, B, and C are collinear, then $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are collinear.

(B) If A, B, and C are noncollinear, then $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are noncollinear.

(C) A, B, and C are collinear iff $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are collinear.

(D) A, B, and C are noncollinear iff $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are noncollinear.

Exercise CAP.2 Proof. (A) If A, B, and C are collinear, by Definition I.0.1 there exists a line $\mathcal{L}$ containing all these points. By elementary mapping theory, $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ all belong to $\varphi(\mathcal{L})$, which by Definition CAP.0(A) is a line. By Definition I.0.1, these points are collinear.

(B) By Theorem CAP.1(D') φ^{-1} is a collineation. If $\varphi(A)$, $\varphi(B)$, and $\varphi(C)$ are collinear. by part (A) and elementary mapping theory, $\varphi^{-1}(\varphi(A)) = A$, $\varphi^{-1}(\varphi(B)) = B$, and $\varphi^{-1}(\varphi(C)) = C$ are collinear. This proves the contrapositive of the assertion.

(C) Part (A) proves one half of the assertion, and the proof of part (B) proves the other half.

(D) Each half of this proof is the contrapositive of half of the proof of part (C).

Note that the proof does not require Axiom PS to be in force. □

Exercise CAP.3* Let φ be a collineation of an affine plane $\mathcal{P}$, $\mathcal{M}$ a line on $\mathcal{P}$ such that every point on $\mathcal{M}$ is a fixed point of φ, and Q a fixed point of φ such that $Q \in (\mathcal{P} \setminus \mathcal{M})$. Then $\varphi = \imath$.

Exercise CAP.3 Proof. Let X be any member of $\mathcal{P}\setminus(\mathcal{M}\cup\{Q\})$, R and S be distinct members of $\mathcal{M}$, $\mathcal{G}=\mathrm{par}(X,\overleftrightarrow{RQ})$ and $\mathcal{H}=\mathrm{par}(X,\overleftrightarrow{SQ})$. By Theorem CAP.4(A), $\overleftrightarrow{QR}$ and $\overleftrightarrow{QS}$ are fixed lines of φ. By part (A) of Theorem CAP.26, $\mathcal{G}$ and $\mathcal{H}$ are fixed lines of φ. By Theorem CAP.4(B), X is a fixed point of φ. Since X is any member of $\mathcal{P}\setminus(\mathcal{M}\cup\{Q\})$, and since each member of $\mathcal{M}\cup\{Q\}$ is a fixed point of φ, the set of fixed points of φ is $\mathcal{P}$. But that means that $\varphi=\imath$. $\square$

Exercise CAP.4* Let $\mathcal{P}$ be an affine plane, $\mathcal{L}_1$, and $\mathcal{L}_2$ be parallel lines on $\mathcal{P}$, O_1 be a member of $\mathcal{L}_1$, O_2 be a member of $\mathcal{L}_2$, and τ be the translation (*cf* Theorem CAP.9) of $\mathcal{P}$ such that $\tau(O_1)=O_2$, then $\tau(\mathcal{L}_1)=\tau(\mathcal{L}_2)$.

Exercise CAP.4 Proof. Since τ is a collineation of $\mathcal{P}$, $\tau(\mathcal{L}_1)$ is a line on $\mathcal{P}$. Since $\tau(O_1)=O_2\in\mathcal{L}_2$, $O_2\in\tau(\mathcal{L}_2)$. By Definition CAP.6, $\tau(\mathcal{L}_1)\parallel\mathcal{L}_1$. Since $\mathcal{L}_2\parallel L_1$, by Axiom PS, $\mathcal{L}_2=\tau(\mathcal{L}_1)$. $\square$

Exercise CAP.5* Let $\mathcal{P}$ be an affine plane, φ be a dilation of $\mathcal{P}$ with fixed point O, and ψ be a stretch of $\mathcal{P}$ with axis $\mathcal{M}$ through O, then $\varphi\circ\psi=\psi\circ\varphi$. (We take Remark CAP.30 as a definition of a stretch.)

Exercise CAP.5 Proof. Let X be any member of $\mathcal{P}$.

(1) If $X\in\mathcal{M}$, then $(\varphi\circ\psi)(X)=\varphi(\psi(X))=\varphi(X)$. Since $\mathcal{M}$ is a fixed line of φ (*cf* Theorem CAP.18), $\psi(\varphi(X))=\varphi(X)$.

(2) Let $X\in(\mathcal{P}\setminus\mathcal{M})$ and let $\mathcal{L}$ be the fixed line of ψ through X (*cf* Theorem CAP.27), then $\psi(X)\neq X$ and $\psi(X)\in\mathcal{L}$. Since $\overleftrightarrow{OX}$ is a fixed line of φ (*cf* Theorem CAP.18), $\varphi(X)\in\overleftrightarrow{OX}$. Furthermore, (*cf* Definition CAP.17), $\varphi(\overleftrightarrow{X\psi(X)})=\mathrm{par}(\varphi(X),\mathcal{L})$. Since $\overleftrightarrow{O\psi(X)}$ is a fixed line of φ, $\varphi(\psi(X))$ is the point of intersection of $\mathrm{par}(\varphi(X),\mathcal{L})$ and $\overleftrightarrow{O\psi(X)}$. Since $\psi(\overleftrightarrow{OX})=\overleftrightarrow{O\psi(X)}=\psi(\overleftrightarrow{O\varphi(X)})=\overleftrightarrow{O\psi(\varphi(X))}$, $\psi(\varphi(X))$ is the point of intersection of $\mathrm{par}(\varphi(X),\mathcal{L})$ and $\overleftrightarrow{O\psi(X)}$. Hence $\varphi(\psi(X))=\psi(\varphi(X))$. $\square$

Chapter 4: Exercises and Answers for Incidence and Betweenness (IB)

Exercise IB.1 If A and B are distinct points, then there exist points E and F such that E–B–A and B–A–F.

Exercise IB.2* Let A, B, C, and D be distinct collinear points, then A–B–C–D iff D–C–B–A.

Exercise IB.2 Proof. By Definition IB.2 A–B–C–D means that A–B–C, A–B–D, A–C–D, and B–C–D. By property B.1 of Definition IB.1 (Symmetric property for betweenness) C–B–A, D–B–A, D–C–A, and D–C–B. By Definition IB.2, D–C–B–A. □

Exercise IB.3 If A and B are any two distinct points, then $\overset{\sqsubset\!\sqsupset}{AB} = \overset{\sqsubset\!\sqsupset}{BA}$ and $\overset{\sqsupset\!\sqsubset}{AB} = \overset{\sqsupset\!\sqsubset}{BA}$.

Exercise IB.4* If A and B are any two distinct points, then $\overset{\sqsupset\!\sqsubset}{AB} \subseteq \overset{\sqsupset\!\sqsupset}{AB} \subseteq \overset{\sqsupset\!\rightarrow}{AB} \subseteq \overleftrightarrow{AB}$, $\overset{\sqsupset\!\sqsubset}{AB} \subseteq \overset{\sqsubset\!\sqsubset}{AB} \subseteq \overset{\sqsubset\!\rightarrow}{AB} \subseteq \overleftrightarrow{AB}$, and $\overset{\sqsubset\!\sqsupset}{AB} \subseteq \overset{\sqsubset\!\rightarrow}{AB} \subseteq \overleftrightarrow{AB}$.

Exercise IB.4 Proof. The proof is direct from Remark IB.4.1 and Theorem IB.5. □

Exercise IB.5 If $\overset{\sqsubset\!\rightarrow}{AB} = \overset{\sqsubset\!\rightarrow}{CD}$ or $\overset{\sqsubset\!\sqsupset}{AB} = \overset{\sqsubset\!\sqsupset}{CD}$, then $\overleftrightarrow{AB} = \overleftrightarrow{CD}$.

Exercise IB.6* Prove Corollary IB.5.2: let A and B be distinct points. Then $\overset{\sqsubset\!\rightarrow}{AB}$ and $\overset{\sqsupset\!\rightarrow}{AB}$ are both proper subsets of $\overleftrightarrow{AB}$, $\overset{\sqsubset\!\sqsupset}{AB}$ is a proper subset of $\overset{\sqsubset\!\rightarrow}{AB}$, and $\overset{\sqsupset\!\sqsubset}{AB}$ and $\overset{\sqsupset\!\sqsupset}{AB}$ are proper subsets of $\overset{\sqsupset\!\rightarrow}{AB}$. (See also Exercise IB.4.)

Exercise IB.6 Proof. By property B.3 of Definition IB.1 there exists a point X such that X–A–B; by the third equality of Theorem IB.5, no such point belongs to $\overset{\sqsubset\!\rightarrow}{AB}$ so this is a proper subset of $\overleftrightarrow{AB}$. $\overset{\sqsupset\!\rightarrow}{AB}$ is a subset of $\overset{\sqsubset\!\rightarrow}{AB}$ so is also a proper subset.

Again by property B.3 there exists a point X such that A–B–X and by the trichotomy property, this means that A–X–B is false. By Definition IB.4 $X \in \overset{\sqsubset\!\rightarrow}{AB}$ and by Definition IB.3 $X \notin \overset{\sqsubset\!\sqsupset}{AB}$, which is therefore a proper subset of $\overset{\sqsubset\!\rightarrow}{AB}$. Also $X \in \overset{\sqsupset\!\rightarrow}{AB}$ and $X \notin \overset{\sqsupset\!\sqsupset}{AB}$, which is thus a proper subset of $\overset{\sqsupset\!\rightarrow}{AB}$; since $\overset{\sqsupset\!\sqsubset}{AB}$ is a subset of $\overset{\sqsupset\!\sqsupset}{AB}$ it is also a proper subset of $\overset{\sqsupset\!\rightarrow}{AB}$. □

Exercise IB.7* Prove the Corollary IB.6.1: for any two distinct points A and B, $[\overrightarrow{AB} \cup [\overrightarrow{BA} = \overleftrightarrow{AB}$.

Exercise IB.7 Proof. Since $[\overrightarrow{AB} = (\overrightarrow{AB} \cup \{A\}$ and $[\overrightarrow{BA} = (\overrightarrow{BA} \cup \{B\}$, by Theorem IB.6 $[\overrightarrow{AB} \cup [\overrightarrow{BA} = (\overrightarrow{AB} \cup (\overrightarrow{BA} \cup \{A, B\} = \overleftrightarrow{AB} \cup \{A, B\} = \overleftrightarrow{AB}$. □

Exercise IB.8* If A and B are any two distinct points, then
(A) $[\overrightarrow{AB} \cap [\overrightarrow{BA} = [\overline{AB}]$,
(B) $(\overrightarrow{AB} \cap (\overrightarrow{BA} = (\overline{AB})$,
(C) $[\overrightarrow{AB} \cap (\overrightarrow{BA} = [\overline{AB})$, and
(D) $(\overrightarrow{AB} \cap [\overrightarrow{BA} = (\overline{AB}]$.

Exercise IB.8 Proof.

(A) By Definition IB.4, $X \in [\overrightarrow{AB}$ iff $X = A$ or A–X–B or $X = B$ or A–B–X, and $X \in [\overrightarrow{BA}$ iff $X = B$ or A–X–B or $X = A$ or B–A–X. Therefore $X \in [\overrightarrow{AB} \cap [\overrightarrow{BA}$ iff $X = A$ or A–X–B or $X = B$, which is true iff $X \in [\overline{AB}]$, by Definition IB.3.

(B) By Definition IB.4, $X \in (\overrightarrow{AB}$ iff A–X–B or $X = B$ or A–B–X, and $X \in (\overrightarrow{BA}$ iff A–X–B or $X = A$ or B–A–X. Therefore $X \in (\overrightarrow{AB} \cap (\overrightarrow{BA}$ iff A–X–B, which is true iff $X \in (\overline{AB})$, by Definition IB.3.

(C) By Definition IB.4, $X \in [\overrightarrow{AB}$ iff A–X–B or $X = A$ or $X = B$ or A–B–X, and $X \in (\overrightarrow{BA}$ iff A–X–B or $X = A$ or B–A–X. Therefore $X \in [\overrightarrow{AB} \cap (\overrightarrow{BA}$ iff A–X–B or $X = A$, which is true iff $X \in [\overline{AB})$, by Definition IB.3.

(D) $X \in (\overline{AB}]$ iff A–X–B or $X = B$ iff $X \in [\overline{BA})$, which by part (C) is true iff $X \in [\overrightarrow{BA} \cap (\overrightarrow{AB}$. □

Exercise IB.9* Let $\mathcal{L}$ be a line, and let A and B be distinct points such that $\mathcal{L} \neq \overleftrightarrow{AB}$. If $(\overline{AB}) \cap \mathcal{L} = \{R\}$, then $\overleftrightarrow{AB} \cap \mathcal{L} = \{R\}$.

Exercise IB.9 Proof. Since $(\overline{AB}) \subseteq \overleftrightarrow{AB}$, $(\overline{AB}) \cap \mathcal{L} = \{R\} \subseteq \overleftrightarrow{AB} \cap \mathcal{L}$. If there were a second point S in $\overleftrightarrow{AB} \cap \mathcal{L}$, then by Exercise I.2 $\overleftrightarrow{AB} = \mathcal{L}$ which contradicts our assumption that $\mathcal{L} \neq \overleftrightarrow{AB}$. □

Exercise IB.10* Prove Corollary IB.14.1: let $\mathcal{P}$, $\mathcal{L}$, P, and Q be as in Theorem IB.14 (that is, $\mathcal{L}$ is a line in plane $\mathcal{P}$, and P and Q are points such that $P \in \mathcal{L}$ and $Q \notin \mathcal{L}$). Then $(\overrightarrow{PQ} \cap \mathcal{L} = \emptyset$.

Exercise IB.10 Proof. By Theorem IB.14 $(\overrightarrow{PQ}$ is a subset of the Q-side of $\mathcal{L}$. By Definition IB.11 the Q-side of $\mathcal{L}$ is $\{X \mid X = Q$ or $(X \in (\mathcal{P} \setminus \{Q\})$ and $[\overline{XQ}] \cap \mathcal{L} = \emptyset\}$; so the Q-side of $\mathcal{L}$ and $\mathcal{L}$ are disjoint. Therefore $(\overrightarrow{PQ} \cap \mathcal{L} = \emptyset$. □

Exercise IB.11* Prove Corollary IB.14.2: let $\mathcal{P}$, $\mathcal{L}$, P, and Q be as in Theorem IB.14 (that is, $\mathcal{L}$ is a line in plane $\mathcal{P}$, and P and Q are points such that $P \in \mathcal{L}$ and $Q \notin \mathcal{L}$). Then $\overline{PQ}$ and $\overline{PQ}$ are subsets of the Q-side of $\mathcal{L}$.

Exercise IB.11 Proof. By Definitions IB.3 and IB.4 $\overline{PQ}$ is a subset of $\overrightarrow{PQ}$. By Theorem IB.13, $\overrightarrow{PQ}$ is a subset of the Q-side of $\mathcal{L}$. $\square$

Exercise IB.12 Prove Corollary IB.14.3: for any triangle $\triangle ABC$, the edges $\overline{AB}$ and $\overline{AC}$ are subsets of $\overrightarrow{BCA}$, $\overline{AB}$ and $\overline{BC}$ are subsets of $\overrightarrow{ACB}$, and $\overline{AC}$ and $\overline{BC}$ are subsets of $\overrightarrow{ABC}$.

Exercise IB.13 Space is convex.

Exercise IB.14 Every plane is convex.

Exercise IB.15* If $\mathcal{G}$ is any collection of convex sets, and if the intersection of the members of $\mathcal{G}$ is nonempty, then the intersection is convex.

Exercise IB.15 Proof. Let P and Q be members of the intersection of all sets in $\mathcal{G}$. This means P and Q belong to every set in $\mathcal{G}$, each of which by Definition IB.9 contains the segment $\overline{AB}$. Thus their intersection contains $\overline{AB}$, and the intersection is convex by Definition IB.9. $\square$

Exercise IB.16 Let $\mathcal{L}$ be a line and let $\mathcal{E}$ be a nonempty proper subset of $\mathcal{L}$ such that $\mathcal{E}$ is not a singleton. Then:

(1) $\mathcal{E}$ is not a segment iff for every pair of distinct points A and B on $\mathcal{L}$, there exists a point U such that A–U–B and $U \notin \mathcal{E}$, or there exists a point V such that A–B–V and $V \in \mathcal{E}$, or there exists a point W such that B–A–W and $W \in \mathcal{E}$.

(2) $\mathcal{E}$ is not a ray iff for every pair of distinct points A and B on $\mathcal{L}$, there exists a point U such that A–U–B and $U \notin \mathcal{E}$, or there exists a point V such that A–B–V and $V \notin \mathcal{E}$, or there exists a point W such that B–A–W and $W \in \mathcal{E}$.

Exercise IB.17* Let $\mathcal{P}$ be an IB plane, $\mathcal{L}$ and $\mathcal{M}$ be lines on $\mathcal{P}$, and O be a point such that $\mathcal{L} \cap \mathcal{M} = \{O\}$, then there exist points P and Q on $\mathcal{L}$ such that P and Q are on opposite sides of $\mathcal{M}$.

Exercise IB.17 Proof. By Axiom I.5, there exists a point P on $\mathcal{L}$ distinct from O. By property B.3 of Definition IB.1 there exists a point Q such that

P–O–Q. By property B.0, Q belongs to $\mathcal{L}$. By Definition IB.11, P and Q are on opposite sides of $\mathcal{M}$. $\square$

Exercise IB.18 (True or False?) Let $\mathcal{P}$ be an IB plane, and let $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ be distinct lines on $\mathcal{P}$ such that $\mathcal{J} \cap \mathcal{L} \neq \emptyset$ and $\mathcal{K} \cap \mathcal{L} \neq \emptyset$. Then if U is a point on $\mathcal{J}$ but not on $\mathcal{L}$, there is a point V on $\mathcal{K}$ such that U and V are on opposite sides of $\mathcal{L}$.

Chapter 5: Exercises and Answers for Pasch Geometry (PSH)

In the following exercises, all points and lines are in a Pasch plane.

Exercise PSH.0* (A) Let $\mathcal{P}$ be a Pasch plane, $\mathcal{L}$ and $\mathcal{M}$ be lines, O be a point on $\mathcal{P}$ such that $\mathcal{L} \cap \mathcal{M} = \{O\}$. If $\mathcal{H}$ is a side of $\mathcal{L}$, then $\mathcal{M} \cap \mathcal{H} \neq \emptyset$.

(B) Let $\mathcal{P}$ be a Pasch plane and let $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ be distinct lines on $\mathcal{P}$ such that $\mathcal{J} \cap \mathcal{L} \neq \emptyset$ and $\mathcal{K} \cap \mathcal{L} \neq \emptyset$. If U is a point on $\mathcal{J}$ but is not on $\mathcal{L}$, then there is a point V on $\mathcal{K}$ such that U and V are on opposite sides of $\mathcal{L}$.

Exercise PSH.0 Proof. (A) By Axiom I.5 there exists a point A on $\mathcal{M}$ distinct from O. Since $\mathcal{L} \cap \mathcal{M} = \{O\}$, by Axiom PSA, A belongs to a side of $\mathcal{L}$. If A belongs to $\mathcal{H}_1$, we are done. If A does not belong to $\mathcal{H}_1$, then we use Theorem PSH.12 (Plane Separation Theorem). It says there exists a side $\mathcal{H}_2$ of $\mathcal{L}$ such that $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ and $\mathcal{P} \setminus \mathcal{L} = \mathcal{H}_1 \cup \mathcal{H}_2$. By property B.3 of Definition IB.1 (Extension Property for betweenness) there exists a point B such that B–O–A. By property B.0 $B \in \mathcal{M}$. By Definition IB.11 A and B are on opposite sides of $\mathcal{L}$. Therefore by Theorem PSA.11 $B \in \mathcal{H}_1$.

(B) Since the lines $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ are distinct lines, $\mathcal{J} \cap \mathcal{L} \neq \emptyset$, and $\mathcal{K} \cap \mathcal{L} \neq \emptyset$, by Exercise I.1 there exist points Q and R such that $\mathcal{J} \cap \mathcal{L} = \{Q\}$ and $\mathcal{K} \cap \mathcal{L} = \{R\}$. By property B.3 of Definition IB.1 there exists a point W such that U–Q–W. By property B.0 $W \in \mathcal{J}$. By Definition IB.11 U and W are on opposite sides of $\mathcal{L}$. Let $\mathcal{H}_1 =$ the U-side of $\mathcal{L}$ and $\mathcal{H}_2 =$ the W-side of $\mathcal{L}$. Then by part (A) there exists a point $V \in \mathcal{K}$ which is on the W-side of $\mathcal{L}$, and this is opposite the U-side of $\mathcal{L}$. $\square$

Exercise PSH.1* Complete the details of the proof of Theorem PSH.8, part (B)(1).

Exercise PSH.1 Proof. By Axiom I.5, there exists a point E on $\mathcal{P}$ not belonging to $\overleftrightarrow{AB}$. By Theorem IB.14, $\overrightarrow{BD} \subseteq$ (the D-side of $\overleftrightarrow{BE}$). Since B–C–D, by Definition IB.4 C belongs to $\overrightarrow{BD}$ and hence to the D-side of $\overleftrightarrow{BE}$. Since A–B–C, the A-side and the C-side $=$ D-side are opposite sides of $\overleftrightarrow{BE}$. Therefore A and D are on opposite sides of $\overleftrightarrow{BE}$. By Axiom PSA, there exists a point Q such that $\overleftrightarrow{AD} \cap \overleftrightarrow{BE} = \{Q\}$ and A–Q–D. But $B \in \overleftrightarrow{AD}$ and $B \in \overleftrightarrow{BE}$, so by Exercise I.1 $Q = B$ and hence A–B–D. $\square$

Exercise PSH.2* (A) Prove Corollary PSH.8.3: Let A, B, C, and D be distinct coplanar points. If A–C–D and B–C–D, then exactly one of the following two statements is true:

$$(1)\ A\text{–}B\text{–}C \text{ and } A\text{–}B\text{–}D; \qquad (2)\ B\text{–}A\text{–}C \text{ and } B\text{–}A\text{–}D.$$

(B) Prove Corollary PSH.8.4: Let A, B, C, and D be distinct coplanar points. If A–B–D and A–C–D, then exactly one of the following two statements is true:

$$(1)\ A\text{–}B\text{–}C \text{ and } B\text{–}C\text{–}D; \qquad (2)\ A\text{–}C\text{–}B \text{ and } C\text{–}B\text{–}D.$$

Exercise PSH.2 Proof. (Proofs for Corollaries PSH.8.3 and PSH.8.4.) (A) If A–C–D, then by property B.1 of Definition IB.1 D–C–A and if B–C–D, then D–C–B. By Corollary PSH.8.2 D–C–A–B or D–C–B–A. By property B.1 of Definition IB.1 either B–A–C–D or A–B–C–D.

(B) By property B.1 of Definition IB.1 D–B–A and D–C–A. By Corollary PSH.8.2 D–B–C–A or D–C–B–A. By property B.1 of Definition IB.1 A–C–B–D or A–B–C–D. □

Exercise PSH.3* Let A, B, C, and D be points such that A–B–C–D and let P and Q be points such that P–A–B and C–D–Q. Then:

(A) $\overleftrightarrow{AB} = \overleftrightarrow{AC} = \overleftrightarrow{AD} = \overleftrightarrow{BC} = \overleftrightarrow{BD} = \overleftrightarrow{CD}$; the points A, B, C, and D are collinear;

(B) $\overleftrightarrow{BC}$ is the union of the disjoint sets $\{B, C\}$, $\overrightarrow{BA}$, $\overline{BC}$, and $\overrightarrow{CD}$;

(C) $\overleftrightarrow{BC}$ is the union of the disjoint sets $\{A, B, C, D\}$, $\overrightarrow{AP}$, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overrightarrow{DQ}$;

(D) $\overline{AD}$ is the union of the disjoint sets $\{A, B, C, D\}$, $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$;

(E) $\overleftrightarrow{AD}$ is the union of the sets $\{X \mid X\text{–}A\text{–}D\}$, $\overline{AD}$, and $\{X \mid A\text{–}D\text{–}X\}$, which are all disjoint.

Exercise PSH.3 Proof. (A) By property B.0 of Definition IB.1 A, B, and C are collinear and B, C, and D are collinear and so A, B, C, and D are collinear. By Exercise I.2 $\overleftrightarrow{AB} = \overleftrightarrow{AC} = \overleftrightarrow{AD} = \overleftrightarrow{BC} = \overleftrightarrow{CD}$.

(B) $\overleftrightarrow{BC} = \{X \mid X\text{–}B\text{–}C \text{ or } X = B \text{ or } B\text{–}X\text{–}C \text{ or } X = C \text{ or } B\text{–}C\text{–}X\}$. By Theorem PSH.13 $\{X \mid X\text{–}B\text{–}C\} = \overrightarrow{BA}$ and

$$\{X \mid B\text{–}C\text{–}X\} = \{X \mid X\text{–}C\text{–}B\} = \overrightarrow{CQ}.$$

Since $\overrightarrow{BA}$, $\overrightarrow{CD}$ and $\overline{BC}$ are disjoint (*cf* property B.2 of Definition IB.1), the proof is complete.

(C) By Definitions IB.3 and IB.4

$\overleftrightarrow{BC} = \overleftrightarrow{AD} = \{X \mid X\text{–}A\text{–}D\} \cup \{A\} \cup \{X \mid A\text{–}X\text{–}D\} \cup \{D\} \cup \{X \mid A\text{–}D\text{–}X\}$.
By Theorem PSH.13 $\{X \mid X\text{–}A\text{–}D\} = \overset{\sqsubset\!\!\longrightarrow}{AD}$ and $\{X \mid A\text{–}D\text{–}X\} = \overset{\sqsubset\!\!\longrightarrow}{DQ}$, so

$$\begin{aligned}\overleftrightarrow{AB} &= \{A\} \cup \overset{\sqsupset\!\!\longrightarrow}{AP} \cup \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AD} \cup \{D\} \cup \overset{\sqsupset\!\!\longrightarrow}{DQ} \\ &= \{A, B, C, D\} \cup \overset{\sqsupset\!\!\longrightarrow}{AP} \cup \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AB} \cup \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{BC} \cup \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{CD} \cup \overset{\sqsupset\!\!\longrightarrow}{DQ}.\end{aligned}$$

(D) Since

$$\begin{aligned}\overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AD} = \{X \mid X = A &\text{ or } A\text{–}X\text{–}B \text{ or } X = B \text{ or} \\ &B\text{–}X\text{–}C \text{ or } X = C \text{ or } C\text{–}X\text{–}D \text{ or } X = D\} \\ &= \{A, B, C, D\} \cup \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB} \cup \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{BC} \cup \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{CD}\end{aligned}$$

and $\{A, B, C, D\} \cap \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB} \cap \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{BC} \cap \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{CD} = \emptyset$, the proof is complete.

(E) By properties B.0, B.1, and B.2 of Definition IB.1

$$\begin{aligned}\overleftrightarrow{AD} &= \{X \mid X\text{–}A\text{–}D \text{ or } X = A \text{ or } A\text{–}X\text{–}D \text{ or } X = D \text{ or } A\text{–}D\text{–}X\} \\ &= \{X \mid X\text{–}A\text{–}D\} \cup \{X = A \text{ or } X = D \text{ or } A\text{–}X\text{–}D\} \cup \{X \mid A\text{–}D\text{–}X\} \\ &= \{X \mid X\text{–}A\text{–}D\} \cup \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AD} \cup \{X \mid A\text{–}D\text{–}X\}.\end{aligned}$$

Moreover the sets forming these unions are all pairwise disjoint. $\square$

Exercise PSH.4* (A) Let A and B be distinct points on the Pasch plane $\mathcal{P}$ and let $\mathcal{E}$ be a nonempty subset of $\overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB}$. Then $\mathcal{E}$ is not a ray.

(B) Let A and B be distinct points on the Pasch plane $\mathcal{P}$ and let $\mathcal{E}$ be a nonempty subset of $\overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AB}$. Then $\mathcal{E}$ is not a ray.

Exercise PSH.4 Proof. (A) Assume $\mathcal{E}$ is a ray; then by Definition IB.4 there exist distinct points D and E belonging to $\mathcal{E}$ such that $\mathcal{E} = \overset{\sqsubset\!\!\longrightarrow}{DE}$ or $\mathcal{E} = \overset{\sqsupset\!\!\longrightarrow}{DE}$. If the latter, choose D' such that $D\text{–}D'\text{–}E$, so that $\overset{\sqsupset\!\!\longrightarrow}{D'E} \subseteq \mathcal{E}$. In either case we can choose D so that $\overset{\sqsupset\!\!\longrightarrow}{DE} \subseteq \mathcal{E} \subseteq \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB}$.

By Theorem PSH.13 $\overset{\sqsupset\!\!\longrightarrow}{DE} = \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{DE} \cup \{X \mid D\text{–}E\text{–}X\}$. Since $\overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{DE}$ is a subset of $\overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB}$, either $A\text{–}D\text{–}E$ or $A\text{–}E\text{–}D$. Choose the notation so that $A\text{–}D\text{–}E$. Since $E \in \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB}$, $A\text{–}E\text{–}B$. By Theorem PSH.8 $A\text{–}D\text{–}E\text{–}B$ so $D\text{–}E\text{–}B$. Since $\overset{\sqsubset\!\!\longrightarrow}{DE}$ is a ray, by Definition IB.4 $B \in \overset{\sqsubset\!\!\longrightarrow}{DE} \subseteq \mathcal{E}$. Since $\mathcal{E} \subseteq \overset{\sqsupset\!\!-\!\!-\!\!\sqsubset}{AB}$, $B \notin \mathcal{E}$. This contradiction proves that our assumption that $\mathcal{E}$ is a ray is false.

(B) Assume $\mathcal{E}$ is a ray; then by Definition IB.4 there exist distinct points D and E belonging to $\mathcal{E}$ such that $\mathcal{E} = \overset{\sqsubset\!\!\longrightarrow}{DE}$ or $\mathcal{E} = \overset{\sqsupset\!\!\longrightarrow}{DE}$. By Theorem PSH.13 $\mathcal{E} = \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{DE} \cup \{X \mid D\text{–}E\text{–}X\}$, or $\mathcal{E} = \overset{\sqsupset\!\!-\!\!-\!\!\sqsupset}{DE} \cup \{X \mid D\text{–}E\text{–}X\}$.

Either D is an endpoint of $\overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AB}$ or it is not. If D is an endpoint, then choose the notation so that $A = D$. Then $A\text{–}E\text{–}B$ or $B = E$. By property B.3 of Definition IB.1 there exists a point C such that $A\text{–}B\text{–}C$. Then $A\text{–}E\text{–}C$, that is, $D\text{–}E\text{–}C$ so that $C \in \overset{\sqsubset\!\!\longrightarrow}{DE} \subseteq \overset{\sqsubset\!\!-\!\!-\!\!\sqsupset}{AB}$, a contradiction to $A\text{–}B\text{–}C$.

If D is not an endpoint either $A\text{–}D\text{–}E\text{–}B$ or $A\text{–}E\text{–}D\text{–}B$. Again choose the notation so that $A\text{–}D\text{–}E\text{–}B$, and again by property B.3 let C be a point such that D–B–C and hence A–D–E–B–C and A–B–C. Then $C \in \overrightarrow{DE} \subseteq \overline{AB}$, a contradiction to A–B–C. Thus our assumption that $\mathcal{E}$ is a ray is false. $\square$

Exercise PSH.5* Let A, B, C, D, and E be points on plane $\mathcal{P}$ such that A, B, and C are noncollinear, A–B–D, and A–C–E. Then $D \in \operatorname{ins} \angle BCE$.

Exercise PSH.5 Proof. By Definition IB.11 and the fact (Theorem PSH.12) that there are exactly two sides to a line, D and E are both on the side of $\overleftrightarrow{BC}$ opposite A, so that D is on the E-side of $\overleftrightarrow{BC}$. By Theorem PSH.12 D and B are on the same side of $\overleftrightarrow{AC}$ because A–B–D so that D is on the B-side of $\overleftrightarrow{AC} = \overleftrightarrow{CE}$, hence by Definition PSH.36 $D \in \operatorname{ins} \angle BCE$. $\square$

Exercise PSH.6 Let A, B, C, D, and E be as in Exercise PSH.5. Then $\overleftrightarrow{AB} \cap (\operatorname{ins} \angle BCE) = \overrightarrow{BD}$ and $\overleftrightarrow{AB} \cap (\operatorname{out} \angle BCE) = \overrightarrow{BA}$.

Exercise PSH.7 Let A, B, C, D, and E be as in Exercise PSH.5. Then there exists a point F such that $\overline{BE} \cap \overline{CD} = \{F\}$.

Exercise PSH.8* Let O, A, B, A', and B' be points on $\mathcal{P}$ such that O, A, and B are noncollinear, B–O–B', and A–O–A'. Let X be any member of $\operatorname{ins} \angle AOB$, and let X' be any point such that X–O–X'. Then $\overleftrightarrow{OX} \cap \operatorname{ins} \angle A'OB' = \overrightarrow{OX'}$.

Exercise PSH.8 Proof. Since $X \in \operatorname{ins} \angle AOB$, by Definition PSH.36 X and A are on the same side of $\overleftrightarrow{OB}$ and X and B are on the same side of $\overleftrightarrow{OA}$. The lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{XX'}$ are concurrent at O. Then A' and A are on opposite sides of $\overleftrightarrow{OB}$, X' and X are on opposite sides of $\overleftrightarrow{OB}$, and X is on the A-side of $\overleftrightarrow{OB}$. Therefore X' is on the A'-side of $\overleftrightarrow{OB}$.

Interchanging A with B and A' with B' in this reasoning, we have that X' is on the B'-side of $\overleftrightarrow{OA}$, hence $X' \in \operatorname{ins} \angle A'OB'$ by Definition PSH.36. By Theorem PSH.38(B) $\overrightarrow{OX'} = \overleftrightarrow{OX} \cap \operatorname{ins} \angle A'OB'$. In this reasoning we have relied heavily on Theorem PSH.12 (Plane Separation Theorem). $\square$

Exercise PSH.9* Let O, A, B, A', and B' be points on the Pasch plane $\mathcal{P}$ such that O, A, and B are noncollinear, B'–O–B, and A'–O–A, let X be any member of $\operatorname{ins} \angle AOB$ and X' be any point such that X'–O–X.

(A) $\overleftrightarrow{OX} \cap \overline{A'B'} = \overrightarrow{OX'} \cap \overline{A'B'}$ is a singleton, *i.e.*, there exists a point Y such that $\overleftrightarrow{OX} \cap \overline{A'B'} = \{Y\}$.

(B) (B) Let X be any point such that $X \in \overline{AB}$; if $X \in \overline{AB}$ define $\Omega(X) = Y$, where Y is as in part (A); if $X = A$ define $Y = A'$, and if $X = B$ define $Y = B'$. Then the mapping Ω maps $\overline{AB}$ onto $\overline{A'B'}$ and is one-to-one, hence is a bijection.

Exercise PSH.9 Proof. (A) By Exercise PSH.8, $X' \in \operatorname{ins}\angle A'OB'$. By Theorem PSH.39 (Crossbar) there is a point Y such that $\overleftrightarrow{OX} \cap \overline{A'B'} = \{Y\}$.

(B) Since Ω maps A to A' and B to B', both A' and B' belong to $\Omega(\overline{AB})$. For any point $Y \in \overline{A'B'}$, by property B.3 of Definition IB.1 there is some X such that Y–O–X. By Exercise PSH.8 $X \in \operatorname{ins}\angle AOB$, and by Theorem PSH.39 there is a point $\hat{X}$ such that $\{\hat{X}\} = \overleftrightarrow{OX} \cap \overline{AB}$. Then $\Omega(\hat{X}) = Y$ so that $Y \in \Omega(\overline{AB})$. This shows that Ω maps $\overline{AB}$ onto $\overline{A'B'}$.

If $W \neq Z$ are two points of $\overline{AB}$, then $\Omega(W) \in \overleftrightarrow{WO}$, $\Omega(Z) \in \overleftrightarrow{ZO}$, W–O–$\Omega(W)$ and Z–O–$\Omega(Z)$. Since $W \neq Z$ $\overleftrightarrow{WO} \neq \overleftrightarrow{ZO}$ and by Exercise I.1 the only point of intersection of these lines is O. Hence $\Omega(W) \neq \Omega(Z)$, and Ω is a bijection. □

Exercise PSH.10 If A and B are distinct points, then $\{A, B\}$ is nonconvex.

Exercise PSH.11 Let $\mathcal{P}$ be a Pasch plane, $\mathcal{L}$ be a line on $\mathcal{P}$, and let $\mathcal{J}$ be a side of $\mathcal{L}$. If $P \in \mathcal{L}$ and $Q \in \mathcal{J}$, then $\overline{PQ} \subseteq \mathcal{J}$.

Exercise PSH.12* Let A, B, and C be noncollinear points on a Pasch plane. If $D \in \operatorname{ins}\angle BAC$, prove that $\overrightarrow{AB} \subseteq \overrightarrow{ADB}$, $\overrightarrow{AC} \subseteq \overrightarrow{ADC}$, $B \in \operatorname{out}\angle CAD$, and $C \in \operatorname{out}\angle BAD$.

Exercise PSH.12 Proof. By Corollary PSH.39.2, B and C are on opposite sides of $\overleftrightarrow{AD}$, so by Theorem PSH.41(C) $B \in \operatorname{out}\angle CAD$ and $C \in \operatorname{out}\angle BAD$. By Theorem IB.14, $\overrightarrow{AB} \subseteq \overrightarrow{ADB}$ and $\overrightarrow{AC} \subseteq \overrightarrow{ADC}$. □

Exercise PSH.13* Let A, B, and C be noncollinear points on a Pasch plane, and let P and Q be members of $\operatorname{ins}\angle BAC$. Then if $P \in \operatorname{ins}\angle BAQ$, $Q \in \operatorname{ins}\angle CAP$.

Exercise PSH.13 Proof. If $P \in \operatorname{ins}\angle BAQ$, by Corollary PSH.39.2, B and Q are on opposite sides of $\overleftrightarrow{AP}$; hence Q is on the side of $\overleftrightarrow{AP}$ opposite B, that is, on the C-side; we already know that Q is on the B-side of $\overleftrightarrow{AC}$, which is the same as the P-side; hence $Q \in \operatorname{ins}\angle CAP$. □

Exercise PSH.14* (Key exercise) (A) Let $\mathcal{E}$ be a convex subset of plane $\mathcal{P}$ and let $\mathcal{L}$ be a line on $\mathcal{P}$. If $\mathcal{E} \cap \mathcal{L} = \emptyset$, then $\mathcal{E}$ is a subset of a side of $\mathcal{L}$.

(B) If a line $\mathcal{M}$, or a segment or a ray does not intersect $\mathcal{L}$, then that line, segment, or ray lies entirely on one side of $\mathcal{L}$.

Exercise PSH.14 Proof. (A) Let P be a point of $\mathcal{E}$. If Q is any other point of $\mathcal{E}$, then $\overline{[PQ]} \subseteq \mathcal{E}$ because $\mathcal{E}$ is convex. By Theorem PSH.12 P and Q are on the same side of $\mathcal{L}$, because $\overline{]PQ[}$ does not intersect $\mathcal{L}$.

(B) By Theorem IB.10 every line is convex; by Theorem PSH.18 all segments and rays are convex. Therefore by part (A) if any of these fail to intersect the line $\mathcal{L}$, they lie entirely on one side of $\mathcal{L}$. $\square$

Exercise PSH.15* Let A, B, and C be noncollinear points on a Pasch plane $\mathcal{P}$ and let $\mathcal{L}$ be a line on $\mathcal{P}$. If $\{A, B, C\} \cap \mathcal{L} = \emptyset$, then either $\mathcal{L} \cap \triangle ABC = \emptyset$ or $\mathcal{L}$ intersects two and only two edges of $\triangle ABC$, in which case $\mathcal{L} \cap \triangle ABC$ is a doubleton.

Exercise PSH.15 Proof. If $\mathcal{L} \cap \triangle ABC \neq \emptyset$, by Theorem PSH.50(B), if $\mathcal{L}$ contains no point of $\{A, B, C\}$ then alternatives (1) and (3) or that theorem are ruled out, and $\mathcal{L}$ intersects $\triangle ABC$ in exactly two points, which are on different edges. $\square$

Exercise PSH.16* The inside of every angle is convex and the inside of every triangle is convex.

Exercise PSH.16 Proof. By Exercise IB.15 any non-empty intersection of two convex sets is convex. By Theorem PSH.9, every side of a line is convex. By Definition PSH.36, the inside of an angle and the inside of a triangle are intersections of sides of lines, hence are convex. $\square$

Exercise PSH.17* Let $\mathcal{P}$ be a Pasch plane and let A, B, and C be noncollinear points on $\mathcal{P}$.

(A) If $D \in \operatorname{ins} \angle BAC$, then $\overrightarrow{]AD} \subseteq \operatorname{ins} \angle BAC$.

(B) $\operatorname{ins} \angle BAC = \bigcup_{D \in \overline{]BC[}} \overrightarrow{]AD}$

Exercise PSH.17 Proof. (A) By Theorem IB.14 $\overrightarrow{]AD} \subseteq B$-side of $\overleftrightarrow{AC}$ and $\overrightarrow{]AD} \subseteq C$-side of $\overleftrightarrow{AB}$. By Definition PSH.36 $\overrightarrow{]AD} \subseteq \operatorname{ins} \angle BAC$.

(B) Let X be any member of $\operatorname{ins} \angle BAC$. By Crossbar (Theorem PSH.39) there exists a member Y of $\overline{]BC[}$ such that $\overrightarrow{]AX} \cap \overline{]BC[} = \{Y\}$. By Theorem

PSH.16 $\overset{\circ}{\overrightarrow{AX}} = \overset{\circ}{\overrightarrow{AY}}$ so $X \in \overset{\circ}{\overrightarrow{AY}}$. Therefore $X \in \bigcup_{Y \in \overset{\circ}{\overline{BC}}} \overset{\circ}{\overrightarrow{AY}}$, and

ins $\angle BAC \subseteq \bigcup_{Y \in \overset{\circ}{\overline{BC}}} \overset{\circ}{\overrightarrow{AY}}$. By part (A), for every $Y \in \overset{\circ}{\overline{BC}}$, $\overset{\circ}{\overrightarrow{AY}} \subseteq$ ins $\angle BAC$,
so ins $\angle BAC \supseteq \bigcup_{Y \in \overset{\circ}{\overline{BC}}} \overset{\circ}{\overrightarrow{AY}}$. $\square$

Exercise PSH.18* (Angle analog of Exercise PSH.32) Let A, B, and C be noncollinear points on the Pasch plane $\mathcal{P}$ and let D be a member of ins $\angle BAC$. Then ins $\angle BAC$ is the union of the disjoint sets $\overset{\circ}{\overrightarrow{AD}}$, ins $\angle BAD$ and ins $\angle DAC$.

Exercise PSH.18 Proof. By Crossbar (Theorem PSH. 39) $\overset{\circ}{\overrightarrow{AD}}$ and $\overset{\circ}{\overline{BC}}$ intersect at a point E. By Exercise PSH.17(B)

$$\text{ins}\,\angle BAE = \bigcup_{Y \in \overset{\circ}{\overline{BE}}} \overset{\circ}{\overrightarrow{AY}},\ \text{ins}\,\angle CAE = \bigcup_{Y \in \overset{\circ}{\overline{CE}}} \overset{\circ}{\overrightarrow{AY}}.$$

Moreover, $\overset{\circ}{\overrightarrow{AD}} = \overset{\circ}{\overrightarrow{AE}}$.
Therefore ins $\angle BAC = \left(\bigcup_{Y \in \overset{\circ}{\overline{BE}}} \overset{\circ}{\overrightarrow{AY}} \right) \cup \left(\bigcup_{Y \in \overset{\circ}{\overline{CE}}} \overset{\circ}{\overrightarrow{AY}} \right) \cup \overset{\circ}{\overrightarrow{AE}}$. $\square$

Exercise PSH.19* Prove parts (3) and (4) of Theorem PSH.48: if A, B, and C are noncollinear points and P is a member of ins $\triangle ABC$, there exists a point $Q \in \overset{\circ}{\overline{BC}}$ such that

(3) $\overset{\circ}{\overline{AQ}} \subseteq$ ins $\triangle ABC$, and
(4) $\overset{\circ}{\overrightarrow{AQ}} \setminus \overline{AQ} \subseteq$ out $\triangle ABC$.

Exercise PSH.19 Proof. We are to show that if A, B, and C be noncollinear points, P is any member of ins $\triangle ABC$ and $Q \in \overset{\circ}{\overline{BC}}$ that (3) $\overset{\circ}{\overline{AQ}} \subseteq$ ins $\triangle ABC$, and (4) $\overset{\circ}{\overrightarrow{AQ}} \setminus \overline{AQ} \subseteq$ out $\triangle ABC$. To show (3), note that A and Q are not on the same edge of $\triangle ABC$; the result follows from Theorem PSH.47. (4) If $X \in \overset{\circ}{\overrightarrow{AQ}} \setminus \overline{AQ}$ then A–Q–X and X is a member of the side of $\overleftrightarrow{BC}$ opposite A, which is a subset of out $\triangle ABC$ by Theorem 46(D). $\square$

Exercise PSH.20* The union of a line $\mathcal{L}$ and one of its sides $\mathcal{H}$ is convex (*i.e.*, a halfplane is convex).

Exercise PSH.20 Proof. Let A and B be distinct members of $\mathcal{L} \cup \mathcal{H}$.

(Case 1) If $A \in \mathcal{L}$ and $B \in \mathcal{L}$, then by Theorem IB.10 $\overline{AB} \subseteq \mathcal{L} \subseteq \mathcal{L} \cup \mathcal{H}$.

(Case 2) If $A \in \mathcal{H}$ and $B \in \mathcal{H}$, then by Definition IB.11 $\overline{AB} \subseteq \mathcal{H} \subseteq \mathcal{L} \cup \mathcal{H}$.

(Case 3) If $A \in \mathcal{H}$ and $B \in \mathcal{L}$, then by Theorem PSH.13 $\overset{\circ}{\overline{AB}} \subseteq \overset{\circ}{\overrightarrow{AB}}$, and by Theorem IB.14 $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \mathcal{L} \cup \mathcal{H}$. If $A \in \mathcal{L}$ and $B \in \mathcal{H}$ they may be relabeled and the same logic applied. □

See also Theorem PSH.9 as used in the proof to Theorem PSH.12.

Exercise PSH.21* Let $\mathcal{A}$ be any subset of plane $\mathcal{P}$ having at least two members and let $\mathcal{B}$ be the union of all segments $\overline{PQ}$ such that $P \in \mathcal{A}$ and $Q \in \mathcal{A}$. Is $\mathcal{B}$ necessarily convex?

Exercise PSH.21 Proof. No. Let A, B, and C be any noncollinear points. Then $\triangle ABC$ (which is the union of all the segments connecting these points) is not convex, by Corollary PSH.47.1. □

Exercise PSH.22* If A, B, and C are noncollinear points, then both $\text{enc}\,\angle ABC$ and $\text{enc}\,\triangle ABC$ are convex sets.

Exercise PSH.22 Proof. By Exercise PSH.16 $\text{ins}\,\angle ABC$ is convex and so is $\text{ins}\,\triangle ABC$.

(I) Proof that $\text{enc}\,\angle ABC$ is convex. Suppose that P and Q are both members of $\text{ins}\,\angle ABC$; then $\overline{PQ} \subseteq \text{ins}\,\angle ABC$. Now let P and Q be members of $\angle ABC$; if both are members of $\overrightarrow{BA}$, $\overline{PQ} \subseteq \overrightarrow{BA}$ since this is convex by Theorem PSH.18. A similar argument shows that if both are members of $\overrightarrow{BC}$, $\overline{PQ} \subseteq \overrightarrow{BC}$.

Finally, suppose $P \in \angle ABC$ and $Q \in \text{ins}\,\angle ABC$. Then by Theorem PSH.43 either alternative (2) or (3) holds; if alternative (2) holds, by Theorem PSH.13, $\overline{PQ} \subseteq \overrightarrow{PQ} \subseteq \text{ins}\,\angle ABC$ so that $\overline{PQ} \subseteq \text{enc}\,\angle ABC$. If alternative (3) holds, and P' is the second point of intersection of $\overleftrightarrow{PQ}$ with $\angle ABC$, then $\overline{PP'} = \overleftrightarrow{PQ} \cap \text{ins}\,\angle ABC$ and hence $Q \in \overline{PP'}$ and $\overline{PQ} \subseteq \overline{PP'} \subseteq \text{ins}\,\angle ABC$ so that $\overline{PQ} \subseteq \text{enc}\,\angle ABC$.

(II) Proof that $\text{enc}\,\triangle ABC$ is convex. Note again that if P and Q are both members of $\text{ins}\,\triangle ABC$ then by convexity $\overline{PQ} \subseteq \text{ins}\,\triangle ABC$. If P and Q are both members of $\triangle ABC$, either alternative (2) or (3) of Theorem PSH.50 holds. If alternative (2) holds, then $\overline{PQ} = \overleftrightarrow{PQ} \cap \text{ins}\,\triangle ABC$ so that $\overline{PQ} \subseteq \text{enc}\,\triangle ABC$. If alternative (3) holds, then $\overleftrightarrow{PQ} \cap \triangle ABC = \emptyset$ and both P and Q are members of the same edge of $\triangle ABC$ which by Theorem PSH.18 is convex, so that $\overline{PQ} \subseteq \text{enc}\,\triangle ABC$.

Finally, if $P \in \triangle ABC$ and $Q \in \text{ins}\,\triangle ABC$, only alternative (2) of Theorem PSH.50 can hold, since it is the only alternative where the line $\overleftrightarrow{PQ}$ intersects $\text{ins}\,\triangle ABC$. Let P' be the second point of intersection of

$\overleftrightarrow{PQ}$ with $\triangle ABC$; then $\overline{PP'} = \overleftrightarrow{PQ} \cap \text{ins}\,\triangle ABC$ and hence $Q \in \overline{PP'}$ and $\overline{PQ} \subseteq \overline{PP'} \subseteq \text{ins}\,\triangle ABC$ so that $\overline{PQ} \subseteq \text{enc}\,\triangle ABC$.

Part (II) could also be proved by observing that if P and Q are any two points of $\text{enc}\,\triangle ABC$, both are points of $\overleftrightarrow{PQ} \cap \text{enc}\,\triangle ABC$ which is a segment by Theorem PSH.50(A). By Theorem PSH.18 a segment is convex, so $\overline{PQ} \subseteq \text{enc}\,\triangle ABC$. $\square$

Exercise PSH.23* Without referring to Theorem PSH.43 (that is, using principally the definitions of inside, outside, and Theorem PSH.41(C)), construct a proof of part (A) of Theorem PSH.44: Let A, B, C, P, and Q be distinct points where A, B, and C are noncollinear; if $P \in \text{ins}\,\angle BAC$ and $Q \in \text{out}\,\angle BAC$, then $\overline{PQ} \cap \angle BAC$ is a singleton.

Exercise PSH.23 Proof. $P \in \text{ins}\,\angle BAC = \overrightarrow{ABC} \cap \overrightarrow{ACB}$, and $Q \in \text{out}\,\angle BAC =$ (side of $\overleftrightarrow{AB}$ opposite C)$\cup$ (side of $\overleftrightarrow{AC}$ opposite B.) We prove the assertion for the case where $Q \in$ side of $\overleftrightarrow{AB}$ opposite C. The other case is similar.

By Theorem IB.12 (or Definition IB.11), there exists R such that $\overline{PQ} \cap \overleftrightarrow{AB} = \{R\}$, so that P–R–Q.

(I) If $R \in \overrightarrow{AB}$ then by Definition PSH.29 $R \in \angle BAC$ and we're done.

(II) If R–A–B then R is on the side of $\overleftrightarrow{AC}$ opposite B so by Theorem IB.12 (or Definition IB.11) there exists a point $S \in \overleftrightarrow{AC}$ such that $\overline{PR} \cap \overleftrightarrow{AC} = \{S\}$ and P–S–R–Q. Since $P \in \overrightarrow{ABC}$ and $R \in \overleftrightarrow{AB}$, by Theorem PSH.38(A) $\overrightarrow{RP} \subseteq \overrightarrow{ABC}$ so that $S \in \overrightarrow{ABC}$ and hence $S \in \overrightarrow{AC}$, and S is the only intersection of $\overleftrightarrow{PQ}$ with $\overleftrightarrow{AC}$ by Exercise I.1. Since $\overleftrightarrow{PQ}$ does not intersect $\overrightarrow{AB}$, S is the only point of intersection of $\overleftrightarrow{PQ}$ and $\overline{PQ}$) with $\angle BAC$. $\square$

Exercise PSH.24* Prove Theorem PSH.47: Let A, B, and C be noncollinear points, and let P and Q belong to $\triangle ABC$. If no edge of $\triangle ABC$ contains both P and Q, then $\overline{PQ} \subseteq \text{ins}\,\triangle ABC$.

Exercise PSH.24 Proof. Note that only one of the points P and Q can be a corner. Without loss of generality we may assume that $P \in \overline{AB}$ and $Q \in \overline{AC}$.

By Theorem PSH.37, $\overline{PQ} \subseteq \text{ins}\,\angle BAC$. If $Q \neq C$ then $\overline{PQ} \subseteq \overrightarrow{BCA}$ because both P and $Q \in \overrightarrow{BCA}$ and $\overline{PQ} \cap \overleftrightarrow{BC} = \emptyset$, and hence $\overline{PQ} \subseteq \text{ins}\,\angle BAC \cap \overrightarrow{BCA} = \text{ins}\,\triangle ABC$ by Definition PSH.36.

If $Q = C$, since $P \in \overrightarrow{BCA}$, we may apply Theorem IB.13 and find that $\overrightarrow{QP} \subseteq \overrightarrow{BCA}$, so that again $\overrightarrow{PQ} \subseteq \text{ins}\,\angle BAC \cap \overrightarrow{BCA} = \text{ins}\,\triangle ABC$. $\square$

Exercise PSH.25* Prove part (2) of Theorem PSH.49, which we restate here for convenience. *Let A, B, and C be noncollinear points and let $\mathcal{L}$ be a line such that $\mathcal{L} \cap \text{ins}\,\triangle ABC \neq \emptyset$ and $\mathcal{L} \cap \{A, B, C\} = \emptyset$. If $P \in \mathcal{L} \cap \text{ins}\,\triangle ABC$ and $Q \neq P$ is any point of $\mathcal{L}$, then*

(1) *$\overrightarrow{PQ}$ intersects exactly one of the segments $\overline{AC}, \overline{BC}$* or *$\overline{AB}$ in exactly one point,*

(2) *$\mathcal{L} = \overleftrightarrow{PQ}$ intersects exactly two of the segments $\overline{AC}, \overline{BC}$ or $\overline{AB}$, and thus $\mathcal{L}$ intersects $\triangle ABC$ in exactly two points D and E, and*

(3) $\overline{DE} \subseteq \text{ins}\,\triangle ABC$.

Exercise PSH.25 Proof. Let $Q' \in \mathcal{L}$ be any point such that Q'–P–Q. By Theorem PSH.15(B), $\mathcal{L} = \overrightarrow{PQ'} \cup \{P\} \cup \overrightarrow{PQ}$. By part (1) of Theorem PSH.49, each of $\overrightarrow{PQ}$ and $\overrightarrow{PQ'}$ intersects exactly one of the segments $\overline{AC}, \overline{BC}$ or $\overline{AB}$ in exactly one point. Since $P \notin \triangle ABC, \mathcal{L}$ intersects $\triangle ABC$ in exactly two points which we may call D and E, and these points are not on the same edge of $\triangle ABC$. $\square$

Exercise PSH.26* Let A, B, and C be noncollinear points, let E be any member of $\overline{AC}$, and let F be any member of $\overline{AB}$. Then $\overline{BE}$ and $\overline{CF}$ intersect in a point O which belongs to $\text{ins}\,\triangle ABC$.

Exercise PSH.26 Proof. By Definition IB.11, A and B are on opposite sides of $\overleftrightarrow{CF}$; by Theorem IB.14, $E \in \overrightarrow{CA} \subseteq$ the A-side of $\overleftrightarrow{CF}$, so that E and B are on opposite sides of $\overleftrightarrow{CF}$. By Theorem PSH.12 there exists a point O such that $\{O\} = \overleftrightarrow{CF} \cap \overline{BE}$. By similar reasoning, there exists a point Q such that $\{Q\} = \overline{CF} \cap \overleftrightarrow{BE}$. Both of these points are intersections of $\overleftrightarrow{CF}$ and $\overleftrightarrow{BE}$, which by Exercise I.1 is a single point, so $O = Q$, and $\{O\} = \overline{CF} \cap \overline{BE}$. Now by Theorem PSH.37 both $F \in \text{ins}\,\angle ACB$ and $E \in \text{ins}\,\angle ABC$, and by Theorem PSH.38, O is a member of both these sets. By Theorem PSH.46(C) $\text{ins}\,\triangle ABC = \text{ins}\,\angle ACB \cap \text{ins}\,\angle ABC$, so $O \in \text{ins}\,\triangle ABC$. $\square$

Exercise PSH.27* Let A, B, and C be noncollinear points on plane $\mathcal{P}$, let Q be a member of $\text{ins}\,\angle ABC$, and R a member of $\text{ins}\,\angle ACB$. Then $\overrightarrow{BQ}$ and $\overrightarrow{CR}$ intersect at a point O which belongs to $\text{ins}\,\angle ABC$.

Exercise PSH.27 Proof. By Theorem PSH.39 (Crossbar) $\overrightarrow{BQ}$ intersects $\overline{AC}$ at some point E, and $\overrightarrow{CR}$ intersects $\overline{AB}$ at some point F. By Exercise

PSH.26, there exists a point O such that $\{O\} = \overline{CF} \cap \overline{BE} \subseteq \overrightarrow{CR} \cap \overrightarrow{BQ}$. $\square$

Exercise PSH.28* Let A, B, and C be noncollinear points and suppose $P \in \overrightarrow{AB}$ and $Q \in \operatorname{ins}\angle BAC$. Then $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle BAC$.

Exercise PSH.28 Proof. Either alternative (2) or (3) of Theorem PSH.43 holds since $\overleftrightarrow{PQ} \cap \operatorname{ins}\angle BAC \neq \emptyset$. If alternative (2) holds, $\overrightarrow{PQ} \subseteq \overrightarrow{PQ} \subseteq \operatorname{ins}\angle BAC$. If alternative (3) holds, let P' be the second point of intersection of $\overleftrightarrow{PQ}$ with $\angle BAC$. Then by Theorem PSH.43 (3) $\overline{PP'} = \overleftrightarrow{PP'} \cap \operatorname{ins}\angle BAC$. Since $Q \in \operatorname{ins}\angle BAC$, $Q \in \overline{PP'} \subseteq \operatorname{ins}\angle BAC$ and $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle BAC$. $\square$

Exercise PSH.29* Let A, B, and C be noncollinear points and suppose $P \in \triangle ABC$ and $Q \in \operatorname{ins}\triangle ABC$. Then $\overrightarrow{PQ} \subseteq \operatorname{ins}\triangle ABC$.

Exercise PSH.29 Proof. If $P \in \triangle ABC$ and is one of the corners, say B, then $P \in \overrightarrow{AB}$ and also $P \in \overrightarrow{CB}$. By Theorem PSH.46 $\operatorname{ins}\triangle ABC = \operatorname{ins}\angle BAC \cap \operatorname{ins}\angle BCA$, so Q is a member of both $\operatorname{ins}\angle BAC$ and $\operatorname{ins}\angle BCA$. By Exercise PSH.28 $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle BAC$ and $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle BCA$, hence $\overrightarrow{PQ} \subseteq \operatorname{ins}\triangle ABC$.

On the other hand, if P is not one of the corners, then it belongs to one of the segments $\overline{AB}$, $\overline{BC}$, or $\overline{AC}$. Without loss of generality assume that $P \in \overline{AB}$. Then $P \in \overrightarrow{AB}$ and also $P \in \overrightarrow{BA}$. By Theorem PSH.46 $\operatorname{ins}\triangle ABC = \operatorname{ins}\angle BAC \cap \operatorname{ins}\angle ABC$, so Q is a member of both $\operatorname{ins}\angle BAC$ and $\operatorname{ins}\angle ABC$. By Exercise PSH.28 $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle BAC$ and $\overrightarrow{PQ} \subseteq \operatorname{ins}\angle ABC$, hence $\overrightarrow{PQ} \subseteq \operatorname{ins}\triangle ABC$. $\square$

Exercise PSH.30* Prove Theorem PSH.42: Let P and Q be distinct points, and let $\mathcal{H}$ be a side of $\overleftrightarrow{PQ}$. Let A and B be members of $\mathcal{H} \cup \overleftrightarrow{PQ}$ such that A, B, and P are noncollinear. Then $\operatorname{ins}\angle APB \subseteq \mathcal{H}$. See figure below.

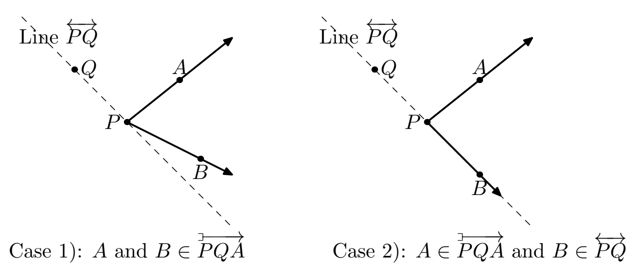

Figure 5.4 for Theorem PSH.42.

Exercise PSH.30 Proof. There are three possibilities: 1) both A and $B \in \mathcal{H}$; 2) $A \in \mathcal{H}$ and $B \in \overleftrightarrow{PQ}$, and 3) $A \in \overleftrightarrow{PQ}$ and $B \in \mathcal{H}$. (It is not possible for both A and B to belong to $\overleftrightarrow{PQ}$, for then A, B, and P would be collinear.) Clearly if we can prove the theorem in case 2), case 3) is also proved.

In either case 1) or 2), $\mathcal{H} = \overset{\sqsupset\!\!\longrightarrow}{PQA}$ is the A-side of $\overleftrightarrow{PQ}$ by Definition IB.11. In case 1), $B \in \overset{\sqsupset\!\!\longrightarrow}{PQA}$ and by Definition IB.11 $\overset{\sqsupset\!\sqsubset}{AB} \subseteq \overline{AB} \subseteq \overset{\sqsupset\!\!\longrightarrow}{PQA}$. In case 2), $B \in \overleftrightarrow{PQ}$ so by Theorem IB.13, $\overset{\sqsupset\!\sqsubset}{AB} \subseteq \overset{\sqsupset\!\!\longrightarrow}{BA} \subseteq \overset{\sqsupset\!\!\longrightarrow}{PQA}$.

In either case, let $R \in \operatorname{ins}\angle APB$. By the Crossbar Theorem PSH.39 $\overset{\sqsupset\!\!\longrightarrow}{PR} \cap \overset{\sqsupset\!\sqsubset}{AB} = \{S\}$ for some point S. Since in either case $\overset{\sqsupset\!\sqsubset}{AB} \subseteq \mathcal{H}, S \in \mathcal{H}$. Now $\overrightarrow{PR} \cap \overleftrightarrow{PQ} = \{P\}$ (because $\overleftrightarrow{PR} \cap \overleftrightarrow{PQ} = \{P\}$ by Exercise I.1) so that $\overset{\sqsupset\!\!\longrightarrow}{PR} \cap \overleftrightarrow{PQ} = \emptyset$. Since R and S both belong to $\overset{\sqsupset\!\!\longrightarrow}{PR}, \overset{\sqsupset\!\sqsubset}{RS} \subseteq \overset{\sqsupset\!\!\longrightarrow}{PR}$ so that $\overset{\sqsupset\!\sqsubset}{RS} \cap \overleftrightarrow{PQ} = \emptyset$. Thus by Definition IB.11 R and S belong to the same side of $\overleftrightarrow{PQ}$, and since $S \in \mathcal{H}$, $R \in \mathcal{H}$. $\square$

Exercise PSH.31* Let P and Q be distinct points on plane $\mathcal{P}$, let $\mathcal{H}$ be a side of $\overleftrightarrow{PQ}$ in $\mathcal{P}$, and let A and B be members of $\mathcal{H}$ such that A, B, and P are noncollinear. Then either $B \in \operatorname{ins}\angle APQ$ or $A \in \operatorname{ins}\angle BPQ$.

Exercise PSH.31 Proof. Since $B \in \overset{\sqsupset\!\!\longrightarrow}{PQA}$, if $B \in \overset{\sqsupset\!\!\longrightarrow}{PAQ}$ then by Definition PSH.36(A) $B \in \operatorname{ins}\angle APQ$. If B is on the side of $\overleftrightarrow{PA}$ opposite Q then by Theorem PSH.38(C), $A \in \operatorname{ins}\angle BPQ$. To see this, in the statement of Theorem PSH.38(C) substitute P for A, Q for B, A for P, and B for C; then the theorem reads: *If A is on the B-side of $\overleftrightarrow{PQ}$, and if Q and B are on opposite sides of $\overleftrightarrow{PA}$, then* $A \in \operatorname{ins}\angle QPB$, thus proving the second alternative. $\square$

Exercise PSH.32 (Side analog for Exercise PSH.18) Let P, O, and Q be points such that P–O–Q, and let R be a point off of $\overleftrightarrow{OP}$. Then $\overset{\sqsupset\!\!\longrightarrow}{OPR}$ is the union of the disjoint sets $\operatorname{ins}\angle POR$, $\overset{\sqsupset\!\!\longrightarrow}{OR}$, and $\operatorname{ins}\angle QOR$.

Exercise PSH.33 Let A, B, and C be noncollinear points and let B' and C' be points such that B–A–B' and C–A–C'. Then $\operatorname{out}\angle BAC$ is the union of the disjoint sets $\overset{\sqsupset\!\!\longrightarrow}{AB'}$, $\overset{\sqsupset\!\!\longrightarrow}{AC'}$, $\operatorname{ins}\angle BAC'$, $\operatorname{ins}\angle CAB'$, and $\operatorname{ins}\angle B'AC'$.

Exercise PSH.34 Let A, B, and C be noncollinear points and let E be a member of $\operatorname{out}\angle BAC$. Then $\overset{\sqsupset\!\!\longrightarrow}{AE}$ is a subset of $\operatorname{out}\angle BAC$.

Exercise PSH.35 Let A, B, and C be noncollinear points and let P and Q be members of $(\operatorname{enc}\angle BAC \setminus \{A\})$ such that P, Q, and A are noncollinear.

Then $\operatorname{ins}\angle PAQ \subseteq \operatorname{ins}\angle BAC$. Note: try solving this before reading the proof of Theorem PSH.41(D).

Exercise PSH.36* Let $\mathcal{L}$ be a line and let $\mathcal{H}$ be a side of $\mathcal{L}$. If A, B, and C are noncollinear members of $\mathcal{H}$, then $\operatorname{enc}\triangle ABC \subseteq \mathcal{H}$.

Exercise PSH.36 Proof. By Theorem PSH.9, $\mathcal{H}$ is convex. Each of A, B, and C is a member of $\mathcal{H}$, so the segments $\overset{[\,]}{AB}$, $\overset{[\,]}{BC}$, and $\overset{[\,]}{AC}$ are all subsets of $\mathcal{H}$, and by Definition IB.7 $\triangle ABC \subseteq \mathcal{H}$.

Now let X be any point of $\operatorname{ins}\triangle ABC$, and let $P \in \triangle ABC$. Then alternative (2) of Theorem 50 applies to $\overleftrightarrow{PX}$, since this is the only alternative where a line intersects the inside of the triangle. There exists a point $Q \neq P$ such that $Q \in \triangle ABC$ and $\overset{(\,)}{PQ} = \overleftrightarrow{PX} \cap \operatorname{ins}\triangle ABC$, hence P–X–Q. By the convexity of $\mathcal{H}$, $X \in \mathcal{H}$. Therefore $\operatorname{enc}\triangle ABC \subseteq \mathcal{H}$. $\square$

Exercise PSH.37 Let A, B, C, R, and S be points such that A, B, and C are noncollinear, $R \in \overset{(\,)}{AB}$, and $S \in \overset{(\,)}{AC}$. Then $\overleftrightarrow{RS} \cap \overset{[\,]}{BC} = \emptyset$ and $\overset{[\,]}{RS} \cap \overleftrightarrow{BC} = \emptyset$.

Exercise PSH.38 Let $\mathcal{T}$ be a triangle, let P be a member of $\operatorname{ins}\mathcal{T}$, and let Q be a point distinct from P. Then there exists a point R such that $\mathcal{T} \cap \overset{[\rightarrow}{PQ} = \{R\}$, $\operatorname{ins}\mathcal{T} \cap \overset{[\rightarrow}{PQ} = \overset{[\,)}{PR}$, and $\operatorname{out}\mathcal{T} \cap \overset{[\rightarrow}{PQ} = \overset{[\rightarrow}{PQ} \setminus \overset{[\,]}{PR}$.

Exercise PSH.39 Let A, B, and C be noncollinear points on plane $\mathcal{P}$, let P be a member of $\triangle ABC$, let Q be a member of $\operatorname{ins}\triangle ABC$, and let R be a point such that Q–P–R. Then $R \in \operatorname{out}\triangle ABC$, $\overset{(\rightarrow}{QP} \cap \operatorname{ins}\triangle ABC = \overset{[\,)}{QP}$, and $\overset{(\rightarrow}{QP} \cap \operatorname{out}\triangle ABC = \overset{(\rightarrow}{QP} \setminus \overset{[\,]}{PQ}$.

Exercise PSH.40 Let $\mathcal{T}$ be a triangle, let P be a member of $\operatorname{ins}\mathcal{T}$ and Q be a member of $\operatorname{out}\mathcal{T}$. Then there exists a point R such that $\overleftrightarrow{PQ} \cap \mathcal{T} = \{R\}$, $\overset{[\,)}{PR} = \overset{[\,]}{PQ} \cap \operatorname{ins}\mathcal{T}$, and $\overset{(\,]}{RQ} = \overset{[\,]}{PQ} \cap \operatorname{out}\mathcal{T}$.

Exercise PSH.41 Let $\mathcal{T}$ be a triangle and let P, Q, and R be noncollinear members of $\operatorname{enc}\mathcal{T}$. Then $\operatorname{ins}\triangle PQR \subseteq \operatorname{ins}\mathcal{T}$.

Exercise PSH.42* Let A, B, and C be noncollinear points and let P, Q, and R be noncollinear members of $\operatorname{ins}\triangle ABC$. Then $\operatorname{enc}\triangle PQR \subseteq \operatorname{ins}\triangle ABC$.

Exercise PSH.42 Proof. Since P, Q, and R are members of $\text{ins}\,\triangle ABC = \overset{]\!\rightarrow}{ABC} \cap \overset{]\!\rightarrow}{ACB} \cap \overset{]\!\rightarrow}{BCA}$, all these points belong to each of the sets $\overset{]\!\rightarrow}{ABC}$, $\overset{]\!\rightarrow}{ACB}$, and $\overset{]\!\rightarrow}{BCA}$. By Exercise PSH.36, $\text{enc}\,\triangle ABC$ is a subset of each of these sets, so that it is a subset of their intersection, that is, of $\text{ins}\,\triangle ABC$. $\square$

Exercise PSH.43 Let A, B, and C be noncollinear points on Pasch plane $\mathcal{P}$, let O be a member of $\text{ins}\,\triangle ABC$, let A' be any point between O and A, let B' be any point between O and B, and let C' be any point between O and C. Then $O \in \text{ins}\,\triangle A'B'C'$, and $\text{enc}\,\triangle A'B'C' \subseteq \text{ins}\,\triangle ABC$.

Exercise PSH.44 Let A, B, and C be noncollinear points. Then:

(a) There exist points P and Q such that A is between P and Q, $\angle BAC \cap \overset{[\,]}{PQ} = \{A\}$, and P and Q are both members of $\text{out}\,\angle BAC$.

(b) If P and Q are any points satisfying the conditions in (a) above, then B and C are on the same side of $\overleftrightarrow{PQ}$.

Exercise PSH.45* Let $\mathcal{E}$ be a nonempty convex subset of the plane $\mathcal{P}$, and let A, B, and C be noncollinear members of $\mathcal{E}$. Then $\text{enc}\,\triangle ABC \subseteq \mathcal{E}$.

Exercise PSH.45 Proof. Since $\mathcal{E}$ is convex, $\triangle ABC \subseteq \mathcal{E}$. Let $X \in \text{ins}\,\triangle ABC$, and let $\mathcal{L}$ be any line containing X. Referring to Theorem PSH.50(B), we observe that only part (2) can apply, since this is the only case in which $\mathcal{L}$ intersects $\text{ins}\,\triangle ABC$. Then by part (2)(b), there are exactly two points P and Q in $\mathcal{L} \cap \triangle ABC$, so that $X \in \overset{]\,[}{PQ} \subseteq \overset{[\,]}{PQ}$; since $\mathcal{E}$ is convex, $X \in \mathcal{E}$, and $\text{enc}\,\triangle ABC \subseteq \mathcal{E}$. $\square$

Exercise PSH.46 Let A, B, and C be noncollinear points and let O be a member of $\text{ins}\,\triangle ABC$. Then

$$\text{ins}\,\triangle ABC = \overset{[\,[}{OA} \cup \overset{]\,[}{OB} \cup \overset{]\,[}{OC} \cup \text{ins}\,\triangle OAB \cup \text{ins}\,\triangle OAC \cup \text{ins}\,\triangle OBC.$$

Exercise PSH.47* Let $\mathcal{P}$ be a Pasch plane and A, B, and U be noncollinear points. Then for every point V in $\mathcal{P}$,

(A) $\overset{]\!\rightarrow}{UV}$ is not a subset of $\overleftrightarrow{AB}$; and

(B) $\overset{]\,[}{UV}$ is not a subset of $\overleftrightarrow{AB}$.

Exercise PSH.47 Proof. Since A, B, and U are noncollinear, $U \notin \overleftrightarrow{AB}$.

(A) Let V be any point on $\mathcal{P}$ distinct from U. By property B.3 of Definition IB.1 there exists a point W such that U–V–W. Then by Definition IB.4

$W \in \overset{\sqsupset\!\!\longrightarrow}{UV}$. By Exercise I.2, if $\overset{\sqsupset\!\!\longrightarrow}{UV} \subseteq \overleftrightarrow{AB}$, both V and W are in $\overleftrightarrow{AB}$ so that $\overleftrightarrow{AB} = \overleftrightarrow{UV} \supseteq \overset{\sqsupset\!\!\longrightarrow}{UV}$, and A, B, and U are collinear, a contradiction.

(B) Let V be any point on $\mathcal{P}$ distinct from U. By two successive applications of Theorem PSH.22 (Denseness) there exist distinct points X and Y belonging to $\overset{\sqsupset\!\!\relbar\!\!\sqsubset}{UV}$. If $\overset{\sqsupset\!\!\relbar\!\!\sqsubset}{UV} \subseteq \overleftrightarrow{AB}$, both X and Y are in $\overleftrightarrow{AB}$ so that by Exercise I.2 $\overleftrightarrow{UV} = \overleftrightarrow{AB}$. Then A, B, and U are collinear, a contradiction. $\square$

Exercise PSH.48* Prove parts 4–6 of Theorem PSH.18: Let A and B be distinct points on the Pasch plane $\mathcal{P}$. Then each of the following sets is convex: (4) $\overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB}$, (5) $\overset{\sqsupset\!\!\relbar\!\!\sqsupset}{AB}$, and (6) $\overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB}$.

Exercise PSH.48 Proof. (4) By Exercise IB.8 $\overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB} = \overset{\sqsubset\!\!\longrightarrow}{AB} \cap \overset{\sqsupset\!\!\longrightarrow}{BA}$, both of which are convex by parts (1) and (2) of Theorem PSH.18. Both these sets contain the point A so are not disjoint, and by Exercise IB.14 their intersection $\overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB}$ is convex.

(5) $\overset{\sqsupset\!\!\relbar\!\!\sqsupset}{AB} = \overset{\sqsubset\!\!\relbar\!\!\sqsubset}{BA}$ is convex by part (4) proved just above.

(6) By Exercise IB.8 $\overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB} = \overset{\sqsubset\!\!\longrightarrow}{AB} \cap \overset{\sqsubset\!\!\longrightarrow}{BA}$, both of which are convex by part (2) of Theorem PSH.18. They both contain the point A so are not disjoint, and by Exercise IB.14 their intersection $\overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB}$ is convex. $\square$

Exercise PSH.49* Prove Theorem PSH.46(B): Let A, B, and C be noncollinear points. Then $\operatorname{ins}\triangle ABC \cup \triangle ABC \cup \operatorname{out}\triangle ABC = \mathcal{P}$ and the sets in this union are pairwise disjoint.

Exercise PSH.49 Proof. That $\operatorname{ins}\triangle ABC \cup \triangle ABC \cup \operatorname{out}\triangle ABC = \mathcal{P}$ is immediate from Definition PSH.36(B). We examine each pair of sets to see that the pair is disjoint:

(1) $\triangle ABC \cap \operatorname{ins}\triangle ABC = \emptyset$ since $\triangle ABC \subseteq \overleftrightarrow{AB} \cup \overleftrightarrow{BC} \cup \overleftrightarrow{CA}$ which is disjoint from $\operatorname{ins}\triangle ABC$ by part (A).

(2) By Definition PSH.36(B), $\operatorname{out}\triangle ABC \cap (\triangle ABC \cup \operatorname{ins}\triangle ABC) = \emptyset$ and therefore $\operatorname{out}\triangle ABC \cap \triangle ABC = \emptyset$ and $\operatorname{out}\triangle ABC \cap \operatorname{ins}\triangle ABC = \emptyset$. $\square$

Exercise PSH.50* Prove Theorem PSH.46(C): Let A, B, and C be noncollinear points. Then $\operatorname{ins}\triangle ABC = \operatorname{ins}\angle BAC \cap \operatorname{ins}\angle ABC = \operatorname{ins}\angle BAC \cap \overset{\sqsupset\!\!\longrightarrow}{BCA}$.

Exercise PSH.50 Proof. By Definition PSH.36(B),

$$\operatorname{ins}\triangle ABC = \overset{\sqsupset\!\!\longrightarrow}{ABC} \cap \overset{\sqsupset\!\!\longrightarrow}{BCA} \cap \overset{\sqsupset\!\!\longrightarrow}{CAB} = \operatorname{ins}\angle BAC \cap \overset{\sqsupset\!\!\longrightarrow}{BCA}$$

since $\operatorname{ins}\angle BAC = \overset{\sqsupset\!\!\longrightarrow}{ABC} \cap \overset{\sqsupset\!\!\longrightarrow}{ACB} = \overset{\sqsupset\!\!\longrightarrow}{ABC} \cap \overset{\sqsupset\!\!\longrightarrow}{CAB}$ by part (A) of the Definition. Also, $\operatorname{ins}\angle ABC = \overset{\sqsupset\!\!\longrightarrow}{ABC} \cap \overset{\sqsupset\!\!\longrightarrow}{BCA}$ by the same definition, so that

$$\operatorname{ins}\angle BAC \cap \operatorname{ins}\angle ABC = \overrightarrow{ABC} \cap \overrightarrow{ACB} \cap \overrightarrow{ABC} \cap \overrightarrow{BCA}$$
$$= \overrightarrow{ABC} \cap \overrightarrow{ACB} \cap \overrightarrow{BCA} = \overrightarrow{ABC} \cap \overrightarrow{BCA} \cap \overrightarrow{CAB} = \operatorname{ins}\triangle ABC. \quad \square$$

Exercise PSH.51* Let $\mathcal{P}$ be a Pasch plane, O, B, and R be noncollinear points on $\mathcal{P}$, C be a member of $\operatorname{ins}\angle ROB$ and B' be a point on $\overleftrightarrow{OB}$ such that B–O–B', then $R \in \operatorname{ins}\angle COB'$.

Exercise PSH.51 Proof. Since $C \in \operatorname{ins}\angle ROB$, by Definition PSH.36 C is on the R-side, that is, R is on the C-side of $\overleftrightarrow{OB} = \overleftrightarrow{OB'} = \overleftrightarrow{BB'}$. By Corollary PSH.39.2 B and R are on opposite sides of $\overleftrightarrow{OC}$. Since B and B' also are on opposite sides of $\overleftrightarrow{OC}$ (*cf* Definition IB.11), R is on the B'-side of $\overleftrightarrow{OC}$. By Definition PSH.36 $R \in \operatorname{ins}\angle COB'$. $\square$

Exercise PSH.52* (A) Let X be any point on the Pasch plane $\mathcal{P}$, then there exists a triangle $\mathcal{T}$ such that $X \in \operatorname{ins}\mathcal{T}$.

(B) Let P and Q be distinct points on plane $\mathcal{P}$. Then there exist triangles $\mathcal{T}$ and $\mathcal{U}$ such that $P \in \operatorname{ins}\mathcal{T}$, $Q \in \operatorname{ins}\mathcal{U}$, $\operatorname{enc}\mathcal{T} \subseteq \operatorname{out}\mathcal{U}$, and $\operatorname{enc}\mathcal{U} \subseteq \operatorname{out}\mathcal{T}$.

Exercise PSH.52 Proof. (A) By Axiom I.5 there exists a point U on $\mathcal{P}$ distinct from X. By property B.3 of Definition IB.1 there is a point B such that B–X–U. By Axiom I.5 there exists a point A not belonging to $\overleftrightarrow{BU}$. By property B.3 of Definition IB.1 there exists a point C such that A–U–C. By Theorem PSH.37 $\overline{BU} \subseteq \operatorname{ins}\angle BAC$ and $\overline{BU} \subseteq \operatorname{ins}\angle BCA$. By Theorem PSH.46(C) $\overline{BU} \subseteq (\operatorname{ins}\angle BAC \cap \operatorname{ins}\angle BCA) = \operatorname{ins}\triangle ABC$. Let $\mathcal{T} = \triangle ABC$. Then $X \in \operatorname{ins}\mathcal{T}$.

(B) We first construct a triangle $\triangle ABC$ with $P \in \operatorname{ins}\triangle ABC$, then we construct a second triangle $\triangle DEF$ with $Q \in \operatorname{ins}\triangle DEF$, in such a way that the enclosures of the triangles are disjoint.

By repeated applications of property B.3 of Definition IB.1 and Theorem PSH.22, there exist points T, A, D, and U such that T–P–A–D–Q–U. As in part (A), let B be a point not on $\overleftrightarrow{PQ}$, and let C and C' be points such that C–T–B–C', and by the argument in part (A), $P \in \operatorname{ins}\triangle ABC$.

Then T–A–D and T–B–C'; applying Exercise PSH.5 we have $D \in \operatorname{ins}\angle ABC'$. Then by Theorem PSH.38 $\overrightarrow{BD} \subseteq \operatorname{ins}\angle ABC'$. By property B.3 there is a point E such that B–D–E and $E \in \operatorname{ins}\angle ABC'$. By Theorem IB.14 $\overrightarrow{AD}$ is a subset of the side of $\overleftrightarrow{AB}$ opposite T, which is the C'-side, and hence both Q and U are on the C'-side of $\overleftrightarrow{AB}$. They are also on the A-side of $\overleftrightarrow{BC'}$ and therefore belong to $\operatorname{ins}\angle ABC'$.

Now by Exercise PSH.16 $\operatorname{ins}\angle ABC'$ is convex, and since both E and U are in $\operatorname{ins}\angle ABC'$, $\overline{EU} \subseteq \operatorname{ins}\angle ABC'$.

By Theorem PSH.43 either the intersection of $\overleftrightarrow{EU}$ with $\angle ABC'$ is a single point (alternative (2)) or exactly two points (alternative (3)). In the first case, either $\overrightarrow{EU}$ intersects $\angle ABC'$ (in which case we define F to be any point with E–U–F) or it intersects it at some point G. $G \notin \overline{EU}$, so by Theorem PSH.22 let F be a point such that E–U–F–G.

In the second case there are two points G and H which are the points of intersection of $\overleftrightarrow{EU}$ with $\angle ABC'$. Then if G–E–U–H let F be a point such that U–F–H; if H–E–U–G then let F be a point such that U–F–G. In all cases E–U–F and $F \in \operatorname{ins}\angle ABC'$.

As in part (A), $Q \in \operatorname{ins}\triangle DEF$. By Exercise PSH.36 $\operatorname{enc}\triangle DEF$ is a subset of the C'-side of $\overleftrightarrow{AB}$. Now C is in the side of $\overleftrightarrow{AB}$ opposite C', and by Definition PSH.36 $\operatorname{ins}\triangle ABC \subseteq \overrightarrow{ABC}$. Also both $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are subsets of $\overrightarrow{ABC}$ by Theorem IB.14, and $\overline{AB} \subseteq \overleftrightarrow{AB}$, so that $\triangle ABC \subseteq (\overrightarrow{ABC} \cup \overleftrightarrow{AB})$ which by Theorem PSH.12 is disjoint from $\overrightarrow{ABC'}$ and thus from $\triangle DEF$.

If we let $\mathcal{T} = \triangle ABC$ and $\mathcal{U} = \triangle DEF$, $P \in \operatorname{ins}\mathcal{T}$, $Q \in \operatorname{ins}\mathcal{U}$, and $\operatorname{enc}\mathcal{T}$ is disjoint from $\operatorname{enc}\mathcal{U}$. By Definition PSH.36, if $\mathcal{T}$ is any triangle in $\mathcal{P}$, $\operatorname{out}\mathcal{T} = \mathcal{P} \setminus \operatorname{enc}\mathcal{T}$, so that by elementary set theory $\operatorname{enc}\mathcal{T} \subseteq \operatorname{out}\mathcal{U}$, and $\operatorname{enc}\mathcal{U} \subseteq \operatorname{out}\mathcal{T}$. $\square$

The reader might try constructing a simpler proof of the above showing only that $\operatorname{enc}\mathcal{T} \cap \operatorname{ins}\mathcal{U} = \operatorname{enc}\mathcal{U} \cap \operatorname{ins}\mathcal{T} = \emptyset$. This would not require that $\mathcal{T} \cap \mathcal{U} = \emptyset$.

Exercise PSH.53* Let $\mathcal{P}$ be a Pasch plane, $\mathcal{L}$ and $\mathcal{L}'$ be distinct lines on $\mathcal{P}$, O be a member of $\mathcal{P} \setminus (\mathcal{L} \cup \mathcal{L}')$, A, B, and C be points on $\mathcal{L}$ such that A–B–C and A', B', and C' be points on $\mathcal{L}'$ such that A–O–A', B–O–B', and C–O–C', then A'–B'–C'.

Exercise PSH.53 Proof. Since A–B–C, by Theorem PSH.37 $B \in \operatorname{ins}\angle AOC$. By Exercise PSH.8 $B' \in \operatorname{ins}\angle A'OC'$. By Theorem PSH.37 $\overline{C'A'} \subseteq \operatorname{ins}\angle C'OA'$. Therefore $B' \in \overline{C'A'}$ and A'–B'–C'. $\square$

Exercise PSH.54* Let A, B, and C be points on the Pasch plane $\mathcal{P}$ such that A–B–C. Then $\overrightarrow{AB} \cap \overrightarrow{CB} = \overline{AC}$.

Exercise PSH.54 Proof. By Theorem PSH.16 $\overrightarrow{AB} = \overrightarrow{AC}$ and $\overrightarrow{CB} = \overrightarrow{CA}$ so $\overrightarrow{AB} \cap \overrightarrow{CB} = \overrightarrow{AC} \cap \overrightarrow{CA}$. The result is immediate from Exercise IB.8(B). $\square$

Exercise PSH.55 (Sets bounded by two parallel lines.) Let $\mathcal{P}$ be the plane containing parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$, let P_1 and P_2 be points on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and let Q_1 and Q_2 be points on $\overleftrightarrow{P_1P_2}$ such that Q_1–P_1–P_2 and P_1–P_2–Q_2, $\mathcal{Q}_1$ be the Q_1-side of $\mathcal{L}_1$, let $\mathcal{Q}_1^*$ be the P_2-side of $\mathcal{L}_1$, let $\mathcal{Q}_2$ be the Q_2-side of $\mathcal{L}_2$, let $\mathcal{Q}_2^*$ be the P_1-side of $\mathcal{L}_2$, and let $\mathcal{Q} = \mathcal{Q}_1^* \cap \mathcal{Q}_2^*$. Then $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \mathcal{Q}_1 \cap \mathcal{Q} = \mathcal{Q}_2 \cap \mathcal{Q} = \emptyset$; each of the sets $\mathcal{Q}_1$, $\mathcal{Q}_1^*$, $\mathcal{Q}_2$, $\mathcal{Q}_2^*$, and $\mathcal{Q}$ is convex; and $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q} = \mathcal{P} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$.

Exercise PSH.56* See Figure 5.13 from Chapter 5, reproduced below. Let O, A, B, A', and B' be distinct points on the Pasch plane $\mathcal{P}$ such that $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'} = \{O\}$ and $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, then

(I) O–A–B iff O–A'–B',

(II) O–B–A iff O–B'–A', and

(III) A–O–B iff A'–O–B'.

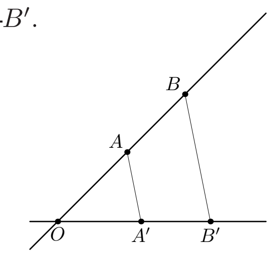

Figure 5.13 for Exercise PSH.56(I).

Exercise PSH.56 Proof. (1) If O–A–B, then O–A'–B'. Since O–A–B, then by Definition IB.11 O and B are on opposite sides of $\overleftrightarrow{AA'}$. Since $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, by Exercise PSH.14 B' is on the B-side of $\overleftrightarrow{AA'}$. Since O and B' are on opposite sides of $\overleftrightarrow{AA'}$, by Axiom PSA there exists a unique point Q such that $\overleftrightarrow{AA'} \cap \overleftrightarrow{OB'} = \{Q\}$ and O–Q–B'. Since $\overleftrightarrow{AA'} \cap \overleftrightarrow{OB'} = \{A'\}$, $Q = A'$, and O–A'–B'.

(2) If O–A'–B', then O–A–B. In (1) interchange "A" and "A'" and interchange "B" and "B'."

(3) If O–B–A, then O–B'–A'. In (1) interchange "A" and "B" and interchange "A'" and "B'."

(4) If O–B'–A', then O–B–A. In (1) interchange "A" and "B'" and interchange "B" and "A'."

(5) If A–O–B, then A'–O–B'. By Theorem IB.5 $B' \in \overleftrightarrow{OA'}$ iff B'–O–A' or $B' = O$ or O–B'–A' or $B' = A'$ or O–A'–B'.

If $B' = O$ then $\overleftrightarrow{BB'} = \overleftrightarrow{BA}$ and $\overleftrightarrow{BB'}$ intersects $\overleftrightarrow{AA'}$ so that $\overleftrightarrow{BB'} \nparallel \overleftrightarrow{AA'}$, which contradicts our hypothesis. Similarly, if $B' = A'$ then $\overleftrightarrow{AA'} \nparallel \overleftrightarrow{BB'}$, again a contradiction.

If O–B'–A' then since O, A, and A' are noncollinear they form a triangle $\triangle OAA'$; by Theorem PSH.6 $\overleftrightarrow{BB'}$ must intersect either $\mathring{\overline{OA}}$, in which case $\overleftrightarrow{AB} = \overleftrightarrow{A'B'}$ which is impossible by hypothesis, or it must intersect $\mathring{\overline{AA'}}$ and $\overleftrightarrow{AA'} \nparallel \overleftrightarrow{BB'}$, again a contradiction.

If O–A'–B', then $\overleftrightarrow{BB'}$ contains points on opposite sides of $\overleftrightarrow{AA'}$ so by Axiom PSA must intersect $\overleftrightarrow{AA'}$, and again $\overleftrightarrow{AA'} \nparallel \overleftrightarrow{BB'}$ which contradicts the hypothesis. Therefore B'–O–A', that is A'–O–B'.

(6) If A'–O–B', then A–O–B. In (5) interchange "A" and "A'" and interchange "B" and "B'". □

Exercise PSH.57* Let $\mathcal{L}$ and $\mathcal{M}$ be distinct lines in a Pasch plane, let A, B, and C be points of $\mathcal{L}$, and let D, E, and F be points of $\mathcal{M}$ such that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BE} \parallel \overleftrightarrow{CF}$. Then A–B–C iff D–E–F.

Exercise PSH.57 Proof. By Exercise PSH.14, A and D are on the same side of $\overleftrightarrow{BE}$ and C and F are on the same side of $\overleftrightarrow{BE}$; by Definition IB.11, if A–B–C then A and C are on opposite sides of $\overleftrightarrow{BE}$ and hence D and F are on opposite sides of the same line, so again by Definition IB.11, D–E–F. Interchanging the roles of A with D and C with F shows the converse. □

Exercise PSH.58* Prove Theorem PSH.34 using the result of Theorem PSH.32. That is, show that if A, B, and C are noncollinear points on a Pasch plane $\mathcal{P}$, then the set of corners of $\triangle ABC$ is $\{A, B, C\}$.

Exercise PSH.58 Proof. The points A, B, and C are all corners of $\triangle ABC$, by Definition IB.7. By the same definition, a corner is a point of the triangle. So to prove this theorem we need only prove that no member of $\mathring{\overline{AB}} \cup \mathring{\overline{BC}} \cup \mathring{\overline{AC}}$ can be a corner of $\triangle ABC$.

We may choose the notation so that $U \in \mathring{\overline{AB}}$, that is, A–U–B, and assume U is a corner of $\triangle ABC$. By Theorem PSH.32 there exist points V and V' such that $\overline{UV} \subseteq \triangle ABC$ and $\mathring{\overline{UV'}} \cap \triangle ABC = \emptyset$ where U, V, and V' are collinear, and since U and $V \in \mathring{\overline{AB}}$, U, V, V', A, and B are all collinear. By property B.2 of Definition IB.1 exactly one of V'–A–B, A–V'–B, or A–B–V' is true.

A–V'–B is impossible for then $\overset{]\,[}{UV'} \subseteq \overset{]\,[}{AB}$ contrary to $\overset{]\,[}{UV'} \cap \triangle ABC = \emptyset$. If V'–A–B, then since A–U–B, by Corollary PSH.8.1, V'–A–U. By Theorem PSH.22 (denseness) let X be such that A–X–U and by the same corollary, V'–A–X–U. Then $X \in \overset{]\,[}{AB}$ and $X \in \overset{]\,[}{UV'}$ so that $X \notin \overset{]\,[}{AB}$, a contradiction. A similar argument shows a contradiction in the case that A–B–V'. Therefore $U \notin \overset{]\,[}{AB}$, and it follows that every corner of $\triangle ABC$ must be either A, B, or C. □

Exercise PSH.59* Let A, B, C, and D be points on the Pasch plane $\mathcal{P}$ such that $\overset{[\,]}{AB} \cup \overset{[\,]}{BC} \cup \overset{[\,]}{CD} \cup \overset{[\,]}{DA}$ is a quadrilateral; then if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, this quadrilateral is rotund.

Exercise PSH.59 Proof. By Exercise PSH.14 both C and D are on the same side of $\overleftrightarrow{AB}$ and both A and B are on the same side of $\overleftrightarrow{CD}$. If the quadrilateral $\square ABCD$ is not rotund, by Theorem PSH.53 one of the corners belongs to the inside of the triangle formed by the remaining three corners. Suppose $A \in \operatorname{ins} \triangle BCD$; then by Theorem PSH.46 $A \in \operatorname{ins} \angle DBC$ and by Theorem PSH.39, $\overrightarrow{BA} \subseteq \overleftrightarrow{AB}$ intersects $\overset{]\,[}{CD} \subseteq \overleftrightarrow{CD}$, which contradicts the parallelism of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Similar proofs will hold for the other corners B, C, and D. □

Exercise PSH.60 Consult a book on projective geometry and compare/contrast those axioms of separation with those involving the open sets used to classify topological spaces.

Chapter 6: Exercises and Answers for Ordering a Line in a Pasch Plane (ORD)

Exercise ORD.1* Let A, B, C, and D be points such that A–B–C–D. If the points on $\overleftrightarrow{AD}$ are ordered so that $A < D$, then $A < B < C < D$.

Exercise ORD.1 Proof. By Theorem ORD.6, A–B–D implies that either $A < B < D$ or $D < B < A$; by hypothesis $A < D$ so $A < B < D$. It follows, since $B < D$, that $B < C < D$, hence $A < B < C < D$. $\square$

Exercise ORD.2 Let O and P be distinct points, and let $\mathcal{E}$ be a nonempty finite subset of $\overleftrightarrow{OP}$ which has n elements. Then there exists a mapping θ of $[1;n]$ onto $\mathcal{E}$ such that for every member k of $[1;n-1]$, $\theta(k) < \theta(k+1)$, and every member of $\{\theta(j)|j \in [1;k]\}$ is less than every member of $\mathcal{E} \setminus \{\theta(j)|j \in [1;k]\}$.

Exercise ORD.3 Let $\mathbb{D}$ be the field of dyadic rational numbers[1], let $\mathbb{D}'$ be equal to $\mathbb{D} \cap [0;1]$, and let A and B be distinct points on the Pasch plane $\mathcal{P}$. Then there exists a mapping θ of $\mathbb{D}$ into $\overset{[\!-\!]}{AB}$ such that, for all members r and s of $\mathbb{D}$, $r < s$ iff $\theta(r) < \theta(s)$.

Exercise ORD.4* Let $\mathcal{E}$ be a convex subset of a line $\mathcal{M}$. If $\mathcal{E}$ is not a singleton, then $\mathcal{E}$ is infinite.

Exercise ORD.5 Let $\mathcal{E}$ be an infinite convex subset of a line $\mathcal{M}$. If A is a member of $\mathcal{E}$, B is a member of $\mathcal{M} \setminus \mathcal{E}$, and C is a point such that A–B–C, then $\overset{[\!\rightarrow}{BC}$ is a subset of $\mathcal{M} \setminus \mathcal{E}$.

Exercise ORD.6* Prove Theorem ORD.7 part (II): let O and P be distinct points on the Pasch plane $\mathcal{P}$ and suppose the points of $\overleftrightarrow{OP}$ are ordered so that $O < P$. If A and B are points on $\overleftrightarrow{OP}$ such that $A < B$, then

$$\overset{]\!-\![}{AB} = \{X|A < X < B\} = \{X|B > X > A\},$$
$$\overset{[\!-\![}{AB} = \{X|A \leq X < B\} = \{X|B > X \geq A\},$$
$$\overset{]\!-\!]}{AB} = \{X|A < X \leq B\} = \{X|B \geq X > A\},$$
$$\overset{[\!-\!]}{AB} = \{X|A \leq X \leq B\} = \{X|B \geq X \geq A\},$$
$$\overset{]\!\rightarrow}{AB} = \{X|A < X\} = \{X|X > A\},$$

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[1] Dyadic Rationals are rational numbers with integer numerators but denominators of the form 2^n where n is a natural number, and the greatest common divisor (gcd) of the numerator and denominator is 1.

$\overset{\sqsubset\!\!\rightarrow}{AB} = \{X|A \leq X\} = \{X|X \geq A\}$;

while if A and B are points on $\overleftrightarrow{OP}$ such that $B < A$, then

$\overset{\sqsupset\!\!\relbar\!\!\sqsubset}{AB} = \{X|A > X > B\} = \{X|B < X < A\}$,
$\overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB} = \{X|A \geq X > B\} = \{X|B < X \leq A\}$,
$\overset{\sqsupset\!\!\relbar\!\!\sqsupset}{AB} = \{X|A > X \geq B\} = \{X|B \leq X < A\}$,
$\overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB} = \{X|A \geq X \geq B\} = \{X|B \leq X \leq A\}$,
$\overset{\sqsupset\!\!\rightarrow}{AB} = \{X|A > X\} = \{X|X < A\}$,
$\overset{\sqsubset\!\!\rightarrow}{AB} = \{X|A \geq X\} = \{X|X \leq A\}$.

Exercise ORD.6 Proof. Let A and B be points on $\overleftrightarrow{OP}$. We will use Theorem ORD.6 repeatedly without further reference.

By Definition IB.3, $X \in \overset{\sqsupset\!\!\relbar\!\!\sqsubset}{AB}$ iff A–X–B. If $A < B$, this is $A < X < B$. If $A > B$, this is $A > X > B$.

$X \in \overset{\sqsubset\!\!\relbar\!\!\sqsubset}{AB}$ iff A–X–B or $X = A$. If $A < B$, this is $A < X < B$ or $X = A$, that is, $A \leq X < B$. If $A > B$, this is $A > X > B$ or $X = A$, that is, $A \geq X > B$.

$X \in \overset{\sqsupset\!\!\relbar\!\!\sqsupset}{AB}$ iff A–X–B or $X = B$. If $A < B$, this is $A < X < B$ or $X = B$, that is, $A < X \leq B$. If $A > B$, this is $A > X > B$ or $X = B$, that is, $A > X \geq B$.

$X \in \overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB}$ iff A–X–B or $X = A$ or $X = B$. If $A < B$, this is $A < X < B$ or $X = A$ or $X = B$, that is, $A \leq X \leq B$. If $A > B$, this is $A > X > B$ or $X = A$ or $X = B$, that is, $A \geq X \geq B$.

By Definition IB.4 $X \in \overset{\sqsupset\!\!\rightarrow}{AB}$ iff A–X–B or $X = B$ or A–B–X. If $A < B$, this is $A < X < B$ or $X = B$ or $A < B < X$, that is, $A < X$. If $A > B$, this is $A > X > B$ or $X = B$ or $A > B > X$, that is, $A > X$.

$X \in \overset{\sqsubset\!\!\rightarrow}{AB}$ iff $X = A$ or A–X–B or $X = B$ or A–B–X. If $A < B$, this is $X = A$ or $A < X < B$ or $X = B$ or $A < B < X$, that is, $A \leq X$. If $A > B$, this is $X = A$ or $A > X > B$ or $X = B$ or $A > B > X$, that is, $A \geq X$. □

Exercise ORD.7* Let A and B be distinct points on the Pasch plane $\mathcal{P}$ and let C and D be distinct members of $\overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB}$, then $\overset{\sqsupset\!\!\relbar\!\!\sqsubset}{CD} \subseteq \overset{\sqsupset\!\!\relbar\!\!\sqsubset}{AB}$ and $\overset{\sqsubset\!\!\relbar\!\!\sqsupset}{CD} \subseteq \overset{\sqsubset\!\!\relbar\!\!\sqsupset}{AB}$.

Exercise ORD.7 Proof. Using Definition ORD.1, we order the points on $\overleftrightarrow{AB}$ so that $A < B$ and we choose the notation so that $C < D$. By Theorem ORD.7(II), $A \leq C \leq B$ and $A \leq D \leq B$ so that $A \leq C < D \leq B$. Let X be any member of $\overset{\sqsupset\!\!\relbar\!\!\sqsubset}{CD}$, then by Definition IB.3, C–X–D. By Theorem ORD.6, $C < X < D$. By Theorem ORD.4, $A \leq C < X < D \leq B$, so that $A < X < B$. By Theorem ORD.7(II), $X \in \overset{\sqsupset\!\!\relbar\!\!\sqsubset}{AB}$. Since X is any member of

$\overset{\circ}{\overline{CD}}$, $\overset{\circ}{\overline{CD}} \subseteq \overset{\circ}{\overline{AB}}$. Since $\{C, D\} \subseteq \overline{AB}$ and $\overline{CD} = \overset{\circ}{\overline{CD}} \cup \{C, D\}$, $\overline{CD} \subseteq \overline{AB}$. $\square$

Exercise ORD.8* Let O, A, B, and C be collinear points on the Pasch plane $\mathcal{P}$ such that $O < A < B$ and $O < A < C$; then there exists a point D such that $D > \max\{B, C\}$.

Exercise ORD.8 Proof. We order the points on $\overleftrightarrow{OA}$ so that $O < A$.

(Case 1: A–B–C.) By property B.3 of Definition IB.1 there exists a point D such that B–C–D. By Theorem PSH.8(B) A–B–C–D. By Theorem ORD.6 $O < A < B < C < D$.

(Case 2: A–C–B.) By property B.3 there exists a point D such that C–B–D. By Theorem PSH.8(B) A–C–B–D. By Theorem ORD.6 $O < A < C < B < D$. $\square$

Exercise ORD.9* Let $\mathcal{P}$ be a Pasch plane, and let $\mathcal{L}$ and $\mathcal{L}'$ be distinct lines on $\mathcal{P}$, O be a member of $\mathcal{P} \setminus (\mathcal{L} \cup \mathcal{L}')$. Suppose further that a line through O intersects $\mathcal{L}$ iff it intersects $\mathcal{L}'$, and that each of the intersections of every such line with $\mathcal{L}$ or $\mathcal{L}'$ is a singleton.

Let A and B be distinct points on $\mathcal{L}$, A' be the point such that $\overleftrightarrow{OA} \cap \mathcal{L}' = \{A'\}$ and B' be the point such that $\overleftrightarrow{OB} \cap \mathcal{L}' = \{B'\}$. Order the points on $\mathcal{L}$ so that $A < B$, and order the points on $\mathcal{L}'$ so that $A' < B'$.

For every $X \in \mathcal{L}$ let $\varphi(X)$ be the point on $\mathcal{L}'$ such that $\overleftrightarrow{OX} \cap \mathcal{L}' = \{\varphi(X)\}$.

(A) φ is a bijection of $\mathcal{L}$ onto $\mathcal{L}'$.

Let X, Y, and Z be any distinct points on $\mathcal{L}$.

(B) X–Y–Z iff $\varphi(X)$–$\varphi(Y)$–$\varphi(Z)$.

(C) $\varphi(\overline{XY}) = \overline{\varphi(X)\varphi(Y)}$.

(D) $\varphi(\overrightarrow{XY}) = \overrightarrow{\varphi(X)\varphi(Y)}$.

(E) $\varphi(\mathcal{L}) = \varphi(\overleftrightarrow{XY}) = \overleftrightarrow{\varphi(X)\varphi(Y)} = \mathcal{L}'$.

(F) If $X < Y$, then $\varphi(X) < \varphi(Y)$.

Exercise ORD.9 Proof. (A) The mapping φ is onto because every line through O intersecting $\mathcal{L}$ also intersects $\mathcal{L}'$. If X and Y are members of $\mathcal{L}$, and $\varphi(X) = \varphi(Y)$ then $\overleftrightarrow{OX} = \overleftrightarrow{OY}$ and $X = Y$ because each line through O that intersects $\mathcal{L}'$ has only one point of intersection with $\mathcal{L}$. Therefore φ is one-to-one, hence is a bijection.

(B) Suppose X–Y–Z; since $\varphi(X) = \overleftrightarrow{XO} \cap \mathcal{L}'$, $\varphi(Y) = \overleftrightarrow{YO} \cap \mathcal{L}'$, and $\varphi(Z) = \overleftrightarrow{ZO} \cap \mathcal{L}'$, by Exercise PSH.53, $\varphi(X)$–$\varphi(Y)$–$\varphi(Z)$. To prove the converse, interchange $\mathcal{L}$ with $\mathcal{L}'$, φ with φ^{-1}, X with $\varphi(X)$, Y with $\varphi(Y)$, and Z with $\varphi(Z)$.

(C) Let X and Y be distinct points on $\mathcal{L}$. By Definition IB.4 $\overline{XY} = \{T \mid T = X \text{ or } X\text{–}T\text{–}Y \text{ or } T = Y\}$. By part (B) $X\text{–}T\text{–}Y$ iff $\varphi(X)\text{–}\varphi(T)\text{–}\varphi(Y)$ so $\varphi(\overline{XY})$ consists of exactly the points $\varphi(X)$, $\varphi(Y)$, and all the points $\varphi(T)$ where $\varphi(X)\text{–}\varphi(T)\text{–}\varphi(Y)$. Therefore $\varphi(\overline{XY}) = \overline{\varphi(X)\varphi(Y)}$.

(D) Let X and Y be distinct points on $\mathcal{L}$. By Definition IB.4 $\overrightarrow{XY} = \{T \mid T = X \text{ or } X\text{–}T\text{–}Y \text{ or } T = Y \text{ or } X\text{–}Y\text{–}T\ \}$. By part (B) $X\text{–}T\text{–}Y$ iff $\varphi(X)\text{–}\varphi(T)\text{–}\varphi(Y)$ and $X\text{–}Y\text{–}T$ iff $\varphi(X)\text{–}\varphi(Y)\text{–}\varphi(T)$ so $\varphi(\overline{XY})$ consists of exactly the points $\varphi(X)$, $\varphi(Y)$, and all the points $\varphi(T)$ where $\varphi(X)\text{–}\varphi(T\text{–}\varphi(Y)$ together with all the points $\varphi(T)$ where $\varphi(X)\text{–}\varphi(Y)\text{–}\varphi(T)$. Therefore $\varphi(\overrightarrow{XY}) = \overrightarrow{\varphi(X)\varphi(Y)}$.

(E) By part (A) φ maps $\overleftrightarrow{AB} = \mathcal{L}$ onto $\overleftrightarrow{A'B'} = \mathcal{L}'$.

(F) By Definition ORD.1 $X < Y$ iff $\overrightarrow{XY} \cap \overrightarrow{AB}$ is a ray. By the fact that φ is a bijection and elementary set theory, together with part (D) $\varphi(\overrightarrow{XY} \cap \overrightarrow{AB}) = \overrightarrow{\varphi(X)\varphi(Y)} \cap \overrightarrow{\varphi(A)\varphi(B)}$. Since $\overrightarrow{XY} \cap \overrightarrow{AB}$ is a ray, by part (D) $\overrightarrow{\varphi(X)\varphi(Y)} \cap \overrightarrow{\varphi(A)\varphi(B)}$ is a ray. By Definition ORD.1 $\varphi(X) < \varphi(Y)$. $\square$

Chapter 7—has no Exercises (COBE)

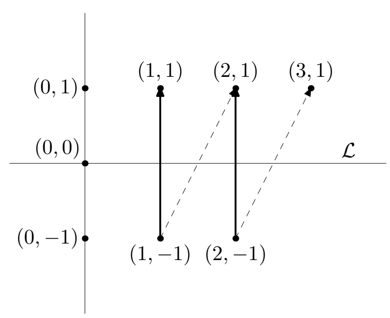
Chapter 8: Exercises and Answers for Neutral Geometry (NEUT)

Strictly speaking, Exercise NEUT.0 is out of place in Chapter 8, because it refers to the coordinate plane, and the development to this point does not show that the incidence, betweenness, and Plane Separation axioms hold on the coordinate plane. This is done in Chapter 21.

Exercise NEUT.0* There can be more than one mirror mapping over a line in the (real) coordinate plane $\mathbb{R}^2$. More specifically, if for each pair (u_1, u_2) of real numbers on the plane, we define $\Phi(u_1, u_2) = (u_1, -u_2)$ and $\Psi(u_1, u_2) = (u_1 - u_2, -u_2)$, both Φ and Ψ are mirror mappings over the x-axis.

Exercise NEUT.0 Proof. Refer to the figure below. It is quite easy to see that Φ is a mirror mapping over $\mathcal{L}$. We give a detailed proof that Ψ is a mirror mapping over the line $\mathcal{L}$.

(A) If $(u_1, u_2) \in \mathcal{L}$ then $u_2 = 0$ and $\Psi(u_1, 0) = (u_1 - 0, 0) = (u_1, 0)$. Thus Ψ satisfies Property (A) of Definition NEUT.1.



Showing action of the mirror mappings Φ (solid arrows) and Ψ (dashed arrows).

(B) $\Psi(u_1, u_2) = (u_1 - u_2, -u_2)$ is on the opposite side of $\mathcal{L}$ from (u_1, u_2) because the midpoint of the segment $\overline{(u_1, u_2)\Psi(u_1, u_2)}$ is $(\frac{u_1 - u_2 + u_1}{2}, \frac{u_2 - u_2}{2}) = (u_1 - \frac{u_2}{2}, 0) \in \mathcal{L}$. Thus Ψ satisfies Property (B) of Definition NEUT.1.

(C) $\Psi(\Psi(u_1, u_2)) = \Psi(u_1 - u_2, -u_2) = (u_1 - u_2 - (-u_2), -(-u_2)) = (u_1, u_2)$. Thus Ψ satisfies Property (C) of Definition NEUT.1.

(D) Let (u_1, u_2) and (v_1, v_2) be points of $\mathbb{R}^2$, and let (x_1, x_2) be a point between them so that (u_1, u_2)–(x_1, x_2)–(v_1, v_2). There exists a real number t such that $0 < t < 1$ such that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 + t(v_1 - u_1) \\ u_2 + t(v_2 - u_2) \end{pmatrix}.$$

Then

$$\begin{aligned} \Psi \begin{pmatrix} u_1 + t(v_1 - u_1) \\ u_2 + t(v_2 - u_2) \end{pmatrix} &= \begin{pmatrix} u_1 + t(v_1 - u_1) - (u_2 + t(v_2 - u_2)) \\ -u_2 - t(v_2 - u_2) \end{pmatrix} \\ &= \begin{pmatrix} (u_1 - u_2) + t(v_1 - v_2 - u_1 + u_2) \\ -u_2 - t(v_2 - u_2) \end{pmatrix} \\ &= \begin{pmatrix} u_1 - u_2 \\ -u_2 \end{pmatrix} + t\left(\begin{pmatrix} v_1 - v_2 \\ -v_2 \end{pmatrix} - \begin{pmatrix} u_1 - u_2 \\ -u_2 \end{pmatrix} \right) \\ &= \Psi(u_1, u_2) + t(\Psi(v_1, v_2) - \Psi(u_1, u_2)), \end{aligned}$$

so that $\Psi(u_1, u_2)$–$\Psi(x_1, x_2)$–$\Psi(v_1, v_2)$. Thus Ψ satisfies Property (D) of Definition NEUT.1, and Ψ is a mirror mapping over $\mathcal{L}$. $\square$

From this point on, the symbol $\mathcal{R}_{\mathcal{L}}$ will denote a reflection over the line $\mathcal{L}$ as defined in Definitions NEUT.1 and NEUT.2.

Exercise NEUT.1* Let $\mathcal{P}$ be a neutral plane and let $\mathcal{L}$ and $\mathcal{M}$ be parallel lines on $\mathcal{P}$, then $\mathcal{R}_{\mathcal{L}}(\mathcal{M})$ is a line which is contained in the side of $\mathcal{L}$ opposite the side containing $\mathcal{M}$ and $\mathcal{M} \parallel \mathcal{R}_{\mathcal{L}}(\mathcal{M})$.

Exercise NEUT.1 Proof. By Theorem NEUT.15, $\mathcal{R}_{\mathcal{L}}(\mathcal{M})$ is a line. By Theorem IB.10 $\mathcal{M}$ is a convex subset of $\mathcal{P}$; by Exercise PSH.14, $\mathcal{M}$ is a subset of a side $\mathcal{H}$ of $\mathcal{L}$. By Definition NEUT.1(B), $\mathcal{R}_{\mathcal{L}}(\mathcal{M})$ is a subset of the side $\mathcal{K}$ of $\mathcal{L}$ opposite $\mathcal{H}$. Hence $\mathcal{R}_{\mathcal{L}}(\mathcal{M}) \cap \mathcal{L} = \emptyset$ and by Definition IP.1, $\mathcal{R}_{\mathcal{L}}(\mathcal{M}) \parallel \mathcal{L}$. $\square$

Exercise NEUT.2* Let $\mathcal{M}$ be any line on the neutral plane $\mathcal{P}$. If X is any point on $\mathcal{P}$ such that $\mathcal{R}_{\mathcal{M}}(X) = X$, then $X \in \mathcal{M}$.

Exercise NEUT.2 Proof. If X were a point off of $\mathcal{M}$, then by Definition NEUT.1(B), X and $\mathcal{R}_{\mathcal{M}}(X)$ would be on opposite sides of $\mathcal{M}$ and thus $\mathcal{R}_{\mathcal{M}}(X)$ would not be equal to X. $\square$

Exercise NEUT.3* Let $\mathcal{P}$ be a neutral plane and let $\mathcal{L}$ and $\mathcal{M}$ be lines on $\mathcal{P}$. If $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{M}}$, then $\mathcal{L} = \mathcal{M}$. This may be restated in its contrapositive form as follows: If $\mathcal{L} \neq \mathcal{M}$, then $\mathcal{R}_{\mathcal{L}} \neq \mathcal{R}_{\mathcal{M}}$.

Exercise NEUT.3 Proof. Let X be any member of $\mathcal{L}$. By Definition NEUT.1(A), $\mathcal{R}_{\mathcal{L}}(X) = X$. Since $\mathcal{R}_{\mathcal{M}}(X) = \mathcal{R}_{\mathcal{L}}(X) = X$, by Exercise NEUT.2 $X \in \mathcal{M}$, so that $\mathcal{L} \subseteq \mathcal{M}$. By reversing the roles of $\mathcal{L}$ and $\mathcal{M}$ in this reasoning we get $\mathcal{M} \subseteq \mathcal{L}$ so $\mathcal{L} = \mathcal{M}$. $\square$

Exercise NEUT.4* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$, then neither $\overleftrightarrow{AB}$ nor $\overleftrightarrow{AC}$ is a line of symmetry of $\angle BAC$.

Exercise NEUT.4 Proof. (I) By Theorem NEUT.15,

$$\mathcal{R}_{\overleftrightarrow{AB}}(\overrightarrow{AC}) = \overrightarrow{\mathcal{R}_{\overleftrightarrow{AB}}(A)\mathcal{R}_{\overleftrightarrow{AB}}(C)} = \overrightarrow{A\mathcal{R}_{\overleftrightarrow{AB}}(C)}.$$

By Definition NEUT.1(B), C and $\mathcal{R}_{\overleftrightarrow{AB}}(C)$ are on opposite sides of $\overleftrightarrow{AB}$. By Theorem IB.14 $\overrightarrow{AC} \subseteq \overrightarrow{ABC}$ and $\overrightarrow{A\mathcal{R}_{\overleftrightarrow{AB}}(C)}$ is a subset of the $\mathcal{R}_{\overleftrightarrow{AB}}(C)$-side of $\overleftrightarrow{AB}$. By Theorem PSH.12 (plane separation) $\overrightarrow{AC}$ and $\overrightarrow{A\mathcal{R}_{\overleftrightarrow{AB}}(C)}$ are disjoint. Hence $\mathcal{R}_{\overleftrightarrow{AB}}(\angle BAC) \neq \angle BAC$ so that by Definition NEUT.4 $\overleftrightarrow{AB}$ is not a line of symmetry of $\angle BAC$.

(II) By interchanging "B" and "C" in the reasoning in (I), we get that $\overleftrightarrow{AC}$ is not a line of symmetry of $\angle BAC$. $\square$

Exercise NEUT.5* Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}$ which has a line $\mathcal{M}$ of symmetry, $\mathcal{H}_1$ and $\mathcal{H}_2$ be the sides of $\mathcal{M}$, $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{H}_1$ and $\mathcal{S}_2 = \mathcal{S} \cap \mathcal{H}_2$, then $\mathcal{R}_{\mathcal{M}}(\mathcal{S}_2) = \mathcal{S}_1$.

Exercise NEUT.5 Proof. (I) Let Y be any member of $\mathcal{R}_{\mathcal{M}}(\mathcal{S}_2)$, then there exists a member X of $\mathcal{S}_2$ such that $Y = \mathcal{R}_{\mathcal{M}}(X)$. By Definition NEUT.1(B) X and Y are on opposite sides of $\mathcal{M}$. Since $\mathcal{M}$ is a line of symmetry for $\mathcal{S}$, $Y = \mathcal{R}_{\mathcal{M}}(X) \in \mathcal{S}$, and hence $Y \in \mathcal{S}_1$. Since Y is arbitrary $\mathcal{R}_{\mathcal{M}}(\mathcal{S}_2) \subseteq \mathcal{S}_1$.

(II) Let X be any member of $\mathcal{S}_1$. By Definition NEUT.1(C) $X = \mathcal{R}_{\mathcal{M}}(\mathcal{R}_{\mathcal{M}}(X))$. By Definition NEUT.1(B) X and $\mathcal{R}_{\mathcal{M}}(X)$ are on opposite sides of $\mathcal{M}$. Thus $\mathcal{R}_{\mathcal{M}}(X)$ belongs to $\mathcal{S}_2$ and $X \in \mathcal{R}_{\mathcal{M}}(\mathcal{S}_2)$. Since X is arbitrary $\mathcal{S}_1 \subseteq \mathcal{R}_{\mathcal{M}}(\mathcal{S}_2)$.

By (I) and (II) $\mathcal{S}_1 = \mathcal{R}_{\mathcal{M}}(\mathcal{S}_2)$. $\square$

Exercise NEUT.6* (A) Let α be an isometry of the neutral plane $\mathcal{P}$ and let $\mathcal{L}$ be a line on $\mathcal{P}$ such that every point on $\mathcal{L}$ is a fixed point of α and no point off of $\mathcal{L}$ is a fixed point of α, then $\alpha = \mathcal{R}_{\mathcal{L}}$.

(B) Let α be an isometry of the neutral plane $\mathcal{P}$ which is also an axial affinity with axis $\mathcal{L}$. Then $\alpha = \mathcal{R}_{\mathcal{L}}$.

Exercise NEUT.6 Proof. (A) By Theorem NEUT.37 either $\alpha = \imath$ or $\alpha = \mathcal{R}_{\mathcal{L}}$. Since no point off of $\mathcal{L}$ is a fixed point of α, $\alpha \neq \imath$. Hence $\alpha = \mathcal{R}_{\mathcal{L}}$.

(B) Let A and B be distinct points of $\mathcal{L}$; by Definition CAP.25, these points are fixed points of α, and α is not the identity map $\imath$. By the contrapositive of Theorem NEUT.24, there can be no fixed point of α that is not on $\mathcal{L}$; by part (A), $\alpha = \mathcal{R}_{\mathcal{L}}$. $\square$

Exercise NEUT.7* Let $\mathcal{L}$ and $\mathcal{M}$ be distinct lines on the neutral plane $\mathcal{P}$, then $\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} \neq \imath$ (the identity mapping of $\mathcal{P}$ onto itself).

Exercise NEUT.7 Proof. If $\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}$ were equal to $\imath$, then $\mathcal{R}_{\mathcal{M}} = \mathcal{R}_{\mathcal{L}}^{-1} = \mathcal{R}_{\mathcal{L}}$ and by Exercise NEUT.3 $\mathcal{L}$ would be equal to $\mathcal{M}$, contrary to the fact that $\mathcal{L}$ and $\mathcal{M}$ are distinct. Hence $\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} \neq \imath$. $\square$

Exercise NEUT.8* If $\mathcal{L}$ and $\mathcal{M}$ are distinct lines on the neutral plane $\mathcal{P}$, then there exists a unique line $\mathcal{J}$ such that $\mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{J}}$. In fact, $\mathcal{J} = \mathcal{R}_{\mathcal{L}}(\mathcal{M})$.

Exercise NEUT.8 Proof. (I: Existence.) Let $\alpha = \mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}$. If α were equal to $\imath$ (the identity mapping of $\mathcal{P}$ onto itself), then $\mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}}$ would be equal to $\mathcal{R}_{\mathcal{L}}$, and $\mathcal{R}_{\mathcal{M}}$ would be equal to $\imath$, contrary to Definition NEUT.1(B)). Hence $\alpha \neq \imath$. Let X be any point on $\mathcal{R}_{\mathcal{L}}(M)$, then there exists a point Y on $\mathcal{M}$ such that $X = \mathcal{R}_{\mathcal{L}}(Y)$. By Definition NEUT.1(C) $Y = \mathcal{R}_{\mathcal{L}}(X)$. Thus $\alpha(X) = \mathcal{R}_{\mathcal{L}}(\mathcal{R}_{\mathcal{M}}(\mathcal{R}_{\mathcal{L}}(\mathcal{R}_{\mathcal{L}}(Y)))) = \mathcal{R}_{\mathcal{L}}(\mathcal{R}_{\mathcal{M}}(Y)) = \mathcal{R}_{\mathcal{L}}(Y) = X$. Hence every point of $\mathcal{R}_{\mathcal{L}}(M)$ is a fixed point of α. By Theorem NEUT.37 $\alpha = \mathcal{R}_{\mathcal{R}_{\mathcal{L}}(\mathcal{M})}$.

(II: Uniqueness.) If $\mathcal{K}$ is a line on $\mathcal{P}$ such that $\alpha = \mathcal{R}_{\mathcal{K}}$, then by Exercise NEUT.3, $\mathcal{K} = \mathcal{J}$. $\square$

Exercise NEUT.9* Let O, A, and B be noncollinear points on the neutral plane $\mathcal{P}$ and let $\mathcal{L}$ be a line such that $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OA}) = \overrightarrow{OB}$. By Remark NEUT.6(B), $\mathcal{L}$ is a line of symmetry of $\angle AOB$, $\mathcal{R}_{\mathcal{L}}$ is an angle reflection for $\angle AOB$, and by Theorem NEUT.20, $\mathcal{R}_{\mathcal{L}}(O) = O$. Construct a proof that $\mathcal{R}_{\mathcal{L}}(O) = O$, using Theorem NEUT.15, but not Theorem NEUT.20 or Theorem PSH.33 (uniqueness of corners).

Exercise NEUT.9 Proof. If $\mathcal{R}_{\mathcal{L}}(O) \neq O$, then for some $X \in \overrightarrow{OB}$, $\mathcal{R}_{\mathcal{L}}(O) = X$, and $\overleftrightarrow{OB} = \overleftrightarrow{OX}$. From Definition NEUT.1(C) $\mathcal{R}_{\mathcal{L}}(X) = \mathcal{R}_{\mathcal{L}}(\mathcal{R}_{\mathcal{L}}(O)) = O$ and $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OB}) = \mathcal{R}_{\mathcal{L}}(\mathcal{R}_{\mathcal{L}}(\overrightarrow{OA})) = \overrightarrow{OA}$. By Definition NEUT.1(D) $\mathcal{R}_{\mathcal{L}}$ is a belineation as well as a collineation. By Theorem COBE.3 or Theorem NEUT.15, $\mathcal{R}_{\mathcal{L}}(\overleftrightarrow{OX}) = \overleftrightarrow{\mathcal{R}_{\mathcal{L}}(O)\mathcal{R}_{\mathcal{L}}(X)} = \overleftrightarrow{XO} = \overleftrightarrow{OX}$, so that $\overleftrightarrow{OX} = \overleftrightarrow{OB}$ is a fixed line for $\mathcal{R}_{\mathcal{L}}$. Then since $\overrightarrow{OA} = \mathcal{R}_{\mathcal{L}}(\overrightarrow{OB}) \subseteq \overleftrightarrow{OB}$ which is a fixed line,

$A \in \overleftrightarrow{OB}$ and O, A, and B are collinear, contradicting our hypothesis. $\square$

Exercise NEUT.10* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$, B' and C' be points such that B–A–B', C–A–C', and $\mathcal{M}$ be a line of symmetry of $\angle BAC$, then $\mathcal{M}$ is a line of symmetry of $\angle B'AC'$.

Exercise NEUT.10 Proof. By Theorem NEUT.20, $A \in \mathcal{M}$, $\mathcal{R}_{\mathcal{M}}(\overrightarrow{AB}) = \overrightarrow{AC}$ and $\mathcal{R}_{\mathcal{M}}(\overrightarrow{AC}) = \overrightarrow{AB}$ so that $\mathcal{R}_{\mathcal{M}}(B) \in \overrightarrow{AC}$ and $\mathcal{R}_{\mathcal{M}}(C) \in \overrightarrow{AB}$. By Theorem NEUT.15 and Definition NEUT.1(A)

$$\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AB}) = \overleftrightarrow{\mathcal{R}_{\mathcal{M}}(A)\mathcal{R}_{\mathcal{M}}(B)} = \overleftrightarrow{AC}$$

and

$$\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AC}) = \overleftrightarrow{\mathcal{R}_{\mathcal{M}}(A)\mathcal{R}_{\mathcal{M}}(C)} = \overleftrightarrow{AB}.$$

By Theorem PSH.15 $\overleftrightarrow{AB}$ is the union of the disjoint sets $\overrightarrow{AB}$, $\{A\}$, and $\overrightarrow{AB'}$ and $\overleftrightarrow{AC}$ is the union of the disjoint sets $\overrightarrow{AC}$, $\{A\}$, and $\overrightarrow{AC'}$. By elementary set theory and the fact that $\mathcal{R}_{\mathcal{M}}$ is one-to-one, $\mathcal{R}_{\mathcal{M}}(\overrightarrow{AB'}) = \overrightarrow{AC'}$ and $\mathcal{R}_{\mathcal{M}}(\overrightarrow{AC'}) = \overrightarrow{AB'}$. By Definition NEUT.3(D) $\mathcal{M}$ is a line of symmetry of $\angle B'AC'$. $\square$

Exercise NEUT.11* Let O, P, and Q be noncollinear points on the neutral plane $\mathcal{P}$ such that $\overleftrightarrow{OP}$ is a line of symmetry of $\overleftrightarrow{OQ}$ and let Q' be a point such that Q'–O–Q. If we let $\mathcal{L} = \overleftrightarrow{OP}$, then $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OQ}) = \overrightarrow{OQ'}$ and $\mathcal{R}_{\mathcal{L}}(Q) \in \overrightarrow{OQ'}$.

Exercise NEUT.11 Proof. By set theory and Theorem NEUT.15, $\mathcal{R}_{\mathcal{L}}(\overleftrightarrow{OQ} \cap \overrightarrow{OPQ}) = (\overleftrightarrow{OQ} \cap \overrightarrow{OPQ'})$. By Theorem PSH.38 $\overleftrightarrow{OQ} \cap \overrightarrow{OPQ} = \overrightarrow{OQ}$ and $\overleftrightarrow{OQ} \cap \overrightarrow{QPQ'} = \overrightarrow{OQ'}$. By Definition NEUT.1(A) $\mathcal{R}_{\mathcal{L}}(O) = O$. Thus $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OQ}) = \overrightarrow{OQ'}$. By Theorem NEUT.15 and Definition NEUT.1(A) $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OQ}) = \overrightarrow{\mathcal{R}_{\mathcal{L}}(O)\mathcal{R}_{\mathcal{L}}(Q)} = \overrightarrow{O\mathcal{R}_{\mathcal{L}}(Q)}$. By Theorem PSH.24 $\mathcal{R}_{\mathcal{L}}(Q) \in \overrightarrow{OQ'}$. $\square$

Exercise NEUT.12* Let $\mathcal{P}$ be a neutral plane and let O, A, A', B, and B' be points such that: (1) A–O–A', (2) B and B' are on opposite sides of $\overleftrightarrow{OA}$, (so that $\{A, O, B\}$ and $\{A', O, B'\}$ are noncollinear), and (3) $\angle AOB \cong \angle A'OB'$. Then B–O–B'.

Exercise NEUT.12 Proof. By Property B.3 of Definition IB.1 there exists a point B'' such that B'–O–B'', and B' and B'' are on opposite sides of $\overleftrightarrow{OA}$; by Theorem PSH.12 (plane separation), B and B'' are on the same side of $\overleftrightarrow{OA}$.

By Theorem NEUT.42 (vertical angles) $\angle A'OB' \cong \angle AOB''$. By Theorem NEUT.14 (congruence is an equivalence relation), $\angle AOB \cong \angle A'OB' \cong$

$\angle AOB''$. By Theorem NEUT.36 $\overrightarrow{OB} = \overrightarrow{OB''}$. By Theorem PSH.24, $B \in \overrightarrow{OB''}$. In Theorem PSH.13, substitute B' for A, O for B, and B'' for C; by part (A), B–O–B'. $\square$

Exercise NEUT.13* Let A, B, C, D, A', B', C', and D' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, $D \in \operatorname{ins}\angle BAC$, $D' \in \overrightarrow{B'A'C'}$, $\angle BAC \cong \angle B'A'C'$, and $\angle BAD \cong \angle B'A'D'$; then $\overrightarrow{A'D'} \subseteq \operatorname{ins}\angle B'A'C'$.

Exercise NEUT.13 Proof. By Theorem NEUT.38 there exists an isometry α of $\mathcal{P}$ such that $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$, $\alpha(\overrightarrow{AC}) = \overrightarrow{A'C'}$, and $\alpha(\angle BAC) = \angle B'A'C'$. By Theorem NEUT.15 $\alpha(\overrightarrow{AB}) = \overrightarrow{\alpha(A)\alpha(B)}$. Since $\overrightarrow{\alpha(A)\alpha(B)} = \overrightarrow{A'B'}$, by Theorem PSH.24 $\alpha(A) = A'$ and $\alpha(B) \in \overrightarrow{A'B'}$. By Theorem NEUT.15 and Definition PSH.29,

$$\begin{aligned}\alpha(\angle BAD) &= \angle\alpha(B)\alpha(A)\alpha(D) = \overrightarrow{\alpha(A)\alpha(B)} \cup \overrightarrow{\alpha(A)\alpha(D)}\\ &= \overrightarrow{A'B'} \cup \overrightarrow{A'\alpha(D)} = \angle B'A'\alpha(D).\end{aligned}$$

By Definition NEUT.3(B) $\angle BAD \cong \angle B'A'\alpha(D)$. Since $\angle BAD \cong \angle B'AD'$, by Theorem NEUT.14 $\angle B'A'\alpha(D) \cong \angle B'AD'$. By Theorem NEUT.15 $\alpha(D) \in \alpha(\operatorname{ins}\angle BAC) = \operatorname{ins}\angle\alpha(B)\alpha(A)\alpha(C) = \operatorname{ins}\angle B'A'C'$. By Definition PSH.36 $\alpha(D) \in \overrightarrow{A'B'C'}$. By assumption, $D' \in \overrightarrow{B'A'C'}$. By Theorem NEUT.36 $\overrightarrow{A'\alpha(D)} = \overrightarrow{A'D'}$. By Theorem PSH.24 $D' \in \overrightarrow{A'\alpha(D)}$. By Exercise PSH.17 $\overrightarrow{A'\alpha(D)} \subseteq \operatorname{ins}\angle B'A'D'$. Hence $\overrightarrow{A'D'} \subseteq \operatorname{ins}\angle B'A'C'$. $\square$

Exercise NEUT.14* Let A, B, C, D, A', B', C', and D' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, $B \in \operatorname{ins}\angle CAD$ (so that by Corollary PSH.39.2 C and D are on opposite sides of $\overleftrightarrow{AB}$), $B' \in \overrightarrow{C'A'D'}$, $\angle CAB \cong \angle C'A'B'$, and $\angle CAD \cong \angle C'A'D'$. Then $B' \in \operatorname{ins}\angle C'A'D'$ (so that C' and D' are on opposite sides of $\overleftrightarrow{A'B'}$.

Exercise NEUT.14 Proof. In Exercise NEUT.13, replace D with B, B with C, C with D; also replace D' with B', B' with C', C' with D'; then $B' \in \operatorname{ins}\angle C'A'D'$. $\square$

Exercise NEUT.15* Let A and B be distinct points on the neutral plane $\mathcal{P}$, M be the midpoint of $\overline{AB}$, C and D be points on the same side of $\overleftrightarrow{AB}$ such that $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{BD} \perp \overleftrightarrow{AB}$ and $\mathcal{M}$ be the perpendicular bisector of $\overline{AB}$, then $\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AC}) = \overleftrightarrow{BD}$ and $\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{BD}) = \overleftrightarrow{AC}$.

Exercise NEUT.15 Proof. By Theorem NEUT.52 $\mathcal{R}_{\mathcal{M}}(A) = B$ and $\mathcal{R}_{\mathcal{M}}(B) = A$. By Theorem NEUT.44 $\angle MAC$ and $\angle MBD$ are right angles. By Theorem NEUT.69 $\angle MAC \cong \angle MBD$. By Theorem NEUT.15 and Definition NEUT.1(A) $\mathcal{R}_{\mathcal{M}}(\angle MAC) = \angle \mathcal{R}_{\mathcal{M}}(M)\mathcal{R}_{\mathcal{M}}(A)\mathcal{R}_{\mathcal{M}}(C) = \angle MB\mathcal{R}_{\mathcal{M}}(C)$. By Definition NEUT.3(B) $\angle MAC \cong \angle MB\mathcal{R}_{\mathcal{M}}(C)$. By Corollary NEUT.44.2 $\angle MB\mathcal{R}_{\mathcal{M}}(C)$ is a right angle. By Theorem NEUT.45 $\overleftrightarrow{B\mathcal{R}_{\mathcal{M}}(C)} \perp \overleftrightarrow{AB}$. Since $\overleftrightarrow{BD} \perp \overleftrightarrow{AB}$, by the uniqueness part of Theorem NEUT.48 $\overleftrightarrow{B\mathcal{R}_{\mathcal{M}}(C)} = \overleftrightarrow{BD}$. By Theorem NEUT.15

$$\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AC}) = \overleftrightarrow{\mathcal{R}_{\mathcal{M}}(A)\mathcal{R}_{\mathcal{M}}(C)} = \overleftrightarrow{B\mathcal{R}_{\mathcal{M}}(C)}.$$

Thus $\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AC}) = \overleftrightarrow{BD}$. By Definition NEUT.1(C)

$$\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{BD}) = \mathcal{R}_{\mathcal{M}}(\mathcal{R}_{\mathcal{M}}(\overleftrightarrow{AC})) = \overleftrightarrow{AC}$$

completing the proof. □

Exercise NEUT.16* Let O, P, Q, and R be points on the neutral plane $\mathcal{P}$ such that $\angle POQ$ is right, $\angle ROQ$ is right, and P and R are on opposite sides of $\overleftrightarrow{OQ}$, then P, O, and R are collinear.

Exercise NEUT.16 Proof. Using Property B.3 of Definition IB.1 let R' be a point such that P–O–R'. Since $\angle POQ$ is right, by Definition NEUT.41(C) $\angle POQ \cong \angle R'OQ$ and $\angle R'OQ$ is right. By Theorem NEUT.69 $\angle R'OQ \cong \angle ROQ$. By Definition IB.11 P and R are on opposite sides of $\overleftrightarrow{OQ}$. By Theorem PSH.12 (plane separation), R and R' are on the same side of $\overleftrightarrow{OQ}$. By Theorem NEUT.36, $\overrightarrow{OR} = \overrightarrow{OR'}$. By Theorem PSH.24 $R \in \overrightarrow{OR'} \subseteq \overleftrightarrow{PO}$. Thus P, O, and R are collinear. □

Exercise NEUT.17* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$. If $\angle ACB$ is right or is obtuse, then $\overline{AC} < \overline{AB}$ and $\overline{BC} < \overline{AB}$.

Exercise NEUT.17 Proof. By Theorem NEUT.84, $\angle BAC$ and $\angle ABC$ are acute angles; by Definition NEUT.81 and transitivity for angles, these angles are both smaller than a right angle, hence smaller than $\angle ACB$. By Theorem NEUT.91, $\overline{BC} < \overline{AB}$ and $\overline{AC} < \overline{AB}$. □

Exercise NEUT.18* Let O, P, and S be noncollinear points on the neutral plane $\mathcal{P}$ such that $\angle POS$ is acute, U be a member of $\overrightarrow{OP}$, and $V = \mathrm{ftpr}(U, \overleftrightarrow{OS})$, then $V \in \overrightarrow{OS}$.

Exercise NEUT.18 Proof. Using Property B.3 of Definition IB.1 let S' be a point such that S'–O–S. By Theorem NEUT.82 $\angle S'OP$ is obtuse. By

Theorem NEUT.44 $\angle OVU$ is right. If V were to belong to $\overrightarrow{OS'}$, then $\triangle OVU$ would have a right angle and an obtuse angle. By Theorem NEUT.84, this is impossible. Hence $V \in \overrightarrow{OS}$. $\square$

Exercise NEUT.19* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$; by Definition NEUT.2 (property R.5) there exists an angle reflection $\mathcal{R}_{\mathcal{M}}$ for $\angle BAC$, and by Theorem NEUT.20(E) a point $P \in \mathcal{M}$ such that $\overrightarrow{AP} \subseteq \text{ins}\,\angle BAC$. By Definition NEUT.3(D) $\overrightarrow{AP}$ is a bisecting ray for $\angle BAC$. Show that $\angle BAP$ is acute.

Exercise NEUT.19 Proof. Using Corollary NEUT.46.1 let $\mathcal{L}$ be the line such that $A \in \mathcal{L}$ and $\mathcal{L} \perp \overleftrightarrow{AP}$. Using Exercise PSH.0 let Q be a point on $\mathcal{L}$ which is on the the B-side of $\overleftrightarrow{AP}$ and let R be a point on $\mathcal{L}$ which is on the C-side of $\overleftrightarrow{AP}$. If $\angle BAP$ were right, then by Definition NEUT.41(C) $\angle CAP$ would be a right angle and by Exercise NEUT.16, B, A, and C would be collinear, contrary to the given fact that B, A, and C are noncollinear. Hence $\angle BAP$ is not a right angle. If $\angle BAP$ were obtuse, then by Definitions NEUT.70 and NEUT.81 Q would belong to $\text{ins}\,\angle BAP$. By Corollary PSH.39.2 B and P would be on opposite sides of $\overleftrightarrow{AQ}$. Similar reasoning shows that C and P would be on opposite sides of $\overleftrightarrow{AQ} = \overleftrightarrow{AR}$. Since B and C would both be on the side $\mathcal{E}$ of $\overleftrightarrow{AQ}$ opposite the P-side, by Theorem PSH.42 $\text{ins}\,\angle BAC$ would be a subset of the side of $\overleftrightarrow{AQ}$ opposite to the P-side . This would contradict the fact that $P \in \text{ins}\,\angle BAC$. Hence $\angle BAP$ is not obtuse. By Theorem NEUT.75 (trichotomy for angles) and Definition NEUT.81 $\angle CAP$ is acute. $\square$

Exercise NEUT.20* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$. If $\angle BAC$ and $\angle ABC$ are both acute, and if $D = \text{ftpr}(C, \overleftrightarrow{AB})$, then $D \in \overline{AB}$.

Exercise NEUT.20 Proof. If D were equal to either A or B, then by Theorem NEUT.44, $\angle BAC$ or $\angle ABC$ would be right. Both of these are impossible by assumption. By Property B.2 of Definition IB.1 there are exactly three mutually exclusive possibilities: (1) B–A–D, (2) A–B–D, and (3) A–D–B. If A were between B and D, then by Theorem NEUT.80 (outside angles) applied to $\triangle ACD$, $\angle BAC$ would be obtuse contrary to the given fact that $\angle BAC$ is acute. Hence A is not between B and D. If B were between A and D then by interchanging "A" and "B" in the above reasoning we would get that $\angle ABC$ is obtuse. Since B–A–D and A–B–D are both false, A–D–B. $\square$

Exercise NEUT.21* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$, if $\overline{AB}$ is the maximal edge of $\triangle ABC$ and if $D = \text{ftpr}(C, \overleftrightarrow{AB})$, then $D \in \overline{AB}$.

Exercise NEUT.21 Proof. If $\angle BAC$ were not acute, then by Definition NEUT.81 it would either be right or obtuse and by Exercise NEUT.17 $\overline{BC}$ would be larger than $\overline{AB}$ contrary to the given fact that $\overline{AB} \geq \overline{BC}$. Hence $\angle BAC$ is acute. If $\angle ABC$ were not acute, then by interchanging "A" and "B" in the reasoning above we would get that $\overline{AC}$ is larger than $\overline{AB}$ contrary to the given fact that $\overline{AB} \geq \overline{AC}$. Hence $\angle ABC$ is acute. By Exercise NEUT.20 $D \in \overline{AB}$. $\square$

Exercise NEUT.22* Let $\mathcal{L}$ be a line on the neutral plane $\mathcal{P}$ and let P be a point such that $P \notin \mathcal{L}$.

(I) Let $Q = \text{ftpr}(P, \mathcal{L})$; if X is any point on $\mathcal{L}$ distinct from Q, then $\overline{PQ} < \overline{PX}$.

(II) If Q is a point on $\mathcal{L}$ with the property that for every point X on $\mathcal{L}$ which is distinct from Q, $\overline{PQ} < \overline{PX}$, then $Q = \text{ftpr}(P, \mathcal{L})$.

Exercise NEUT.22 Proof. (I) By Theorem NEUT.44 $\angle PQX$ is right. By Exercise NEUT.17 $\overline{PQ} < \overline{PX}$.

(II) Assume $\angle PQX$ is not right, then by Theorem NEUT.44, $\overleftrightarrow{PQ}$ and $\mathcal{L}$ are not perpendicular to each other. Using Theorem NEUT.48(A), let $\mathcal{M}$ be the line such that $P \in \mathcal{M}$ and $M \perp \mathcal{L}$. By Theorem NEUT.44 there exists a point R such that $\mathcal{M} \cap \mathcal{L} = \{R\}$, and by the same theorem $\angle PRQ$ is a right angle. By Part I above, $\overline{PR} < \overline{PQ}$, *i.e.*, $\overline{PQ} > \overline{PR}$. This contradicts the given fact that for every point X on $\mathcal{L}$ distinct from Q, $\overline{PQ} < \overline{PX}$. Hence the assumption that $\angle PQX$ is not right is false, so $\angle PQX$ is right. $\square$

Exercise NEUT.23* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$, P be a member of ins$\,\angle BAC$, $Q = \text{ftpr}(P, \overleftrightarrow{AB})$, and $R = \text{ftpr}(P, \overleftrightarrow{AC})$.

(1) If $\overrightarrow{AP}$ is the bisecting ray of $\angle BAC$, then $\overline{PQ} \cong \overline{PR}$.

(2) If $Q \in \overrightarrow{AB}$, $R \in \overrightarrow{AC}$, and $\overline{PQ} \cong \overline{PR}$, then $\overrightarrow{AP}$ is the bisecting ray of $\angle BAC$.

Exercise NEUT.23 Proof. We will use Theorem NEUT.15 several times without further reference. (1) If $\overrightarrow{AP}$ is the bisecting ray, then $\mathcal{R}_{\overleftrightarrow{AP}}(\overleftrightarrow{AB}) =$

$\overleftrightarrow{AC}$ by Theorem NEUT.39. Then $\mathcal{R}_{\overleftrightarrow{AP}}(Q) \in \overleftrightarrow{AC}$ and $\mathcal{R}_{\overleftrightarrow{AP}}(\angle AQP) = \angle(\mathcal{R}_{\overleftrightarrow{AP}}(A))(\mathcal{R}_{\overleftrightarrow{AP}}(Q))(\mathcal{R}_{\overleftrightarrow{AP}}(P)) = \angle A(\mathcal{R}_{\overleftrightarrow{AP}}(Q))P$ which is a right angle by Corollary NEUT.44.2.

Therefore by Theorem NEUT.47(B), $\overleftrightarrow{PR} = \overleftrightarrow{P(\mathcal{R}_{\overleftrightarrow{AP}}(Q))}$ so that $\mathcal{R}_{\overleftrightarrow{AP}}(Q) = R$. Thus $\mathcal{R}_{\overleftrightarrow{AP}}(\overline{PQ}) = \overline{(\mathcal{R}_{\overleftrightarrow{AP}}(P))(\mathcal{R}_{\overleftrightarrow{AP}}(Q))} = \overline{PR}$ and $\overline{PQ} \cong \overline{PR}$.

(2) Since $\overline{AP} \cong \overline{AP}$ and $\overline{PQ} \cong \overline{PR}$ and both $\angle PQA$ and $\angle PRA$ are right, by Theorem NEUT.96 (hypotenuse-leg) $\angle PAQ \cong \angle PAR$. By Theorem NEUT.39 $\overrightarrow{AP}$ is the bisecting ray of $\angle BAC$. $\square$

Exercise NEUT.24* Let $\mathcal{P}$ be a neutral plane and let A, B, C, and D be points on $\mathcal{P}$ such that $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ is a quadrilateral, and suppose that $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Then

(1) $\square ABCD$ is rotund;

(2) $\overline{BC} \cong \overline{AD}$ iff $\angle ADC \cong \angle BCD$; and

(3) $\overline{BC} < \overline{AD}$ iff $\angle ADC < \angle BCD$.

Exercise NEUT.24 Proof. (1) By Theorem NEUT.47 $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Thus by Theorem IB.9 and Exercise PSH.14 every point of $\overleftrightarrow{AD}$ is on the A-side of $\overleftrightarrow{BC}$ and every point of $\overleftrightarrow{BC}$ is on the B-side of $\overleftrightarrow{AD}$. By Theorem IB.14 $\overrightarrow{CD} \subseteq \overrightarrow{BCA}$ and $\overrightarrow{DC} \subseteq \overrightarrow{ADB}$ so that by Exercise IB.8 $\overrightarrow{CD} \subseteq \overrightarrow{BCA} \cap \overrightarrow{ADB}$.

Then if D and C are on opposite sides of $\overleftrightarrow{AB}$, by Theorem PSH.11 (PSA) $\overrightarrow{CD} \cap \overleftrightarrow{AB} \neq \emptyset$ and this point of intersection must lie between A and B, that is, $\overrightarrow{CD} \cap \overline{AB} \neq \emptyset$ which contradicts the assumption that $\square ABCD$ is a quadrilateral (*cf* Definition PSH.31).

Therefore both C and D are on the same side of $\overleftrightarrow{AB}$. By interchanging "A" with "C" and "B" with "D" in the above argument, we see that A and B must be on the same side of $\overleftrightarrow{CD}$. Since $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are parallel, A is on the D-side of $\overline{BC}$ and C is on the B-side of $\overline{AD}$, so that by Definition PSH.31 $\square ABCD$ is rotund.

Part (2) follows from (A) and (D) below; part (3) follows from (B) and (E) below:

(A) By Theorem NEUT.44 both $\angle BAD$ and $\angle ABC$ are right. By Theorem NEUT.69, $\angle BAD \cong \angle ABC$. If $\overline{AD} \cong \overline{BC}$, then by Theorem NEUT.64 (EAE) applied to $\triangle BAD$ and $\triangle ABC$ $\overline{AC} \cong \overline{BD}$. Hence by Theorem NEUT.62 (EEE) applied to $\triangle ADC$ and $\triangle BCD$ $\angle ADC \cong \angle BCD$.

(B) If $\overline{BC} < \overline{AD}$, then by Definition NEUT.70 there exists a point P such that A–P–D and $\overline{BC} \cong \overline{AP}$. By part (A) $\angle APC \cong \angle BCP$. By Theorem NEUT.80 (outside angles) $\angle ADC < \angle APC$. By Theorem NEUT.76 (transitivity for angles) $\angle ADC < \angle BCD$.

(C) If $\overline{BC} > \overline{AD}$, then by Definition NEUT.70 $\overline{AD} < \overline{BC}$. Interchanging A and B, and interchanging C and D in part (B), $\angle BCD < \angle ADC$.

(D) Suppose $\angle ADC \cong \angle BCD$. By Theorem NEUT.72 (trichotomy for segments) one and only one of the following statements holds: $\overline{AD} \cong \overline{BC}$, $\overline{AD} < \overline{BC}$, or $\overline{AD} > \overline{BC}$. If $\overline{AD} < \overline{BC}$, then by part (C) $\angle BCD < \angle ADC$ contrary to the given fact that $\angle ADC \cong \angle BCD$. If $\overline{AD} > \overline{BC}$, then by part (B) $\angle BCD > \angle ADC$ contrary to the given fact that $\angle ADC \cong \angle BCD$. Hence $\overline{AD} \cong \overline{BC}$.

(E) If $\angle ADC < \angle BCD$, again, as in part (D) we use trichotomy for segments. $\overline{AD} \cong \overline{BC}$ is ruled out by part (A), and $\overline{AD} < \overline{BC}$ is ruled out by part (C). Therefore $\overline{BC} < \overline{AD}$. $\square$

Exercise NEUT.25* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, both $\angle ACB$ and $\angle A'C'B'$ are right, $\overline{BC} \cong \overline{B'C'}$ and $\overline{AC} < \overline{A'C'}$, then $\angle ABC < \angle A'B'C'$, $\overline{AB} < \overline{A'B'}$, and $\angle B'A'C' < \angle BAC$.

Exercise NEUT.25 Proof. Using Theorem NEUT.67 (segment construction) let A'' be the point on $\overrightarrow{CA}$ such that $\overline{CA''} \cong \overline{C'A'}$.

By Theorem NEUT.73 (transitivity for segments) $\overline{CA} < \overline{CA''}$. In Theorem NEUT.74, substitute C for O, A for X, and A'' for Q to get C–A–A''. By Definition IB.3 $A \in \overline{CA''}$. By Theorem PSH.37 $A \in \text{ins}\,\angle CBA''$. By Theorem NEUT.69 $\angle A''CB \cong \angle A'C'B'$. By Theorem NEUT.64 (EAE) $\triangle A''BC \cong \triangle A'B'C'$ so that $\angle A''BC \cong \angle A'B'C'$, $\angle BA''C \cong \angle B'A'C'$ and $\overline{A''B} \cong \overline{A'B'}$.

By Definition NEUT.70 $\angle ABC < \angle A''BC$; by transitivity for angles (Theorem NEUT.76), $\angle ABC < \angle A'B'C'$.

Applying Theorem NEUT.80 (outside angles) to $\triangle ABA''$, $\angle BA''C < \angle BAC$, so by transitivity for angles $\angle B'A'C' < \angle BAC$.

By Theorem NEUT.93 $\overline{A''B} > \overline{BC}$; by Theorem NEUT.95 $\overline{AB} < \overline{A''B}$ so by transitivity for segments (Theorem NEUT.73) $\overline{AB} < \overline{A'B'}$. $\square$

Exercise NEUT.26* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, $\angle ACB$ and $\angle A'C'B'$ are both right, $\overline{BC} < \overline{B'C'}$ and $\overline{AC} > \overline{A'C'}$, then $\angle ABC > \angle A'B'C'$ and $\angle BAC < \angle B'A'C'$.

Exercise NEUT.26 Proof. Using Theorem NEUT.67 (Segment Construction) let A'' be the member of $\overrightarrow{CA}$ such that $\overline{CA''} \cong \overline{C'A'}$ and let B'' be the member of $\overrightarrow{C'B'}$ such that $\overline{CB''} \cong \overline{C'B'}$. By Theorem NEUT.64 (EAE) $\triangle A''B''C \cong \triangle A'B'C'$. By Theorem NEUT.73 (transitivity for segments) $\overline{A''C} < \overline{AC}$ and $\overline{BC} < \overline{B''C}$. Again, substituting appropriately in Theorem NEUT.74, we get A–A''–C and B''–B–C. By Theorem IB.14 $\overrightarrow{AC} \subseteq \overrightarrow{ABC}$ so that $A'' \in \overrightarrow{ABC}$. By Definition IB.11 B'' and C are on opposite sides of $\overleftrightarrow{AB}$. By Theorem PSH.12 (plane separation) B'' and A'' are on opposite sides of $\overleftrightarrow{AB}$. By Axiom PSA $\overline{A''B''}$ and $\overleftrightarrow{AB}$ intersect at a point D. Similar reasoning shows A and B are on opposite sides of $\overleftrightarrow{A''B''}$. By Axiom PSA $\overline{AB}$ and $\overleftrightarrow{A''B''}$ intersect at a point D'. By Corollary IB.5.2 $\overline{AB} \subseteq \overleftrightarrow{AB}$ and $\overline{A''B''} \subseteq \overleftrightarrow{A''B''}$. Hence by Exercise I.1 $D' = D$. By Theorem NEUT.80 (outside angles) applied to $\triangle ADA''$ we get $\angle BAC < \angle B''A''C$. By the same theorem applied to $\triangle BDB''$ we get $\angle ABC > \angle A''B''C$. By Theorem NEUT.76 (transitivity for angles) $\angle ABC > \angle A'B'C'$ and $\angle BAC < \angle B'A'C'$. $\square$

Exercise NEUT.27* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, both $\angle ACB$ and $\angle A'C'B'$ are right, $\overline{BC} < \overline{B'C'}$, and $\overline{AC} < \overline{A'C'}$, then $\overline{AB} < \overline{A'B'}$.

Exercise NEUT.27 Proof. Using Theorem NEUT.67 (segment construction) let B'' be the point on $\overrightarrow{CB}$ such that $\overline{B''C} \cong \overline{B'C'}$. Applying Exercise NEUT.25 to $\triangle ABC$ and $\triangle AB''C$, since $\overline{AC} \cong \overline{AC}$ and $\overline{BC} < \overline{B''C}$, we get $\overline{AB} < \overline{AB''}$. Applying Exercise NEUT.25 to $\triangle AB''C$ and $\triangle A'B'C'$, since $\overline{B''C} \cong \overline{B'C'}$ and $\overline{AC} < \overline{A'C'}$, we get $\overline{AB''} < \overline{A'B'}$. By Theorem NEUT.73 (transitivity for segments) $\overline{AB} < \overline{AB''} < \overline{A'B'}$. $\square$

Exercise NEUT.28* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, both $\angle ACB$ and $\angle A'C'B'$ are right, $\overline{BC} < \overline{B'C'}$ and $\overline{AB} \cong \overline{A'B'}$, then $\overline{A'C'} < \overline{AC}$, $\angle BAC < \angle B'A'C'$ and $\angle ABC > \angle A'B'C'$.

Exercise NEUT.28 Proof. If $\overline{A'C'}$ were larger than or were congruent to $\overline{AC}$, then by Exercise NEUT.27 or Exercise NEUT.25 $\overline{A'B'}$ would be larger than $\overline{AB}$, contrary to the given fact that $\overline{AB} \cong \overline{A'B'}$. Hence by Theorem NEUT.72 (trichotomy for segments) $\overline{A'C'} < \overline{AC}$. By Exercise NEUT.26 $\angle BAC < \angle B'A'C'$ and $\angle ABC > \angle A'B'C'$. $\square$

Exercise NEUT.29* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that both $\angle ACB$ and $\angle A'C'B'$ are right, $\overline{AB} \cong \overline{A'B'}$ and $\angle A'B'C' < \angle ABC$, then $\overline{A'C'} < \overline{AC}$, $\overline{BC} < \overline{B'C'}$ and $\angle BAC < \angle B'A'C'$.

Exercise NEUT.29 Proof. Using Theorem NEUT.67 (segment construction) let C'' be the point on $\overrightarrow{B'C'}$ such that $\overline{B'C''} \cong \overline{BC}$. Using Theorem NEUT.48 let $\mathcal{L}$ be the line such that $C'' \in \mathcal{L}$ and $\mathcal{L} \perp \overleftrightarrow{B'C'}$. Using Exercise PSH.0 let T be a member of $\mathcal{L} \cap \overrightarrow{B'C'A'}$. Using Theorem NEUT.67 (segment construction) let A'' be the point on $\overrightarrow{C''T}$ such that $\overline{C''A''} \cong \overline{CA}$. By Theorem NEUT.44 $\angle A''C''B'$ is right. By Theorem NEUT.69 $\angle ACB \cong \angle A''C''B'$. By Theorem NEUT.64 (EAE) $\triangle B'A''C'' \cong \triangle BAC$ and thus $\angle B'A''C'' \cong \angle BAC$, $\angle A''B'C'' \cong ABC$, and $\overline{A''B'} \cong \overline{AB}$. By the assumption that $\angle A'B'C' < \angle ABC$ and Theorem NEUT.76 (transitivity for angles), $\angle A'B'C' < \angle A''B'C'' = \angle A''B'C'$.

By Theorem NEUT.78 $A' \in \text{ins}\angle A''B'C'$. By Theorem PSH.39 (Crossbar), $\overrightarrow{B'A'}$ and $\overrightarrow{A''C''}$ intersect at a point S such that A''–S–C'', that is, C''–S–A''. By Theorem NEUT.74 $\overline{C''S} < \overline{C''A''}$. By Theorem NEUT.93 and Theorem NEUT.95 (applied to $\triangle A''B'C''$), $\overline{B'S} < \overline{B'A''}$. By Theorem NEUT.73 (transitivity for segments), $\overline{B'S} < \overline{B'A'}$. By Theorem NEUT.74 B'–S–A'. By Definition IB.11 B' and A' are on opposite sides of $\overleftrightarrow{C''A''}$. By Theorem NEUT.44 $\overleftrightarrow{C'A'} \perp \overleftrightarrow{B'C'}$. By Theorem NEUT.47(A) $\overleftrightarrow{C'A'} \cap \overleftrightarrow{C''A''} = \emptyset$. By Theorem IB.10 and Exercise PSH.14 $\overleftrightarrow{C'A'} \subseteq \overrightarrow{C''A''A'}$, so that C' and A' are on the same side of $\overleftrightarrow{C''A''}$. By Theorem PSH.12 (plane separation) B' and C' are on opposite sides of $\overleftrightarrow{C''A''}$. By Axiom PSA, Exercise I.1, and Corollary IB.5.2 B'–C''–C'. By Theorem NEUT.74 $\overline{B'C''} < \overline{B'C'}$. By Theorem NEUT.73 (transitivity for segments) $\overline{BC} < \overline{B'C'}$. By Exercise NEUT.28 $\overline{A'C'} < \overline{AC}$, $\angle ABC > \angle A'B'C'$, and $\angle BAC < \angle B'A'C'$. $\square$

Exercise NEUT.30* Let A, B, C, A', B', and C' be points on the neutral plane $\mathcal{P}$ such that A, B, and C are noncollinear, A', B', and C' are noncollinear, both $\angle ACB$ and $\angle A'C'B'$ are right, $\overline{BC} \cong \overline{B'C'}$, and $\angle ABC < \angle A'B'C'$, then $\overline{AC} < \overline{A'C'}$, $\overline{AB} < \overline{A'B'}$, and $\angle B'A'C' < \angle BAC$.

Exercise NEUT.30 Proof. If $\overline{AC} \cong \overline{A'C'}$, by Theorem NEUT.64 (EAE) $\angle ABC \cong \angle A'B'C'$. By Theorem NEUT.75 (trichotomy for angles) this is contrary to the assumption that $\angle ABC < \angle A'B'C'$. Also, if $\overline{A'C'} < \overline{AC}$, it follows from Exercise NEUT.25 that $\angle A'B'C' < \angle ABC$, again contrary to this assumption. Hence by Theorem NEUT.72 (trichotomy for segments) $\overline{AC} < \overline{A'C'}$. Applying Exercise NEUT.25, we have $\angle ABC < \angle A'B'C'$,

$\overline{AB} < \overline{A'B'}$, and $\angle B'A'C' < \angle BAC$. $\square$

Exercise NEUT.31* Let P, O, and T be noncollinear points on the neutral plane $\mathcal{P}$, let S be a member of $\text{ins}\,\angle POT$ such that $\angle POS < \angle TOS$, and let M be a member of $\text{ins}\,\angle POT$ such that $\overrightarrow{OM}$ is the bisecting ray of $\angle POT$, then $M \in \text{ins}\,\angle TOS$.

Exercise NEUT.31 Proof. (I) If M were a member of $\overrightarrow{OS}$, then by Theorem PSH.16 $\overrightarrow{OM}$ would be equal to $\overrightarrow{OS}$ and by Definition PSH.29 $\angle POM$ would be equal to $\angle POS$. By Theorem NEUT.39 $\angle POS$ would be congruent to $\angle TOS$. By Theorem NEUT.75 (trichotomy for angles) this would contradict the given fact that $\angle POS < \angle TOS$. Hence $M \notin \overrightarrow{OS}$.

(II) If M were a member of $\text{ins}\,\angle POS$, by Definition NEUT.70 $\angle POM < \angle POS$. By Exercise PSH.13, $S \in \text{ins}\,\angle TOM$, and hence by Definition NEUT.70 $\angle TOS < \angle TOM$ By hypothesis $\angle POS < \angle TOS$. Putting this all together by Theorem NEUT.76 (transitivity for angles) we have $\angle POM < \angle POS < \angle TOS < \angle TOM$ which contradicts the given fact that $\angle POM \cong \angle TOM$. Hence $M \notin \text{ins}\,\angle POS$, and by part (I) $M \notin \overrightarrow{OS}$, so by Exercise PSH.18, $M \in \text{ins}\,\angle TOS$. $\square$

Exercise NEUT.32* Let P, O, and T be noncollinear points on the neutral plane $\mathcal{P}$, S and V be members of $\text{ins}\,\angle POT$ such that $\angle POS < \angle TOS$ and $\angle POV \cong \angle TOS$, and M be a member of $\text{ins}\,\angle POT$ such that $\overrightarrow{OM}$ is the bisecting ray of $\angle POT$. Then

(1) $S \in \text{ins}\,\angle POV$ and $V \in \text{ins}\,\angle TOS$,
(2) $\overrightarrow{OM}$ is the bisecting ray of $\angle SOV$,
(3) $\angle TOV \cong \angle POS$, and
(4) $M \in \text{ins}\,\angle TOS \cap \text{ins}\,\angle POV$.

Exercise NEUT.32 Proof. Since both S and V are members of $\text{ins}\,\angle POT$, by Definition PSH.36 $S \in \overrightarrow{OPT}$ and $V \in \overrightarrow{OPT}$. By Definition IB.11 $S \in \overrightarrow{OPV}$. Since $\angle POS < \angle TOS$ by Exercise NEUT.31 $M \in \text{ins}\,\angle TOS$.

By Theorem NEUT.39 $\mathcal{L} = \overleftrightarrow{OM}$ is the line of symmetry of $\angle POT$, so that $\mathcal{R}_{\mathcal{L}}(\overrightarrow{OT}) = \overrightarrow{OP}$. By Theorem NEUT.15

$$\mathcal{R}_{\mathcal{L}}(\angle TOS) = \angle \mathcal{R}_{\mathcal{L}}(T)\mathcal{R}_{\mathcal{L}}(O)\mathcal{R}_{\mathcal{L}}(S) = \angle PO\mathcal{R}_{\mathcal{L}}(S),$$

so that $\angle TOS \cong \angle PO\mathcal{R}_{\mathcal{L}}(S)$. By hypothesis, $\angle POV \cong \angle TOS$, so by Theorem NEUT.76 (transitivity for angles) $\angle POV \cong \angle PO\mathcal{R}_{\mathcal{L}}(S)$. Since $\mathcal{R}_{\mathcal{L}}(S)$ and V are on the same side of $\overleftrightarrow{OP}$, by Theorem NEUT.36, $\overrightarrow{OV} = \overrightarrow{O\mathcal{R}_{\mathcal{L}}(S)}$. This shows that $\mathcal{L}$ is the line of symmetry for $\angle SOV$, and hence $\overrightarrow{OM}$

is its bisecting ray, proving (2). Also $\mathcal{R}_{\mathcal{L}}(\angle POS) = \angle TOV$ and hence $\angle TOV \cong \angle POS$, proving (3).

To prove (1), note that $\angle POS < \angle TOS \cong \angle POV$ so that by Theorem NEUT.78, $S \in \operatorname{ins}\angle POV$. Since $\overrightarrow{O\mathcal{R}_{\mathcal{L}}(S)} = \overrightarrow{OV}$, $\overrightarrow{O\mathcal{R}_{\mathcal{L}}(T)} = \overrightarrow{OP}$, and $\overrightarrow{O\mathcal{R}_{\mathcal{L}}(P)} = \overrightarrow{OT}$, we have $\mathcal{R}_{\mathcal{L}}(\angle POS) = \angle\mathcal{R}_{\mathcal{L}}(P)\mathcal{R}_{\mathcal{L}}(O)\mathcal{R}_{\mathcal{L}}(S) = \angle TOV$ and $\mathcal{R}_{\mathcal{L}}(\angle TOS) = \angle POV$. Thus $\angle TOV \cong \angle POS < \angle POV \cong \angle TOS$ so that by Theorem NEUT.78, $V \in \operatorname{ins}\angle TOS$; this proves (1).

Since by hypothesis $\angle TOS \cong \angle POV$ and $\angle POS < \angle TOS$ we have $\angle TOV \cong \angle POS < \angle TOS \cong \angle POV$, or $\angle TOV < \angle POV$. Again by Exercise NEUT.31, $M \in \operatorname{ins}\angle POV$. Thus $M \in \operatorname{ins}\angle TOS \cap \operatorname{ins}\angle POV$, proving (4). $\square$

Exercise NEUT.33* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, M_1, A_2, B_2, and M_2 be points on $\mathcal{P}$ such that $A_1 \neq B_1$ and $A_2 \neq B_2$, M_1 is the midpoint of $\overline{A_1B_1}$ and M_2 is the midpoint of $\overline{A_2B_2}$, then $\overline{A_1B_1} \cong \overline{A_2B_2}$ iff $\overline{A_1M_1} \cong \overline{A_2M_2}$.

Exercise NEUT.33 Proof. (I: If $\overline{A_1B_1} \cong \overline{A_2B_2}$, then $\overline{A_1M_1} \cong \overline{A_2M_2}$.) Using Theorem NEUT.56 let α be an isometry of $\mathcal{P}$ such that $\alpha(\overline{A_1B_1}) = \overline{A_2B_2}$, $\alpha(A_1) = A_2$, and $\alpha(B_1) = B_2$. By Definition NEUT.3(C) $\overline{A_1M_1} \cong \overline{M_1B_1}$. By Theorem NEUT.13 $\alpha(\overline{A_1M_1}) \cong \alpha(\overline{M_1B_1})$.

By Theorem NEUT.15

$$\alpha(\overline{A_1M_1}) = \overline{\alpha(A_1)\alpha(M_1)} = \overline{A_2\alpha(M_1)} \text{ and}$$
$$\alpha(\overline{M_1B_1}) = \overline{\alpha(M_1)\alpha(B_1)} = \overline{\alpha(M_1)B_2}$$

so that $\overline{A_2\alpha(M_1)} \cong \overline{\alpha(M_1)B_1}$. By Definition NEUT.3(C) A_1–M_1–B_1.

By Definition NEUT.1(D) $\alpha(A_1)$–$\alpha(M_1)$–$\alpha(B_1)$, *i.e.,* A_2–$\alpha(M_1)$–B_2. By the uniqueness part of Theorem NEUT.50 $\alpha(M_1) = M_2$. Thus $\alpha(\overline{A_1M_1}) = \overline{A_2M_2}$. By Definition NEUT.3(B) $\overline{A_1M_1} \cong \overline{A_2M_2}$.

(II: If $\overline{A_1M_1} \cong \overline{A_2M_2}$), then $\overline{A_1B_1} \cong \overline{A_2B_2}$. Using Theorem NEUT.56 let γ be the isometry of $\mathcal{P}$ such that $\gamma(\overline{A_1M_1}) = \overline{A_2M_2}$, $\gamma(A_1) = A_2$, and $\gamma(M_1) = M_2$. By Definition NEUT.3(C) $\overline{A_1M_1} \cong \overline{M_1B_1}$. By Theorem NEUT.13 $\gamma(\overline{A_1M_1}) \cong \gamma(\overline{M_1B_1})$. By Theorem NEUT.15

$$\gamma(\overline{A_1M_1} = \overline{\gamma(A_1)\gamma(M_1)} = \overline{A_2M_2} \text{ and}$$
$$\gamma(\overline{M_1B_1}) = \overline{\gamma(M_1)\gamma(B_1)} = \overline{M_2\gamma(B_1)}.$$

By Theorem NEUT.14 and Definition NEUT.3(B) $\overline{A_2M_2} \cong \overline{M_2\gamma(B_1)}$.

By Definition NEUT.3(C) $\overline{A_2M_2} \cong \overline{M_2B_2}$. It follows from Theorem NEUT.14 that $\overline{M_2\gamma(B_1)} \cong \overline{M_2B_2}$. By Definition NEUT.3(C) A_1–M_1–B_1. By Definition NEUT.1(D) $\gamma(A_1)$–$\gamma(M_1)$–$\gamma(B_1)$, that is to say, A_2–M_2–$\gamma(B_1)$.

By Theorem PSH.13 $\gamma(B_1) \in \overrightarrow{M_2B_2}$. By Property R.4 of Definition NEUT.2, $\gamma(B_1) = B_2$. Thus by Theorem NEUT.15 $\gamma(\overline{A_1B_1}) = \overline{\gamma(A_1)\gamma(B_1)} = \overline{A_2B_2}$. By Definition NEUT.3(B) $\overline{A_1B_1} \cong \overline{A_2B_2}$. $\square$

Exercise NEUT.34* Let $\mathcal{P}$ be a neutral plane, O and P be distinct points on $\mathcal{P}$, let the points on $\overleftrightarrow{OP}$ be ordered so that $O < P$, and let A and B be distinct points on $\overrightarrow{OP}$. Let M be the midpoint of $\overline{OA}$ and N be the midpoint of $\overline{OB}$, then $A < B$ iff $M < N$.

Exercise NEUT.34 Proof. (I: If $M < N$, then $A < B$.). If $M < N$, then in Theorem NEUT.74 substitute M for X and N for Q to get $\overline{OM} < \overline{ON}$. By Definition NEUT.3(C) O–N–B; by Theorem ORD.6 O–M–N so that O–M–N–B, and hence B–N–M. Again in Theorem NEUT.74 substitute B for O, N for X, and M for Q to get $\overline{BN} < \overline{BM}$.

By Definition NEUT.3(C) $\overline{OM} \cong \overline{MA}$ and $\overline{ON} \cong \overline{NB}$. $\overline{MA} \cong \overline{OM} < \overline{ON} \cong \overline{NB}$, so by Theorem NEUT.73 (transitivity for segments), $\overline{MA} < \overline{NB}$, and since $\overline{NB} < \overline{MB}$, $\overline{MA} < \overline{MB}$. Then by Theorem NEUT.74, $A < B$.

(II: If $A < B$, then $M < N$.) If $N = M$, then by Exercise NEUT.33 $\overline{OA} \cong \overline{OB}$ and by Property R.4 of Definition NEUT.2, $A = B$, contrary to the given fact that $A < B$ (see Theorem ORD.5). If $N < M$, then by part (I) $B < A$ contrary to the given fact that $A < B$. Hence $N < M$ is false. By Theorem ORD.5 (trichotomy for ordering) $M < N$. $\square$

Exercise NEUT.35* Let $\mathcal{P}$ be a neutral plane, O and P be distinct points on $\mathcal{P}$, A and B be distinct members of $\overrightarrow{OP}$, M be the midpoint of $\overline{OA}$, and N be the midpoint of $\overline{OB}$, then O–A–B iff O–M–N.

Exercise NEUT.35 Proof. By Theorem ORD.6 $O < A < B$ iff O–A–B and $O < M < N$ iff O–M–B. Hence by Exercise NEUT.34 O–A–B iff O–M–N. $\square$

Exercise NEUT.36* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, M_1, A_2, B_2, and M_2 be points on $\mathcal{P}$ such that $A_1 \neq B_1$, $A_2 \neq B_2$, M_1 be the midpoint of $\overline{A_1B_1}$ and M_2 be the midpoint of $\overline{A_2B_2}$, then $\overline{A_1B_1} < \overline{A_2B_2}$ iff $\overline{A_1M_1} < \overline{A_2M_2}$.

Exercise NEUT.36 Proof. By Theorem NEUT.67 (segment construction) there exists a point S such that $S \in \overrightarrow{A_2B_2}$ and $\overline{A_2S} \cong \overline{A_1B_1}$. Let M be the midpoint of $\overline{A_2S}$. By Definition NEUT.70 A_2–S–B_2 iff $\overline{A_2S} < \overline{A_2B_2}$ and A_1–M–M_2 iff $\overline{A_2M} < \overline{A_2M_2}$. By Exercise NEUT.35 A_2–S–B_2 iff A_2–M–M_2.

Hence $\overline{A_2S} < \overline{A_2B_2}$ and $\overline{A_2M} < \overline{A_2M_2}$. Since $\overline{A_1B_1} \cong \overline{A_2S}$ and $\overline{A_1M_1} \cong \overline{A_2M}$ by Theorem NEUT.73 (transitivity for segments) $\overline{A_1B_1} < \overline{A_2B_2}$ iff $\overline{A_1M_2} < \overline{A_2M_2}$. $\square$

Exercise NEUT.37* Let A_1, B_1, A_2, and B_2 be points on the neutral plane $\mathcal{P}$ such that $A_1 \neq B_1$, $A_2 \neq B_2$, and $\overline{A_1B_1} \cong \overline{A_2B_2}$ and let C_1 and C_2 be points such that A_1–C_1–B_1, $C_2 \in \overrightarrow{A_2B_2}$ and $\overline{A_1C_1} \cong \overline{A_2C_2}$, then A_2–C_2–B_2.

Exercise NEUT.37 Proof. By Theorem NEUT.56 there exists an isometry α of $\mathcal{P}$ such that $\alpha(\overline{A_1B_1}) = \overline{A_2B_2}$, $\alpha(A_1) = A_2$ and $\alpha(B_1) = B_2$. By Definition NEUT.1(D) $\alpha(A_1)$–$\alpha(C_1)$–$\alpha(B_1)$ *i.e.,* A_2–$\alpha(C_1)$–B_2. By Definition IB.4 $\alpha(C_1) \in \overrightarrow{A_2B_2}$. By Theorem NEUT.15 $\alpha(\overline{A_1C_1}) = \overline{\alpha(A_1)\alpha(C_1)} = \overline{A_2\alpha(C_1)}$. By Definition NEUT.3(B) $\overline{A_1C_1} \cong \overline{A_2\alpha(C_1)}$. Since $\overline{A_1C_1} \cong \overline{A_2C_2}$, by Theorem NEUT.14, $\overline{A_2\alpha(C_1)} \cong \overline{A_2C_2}$. By Property R.4 of Definition NEUT.2, $\alpha(C_1) = C_2$, so A_2–C_2–B_2. $\square$

Exercise NEUT.38* Let A_1, B_1, A_2, B_2, C_1, and C_2 be points on the neutral plane $\mathcal{P}$ such that $A_1 \neq B_1$, $A_2 \neq B_2$, $C_1 \in \overline{A_1B_1}$, and $C_2 \in \overline{A_2B_2}$.

(A) If $\overline{A_1C_1} \cong \overline{A_2C_2}$ and $\overline{C_1B_1} \cong \overline{C_2B_2}$, then $\overline{A_1B_1} \cong \overline{A_2B_2}$.

(B) If $\overline{A_1C_1} \cong \overline{A_2C_2}$ and $\overline{A_1B_1} \cong \overline{A_2B_2}$, then $\overline{C_1B_1} \cong \overline{C_2B_2}$.

Exercise NEUT.38 Proof. (A) Since $\overline{A_1C_1} \cong \overline{A_2C_2}$ by Theorem NEUT.56 there exists an isometry α of $\mathcal{P}$ such that $\alpha(\overline{A_1C_1}) = \overline{A_2C_2}$, $\alpha(A_1) = A_2$, and $\alpha(C_1) = C_2$. By Definition IB.3 A_1–C_1–B_1. By Definition NEUT.1(D) $\alpha(A_1)$–$\alpha(C_1)$–$\alpha(B_1)$, *i.e.,* A_2–C_2–$\alpha(B_1)$. By Theorem PSH.13 $\alpha(B_1) \in \overrightarrow{C_2B_2}$. By Theorem NEUT.15 $\alpha(\overline{C_1B_1}) = \overline{\alpha(C_1)\alpha(B_1)} = \overline{C_2\alpha(B_1)}$. By Definition NEUT.3(B) $\overline{C_1B_1} \cong \overline{C_2\alpha(B_1)}$. Since $\overline{C_1B_1} \cong \overline{C_2B_2}$, by Theorem NEUT.14 (congruence is an equivalence relation), $\overline{C_2\alpha(B_1)} \cong \overline{C_2B_2}$. Since $\alpha(B_1) \in \overrightarrow{C_2B_2}$ by Property R.4 of Definition NEUT.2, $\alpha(B_1) = B_2$. By Theorem NEUT.15 $\alpha(\overline{A_1B_1}) = \overline{\alpha(A_1)\alpha(B_1)} = \overline{A_2B_2}$. By Definition NEUT.3(B) $\overline{A_1B_1} \cong \overline{A_2B_2}$.

(B) Since $\overline{A_1C_1} \cong \overline{A_2C_2}$ by Theorem NEUT.56 there exists an isometry γ of $\mathcal{P}$ such that $\gamma(\overline{A_1C_1}) = \overline{A_2C_2}$, $\gamma(A_1) = A_2$, and $\gamma(C_1) = C_2$. By Definition IB.3 A_1–C_1–B_1. By Definition NEUT.1(D) $\gamma(A_1)$–$\gamma(C_1)$–$\gamma(B_1)$, *i.e.,* A_2–C_2–$\gamma(B_1)$. By Definition IB.4 $\gamma(B_1) \in \overrightarrow{A_2B_2}$. By Theorem NEUT.15 $\gamma(\overline{A_1B_1}) = \overline{\gamma(A_1)\gamma(B_1)} = \overline{A_2\gamma(B_1)}$. By Definition NEUT.3(B) $\overline{A_1B_1} \cong \overline{A_2\gamma(B_1)}$. Since $\overline{A_1B_1} \cong \overline{A_2B_2}$, by Theorem NEUT.14 (congruence is an equivalence relation) $\overline{A_1B_1} \cong \overline{A_2\gamma(B_1)}$. By Definition NEUT.3(B) $\overline{A_2\gamma(B_1)} \cong$

$\overline{A_2B_2}$. Since $\gamma(B_1) \in \overline{A_2B_2}$, by Property R.4 of Definition NEUT.2, $\gamma(B_1) = B_2$. By Theorem NEUT.15 $\gamma(\overline{C_1B_1}) = \overline{\gamma(C_1)\gamma(B_1)} = \overline{C_2B_2}$. By Definition NEUT.3(B) $\overline{C_1B_1} \cong \overline{C_2B_2}$. $\square$

Exercise NEUT.39* Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, A', B', C', and D' be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) A', B', and C' are noncollinear, (3) $\overrightarrow{AD}$ is the bisecting ray of $\angle BAC$, (4) $\overrightarrow{A'D'}$ is the bisecting ray of $\angle B'A'C'$. Then $\angle BAC \cong \angle B'A'C'$ iff $\angle BAD \cong \angle B'A'D'$.

Exercise NEUT.39 Proof. (I: If $\angle BAC \cong \angle B'A'C'$, then $\angle BAD \cong \angle B'A'D'$.) By Theorem NEUT.38 there exists an isometry α of $\mathcal{P}$ such that $\alpha(\angle BAC) = \angle B'A'C'$, $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$, and $\alpha(\overrightarrow{AC}) = \overrightarrow{A'C'}$. By Theorem NEUT.15 $\alpha(\overrightarrow{AB}) = \overrightarrow{\alpha(A)\alpha(B)}$ and $\alpha(\overrightarrow{AC}) = \overrightarrow{\alpha(A)\alpha(C)}$. By Theorem PSH.24 $\alpha(A) = A'$, $\alpha(B) \in \overrightarrow{A'B'}$, and $\alpha(C) \in \overrightarrow{A'C'}$. By Definition NEUT.3(D) $D \in \operatorname{ins}\angle BAC$ and $D' \in \operatorname{ins}\angle B'A'C'$. By Theorem NEUT.15 $\alpha(D) \in \operatorname{ins}\alpha(\angle BAC) = \operatorname{ins}\angle B'A'C'$. By the same theorem

$$\begin{aligned}\alpha(\angle BAD) &= \alpha(\overrightarrow{AB} \cup \overrightarrow{AD}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{AD})\\ &= \overrightarrow{A'B'} \cup \overrightarrow{A'\alpha(D)} = \angle B'A'\alpha(D).\end{aligned}$$

By Definition NEUT.3(B) $\angle BAD \cong \angle B'A'\alpha(D)$. By similar reasoning $\angle CAD \cong \angle C'A'\alpha(D)$. Since $\angle BAD \cong \angle CAD$ by Theorem NEUT.14 (congruence is an equivalence relation) $\angle B'A'\alpha(D) \cong \angle C'A'\alpha(D)$. By Theorem NEUT.39 and Definition NEUT.3(D) $\overrightarrow{A'\alpha(D)}$ is a bisecting ray of $\angle B'A'C'$, and hence by Theorem NEUT.26, $\overrightarrow{A'\alpha(D)} = \overrightarrow{A'D'}$. By Theorem PSH.24 $\alpha(D) \in \overrightarrow{A'D'}$. Since $\angle BAD \cong \angle B'A'\alpha(D)$, $\angle BAD \cong \angle B'A'D'$.

(II: If $\angle BAD \cong \angle B'A'D'$, then $\angle BAC \cong \angle B'A'C'$.) By Theorem NEUT.38 there exists an isometry γ of $\mathcal{P}$ such that $\gamma(\angle BAD) = \angle B'A'D'$, $\gamma(\overrightarrow{AB}) = \overrightarrow{A'B'}$, and $\gamma(\overrightarrow{AD}) = \overrightarrow{A'D'}$. By Theorem NEUT.15

$$\begin{aligned}\gamma(\angle BAD) &= \gamma(\overrightarrow{AB} \cup \overrightarrow{AD}) = \gamma(\overrightarrow{AB}) \cup \gamma(\overrightarrow{AD})\\ &= \overrightarrow{\gamma(A)\gamma(B)} \cup \overrightarrow{\gamma(A)\gamma(D)} = \angle\gamma(B)\gamma(A)\gamma(D).\end{aligned}$$

Since $\overrightarrow{\gamma(A)\gamma(B)} = \overrightarrow{A'B'}$ and $\overrightarrow{\gamma(A)\gamma(D)} = \overrightarrow{A'D'}$, by Theorem PSH.24 $\gamma(A) = A'$, $\gamma(B) \in \overrightarrow{AB'}$ and $\gamma(D) \in \overrightarrow{A'D'}$. By Theorem NEUT.15

$$\begin{aligned}\gamma(\angle CAD) &= \gamma(\overrightarrow{AC} \cup \overrightarrow{AD}) = \gamma(\overrightarrow{AC}) \cup \overrightarrow{AD})\\ &= \overrightarrow{\gamma(A)\gamma(C)} \cup \overrightarrow{\gamma(A)\gamma(D)} = \overrightarrow{A'\gamma(C)} \cup \overrightarrow{A'D'} = \angle\gamma(C)A'D'.\end{aligned}$$

By Definition NEUT.3(B) $\angle CAD \cong \angle\gamma(C)A'D'$. By Theorem NEUT.39 $\angle BAD \cong \angle CAD$ and $\angle B'A'D' \cong \angle C'A'D'$. Since $\angle BAD \cong \angle B'A'D'$, by Theorem NEUT.14 (congruence is an equivalence relation), $\angle\gamma(C)A'D' \cong \angle C'A'D'$. Since $D \in \operatorname{ins}\angle BAC$, by Theorem NEUT.15

$$\gamma(D) \in \operatorname{ins}\gamma(\angle BAC) = \operatorname{ins}\angle B'A'\gamma(C).$$

Since $\gamma(D) \in \overrightarrow{A'D'}$, $D' \in \operatorname{ins}\angle B'A'\gamma(C)$. By Definition PSH.36 D' and $\gamma(C)$ are on the same side of $\overleftrightarrow{A'B'}$. By Theorem NEUT.36 $\overrightarrow{A'\gamma(C)} = \overrightarrow{A'C'}$. By Theorem PSH.24 $\gamma(C) \in \overrightarrow{A'C'}$. By Theorem NEUT.15

$$\gamma(\angle BAC) = \gamma(\overrightarrow{AB} \cup \overrightarrow{AC}) = \gamma(\overrightarrow{AB}) \cup \gamma(\overrightarrow{AC})$$
$$= \overrightarrow{\gamma(A)\gamma(B)} \cup \overrightarrow{\gamma(A)\gamma(C)} = \overrightarrow{A'B'} \cup \overrightarrow{A'C'} = \angle B'A'C'.$$

By Definition NEUT.3(B) $\angle BAC \cong \angle B'A'C'$. $\square$

Exercise NEUT.40* Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, A', B', C', and D' be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) A', B', and C' are noncollinear, (3) $D \in \operatorname{ins}\angle BAC$ and $D' \in \operatorname{ins}\angle B'A'C'$.

(A) If $\angle BAD \cong \angle B'A'D'$ and $\angle CAD \cong \angle C'A'D'$, then $\angle BAC \cong \angle B'A'C'$.

(B) If $\angle BAD \cong \angle B'A'D'$ and $\angle BAC \cong \angle B'A'C'$, then $\angle CAD \cong \angle C'A'D'$.

Exercise NEUT.40 Proof. (A) By Theorem NEUT.38 there exists an isometry α of $\mathcal{P}$ such that $\alpha(\angle BAD) = \angle B'A'D'$, $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$ and $\alpha(\overrightarrow{AD}) = \overrightarrow{A'D'}$. By Definition PSH.29, Theorem NEUT.15, and elementary mapping theory

$$\alpha(\angle BAD) = \alpha(\overrightarrow{AB} \cup \overrightarrow{AD}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{AD}) = \overrightarrow{\alpha(A)\alpha(B)} \cup \overrightarrow{\alpha(A)\alpha(D)}.$$

By Theorem PSH.24 $\alpha(A) = A'$, $\alpha(B) \in \overrightarrow{A'B'}$ and $\alpha(D) \in \overrightarrow{A'D'}$. By Definition PSH.29, Theorem NEUT.15, and elementary mapping theory,

$$\alpha(\angle CAD) = \alpha(\overrightarrow{AC} \cup \overrightarrow{AD}) = \alpha(\overrightarrow{AC}) \cup \alpha(\overrightarrow{AD})$$
$$= \overrightarrow{\alpha(A)\alpha(C)} \cup \overrightarrow{\alpha(A)\alpha(D)} = \overrightarrow{A'\alpha(C)} \cup \overrightarrow{A'D'} = \angle\alpha(C)A'D'.$$

By Definition NEUT.3(B) $\angle CAD \cong \angle\alpha(C)A'D'$. Since $\angle CAD \cong \angle C'A'D'$, by Theorem NEUT.14 (congruence is an equivalence relation), $\alpha(C)A'D' \cong \angle C'A'D'$. Since $\alpha(D) \in \overrightarrow{A'D'}$, by Theorem PSH.16 $\overrightarrow{A'\alpha(D)} = \overrightarrow{A'D'}$. Since $D \in \operatorname{ins}\angle BAD$, by Theorem NEUT.15 $\alpha(D) \in \operatorname{ins}\angle\alpha(B)\alpha(A)\alpha(C) = \operatorname{ins}\angle B'A'\alpha(C)$. By Corollary PSH.39.2 B' and $\alpha(C)$ are on opposite sides of $\overleftrightarrow{A'\alpha(D)} = \overleftrightarrow{A'D'}$.

Since $D' \in \operatorname{ins}\angle B'A'C'$, B' and C' are on opposite sides of $\overleftrightarrow{A'D'}$. By Theorem PSH.12 (plane separation), C' and $\alpha(C)$ are on the same side of $\overleftrightarrow{A'D'}$. By Theorem NEUT.36 $\overrightarrow{A'C'} = \overrightarrow{A'\alpha(C)}$. By Definition PSH.29, Theorem NEUT.15, and elementary mapping theory

$$\alpha(\angle BAC) = \alpha(\overrightarrow{AB} \cup \overrightarrow{AC}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{AC})$$
$$= \overrightarrow{\alpha(A)\alpha(B)} \cup \overrightarrow{\alpha(A)\alpha(C)} = \overrightarrow{A'B'} \cup \overrightarrow{A'C'} = \angle B'A'C'.$$

By Definition NEUT.3(B) $\angle BAC \cong B'A'C'$.

(B) By Theorem NEUT.38 there exists an isometry γ of $\mathcal{P}$ such that $\gamma(\angle BAD) \cong \angle B'A'D'$, $\gamma(\overrightarrow{AB}) = \overrightarrow{A'B'}$ and $\gamma(\overrightarrow{AD}) = \overrightarrow{A'D'}$. Reasoning as in part (A) we get $\gamma(\angle BAD) = \overrightarrow{\gamma(A)\gamma(B)} \cup \overrightarrow{\gamma(A)\gamma(D)}$, $\gamma(A) = A'$, $\gamma(B) \in \overrightarrow{A'B'}$, and $\gamma(D) \in \overrightarrow{A'D'}$. Furthermore $\gamma(\angle BAC) = \angle B'A'\gamma(C)$. So that $\angle BAC \cong \angle B'A'\gamma(C)$ and thus, $\angle B'A'\gamma(C) \cong \angle B'A'C'$, and so $\angle B'A'\gamma(C) \cong \angle B'A'C'$. Since $D \in \operatorname{ins}\angle BAC$, by Theorem NEUT.15

$$\gamma(D) \in \operatorname{ins}\gamma(\angle BAC) = \operatorname{ins}\angle B'A'\gamma(C) = \operatorname{ins}\angle B'A'C'.$$

Since $D' \in \operatorname{ins}\angle B'A'C'$ and $\gamma(D) \in \operatorname{ins}\angle B'A'C'$ and $\gamma(D) \in \overrightarrow{A'D'}$ so that by Theorem PSH.16 $\overrightarrow{A'\gamma(D)} = \overrightarrow{A'D'}$. By Definition PSH.36 D' and C' are on the same side of $\overleftrightarrow{A'B'}$ and $\gamma(D)$ and C' are on the same side of $\overleftrightarrow{A'B'}$. Thus by Theorem PSH.12 (plane separation) C' and $\gamma(C)$ are on the same side of $\overleftrightarrow{A'B'}$. By Theorem NEUT.36 $\overrightarrow{A'C'} = \overrightarrow{A'\gamma(C)}$. By Definition PSH.29, Theorem NEUT.14, and elementary mapping theory

$$\gamma(\angle CAD) = \gamma(\overrightarrow{AC} \cup \overrightarrow{AD}) = \gamma(\overrightarrow{AC}) \cup \gamma(\overrightarrow{AD}) = \overrightarrow{A'C'} \cup \overrightarrow{A'D'} = \angle C'A'D'.$$

By Definition NEUT.3(B) $\angle CAD \cong \angle C'A'D'$. $\square$

Exercise NEUT.41* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 be points on $\mathcal{P}$ such that: (1) A_1, B_1, and C_1 are noncollinear, (2) $D_1 \in \operatorname{ins}\angle B_1A_1C_1$, (3) A_2, B_2, and C_2 are noncollinear, (4) $D_2 \in \operatorname{ins}\angle B_2A_2C_2$, and (5) $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$. Then $\angle B_1A_1C_1 < \angle B_2A_2C_2$ iff $\angle D_1A_1C_1 < \angle D_2A_2C_2$.

Exercise NEUT.41 Proof. (I: If $\angle B_1A_1C_1 < \angle B_2A_2C_2$, then $\angle D_1A_1C_1 < \angle D_2A_2C_2$.) By Definition NEUT.70 $\angle B_1A_1D_1 < \angle B_1A_1C_1$, $\angle B_2A_2D_2 < \angle B_2A_2C_2$ and there exists a point S belonging to $\operatorname{ins}\angle B_2A_2C_2$ such that $\angle B_1A_1C_1 \cong \angle B_2A_2S$. Since $\angle B_1A_1D_1 < \angle B_1A_1C_1$, $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$, and $\angle B_1A_1C_1 \cong \angle B_2A_2S$, by Theorem NEUT.76 (transitivity for angles), $\angle B_2A_2D_2 < \angle B_2A_2S$. By Definition NEUT.70 $D_2 \in \operatorname{ins}\angle B_2A_2S$. Since $\angle B_1A_1C_1 < \angle B_2A_2C_2$ and $\angle B_1A_1C_1 \cong \angle B_2A_2S$, by Theorem NEUT.76 (transitivity for angles) $\angle B_2A_2S < \angle B_2A_2C_2$. By Theorem NEUT.78 $S \in \operatorname{ins}\angle B_2A_2C_2$. By Exercise PSH.18 $\operatorname{ins}\angle B_2A_2C_2$ is the union of the disjoint sets $\overrightarrow{A_2D_2}$, $\operatorname{ins}\angle B_2A_2D_2$, and $\operatorname{ins}\angle D_2A_2C_2$. Since $D_2 \in \operatorname{ins}\angle B_2A_2S$, by Exercise PSH.12 $S \in \operatorname{out}\angle B_2A_2D_2$, so that $S \in \operatorname{ins}\angle D_2A_2C_2$. By Definition NEUT.70 $\angle D_2A_2S < \angle D_2A_2C_2$. We know that $\angle B_1A_1C_1 \cong \angle B_2A_2S$ and $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$, so that by Exercise NEUT.40, $\angle D_1A_1C_1 \cong \angle D_2A_2S$. Since $\angle D_2A_2S < \angle D_2A_2C_2$ by Theorem NEUT.76 (transitivity for angles), $\angle D_1A_1C_1 < \angle D_2A_2C_2$.

(II: If $\angle D_1A_1C_1 < \angle D_2A_2C_2$, then $\angle B_1A_1C_1 < \angle B_2A_2C_2$.) By Definition NEUT.70 there exists a point T belonging to the ins$\angle D_2A_2C_2$ such that $\angle D_1A_1C_1 \cong \angle D_2A_2T$. By Exercise PSH.18 ins$\angle B_2A_2C_2$ is the union of the disjoint sets $\overrightarrow{A_2D_2}$, ins$\angle B_2A_2D_2$, and ins$\angle D_2A_2C_2$, so that $T \in$ ins$\angle B_2A_2C_2$. By Definition NEUT.70 $\angle B_2A_2T < \angle B_2A_2C_2$. Since $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$ and $\angle D_1A_1C_1 \cong \angle D_2A_2T$ by Exercise NEUT.40 $\angle B_1A_1C_1 \cong \angle B_2A_2T$.

Since $\angle B_2A_2T < \angle B_2A_2C_2$ and $\angle B_1A_1C_1 \cong \angle B_2A_2T$, by Theorem NEUT.76 (transitivity for angles) $\angle B_1A_1C_1 < \angle B_2A_2C_2$. $\square$

Exercise NEUT.42* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 be points on $\mathcal{P}$ such that: (1) A_1, B_1, and C_1 are noncollinear, (2) $D_1 \in$ ins$\angle B_1A_1C_1$, (3) A_2, B_2, and C_2 are noncollinear, and (4) $D_2 \in$ ins$\angle B_2A_2C_2$. Then if $\angle B_1A_1D_1 < \angle B_2A_2D_2$ and $\angle D_1A_1C_1 < \angle D_2A_2C_2$, $\angle B_1A_1C_1 < \angle B_2A_2C_2$.

Exercise NEUT.42 Proof. Let A_3 and B_3 be distinct points on $\mathcal{P}$ and let $\mathcal{H}$ be a side of $\overleftrightarrow{A_3B_3}$. By Theorem NEUT.68 (angle construction) there exists a point D_3 belonging to $\mathcal{H}$ such that $\angle B_3A_3D_3 \cong \angle B_1A_1D_1$. By the same theorem there exists a point C_3 on the side of $\overleftrightarrow{A_3D_3}$ opposite the B_2-side such that $\angle D_3A_3C_3 \cong \angle D_2A_2C_2$.

Since $\angle B_3A_3D_3 \cong \angle B_1A_1D_1$ and $\angle C_1A_1D_1 < \angle C_3A_3D_3$, by Exercise NEUT.41 $\angle B_1A_1C_1 < \angle B_3A_3C_3$.

Since $\angle D_3A_3C_3 \cong \angle D_2A_2C_2$ and $\angle B_3A_3D_3 < \angle B_2A_2D_2$, by the same exercise $\angle B_3A_3C_3 < \angle B_2A_2C_2$.

Then $\angle B_1A_1C_1 < \angle B_3A_3C_3 < \angle B_2A_2C_2$ so that by Theorem NEUT.76 (transitivity for angles) $\angle B_1A_1C_1 < \angle B_2A_2C_2$. $\square$

Exercise NEUT.43* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 be points on $\mathcal{P}$ such that: (1) A_1, B_1, and C_1 are noncollinear, (2) $\overrightarrow{A_1D_1}$ is the bisecting ray of $\angle B_1A_1C_1$, (3) A_2, B_2, and C_2 are noncollinear, and (4) $\overrightarrow{A_2D_2}$ is the bisecting ray of $\angle B_2A_2C_2$. Then $\angle B_1A_1C_1 < \angle B_2A_2C_2$ iff $\angle B_1A_1D_1 < \angle B_2A_2D_2$.

Exercise NEUT.43 Proof. By Theorem NEUT.39

$$\angle B_1A_1D_1 \cong \angle D_1A_1C_1 \text{ and } \angle B_2A_2D_2 \cong \angle D_2A_2C_2.$$

(I) If $\angle B_1A_1D_1 < \angle B_2A_2D_2$, then by Theorem NEUT.76 (transitivity of angles) $\angle D_1A_1C_1 < \angle D_2A_2C_2$. By Exercise NEUT.42, $\angle B_1A_1C_1 < \angle B_2A_2C_2$.

(II) Conversely, suppose that $\angle B_1A_1C_1 < \angle B_2A_2C_2$. By Theorem NEUT.75 (trichotomy for angles), exactly one of $\angle B_1A_1D_1 < \angle B_2A_2D_2$, $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$, or $\angle B_1A_1D_1 > \angle B_2A_2D_2$ holds.

If $\angle B_1A_1D_1 \cong \angle B_2A_2D_2$, then by Exercise NEUT.39 $\angle B_1A_1C_1 \cong \angle B_2A_2C_2$, which is contrary to our assumption.

If $\angle B_1A_1D_1 > \angle B_2A_2D_2$, then by part (I) (interchanging the subscripts 1 and 2) $\angle B_1A_1C_1 > \angle B_2A_2C_2$, which is contrary to our assumption.

Therefore $\angle B_1A_1D_1 < \angle B_2A_2D_2$, completing the proof. □

Exercise NEUT.44* Let $\mathcal{P}$ be a neutral plane and let A, B, C, P, and Q be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) $P \in \operatorname{ins}\angle BAC$, and (3) $Q \in \operatorname{ins}\angle BAP$. Then $\angle QAP < \angle BAC$.

Exercise NEUT.44 Proof. Since $P \in \operatorname{ins}\angle BAC$ by Definition NEUT.70 $\angle BAP < \angle BAC$. By the same definition $\angle QAP < \angle BAP$. By Theorem NEUT.76 (transitivity for angles) $\angle QAP < \angle BAC$. □

The reader will note that the next exercise is identical to Exercise NEUT.42, although somewhat disguised by the use of different notation. At one point we thought to eliminate it. We decided to leave it in, since the method of proof is different from that for Exercise NEUT.42.

Exercise NEUT.45* Use Exercise NEUT.44 to prove the following: Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, A', B', C', and D' be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) A', B', and C' are noncollinear, (3) $D \in \operatorname{ins}\angle BAC$ and (4) $D' \in \operatorname{ins}\angle B'A'C'$. If $\angle BAD < \angle B'A'D'$ and $\angle CAD < \angle C'A'D'$, then $\angle BAC < \angle B'A'C'$.

Exercise NEUT.45 Proof. By Definition NEUT.70 there exist points P and Q such that $P \in \operatorname{ins}\angle C'A'D'$, $Q \in \operatorname{ins}\angle B'A'D'$, $\angle D'A'Q \cong \angle BAD$ and $\angle D'A'P \cong \angle DAC$.

By Exercise PSH.18, P, Q, and D' are members of $\operatorname{ins}\angle B'A'C'$. Since $P \in \operatorname{ins}\angle C'A'D'$, by Exercise PSH.13 $D' \in \operatorname{ins}\angle B'A'P$; then $Q \in \operatorname{ins}\angle B'A'D' \subseteq \operatorname{ins}\angle B'A'P'$, again by Exercise PSH.18.

We may now apply Exercise NEUT.44 to get $\angle QA'P < \angle B'A'C'$. By Exercise NEUT.40 $\angle QA'P \cong \angle BAC$. By Theorem NEUT.76 (transitivity for angles) $\angle BAC < \angle B'A'C'$. □

Exercise NEUT.46* Let A and B be distinct points on the neutral plane $\mathcal{P}$, $\mathcal{L}$ be the perpendicular bisector of $\overline{AB}$, and α be an isometry of $\mathcal{P}$ such that $\alpha(\overline{AB}) = \overline{AB}$, then one and only one of the following statements is true: (A) α is the identity mapping $\imath$ of $\mathcal{P}$ onto itself, (B) $\alpha = \mathcal{R}_{\overleftrightarrow{AB}}$, (C) $\alpha = \mathcal{R}_{\mathcal{L}}$, or (D) $\alpha = \mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\overleftrightarrow{AB}}$.

Exercise NEUT.46 Proof. By Theorem NEUT.15 $\alpha(\overline{AB}) = \overline{\alpha(A)\alpha(B)}$ and $\alpha(\overrightarrow{AB}) = \overrightarrow{\alpha(A)\alpha(B)}$. Thus $\{\alpha(A), \alpha(B)\} = \{A, B\}$, so that either $\alpha(A) = A$ and $\alpha(B) = B$, or $\alpha(A) = B$ and $\alpha(B) = A$. If $\alpha(A) = A$ and $\alpha(B) = B$, then by Theorem NEUT.37 either $\alpha = \imath$ (the identity mapping of $\mathcal{P}$ onto itself) or $\alpha = \mathcal{R}_{\overleftrightarrow{AB}}$. If $\alpha(A) = B$ and $\alpha(B) = A$ let $\gamma = \mathcal{R}_{\mathcal{L}} \circ \alpha$, then by Theorem NEUT.11 α is an isometry of $\mathcal{P}$. Moreover $\gamma(A) = A$ and $\gamma(B) = B$ so by Theorem NEUT.37 either $\gamma = \imath$ or $\gamma = \mathcal{R}_{\overleftrightarrow{AB}}$. If $\gamma = \mathcal{R}_{\mathcal{L}} \circ \alpha = \imath$, then by Definition NEUT.1(C) $\alpha = \mathcal{R}_{\mathcal{L}}$. If $\gamma = \mathcal{R}_{\mathcal{L}} \circ \alpha = \mathcal{R}_{\overleftrightarrow{AB}}$, $\alpha = \mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\overleftrightarrow{AB}}$. $\square$

Exercise NEUT.47* Let A, B, and C be distinct points on the neutral plane $\mathcal{P}$ and let α be an isometry of $\mathcal{P}$ such that A is a fixed point of α and B is not a fixed point of α. Then A is the midpoint of $\overline{BC}$ iff B–A–C and $\alpha(B) = C$.

Exercise NEUT.47 Proof. (I: If A is the midpoint of $\overline{BC}$, then B–A–C and $\alpha(B) = C$.) By Definition NEUT.3(C) B–A–C and $\overline{BA} \cong \overline{CA}$. By Definition NEUT.1(D) $\alpha(B)$–$\alpha(A)$–$\alpha(C)$, *i.e.*, $\alpha(B)$–A–$\alpha(C)$. By Property B.0 of Definition IB.1 $\alpha(B) \neq A$. By Theorem PSH.15 $\overleftrightarrow{AB} \setminus \{A\}$ is the union of the disjoint rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. By Theorem NEUT.15 $\alpha(\overline{AB}) = \overline{\alpha(A)\alpha(B)} = \overline{A\alpha(B)}$ so that by Definition NEUT.3(B) $\overline{AB} \cong \overline{A\alpha(B)}$. If $\alpha(B)$ were a member of $\overrightarrow{AB}$, then by Property R.4 of Definition NEUT.2 $\alpha(B)$ would equal B, *i.e.*, B would be a fixed point of α. This would contradict the given fact that B is not a fixed point of α. Hence $\alpha(B) \in \overrightarrow{AC}$. Since $\overline{AB} \cong \overline{AC}$ by Theorem NEUT.14 (congruence is an equivalence relation) $\overline{AC} \cong \overline{A\alpha(B)}$. By Property R.4 of Definition NEUT.2, $\alpha(B) = C$.

(II: If B–A–C and $\alpha(B) = C$, then A is the midpoint of $\overline{BC}$.) By Theorem NEUT.15 $\alpha(\overline{BA}) = \overline{\alpha(B)\alpha(A)} = \overline{CA}$. By Definition NEUT.3(B) $\overline{BA} \cong \overline{CA}$. By Definition NEUT.3(C) A is the midpoint of $\overline{BC}$. $\square$

Exercise NEUT.48* Let $\mathcal{P}$ be a neutral plane, let $\mathcal{L}$ and $\mathcal{M}$ be distinct lines on $\mathcal{P}$ through the point O, and let $\mathcal{L}_1$ and $\mathcal{M}_1$ be lines on $\mathcal{P}$ such that $\mathcal{L}_1 \perp \mathcal{L}$ and $\mathcal{M}_1 \perp \mathcal{M}$, then $\mathcal{L}_1$ and $\mathcal{M}_1$ are distinct.

Exercise NEUT.48 Proof. If $\mathcal{L}_1$ and $\mathcal{M}_1$ were equal then there would exist distinct lines (namely $\mathcal{L}$ and $\mathcal{M}$) through O each of which is perpendicular to $\mathcal{L}_1$, contrary to Theorem NEUT.47(B). Hence $\mathcal{L}_1 \neq \mathcal{M}_1$. $\square$

Exercise NEUT.49* Let P, O, and T be noncollinear points on the neutral plane $\mathcal{P}$ and let S and V be members of $\text{ins}\,\angle POT$ such that $\angle POS < \angle TOS$ and $\angle POV \cong \angle TOS$. Furthermore, let X be any member of $\text{ins}\,\angle TOV$ and let W be a point such that $\angle POW < \angle POX$ and $\angle XOW \cong \angle POS$, then $W \in \text{ins}\,\angle POV$.

Exercise NEUT.49 Proof. By Exercise NEUT.32 $S \in \text{ins}\,\angle POV$, $V \in \text{ins}\,\angle TOS$, and $\angle TOV \cong \angle POS$. Since $\angle XOW \cong \angle POS$ and $\angle POS \cong \angle TOV$, by Theorem NEUT.14 $\angle XOW \cong \angle TOV$. Since $\angle POS < \angle TOS$, by Theorem NEUT.76 (transitivity for angles) $\angle TOV < \angle TOS$. Since $S \in \text{ins}\,\angle POV$ by Definition NEUT.70 $\angle POS < \angle POV$. By Definition PSH.36 V and S are on the same side of $\overleftrightarrow{OT}$ and V and S are on the same side of $\overleftrightarrow{OP}$. By Theorem NEUT.78 $V \in \text{ins}\,\angle TOS$ and $S \in \text{ins}\,\angle POV$. If W were to belong to $\overrightarrow{OV}$ or to $\text{ins}\,\angle TOV$, then by Exercise NEUT.44 $\angle XOW$ would be smaller than $\angle TOV$. By Theorem NEUT.75 (trichotomy for angles), this contradicts the established fact that $\angle XOW \cong \angle TOV$. Hence $W \notin (\overrightarrow{OV} \cup \text{ins}\,\angle TOV)$. By Exercise PSH.18 $W \in \text{ins}\,\angle POV$. $\square$

Exercise NEUT.50* Let $\mathcal{P}$ be a neutral plane, $\mathcal{L}$ and $\mathcal{M}$ be lines on $\mathcal{P}$ such that $\mathcal{L} \perp \mathcal{M}$, and $\mathcal{E}$ be a side of $\mathcal{L}$. Then $\mathcal{M}$ is a line of symmetry of $\mathcal{E}$.

Exercise NEUT.50 Proof. By Theorem NEUT.10 we need only show that if X is any member of $\mathcal{E}$, then $\mathcal{R}_{\mathcal{M}}(X) = X$. If $X \in \mathcal{M}$, then by Definition NEUT.1(A) $\mathcal{R}_{\mathcal{M}}(X) \in \mathcal{E}$. If $X \in (\mathcal{P} \setminus \mathcal{M})$, then by Theorem NEUT.48(A) $\overline{X\mathcal{R}_{\mathcal{M}}(X)} \perp \mathcal{M}$. By Theorem NEUT.47(A) $\overline{X\mathcal{R}_{\mathcal{M}}(X)} \parallel \mathcal{L}$. By Theorem IB.10 and Exercise PSH.14 $\mathcal{R}_{\mathcal{M}}(X) \in \mathcal{E}$. $\square$

Exercise NEUT.51* Let $\mathcal{P}$ be a neutral plane and let A, B, and C be noncollinear points on $\mathcal{P}$ such that $\angle ACB$ is a maximal angle of $\triangle ABC$.

(A) If D is any member of $\overline{BC}$, then $\overline{AD} < \overline{AB}$.

(B) If $\angle ACB$ is acute, there exists a point $D \in \overline{BC}$ such that $\overline{AC} > \overline{AD}$.

(C) If $\angle ACB$ is right or obtuse then for every $D \in \overline{BC}$, $\overline{AC} < \overline{AD}$.

Exercise NEUT.51 Proof. (A) By Theorem NEUT.92 $\overline{AC} \leq \overline{AB}$. By Theorem NEUT.95 $\overline{AD} < \overline{AB}$.

(B) By Theorem NEUT.86 $\angle ABC$ is acute. By assumption $\angle ACB$ is acute. Let $D = \text{ftpr}(A, \overleftrightarrow{BC})$. By Exercise NEUT.20, $D \in \overline{BC}$. Then $\triangle ADC$ is right and $\angle ADC$ is a right angle, hence by Theorem NEUT.93 $\overline{AC} > \overline{AD}$.

(C)(1) If $\angle ACB$ is right then for any $D \in \overline{BC}$, $\angle ACD$ is right and by Theorem NEUT.93 $\overline{AD} > \overline{AC}$.

(C)(2) If $\angle ACB$ is obtuse, then let $E = \text{ftpr}(A, \overleftrightarrow{CB})$. Then $\triangle AED$ is a right triangle, and $\angle AEB$ is maximal by Theorem NEUT.84 so that by part (A) above, $\overline{AD} > \overline{AC}$. $\square$

Exercise NEUT.52* Let $\mathcal{P}$ be a neutral plane, A, B, and C be points on $\mathcal{P}$ such that B–A–C, and D be a member of $\mathcal{P} \setminus \overleftrightarrow{AB}$ such that $\angle BAD < \angle CAD$, then $\angle BAD$ is acute and $\angle CAD$ is obtuse.

Exercise NEUT.52 Proof. By Corollary NEUT.46.1 there exists a point P such that $P \in \overrightarrow{ABD}$ and $\angle BAP$ is right. If D were a member of $\overrightarrow{AP}$, then by Theorem NEUT.44, $\angle BAD$ and $\angle CAD$ would be congruent. By Theorem NEUT.75 (trichotomy for angles) this would contradict the given fact that $\angle BAD < \angle CAD$. Hence $D \in (\overrightarrow{ABP} \setminus \overrightarrow{AP})$.

By Exercise PSH.31 either $D \in \text{ins}\,\angle BAP$ or $D \in \text{ins}\,\angle CAP$. If D were a member of $\text{ins}\,\angle CAP$, then by Definition NEUT.70 $\angle CAD < \angle CAP$. By Exercise PSH.51 P is a member of $\text{ins}\,\angle BAD$ and thus by Definition NEUT.70 $\angle BAP < \angle BAD$. Since $\angle BAP$ is right, by Definition NEUT.41(C) $\angle BAP \cong \angle CAP$. By Theorem NEUT.76 (transitivity for angles) $\angle CAD < \angle BAD$, contrary to the given fact that $\angle BAD < \angle CAD$.

Hence $D \in \text{ins}\,\angle BAP$. By Definition NEUT.70 $\angle BAD < \angle BAP$. By Definition NEUT.81 $\angle BAD$ is acute. By Exercise PSH.51 $P \in \text{ins}\,\angle CAD$. By Definition NEUT.70 $\angle CAP < \angle CAD$. By Definition NEUT.81 $\angle CAD$ is obtuse. $\square$

Exercise NEUT.53* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$ such that $\overline{AC} < \overline{AB}$ and D be the point of intersection of the bisecting ray of $\angle BAC$ and $\overline{BC}$ (so $\angle BAD \cong \angle CAD$), then $\angle ADC$ is acute, $\angle ADB$ is obtuse, and $\overline{DC} < \overline{DB}$.

Exercise NEUT.53 Proof. (A) By Definition NEUT.70 there exists a point E belonging to $\overline{AB}$ such that $\overline{AE} \cong \overline{AC}$. By Theorem NEUT.64 (EAE) applied to $\triangle ADE$ and $\triangle ADC$, $\angle ADE \cong \angle ADC$, $\angle AEC \cong \angle ACD$, and $\overline{DE} \cong \overline{DC}$. By Theorem PSH.37 $E \in \text{ins}\,\angle ADB$. By Definition NEUT.70

$\angle ADE < \angle ADB$. By Theorem NEUT.76 (transitivity for angles) $\angle ADC < \angle ADB$. By Exercise NEUT.52 $\angle ADC$ is acute and $\angle ADB$ is obtuse.

By Property B.3 of Definition IB.1 there exists a point F' such that A–C–F'. By Theorem NEUT.67 (segment construction) there exists a point F belonging to $\overrightarrow{CF'}$ such that $\overline{CF} \cong \overline{EB}$. By Theorem NEUT.43 $\angle DEB \cong \angle DCF$. By Theorem NEUT.65 (AEA) applied to $\triangle DEB$ and $\triangle DCF$, $\angle EDB \cong \angle CDF$. By Exercise NEUT.12 E–D–F. By Exercise NEUT.38 $\overline{AB} \cong \overline{AF}$. By Theorem NEUT.64 (EAE) applied to $\triangle ABC$ and $\triangle AFE$, $\angle ACB \cong \angle AEF$, $\angle ABC \cong \angle AFE$. By Theorem NEUT.80 (outside angles) applied to $\triangle ABC$, $\angle ABC < \angle BCF$. By Theorem NEUT.91 applied to $\triangle DCF$ $\overline{DC} < \overline{DF}$. By Theorem NEUT.73 (transitivity for segments) $\overline{DC} < \overline{DB}$. $\square$

Exercise NEUT.54* Let $\mathcal{P}$ be a neutral plane and let A, B, and M be distinct collinear points on $\mathcal{P}$ such that $\overline{AM} \cong \overline{BM}$, then M is the midpoint of $\overline{AB}$.

Exercise NEUT.54 Proof. By Property B.2 of Definition IB.1 one and only one of the following statements is true: A–M–B; M–A–B; A–B–M. If M–A–B were true, then by Definition NEUT.70 $\overline{AM}$ would be smaller than $\overline{BM}$. If M–B–A were true, then by the same definition $\overline{BM}$ would be smaller than $\overline{AM}$. Each of these situations contradicts Theorem NEUT.72 (trichotomy for segments). Hence A–M–B. By Definition NEUT.3(C) M is the midpoint of $\overline{AB}$. $\square$

Exercise NEUT.55* Let $\mathcal{P}$ be a neutral plane, A and B be distinct points on $\mathcal{P}$, M be the midpoint of $\overline{AB}$, and C be a member of $\overline{AB}$. Then $C \in \overline{AM}$ iff $\overline{AC} < \overline{BC}$.

Exercise NEUT.55 Proof. (I: If $C \in \overline{AM}$, then $\overline{AC} < \overline{BC}$.) By Definition NEUT.70 $\overline{AC} < \overline{AM}$. By Definition NEUT.3(C) $\overline{AM} \cong \overline{BM}$. By Theorem NEUT.73 (transitivity for segments) $\overline{AC} < \overline{BM}$. By Definition NEUT.70 $\overline{BM} < \overline{BC}$. By Theorem NEUT.73 $\overline{AC} < \overline{BC}$.

(II: If $\overline{AC} < \overline{BC}$, then $C \in \overline{AM}$.) By Theorem PSH.15(D) one and only one of the following possibilities is true: $C = M$; $C \in \overline{BM}$; $C \in \overline{AM}$. If C were equal to M, then by Definition NEUT.3(C) $\overline{AC}$ would be congruent to $\overline{CB}$. If C were a member of $\overline{MB}$, then by part (I) $\overline{BC}$ would be smaller than $\overline{AC}$. By Theorem NEUT.72 (trichotomy for segments) each of these two possibilities contradicts the given fact that $\overline{AC} < \overline{BC}$. Hence $C \in \overline{AM}$. $\square$

Exercise NEUT.56* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$, P be a member of $\text{ins}\,\angle BAC$ such that $\overrightarrow{AP}$ is the bisecting ray of $\angle BAC$, and let Q also be a member of $\text{ins}\,\angle BAC$. Then $Q \in \text{ins}\,\angle BAP$ iff $\angle BAQ < \angle CAQ$.

Exercise NEUT.56 Proof. (I: If $Q \in \text{ins}\,\angle BAP$, then $\angle BAQ < \angle CAQ$.) By Definition NEUT.70 $\angle BAQ < \angle BAP$. By Theorem NEUT.39 $\angle BAP \cong \angle CAP$. By Theorem NEUT.76 (transitivity for angles) $\angle BAQ < \angle CAP$. By Exercise PSH.13, since $Q \in \text{ins}\,\angle BAP$, $P \in \text{ins}\,\angle CAQ$. By Definition NEUT.70 $\angle CAP < \angle CAQ$. By Theorem NEUT.76 (transitivity for angles) $\angle BAQ < \angle CAQ$.

(II: If $\angle BAQ < \angle CAQ$, then $Q \in \text{ins}\,\angle BAP$.) By Exercise PSH.18 one and only one of the following possibilities holds: $Q \in \overrightarrow{AP}$; $Q \in \text{ins}\,\angle BAP$; $Q \in \text{ins}\,\angle CAP$. If Q were a member of $\overrightarrow{AP}$, then by Theorem PSH.16 $\overrightarrow{AQ}$ would be equal to $\overrightarrow{AP}$ and by Definition NEUT.39 $\angle BAQ$ and $\angle CAQ$ would be congruent. If Q were a member of $\text{ins}\,\angle CAP$, then by part (I) $\angle CAQ$ would be smaller than $\angle BAQ$. By Theorem NEUT.72 (trichotomy for segments), each of these two possibilities contradicts the given fact that $\angle BAQ < \angle CAQ$. Hence $Q \in \text{ins}\,\angle BAP$. $\square$

Exercise NEUT.57* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$ such that $\overline{AC} < \overline{AB}$, and D be the midpoint of $\overline{BC}$.

(A) $\angle ADC$ is acute and $\angle ADB$ is obtuse.

(B) If E is the point of intersection of the bisecting ray of $\angle BAC$ and segment $\overline{BC}$, then C–E–D–B and $\angle BAD < \angle CAD$.

Exercise NEUT.57 Proof. (A) By Theorem NEUT.98 (Hinge) applied to $\triangle DAB$ and $\triangle DAC$, $\angle ADC < \angle ADB$. By Exercise NEUT.52 $\angle ADC$ is acute and $\angle ADB$ is obtuse.

(B) By Exercise NEUT.53 $\overline{EC} < \overline{EB}$. By Exercise NEUT.55 $E \in \overline{DC}$. By Definition IB.3 C–E–D. By Definition NEUT.3(C) C–D–B. By Theorem PSH.8 C–E–D–B. By Theorem PSH.37 $D \in \text{ins}\,\angle BAE$, so by Exercise NEUT.56 $\angle BAD < \angle CAD$. $\square$

Exercise NEUT.58* Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, E, and F be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) D, E, and F are noncollinear, (3) $\angle BAC \cong \angle EDF$ and $\angle CBA \cong \angle FED$, and (4) $\overline{AB} < \overline{DE}$. Then $\overline{AC} < \overline{DF}$ and $\overline{BC} < \overline{EF}$.

Exercise NEUT.58 Proof. By Definition NEUT.70 there exists a point B' belonging to $\overline{DE}$ such that $\overline{DB'} \cong \overline{AB}$. By Theorem NEUT.68 (angle construction) there exists a point U on the F-side of $\overleftrightarrow{DE}$ such that $\angle DB'U \cong \angle ABC$. Let V be a point such that V–B'–U. Then by Theorem NEUT.42 (vertical angles) $\angle DB'U \cong \angle VB'E$. Since by assumption $\angle ABC \cong \angle DEF$, by Theorem NEUT.14 (congruence is an equivalence relation) $\angle DEF \cong \angle ABC \cong \angle DB'U \cong \angle VB'E$. Thus by Theorem NEUT.87 (alternate interior angles) $\overleftrightarrow{UV} \parallel \overleftrightarrow{FE}$.

By the Postulate of Pasch $\overleftrightarrow{UV}$ intersects either $\overline{DF}$ or $\overline{EF}$; the latter can't be true because $\overleftrightarrow{UV} \parallel \overleftrightarrow{FE}$, so there is a point C' such that $\overline{DF} \cap \overleftrightarrow{UV} = \{C'\}$, and since $C' \in \overline{DF}$, F–C'–D. By Theorem NEUT.65 (AEA) $\triangle DB'C' \cong \triangle ABC$ hence $\overline{DC'} \cong AC$ and from Definition NEUT.70 $\overline{AC} \cong \overline{DC'} < \overline{DF}$. Interchanging "A" with "B" and interchanging "D" with "E" in the above argument shows that $\overline{BC} < \overline{EF}$. □

Exercise NEUT.59* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$, F be the midpoint of $\overline{AB}$, E be the midpoint of $\overline{AC}$, and O be the point of intersection of $\overline{BE}$ and $\overline{CF}$. If $\overline{AB} \cong \overline{AC}$, then $\overline{BE} \cong \overline{CF}$, $\angle CBE \cong \angle BCF$, $\angle ABE \cong \angle ACF$, $\overleftrightarrow{AO}$ is the perpendicular bisector of $\overline{BC}$ and $\overrightarrow{AO}$ is the bisecting ray of $\angle BAC$.

Exercise NEUT.59 Proof. By Exercise NEUT.33 $\overline{BF} \cong \overline{CE}$. By Theorem NEUT.40(A) (*Pons Asinorum*) $\angle ABC \cong \angle ACB$. By Theorem NEUT.64 (EAE) $\overline{BE} \cong \overline{CF}$ and $\angle BCF \cong \angle CBE$. By Theorem NEUT.40(B) (the converse of *Pons Asinorum*) $\overline{OB} \cong \overline{OC}$. Let $\mathcal{L}$ be the perpendicular bisector (See Definition NEUT.51) of $\overline{BC}$. By definition $\mathcal{L}$ intersects $\overline{BC}$ at its midpoint. By Theorem NEUT.52 $\mathcal{R}_{\mathcal{L}}(B) = C$; by Theorem NEUT.55 $\mathcal{L}$ is identical with the line of symmetry of $\angle BAC$. By Theorem NEUT.20 $A \in \mathcal{L}$.

Since $\mathcal{R}_{\mathcal{L}}(B) = C$ and $A \in \mathcal{L}$, $\mathcal{R}_{\mathcal{L}}(\overline{AB}) = \overline{AC}$ and by Exercise NEUT.33 $\mathcal{R}_{\mathcal{L}}(E) = F$, so that $\mathcal{R}_{\mathcal{L}}(\overline{BE}) = \overline{CF}$. Let Q be the point of intersection of $\mathcal{L}$ and $\overline{BE}$; since $Q \in \mathcal{L}$, $\mathcal{R}_{\mathcal{L}}(Q) = Q \in \mathcal{R}_{\mathcal{L}}(\overline{BE}) = \overline{CF}$, so that Q is the point of intersection of these two segments, that is, $Q = O$; therefore $O \in \mathcal{L}$, and $\mathcal{L} = \overleftrightarrow{AO}$. We defined $\mathcal{L}$ to be the perpendicular bisector of $\overline{BC}$, and since $O \in \operatorname{ins} \angle BAC$, $\overrightarrow{AO}$ is its bisecting ray. □

Exercise NEUT.60* Let $\mathcal{P}$ be a neutral plane and let A, B, and C be noncollinear points on $\mathcal{P}$, E be the midpoint of $\overline{AC}$ and F be the midpoint of $\overline{AB}$. If $\overline{AC} < \overline{AB}$, then $\angle ABE < \angle ACF$.

Exercise NEUT.60 Proof. By Definition NEUT.70 there exists a point B' belonging to $\overrightarrow{AB}$ such that $\overline{AB'} \cong \overline{AC}$. Let F' be the midpoint of $\overline{AB'}$. By Exercise NEUT.59 $\angle ACF' \cong \angle AB'E$. By Theorem NEUT.80 (outside angles) applied to $\triangle EB'B$, $\angle ABE < \angle AB'E$. By Exercise NEUT.36 $\overline{AF'} < \overline{AF}$ because $\overline{AB'} < \overline{AB}$. By Exercise NEUT.55 $F' \in \overrightarrow{AF}$; by Theorem PSH.37 $F' \in \operatorname{ins} ACF$ and by Definition NEUT.70 $\angle ACF' < \angle ACF$. By Theorem NEUT.76 $\angle ABE < \angle AB'E \cong \angle ACF' < \angle ACF$. $\square$

Exercise NEUT.61* Let $\mathcal{P}$ be a neutral plane and let A, B, C, E, and F be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) E is the point where the bisecting ray of $\angle ABC$ and $\overrightarrow{AC}$ intersect, (3) F is the point where the bisecting ray of $\angle ACB$ and $\overrightarrow{AB}$ intersect. If $\overline{AB} < \overline{AC}$, then $\overline{BE} < \overline{CF}$.

Exercise NEUT.61 Proof. In this proof a carefully sketched and labeled figure will be of great assistance in keeping things straight.

Since $\overline{AB} < \overline{AC}$, by Theorem NEUT.90 $\angle ACB < \angle ABC$. By Exercise NEUT.43 $\angle ACF < \angle ABE$ and $\angle BCF < \angle CBE$. By Theorem NEUT.68 (angle construction) there exists a point U on the A-side of $\overleftrightarrow{BE}$ such that $\angle EBU \cong \angle ECF = \angle ACF$. Since $\angle ACF < \angle ABE$ by Theorem NEUT.76 (transitivity for angles) $\angle EBU < \angle ABE$.

By Theorem NEUT.78 $U \in \operatorname{ins} \angle ABE$. By Theorem PSH.39 (Crossbar) $\overrightarrow{BU}$ and $\overrightarrow{AE}$ intersect at a point A'. Let O be the point (See Exercise PSH.26) of intersection of $\overrightarrow{BE}$ and $\overrightarrow{CF}$. Since $U \in \operatorname{ins} \angle OBF$, by Theorem PSH.39 (Crossbar) $\overrightarrow{BU}$ and $\overrightarrow{OF}$ intersect at a point F'; $F' \in \overrightarrow{OF} \subseteq \overrightarrow{CF}$, so that C–F'–F, and by Definition NEUT.70, $\overline{CF'} < \overline{CF}$.

Then $\angle A'CF' = \angle ACF \cong \angle EBU = \angle EBA'$ and again by Exercise NEUT.43 $\angle BCF' < \angle EBC$. By Theorem NEUT.76 (transitivity for angles) $\angle A'CB < \angle A'BC$, and by Theorem NEUT.91 $\overline{A'B} < \overline{A'C}$.

Now compare $\triangle A'BE$ and $\triangle A'CF'$, where A' corresponds to A', B corresponds to C, and E corresponds to F'. We see that

(1) $\angle CA'B = \angle EA'B = \angle F'A'C$ is common to both triangles,
(2) $\angle A'CF' \cong \angle A'BE$, and
(3) $\overline{A'B} < \overline{A'C}$.

Hence by Exercise NEUT.58 $\overline{BE} < \overline{CF'} < \overline{CF}$, so that $\overline{BE} < \overline{CF}$. □

Exercise NEUT.62* (Steiner-Lehmus) Let $\mathcal{P}$ be a neutral plane and let A, B, C, E, and F be points on $\mathcal{P}$ such that:

(1) A, B, and C are noncollinear,

(2) E is the point of intersection of the bisecting ray of $\angle ABC$, and $\overline{AC}$, and

(3) F is the point of intersection of the bisecting ray of $\angle ACB$ and $\overline{AB}$.

If $\overline{BE} \cong \overline{CF}$, then $\overline{AB} \cong \overline{AC}$.

Exercise NEUT.62 Proof. We prove the contrapositive, which is equivalent. If $\overline{AB}$ and $\overline{AC}$ are not congruent, then $\overline{BE}$ and $\overline{CF}$ are not congruent. By Theorem NEUT.72 (trichotomy for segments) we can choose the notation so that $\overline{AB} < \overline{AC}$. By Exercise NEUT.61 $\overline{BE} < \overline{CF}$. By Theorem NEUT.72 $\overline{BE}$ and $\overline{CF}$ are not congruent. □

Exercise NEUT.63* (A) Let $\mathcal{P}$ be a neutral plane and let A, B, C, and D be points on $\mathcal{P}$ such that:

(1) A, B, and C are noncollinear,

(2) $\angle BAC$ is acute,

(3) B and D are on opposite sides of $\overleftrightarrow{AC}$,

(4) $\angle CAD \cong \angle CAB$.

Then D is on the C-side of $\overleftrightarrow{AB}$.

(B) Let $\mathcal{P}$ be a neutral plane and let A, B, C, and D be points on $\mathcal{P}$ such that:

(1) A, B, and C are noncollinear,

(2) $\angle BAC$ is acute,

(3) B and D are on opposite sides of $\overleftrightarrow{AC}$,

(4) $\angle CAD$ is acute or right.

Then D is on the C-side of $\overleftrightarrow{AB}$.

Exercise NEUT.63 Proof. (A) Using Property B.3 of Definition IB.1 let B' be a point such that B'–A–B. By Theorem NEUT.83 $\angle CAD$ is acute. By Theorem NEUT.82 $\angle CAB'$ is obtuse. By Definition IB.11 B and B' are on opposite sides of $\overleftrightarrow{AC}$. By Theorem PSH.12 (plane separation) B' and D are on the same side of $\overleftrightarrow{AC}$. Since $\angle CAD$ is acute and $\angle CAB'$ is obtuse, by Theorem NEUT.83 $\angle CAD < \angle CAB'$. By Theorem NEUT.78 $D \in \text{ins}\,\angle CAB'$. By Definition PSH.36 C and D are on the same side of $\overleftrightarrow{AB}$, *i.e.*, D is on the C-side of $\overleftrightarrow{AB}$.

(B) Using Property B.3 of Definition IB.1 let B' be a point such that B'–A–B. Let E be a point on the C-side of $\overleftrightarrow{AB}$ such that $\overleftrightarrow{AE} \perp \overleftrightarrow{AB}$. Since it is acute, $\angle BAC < \angle BAE$ which is right, and since E and C are on the same side of $\overleftrightarrow{AB}$, by Theorem NEUT.78, $C \in \operatorname{ins} \angle BAE$.

By Corollary PSH.39.2 E and B are on opposite sides of $\overleftrightarrow{AC}$. D and B are on opposite sides of $\overleftrightarrow{AC}$, so D and E are on the same side of $\overleftrightarrow{AC}$. Now $\angle B'AC$ is obtuse because it is a supplement of $\angle BAC$ (*cf* Theorem NEUT.82).

If $\angle CAD$ is either a right or an acute angle, $\angle CAD < \angle B'AC$, and by Definition NEUT.70, $D \in \operatorname{ins} \angle B'AC$. By Definition PSH.36 D is on the C-side of $\overleftrightarrow{AB'} = \overleftrightarrow{AB}$. $\square$

Exercise NEUT.64* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, C_1, D_1, A_2, B_2, C_2, and D_2 be points on $\mathcal{P}$ such that:

(1) A_1, B_1, and C_1 are noncollinear,

(2) A_2, B_2, and C_2 are noncollinear,

(3) B_1 and D_1 are on opposite sides of $\overleftrightarrow{A_1C_1}$,

(4) B_2 and D_2 are on opposite sides of $\overleftrightarrow{A_2C_2}$,

(5) $\angle D_1A_1C_1 \cong \angle B_1A_1C_1$,

(6) $\angle D_2A_2C_2 \cong \angle B_2A_2C_2$,

(7) $\angle B_1A_1C_1 < \angle B_2A_2C_2$, and $\angle B_2A_2C_2$ is acute.

Then $\angle B_1A_1D_1 < \angle B_2A_2D_2$.

Exercise NEUT.64 Proof. By Theorem NEUT.83 $\angle B_1A_1C_1$ is acute. By Theorem NEUT.76 (transitivity for angles) $\angle D_1A_1C_1 \cong \angle B_1A_1C_1 < \angle B_2A_2C_2 \cong \angle D_2A_2C_2$. By Exercise NEUT.63 C_1 and D_1 are on the same side of $\overleftrightarrow{A_1B_1}$ and C_2 and D_2 are on the same side of $\overleftrightarrow{A_2B_2}$. By Exercise NEUT.42 $\angle B_1A_1D_1 < \angle B_2A_2D_2$. $\square$

Exercise NEUT.65* Let $\mathcal{P}$ be a neutral plane and let A, B, and C be noncollinear points on $\mathcal{P}$ such that each angle of $\triangle ABC$ is acute, $D = \operatorname{ftpr}(B, \overleftrightarrow{AC})$ and $E = \operatorname{ftpr}(C, \overleftrightarrow{AB})$, then $\overline{BD}$ and $\overline{CE}$ intersect at a point O which belongs to $\operatorname{ins} \triangle ABC$.

Exercise NEUT.65 Proof. By Exercise NEUT.20 $D \in \overline{AC}$ and $E \in \overline{AB}$. By Exercise PSH.26 $\overline{BD}$ and $\overline{CE}$ intersect at a point O which belongs to $\operatorname{ins} \triangle ABC$. $\square$

Exercise NEUT.66* Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, E, and F be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, $\angle ABC$

and $\angle ACB$ are both acute, and $\overline{AC} < \overline{AB}$, (2) D is the midpoint of $\overline{BC}$, E is the point of intersection of the bisecting ray of $\angle BAC$ and $\overline{BC}$, and $F = \text{ftpr}(A, \overleftrightarrow{BC})$. If the points on $\overleftrightarrow{BC}$ are ordered so that $B < C$, then $B < D < E < F < C$. Moreover, $\overline{AF} < \overline{AE} < \overline{AD} < \overline{AB}$.

Exercise NEUT.66 Proof. By Exercise NEUT.57 and Theorem ORD.6 $B < D < E < C$. By Exercise NEUT.53 $\angle AEB$ is obtuse so that by Theorem NEUT.44 $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$ are not perpendicular to each other and thus $F \neq E$. By Theorem NEUT.82 $\angle AEC$ is acute. By Exercise NEUT.20 $F \in \overline{EC}$.

By Theorem ORD.6 $B < D < E < F < C$. By Exercise NEUT.22 $\overline{AF} < \overline{AB}$. Applying Exercise NEUT.51(C) successively to $\triangle ABF$ and $\triangle ABE$ we have $\overline{AF} < \overline{AE} < \overline{AD} < \overline{AB}$. $\square$

Exercise NEUT.67* Let $\mathcal{P}$ be a neutral plane and let A, B, C, D, E, and F be points on $\mathcal{P}$ such that: (1) A, B, and C are noncollinear, (2) D is the midpoint of $\overline{BC}$, (3) E is the point of intersection of the bisecting ray of $\angle BAC$ and $\overline{BC}$, and (4) $F = \text{ftpr}(A, \overleftrightarrow{BC})$. If $\overline{AB} \cong \overline{AC}$, then $D = E = F$.

Exercise NEUT.67 Proof. Let $\mathcal{L} = \overleftrightarrow{AE}$. Then $\mathcal{R}_{\mathcal{L}}(B)$ is a point on $\overrightarrow{AC}$ and $\mathcal{R}_{\mathcal{L}}(\overline{AB}) \cong \overline{AB} \cong \overline{AC}$ so that by Property R.4 of Definition NEUT.2, $\mathcal{R}_{\mathcal{L}}(B) = C$. By Theorem NEUT.20, $D = E$ and by Theorem NEUT.48(A) $\mathcal{L} \perp \overline{BC}$ so that $E = F$. $\square$

The following exercise will strike the reader as decidedly odd, because we can hardly imagine a triangle such that the perpendicular bisectors of the sides do not intersect. But this is all we can prove at this stage of our development. The issue is resolved in Chapter 11, Theorem EUC.9.

Exercise NEUT.68* Let $\mathcal{P}$ be a neutral plane, A, B, and C be noncollinear points on $\mathcal{P}$. Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be the perpendicular bisectors of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ respectively. Then either (1) $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are concurrent at a point O, or (2) $\mathcal{L} \parallel \mathcal{M}$, $\mathcal{L} \parallel \mathcal{N}$, and $\mathcal{M} \parallel \mathcal{N}$.

Exercise NEUT.68 Proof. (Case 1: Two of the three lines intersect at a point O) We choose the notation so that $\mathcal{L}$ and $\mathcal{M}$ intersect at O. Both $O \in \mathcal{L}$ and $O \in \mathcal{M}$, so by Theorem NEUT.53 $\overline{OA} \cong \overline{OB}$ and $\overline{OA} \cong \overline{OC}$. By Theorem NEUT.14 (congruence is an equivalence relation) $\overline{OB} \cong \overline{OC}$. By Theorem NEUT.55 the line of symmetry of $\angle BOC$ is the line of symmetry, hence the perpendicular bisector of $\overline{BC}$, which is $\mathcal{N}$. Thus $O \in \mathcal{N}$. By Exercise I.1 $\mathcal{L} \cap \mathcal{M} \cap \mathcal{N} = \{O\}$.

(Case 2: $\mathcal{L} \parallel \mathcal{M}$) If any two of the lines $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ were concurrent, then by Case 1 $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ would all be concurrent. Hence $\mathcal{L} \parallel \mathcal{M}$, $\mathcal{L} \parallel \mathcal{N}$, and $\mathcal{M} \parallel \mathcal{N}$. $\square$

Exercise NEUT.69* Let $\mathcal{L}$ be a line on a neutral plane $\mathcal{P}$; let A, B, and C be points on $\mathcal{L}$ such that B–A–C, and let $\mathcal{M}$ be the line such that $A \in \mathcal{M}$ and $\mathcal{M} \perp \mathcal{L}$. We order the points on $\mathcal{L}$ such that $A < B$. Let X and Y be points on $\mathcal{L}$. Then $X < Y$ iff $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$.

Exercise NEUT.69 Proof. Since $A < B$ and B–A–C, by Theorem ORD.6 $C < A < B$. By Theorem PSH.38 $\overrightarrow{AB} = \overleftrightarrow{AB} \cap$ the B-side of $\mathcal{M}$ and $\overrightarrow{AC} = \overleftrightarrow{AB} \cap$ the C-side of $\mathcal{M}$. By Definition NEUT.1(D) $\mathcal{R}_{\mathcal{M}}(C)$–$\mathcal{R}_{\mathcal{M}}(A)$–$\mathcal{R}_{\mathcal{M}}(B)$. By Definition NEUT.1(A) $\mathcal{R}_{\mathcal{M}}(A) = A$. Now assume that $X < Y$. (Case 1: $Y = A$). Since $\mathcal{R}_{\mathcal{M}}(A) = A$, $\mathcal{R}_{\mathcal{M}}(C)$–$A$–$\mathcal{R}_{\mathcal{M}}(B)$. Since $X < A < B$ by Theorem ORD.8 $X \in \overrightarrow{AC}$. By Definition NEUT.1(B) X and $\mathcal{R}_{\mathcal{M}}(X)$ are on opposite sides of $\mathcal{M}$, thus $\mathcal{R}_{\mathcal{M}}(X) \in \overrightarrow{AB}$. By Theorem ORD.8 $\mathcal{R}_{\mathcal{M}}(X) > A = \mathcal{R}_{\mathcal{M}}(Y)$. By Definition ORD.1 $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$.

(Case 2: $Y < A$). Since $X < Y$ by Theorem ORD.4 $X < Y < A$. By Theorem ORD.8 $Y \in \overrightarrow{AC}$ and $X \in \overrightarrow{AC}$. By Definition NEUT.1(B) Y and $\mathcal{R}_{\mathcal{M}}(Y)$ are on opposite sides of $\mathcal{M}$, so that $\mathcal{R}_{\mathcal{M}}(Y) \in \overrightarrow{AB}$ and $\mathcal{R}_{\mathcal{M}}(X) \in \overrightarrow{AB}$. Since $X < Y < A$, by Theorem ORD.6 X–Y–A. By Definitions NEUT.1(A) and (D) $\mathcal{R}_{\mathcal{M}}(X)$–$\mathcal{R}_{\mathcal{M}}(Y)$–$A$. Since $\mathcal{R}_{\mathcal{M}}(Y) > A$, by Theorem ORD.6 $\mathcal{R}_{\mathcal{M}}(X) > \mathcal{R}_{\mathcal{M}}(Y)$. By Definition ORD.1 $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$.

(Case 3: $X = A$). The proof is similar to Case 1.

(Case 4: $A < Y$ and $X < A$) $X \in \overrightarrow{AC}$ and $Y \in \overrightarrow{AB}$, so that X and Y are on opposite sides of $\mathcal{M}$. Since X and $\mathcal{R}_{\mathcal{M}}(X)$ are on opposite sides of $\mathcal{M}$ and Y and $\mathcal{R}_{\mathcal{M}}(Y)$ are on opposite sides of $\mathcal{M}$, $\mathcal{R}_{\mathcal{M}}(X) \in \overrightarrow{AB}$ and $\mathcal{R}_{\mathcal{M}}(Y) \in \overrightarrow{AC}$ and thus by Theorem ORD.8 $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$.

(Case 5: $A < X < Y$) By Theorem ORD.6 A–X–Y. By Definition NEUT.1(A) and (D) A–$\mathcal{R}_{\mathcal{M}}(X)$–$\mathcal{R}_{\mathcal{M}}(Y)$. Since X and Y are both members of $\overrightarrow{AB}$ by Definition NEUT.1(B) $\mathcal{R}_{\mathcal{M}}(X)$ and $\mathcal{R}_{\mathcal{M}}(Y)$ are both members of $\overrightarrow{AC}$ and by Theorem ORD.8 $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$.

This shows that if $X < Y$, $\mathcal{R}_{\mathcal{M}}(Y) < \mathcal{R}_{\mathcal{M}}(X)$. The converse follows immediately from Definition NEUT.1(C), which says that $\mathcal{R}_{\mathcal{M}} = \mathcal{R}_{\mathcal{M}}^{-1}$. $\square$

Exercise NEUT.70* Let $\mathcal{P}$ be a neutral plane, $\mathcal{L}$ and $\mathcal{M}$ be lines on $\mathcal{P}$ which intersect at the point O, A be a point on $\mathcal{L}$ distinct from O, and X and Y be points on $\mathcal{M}$ distinct from O such that X and Y are on the same side of $\mathcal{L}$. Let the points on $\mathcal{M}$ be ordered so that $O < X$. Then $O < X < Y$ iff $\angle OAX < \angle OAY$.

Exercise NEUT.70 Proof. (I) If $O < X < Y$, then by Theorem ORD.6 O–X–Y. By Definition IB.3 $X \in \overrightarrow{OY}$. By Theorem PSH.37 $X \in \text{ins}\,\angle OAY$. By Definition NEUT.70 $\angle OAX < \angle OAY$.

(II) Since X and Y are on the same side of $\mathcal{L}$, $O < Y$. Now suppose $\angle OAX < \angle OAY$. By Theorem ORD.5 (trichotomy for ordering) one and only one of the following statements is true: $X = Y$; $X < Y$; $Y < X$. By Theorem NEUT.75 (trichotomy for angles), one and only one of the following statements is true: $\angle OAX \cong \angle OAY$; $\angle OAX < \angle OAY$; $\angle OAY < \angle OAX$. If X were equal to Y, then $\angle OAX$ would be equal (and therefore congruent) to $\angle OAY$. This contradicts the fact that $\angle OAX < \angle OAY$. If Y were less than X, then by Part I, $\angle OAY$ would be smaller than $\angle OAX$. This contradicts the fact that $\angle OAX < \angle OAY$. Hence $X < Y$. □

Exercise NEUT.71* Let $\mathcal{P}$ be a neutral plane, A, B, and C be non-collinear points on $\mathcal{P}$, and D be a member of $\overleftrightarrow{BC} \setminus \{B, C\}$. Then B–D–C iff $\angle ACB < \angle ADB$ and $\angle ABC < \angle ADC$.

Exercise NEUT.71 Proof. (I: If B–D–C, then $\angle ACB < \angle ADB$ and $\angle ABC < \angle ADC$.) By Theorem NEUT.80 (outside angles) applied to $\triangle ADC$, $\angle ACB < \angle ADB$. By the same theorem applied to $\triangle ABD$ $\angle ABC < \angle ADC$.

(II: If $\angle ACB < \angle ADB$ and $\angle ABC < \angle ADC$, then B–D–C.) Suppose $\angle ACB < \angle ADB$ and $\angle ABC < \angle ADC$. By Property B.2 of Definition IB.1 (trichotomy for betweenness) one and only one of the following statements is true: C–B–D; B–C–D; B–D–C. By Theorem NEUT.75 (trichotomy for angles) one and only one of the following statements is true: $\angle ACB \cong \angle ADB$; $\angle ACB < \angle ADB$; $\angle ADB < \angle ACB$, and one and only one of the following statements is true: $\angle ABC \cong \angle ADC$; $\angle ABC < \angle ADC$; $\angle ADC < \angle ABC$. If C–B–D were true, then by Theorem NEUT.80 (outside angles) applied to $\triangle ADB$, $\angle ADC$ would be smaller than $\angle ABC$, contrary to the given fact that $\angle ABC < \angle ADC$. If B–C–D were true, then by Theorem NEUT.80 (outside angles) applied to $\triangle ADC$, $\angle ADB$ would be smaller than $\angle ACB$, contrary to the given fact that $\angle ACB < \angle ADB$. Hence B–D–C. □

Exercise NEUT.72* Let A, B, C, and M be points on the neutral plane $\mathcal{P}$ such that $A \neq B$, $A \neq C$, M is the midpoint of $\overline{AB}$ and M is the midpoint of $\overline{AC}$. Then $B = C$.

Exercise NEUT.72 Proof. Since $\overline{AM} \cong \overline{AM}$, by Exercise NEUT.33 $\overline{AB} \cong \overline{AC}$. By Definition NEUT.3(C) A–M–B and A–M–C. By Theorem PSH.16 $\overrightarrow{AM} = \overrightarrow{AB} = \overrightarrow{AC}$. By Theorem PSH.24 $C \in \overrightarrow{AB}$. By Property R.4 of Definition NEUT.2, $B = C$. $\square$

Exercise NEUT.73* Let A and M be distinct points on the neutral plane $\mathcal{P}$. Then there exists a unique point B such that M is the midpoint of $\overline{AB}$.

Exercise NEUT.73 Proof. (I: Existence.) By Property B.3 of Definition IB.1 there exists a point D such that A–M–D. By Theorem NEUT.67 (segment construction) there exists a unique point B belonging to $\overrightarrow{MD}$ such that $\overline{MB} \cong \overline{AM}$. By Theorem PSH.13 $\{X \mid A\text{–}M\text{–}X\} = \overrightarrow{MD}$. Hence A–M–B. By Definition NEUT.3(C) M is the midpoint of $\overline{AB}$.

(II: Uniqueness.) This is Exercise NEUT.72 above. $\square$

Exercise NEUT.74* Let $\mathcal{P}$ be a neutral plane, $\mathcal{L}$ be a line on $\mathcal{P}$, and θ be the mapping of $\mathcal{P}$ into $\mathcal{P}$ such that: (1) For every member X of $\mathcal{L}$, $\theta(X) = X$. (2) For every member X of $\mathcal{P} \setminus \mathcal{L}$, $\theta(X)$ is the point such that . Then $\theta = \mathcal{R}_{\mathcal{L}}$.

Exercise NEUT.74 Proof. (Case 1: $X \in (\mathcal{P} \setminus \mathcal{L})$) By Theorem NEUT.55 $\operatorname{ftpr}(X, \mathcal{L})$ is the midpoint of $\overline{X\mathcal{R}_{\mathcal{L}}(X)}$; by hypothesis it is also the midpoint of $\overline{X\theta(X)}$. By Exercise NEUT.72, $\theta(X) = \mathcal{R}_{\mathcal{L}}(X)$.

(Case 2: $X \in \mathcal{L}$) By Property (A) of Definition NEUT.1, $\mathcal{R}_{\mathcal{L}}(X) = X = \theta(X)$. Hence $\theta = \mathcal{R}_{\mathcal{L}}$. $\square$

Exercise NEUT.75* Let $\mathcal{P}$ be a neutral plane and let θ be an isometry of $\mathcal{P}$. Then:

(A) If A and B are distinct points of $\mathcal{P}$ and if M is the midpoint of $\overline{AB}$, then $\theta(M)$ is the midpoint of $\overline{\theta(A)\theta(B)}$.

(B) Let A, B, and C be noncollinear points on $\mathcal{P}$. If H is a member of $\operatorname{ins}\angle BAC$ such that $\overrightarrow{AH}$ is the bisecting ray of $\angle BAC$, then $\overrightarrow{\theta(A)\theta(B)}$ is the bisecting ray of $\angle\theta(B)\theta(A)\theta(C)$ and if D is the point of intersection of $\overrightarrow{AH}$ and $\overleftrightarrow{BC}$, then $\theta(D)$ is the point of intersection of $\overrightarrow{\theta(A)\theta(H)}$ and $\overleftrightarrow{\theta(B)\theta(C)}$.

(C) If $\mathcal{L}$ is line on $\mathcal{P}$, Q is a member of $\mathcal{P} \setminus \mathcal{L}$, $\mathcal{M} = \mathrm{pr}(Q, \mathcal{L})$, and $F = \mathrm{ftpr}(Q, \mathcal{L})$, then $\theta(\mathcal{M}) = \mathrm{pr}(\theta(Q), \theta(\mathcal{L}))$ and $\theta(F) = \mathrm{ftpr}(\theta(Q), \theta(\mathcal{L}))$.

Exercise NEUT.75 Proof. (A) By Definition NEUT.3(C) A–M–B and $\overline{AM} \cong \overline{BM}$. By Definition NEUT.1(D) $\theta(A)$–$\theta(M)$–$\theta(B)$. By Theorem NEUT.13 $\theta(\overline{AM}) \cong \theta(\overline{BM})$. By Theorem NEUT.15 $\theta(\overline{AM}) = \overline{\theta(A)\theta(M)}$ and $\theta(\overline{BM}) = \overline{\theta(B)\theta(M)}$. Hence $\overline{\theta(A)\theta(M)} \cong \overline{\theta(B)\theta(M)}$. By Definition NEUT.3(C) $\theta(M)$ is the midpoint of $\overline{\theta(A)\theta(B)}$.

(B) By Theorem NEUT.39 $\angle BAH \cong \angle CAH$. By Theorem NEUT.13 $\theta(\angle BAH) \cong \theta(\angle CAH)$. By Theorem NEUT.15 $\theta(\angle BAH) = \angle\theta(B)\theta(A)\theta(H)$ and $\theta(\angle CAH) = \angle\theta(C)\theta(A)\theta(H)$. By Theorem NEUT.39 and Theorem NEUT.15 $\overrightarrow{\theta(A)\theta(H)}$ is the bisecting ray of $\angle\theta(B)\theta(A)\theta(C)$. By Theorems NEUT.11 and NEUT.15 θ is a bijection of $\overrightarrow{AH}$ onto $\overrightarrow{\theta(A)\theta(H)}$ and is also a bijection of $\overline{BC}$ onto $\overline{\theta(B)\theta(C)}$. By Theorem NEUT.15 $\theta(H) \in \mathrm{ins}\,\angle\theta(B)\theta(A)\theta(C)$. By Theorem PSH.39 (Crossbar) $\overrightarrow{\theta(A)\theta(H)} \cap \overline{\theta(B)\theta(C)}$ is a singleton. Since $\theta(D)$ is a member of both $\overrightarrow{\theta(A)\theta(H)}$ and $\overline{\theta(B)\theta(C)}$, $\overrightarrow{\theta(A)\theta(H)} \cap \overline{\theta(B)\theta(C)} = \{\theta(D)\}$.

(C) Let G and K be points on $\mathcal{L}$ such that G–F–K. By Definition NEUT.1(D) $\theta(G)$–$\theta(F)$–$\theta(K)$. By Theorem NEUT.11 and NEUT.15 θ is a bijection of $\mathcal{L}$ onto $\theta(\mathcal{L})$ and of $\mathcal{M}$ onto $\theta(\mathcal{M})$. By Corollary NEUT.44.1, since $\mathcal{L} \perp \mathcal{M}$, $\theta(\mathcal{L}) \perp \theta(\mathcal{M})$. By elementary set theory $\theta(\mathcal{L}) \cap \theta(\mathcal{M}) = \{\theta(F)\}$, so that $\theta(F) = \mathrm{ftpr}(\theta(Q), \theta(\mathcal{L}))$. $\square$

Exercise NEUT.76* Let $\mathcal{P}$ be a neutral plane and let A_1, B_1, C_1, D_1, E_1, F_1, A_2, B_2, C_2, D_2, E_2, and F_2 be points on $\mathcal{P}$ such that:

(1) A_1, B_1, and C_1 are noncollinear; A_2, B_2, and C_2 are noncollinear; and $\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2$.

(2) θ is an isometry of $\mathcal{P}$ such that $\theta(\triangle A_1B_1C_1) = \triangle A_2B_2C_2$, $\theta(A_1) = A_2$, $\theta(B_1) = B_2$, and $\theta(C_1) = C_2$.

(3) D_1 is the midpoint of $\overline{B_1C_1}$ and D_2 is the midpoint of $\overline{B_2C_2}$.

(4) E_1 is the point of intersection of the bisecting ray of $\angle B_1A_1C_1$ and $\overline{B_1C_1}$; and E_2 is the point of intersection of the bisecting ray of $\angle B_2A_2C_2$ and $\overline{B_2C_2}$.

(5) $F_1 = \mathrm{ftpr}(A_1, \overleftrightarrow{B_1C_1})$ and $F_2 = \mathrm{ftpr}(A_2, \overleftrightarrow{B_2C_2})$.

Then $\theta(D_1) = D_2$, $\theta(E_1) = E_2$, and $\theta(F_1) = F_2$.

Exercise NEUT.76 Proof. By Exercise NEUT.75(A) $\theta(D_1)$ is the midpoint of $\overline{B_1C_1}$ and so by Theorem NEUT.50 $\theta(D_1) = D_2$. By Exercise NEUT.75(B) $\theta(E_1)$ is the point of intersection of the bisecting ray of

$\angle B_2A_2C_2$ and $\overline{BC}$ so $\theta(E_1) = E_2$. By Exercise NEUT.75 part (C), $\theta(F_1) = \text{ftpr}(\theta(A_1), \overleftrightarrow{\theta(B_1)\theta(C_1)}) = F_2$. $\square$

Exercise NEUT.77* Let A, B, C, D, and E be points on the neutral plane $\mathcal{P}$ such that A–B–C, A–B–D, A–D–E, and $\overline{BC} \cong \overline{DE}$, then A–C–E.

Exercise NEUT.77 Proof. Since A–B–C and A–B–D, by Corollary PSH.8.2 exactly one of $C = D$, B–D–C, or B–C–D holds. In the the rest of the proof we may invoke Theorem PSH.8 and its corollaries without further citation.

(Case 1: $D = C$). A–D–E yields A–C–E.

(Case 2: B–C–D). Since A–B–C and B–C–D, A–B–C–D so that A–C–D. Since A–D–E, we have A–C–D–E, so that A–C–E.

(Case 3: B–D–C). Since A–B–C, A–B–D–C, and hence A–D–C. Since A–D–E, by Corollary PSH.8.2, either $E = C$, D–E–C, or D–C–E. In the first two of these alternatives, $E \in \overline{AC}$; if this were true, by Definition NEUT.70 $\overline{DE}$ would be smaller than $\overline{BC}$. By Theorem NEUT.72 (trichotomy for segments) this would contradict the fact that $\overline{DE} \cong \overline{AC}$. Therefore A–B–D–C–E, hence A–C–E. $\square$

Exercise NEUT.78* Let $\mathcal{P}$ be a neutral plane and let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be distinct lines on $\mathcal{P}$ concurrent at the point O such that no two of them are perpendicular to each other, Q be a member of $\mathcal{F} \setminus \{O\}$, $R = \text{ftpr}(Q, \mathcal{G})$, $S = \text{ftpr}(R, \mathcal{H})$ and $T = \text{ftpr}(Q, \mathcal{H})$. Then $S \neq T$.

Exercise NEUT.78 Proof. If $S = T$ then $\overleftrightarrow{TQ} = \overleftrightarrow{SR}$ because there is only one perpendicular to a line at a point on that line, by Theorem NEUT.47(B); $R \in \overleftrightarrow{TQ}$, and hence $\overleftrightarrow{TQ} = \overleftrightarrow{RQ}$. Then both $\mathcal{G} \perp \overleftrightarrow{TQ}$ and $\mathcal{H} \perp \overleftrightarrow{TQ}$ so that by Theorem NEUT.47(A) $\mathcal{G} \parallel \mathcal{H}$ which is impossible because $\mathcal{G}$ and $\mathcal{H}$ intersect at the point O. $\square$

Exercise NEUT.79* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$ and Q be a member of ins $\angle BAC$. Then $\overrightarrow{AQ}$ is the bisecting ray of $\angle BAC$ iff for every member T of $\overrightarrow{AQ}$, $\overline{TD} \cong \overline{TE}$, where $D = \text{ftpr}(T, \overleftrightarrow{AB})$ and $E = \text{ftpr}(T, \overleftrightarrow{AC})$.

Exercise NEUT.79 Proof. (I: If $\overrightarrow{AQ}$ is the bisecting ray of $\angle BAC$, then $\overline{TD} \cong \overline{TE}$.) By Exercise NEUT.19, $\angle BAT$ and $\angle CAT$ are acute, and by Exercise NEUT.18, $D \in \overrightarrow{AB}$ and $E \in \overrightarrow{AC}$. By Theorem PSH.16, $\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{AC} = \overrightarrow{AE}$ so that $\angle TAB = \angle TAD$ and $\angle TAC = \angle TAE$.

By Definition NEUT.99 $\angle ADT$ and $\angle AET$ are right. Then by Theorem NEUT.69 $\angle ADT \cong \angle AET$. By Theorem NEUT.39 $\angle TAD \cong \angle TAE$. By Theorem NEUT.60 (Kite) $\overleftrightarrow{AT}$ is the line of symmetry of $\angle DAE$, $\angle DTE$, $\overline{DE}$, and $\square ADTE$ so that $\mathcal{R}_{\overleftrightarrow{AT}}(D) = E$. By Theorem NEUT.15, $\mathcal{R}_{\overleftrightarrow{AT}}(\overline{TD}) = \overline{\mathcal{R}_{\overleftrightarrow{AT}}(T)\mathcal{R}_{\overleftrightarrow{AT}}(D)} = (\overline{TE})$ so that $\overline{TD} \cong \overline{TE}$.

(II: If $\overline{TD} \cong \overline{TE}$, then $\overrightarrow{AT} = \overrightarrow{AQ}$ is the bisecting ray of $\angle BAC$) Since $\overline{AT} = \overline{AT}$ and $\overline{TD} \cong \overline{TE}$, by Theorem NEUT.96 there exists an isometry α such that $\alpha(A) = A$, $\alpha(T) = T$, and $\alpha(D) = E$. Then by Theorem NEUT.15 $\alpha(\angle DAT) = \angle\alpha(D)\alpha(A)\alpha(T) = \angle EAT$ so that by Theorem NEUT.39, $\overrightarrow{AQ}$ is the bisecting ray of $\angle BAC$. $\square$

Exercise NEUT.80* Prove parts (B), (C), and (D) in Theorem NEUT.83: Let $\mathcal{P}$ be a neutral plane; then

(B) every angle on $\mathcal{P}$ congruent to an obtuse angle is obtuse;

(C) every angle on $\mathcal{P}$ smaller than an acute angle is acute; and

(D) every acute angle on $\mathcal{P}$ is smaller than every obtuse angle on $\mathcal{P}$.

Exercise NEUT.80 Proof. (B) Suppose $\angle BAC$ is obtuse and $\angle BAC \cong \angle B'A'C'$. By Theorem NEUT.38 there exists an isometry α such that $\alpha(\angle BAC) = \angle B'A'C'$, $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$, and $\alpha(\overrightarrow{AC}) = \overrightarrow{A'C'}$. Let D be a point such that D–A–C; then $\angle BAD$ is supplementary to $\angle BAC$ and thus by Theorem NEUT.82 $\angle BAD$ is acute.

Let $D' = \alpha(D)$; by Theorem NEUT.15, $\alpha(\overrightarrow{AD}) = \overrightarrow{\alpha(A)\alpha(D)} = \overrightarrow{A'D'}$. Then

$$\alpha(\angle BAD) = \alpha(\overrightarrow{AB} \cup \overrightarrow{AD}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{AD}) = \overrightarrow{A'B'} \cup \overrightarrow{A'D'} = \angle B'A'D'.$$

Also,

$$\alpha(\angle BAC) = \alpha(\overrightarrow{AB} \cup \overrightarrow{AC}) = \alpha(\overrightarrow{AB}) \cup \alpha(\overrightarrow{AC}) = \overrightarrow{A'B'} \cup \overrightarrow{A'C'} = \angle B'A'C'.$$

By Definition NEUT.1(D), $\alpha(D)$–$\alpha(A)$–$\alpha(C)$; since $\alpha(\overrightarrow{AC}) = \overrightarrow{A'C'}$, D'–A'–C'. Thus $\angle B'A'C'$ and $\angle B'A'D'$ are supplements. By part (A), $\angle B'A'D'$ is acute; therefore $\angle B'A'C'$ is obtuse.

(C) Let $\mathcal{A}$ be an acute angle and let $\mathcal{C}$ be a right angle. Then by Definition NEUT.81, $\mathcal{A} < \mathcal{C}$. If $\mathcal{B}$ is an angle and $\mathcal{B} < \mathcal{A}$, $\mathcal{B} < \mathcal{A} < \mathcal{C}$ so by Theorem NEUT.76 (transitivity for angles), $\mathcal{B} < \mathcal{C}$ and $\mathcal{B}$ is acute.

(D) Let $\mathcal{A}$ be an acute angle and $\mathcal{B}$ an obtuse angle. Let $\mathcal{C}$ be a right angle. Then by Definition NEUT.81 $\mathcal{A} < \mathcal{C} < \mathcal{B}$ so that by Theorem NEUT.76 $\mathcal{A} < \mathcal{B}$. $\square$

Exercise NEUT.81* Without invoking Theorem NEUT.15 parts (4) through (7), prove that if $A \neq B$ are points in a neutral plane,

(A) $[AB] \ncong [AB)$ and $[AB] \ncong (AB]$;

(B) $[AB] \ncong (AB)$; and

(C) $(AB) \ncong [AB)$ and $(AB) \ncong (AB]$.

Exercise NEUT.81 Proof. We may invoke the fact that an isometry is a belineation, that is, preserves betweenness.

(A) We show that $[AB] \ncong [AB)$. Suppose α is an isometry such that $\alpha([AB)) = [AB]$. $\alpha(A)$ is either A or B, for if $A{-}\alpha(A){-}B$ then $\alpha^{-1}(A){-}A{-}\alpha^{-1}(B)$ which is impossible since both $\alpha^{-1}(A)$ and $\alpha^{-1}(B)$ are members of $[AB)$.

If $\alpha(A) = B$ then there exists a point $C \in [AB)$ such that $\alpha(C) = A$. By Theorem PSH.22 we may pick D such that $C{-}D{-}B$; then $A{-}C{-}D$ and since α preserves betweenness, $\alpha(A){-}\alpha(C){-}\alpha(D)$, that is, $B{-}A{-}\alpha(D)$, so that $\alpha(D) \notin [AB]$. But α maps $[AB)$ onto $[AB]$, so that $\alpha(D) \in [AB]$, a contradiction.

If $\alpha(A) = A$ then there exists a point $C \in [AB)$ such that $\alpha(C) = B$. Since $A{-}C{-}B$, by Theorem PSH.22 we may pick D such that $C{-}D{-}B$ so that $A{-}C{-}D$; then since α preserves betweenness, $\alpha(A){-}\alpha(C){-}\alpha(D)$, that is, $A{-}B{-}\alpha(D)$, so that $\alpha(D) \notin [AB]$. But α maps $[AB)$ onto $[AB]$, so that $\alpha(D) \in [AB]$, a contradiction.

A similar proof shows that $[AB] \ncong (AB]$.

(B) We show that $[AB] \ncong (AB)$. Suppose α is an isometry such that $\alpha((AB)) = [AB]$. Then there exists a point $C \in (AB)$ such that $\alpha(C) = A$. Since $A{-}C{-}B$, $\alpha(A){-}\alpha(C){-}\alpha(D)$, that is to say, $\alpha(A){-}A{-}\alpha(D)$, where both $\alpha(A)$ and $\alpha(D)$ are members of $[AB]$, and $\alpha(A) \neq A \neq \alpha(D)$ (betweenness implies distinctness).

Neither $\alpha(A)$ nor $\alpha(D)$ can equal A, for this would contradict distinctness.

Suppose $\alpha(A) = B$; we know already that $\alpha(D) \in [AB]$ and that $\alpha(D) \neq A$; by distinctness $\alpha(D) \neq B = \alpha(A)$. Thus $A{-}\alpha(D){-}B$ and $\alpha(A){-}A{-}\alpha(D){-}B$, that is $B{-}A{-}\alpha(D){-}B$ which is impossible.

If $\alpha(D) = B$ then by similar reasoning, $A{-}\alpha(A){-}B$ and $A{-}\alpha(A){-}A{-}B$ which is impossible.

If neither $\alpha(A)$ or $\alpha(D)$ is B, then both $A{-}\alpha(A){-}B$ and $A{-}\alpha(D){-}B$ so that $A{-}\alpha(A){-}A{-}\alpha(D){-}B$ which again is impossible.

(C) We show that $(AB) \ncong [AB)$. Suppose α is an isometry such that $\alpha((AB)) = [AB)$. Then there exists a point $C \in (AB)$ such that $\alpha(C) = A$. Since $A{-}C{-}B$, $\alpha(A){-}\alpha(C){-}\alpha(D)$, that is to say, $\alpha(A){-}A{-}\alpha(D)$, where both

$\alpha(A)$ and $\alpha(D)$ are members of $\overline{AB}$, and $\alpha(A) \neq A \neq \alpha(D)$ (betweenness implies distinctness).

Then both A–$\alpha(A)$–B and A–$\alpha(D)$–B so that A–$\alpha(A)$–A–$\alpha(D)$–B which is impossible. A similar proof shows that $\overline{AB} \ncong \overline{AB}$. $\square$

Exercise NEUT.82* Let A, B, and C be points on a neutral plane such that $A \neq B$, $C \in \overrightarrow{AB}$, and $\overline{AB} \cong \overline{AC}$. Let φ be the isometry such that $\varphi(\overline{AB}) = \overline{AC}$. (A) Using only NEUT.1 through NEUT.20, show that if φ is its own inverse, then $B = C$. (B) Discuss why this type of proof will not work in the general case, where φ is not necessarily its own inverse. If it did, we could prove Property R.4 of Definition NEUT.2 as a theorem.

Exercise NEUT.82(A) Proof. Assume that $B \neq C$. Then either A–B–C or A–C–B. If A–B–C, then $\varphi^{-1}(\overline{AC}) = \overline{AB}$. Let us assume that A–C–B. By Remark NEUT.16 either $\varphi(A) = A$ and $\varphi(B) = C$, or $\varphi(A) = C$ and $\varphi(B) = A$.

If $\varphi(A) = A$ and $\varphi(B) = C$, then since φ is its own inverse, $\varphi(C) = B$. Since φ preserves betweenness, $\varphi(A)$–$\varphi(C)$–$\varphi(B)$, which is to say A–B–C which contradicts our assumption.

If $\varphi(A) = C$ and $\varphi(B) = A$, then since φ is its own inverse, $\varphi(C) = A$, which contradicts the assumption that φ is a 1-1 mapping. $\square$

Exercise NEUT.83* Let $\mathcal{L}$ be a line on a neutral plane $\mathcal{P}$. Let φ be a mapping obeying properties (B) through (D) of Definition NEUT.1. Then if every point O of $\mathcal{L}$ is contained in some line $\overleftrightarrow{A\varphi(A)}$, where $A \notin \mathcal{L}$, Property (A) of Definition NEUT.1 holds for φ.

Exercise NEUT.83 Proof. By Property (B), $\varphi(A)$ is on the opposite side of $\mathcal{L}$ from A, so the point O of intersection of $\mathcal{L}$ and $\overleftrightarrow{A\varphi(A)}$ satisfies A–O–$\varphi(A)$. By Property (D) $\varphi(A)$–$\varphi(O)$–$\varphi(\varphi(A))$, that is, $\varphi(A)$–$\varphi(O)$–A. If $\varphi(O) \neq O$, $\varphi(O) \notin \mathcal{L}$, so $\varphi(\varphi(O)) = O \notin \mathcal{L}$, a contradiction. Therefore $\varphi(O) = O$. $\square$

The following scrap came from some attempts to show that there is a Pasch plane in which there is a line over which there exists no reflection. It seemed, somehow, worth saving, as it gives some insight into the structure of fixed lines.

Exercise NEUT.84* Let $\mathcal{L}$ be a line in a neutral plane $\mathcal{P}$, and let A and B be distinct points on the same side of $\mathcal{L}$. Then if φ is a reflection over $\mathcal{L}$, the lines $\overleftrightarrow{\varphi(A)B}$ and $\overleftrightarrow{A\varphi(B)}$ intersect at a point $P \in \mathcal{L}$.

Exercise NEUT.84 Proof. (A) By Theorem NEUT.15 φ is a collineation, so maps lines into lines. By Theorem NEUT.22 the lines $\overleftrightarrow{A\varphi(A)}$ and $\overleftrightarrow{B\varphi(B)}$ do not intersect. By Property (B) of Definition NEUT.1, A and $\varphi(B)$ are on opposite sides of $\mathcal{L}$, so by Axiom PSA the segment $\overline{A\varphi(B)}$ intersects $\mathcal{L}$ at some point P.

It follows from Theorem NEUT.15 and Property (A) of Definition NEUT.1 that $\varphi(\overline{A\varphi(B)}) = \overline{\varphi(A)B}$ and thus $P \in \overline{\varphi(A)B}$. Since these two segments are distinct, they have only one point of intersection, which must be P. $\square$

Chapter 9: Exercises and Answers for 9 Free Segments of a Neutral Plane (FSEG)

Exercise FSEG.1* Let A, B, C, and D be points on the neutral plane $\mathcal{P}$ such that $A \neq B$ and $C \neq D$. Then $[\overline{AB}] = [\overline{CD}]$ iff $\overline{AB} \cong \overline{CD}$.

Exercise FSEG.1 Proof. (I: If $\overline{AB} \cong \overline{CD}$, then $[\overline{AB}] = [\overline{CD}]$.) (A) Let $[\overline{AB}]$ be a free segment; using Definition FSEG.2, let $\overline{XY} \in [\overline{AB}]$; then $\overline{XY} \cong \overline{AB}$. Since $\overline{AB} \cong \overline{CD}$, by Theorem NEUT.14 $\overline{XY} \cong \overline{CD}$, and by Definition FSEG.2, $\overline{XY} \in [\overline{CD}]$. Therefore $[\overline{AB}] \subseteq [\overline{CD}]$. (B) If we interchange "A" with "C" and "B" with "D" in part (A) we get $[\overline{CD}] \subseteq [\overline{AB}]$. Thus $[\overline{AB}] = [\overline{CD}]$.

(II: Conversely, if $[\overline{AB}] = [\overline{CD}]$, then $\overline{AB} \cong \overline{CD}$.) Since $[\overline{AB}] = [\overline{CD}]$, $\overline{AB} \in [\overline{CD}]$, and thus $\overline{AB} \cong \overline{CD}$. $\square$

Exercise FSEG.2* Let A, B, C, and D be points on the neutral plane $\mathcal{P}$ such that $A \neq B$ and $C \neq D$. Then $[\overline{AB}] < [\overline{AB}] \oplus [\overline{CD}]$.

Exercise FSEG.2 Proof. By Theorem FSEG.1 there exists a point E on $\mathcal{P}$ such that A–B–E and $\overline{BE} \cong \overline{CD}$. By Definition NEUT.70, $\overline{AB} < \overline{AE}$, so that by Definition FSEG.3 $[\overline{AB}] < [\overline{AE}] = [\overline{AB}] \oplus [\overline{CD}]$. $\square$

Exercise FSEG.3* Let A and B be distinct points on the neutral plane $\mathcal{P}$ and let m and n be natural numbers. For the purposes of this exercise, we define certain rational multiples of free segments, using induction, as follows:

Definition (1): Define $1[\overline{AB}] = [\overline{AB}]$. For any n, if a point C has been determined so that $n[\overline{AB}] = [\overline{AC}]$ define $(n+1)[\overline{AB}] = [\overline{AC}] \oplus [\overline{AB}]$.

Definition (2): Using Theorem NEUT.50, let M be the midpoint of $\overline{AB}$. Then define $\frac{1}{2}[\overline{AB}] = [\overline{AM}]$. If for any m, C has been determined so that $\frac{1}{2^m}[\overline{AB}] = [\overline{AC}]$, let D be the midpoint of $\overline{AC}$ and define $\frac{1}{2^{m+1}}[\overline{AB}] = [\overline{AD}]$.

Definition (3): For any n and m, define $\frac{n}{2^m}[\overline{AB}] = \frac{1}{2^m}\left(n[\overline{AB}]\right)$.

Let A, B, C, and D be points on the neutral plane such that $A \neq B$ and $C \neq D$; use the definitions above to show the following:

(I) If $[\overline{AB}] < [\overline{CD}]$, then for any natural numbers n and m,

(A) $n[\overline{AB}] < n[\overline{CD}]$,

(B) $\frac{1}{2^m}[\overline{AB}] < \frac{1}{2^m}[\overline{CD}]$, and

(C) $\frac{n}{2^m}[\overline{AB}] < \frac{n}{2^m}[\overline{CD}]$.

(II) $\frac{n}{2^m}([\overline{AB}] \oplus [\overline{CD}]) = \frac{n}{2^m}[\overline{AB}] \oplus \frac{n}{2^m}[\overline{CD}]$.

Exercise FSEG.3 Proof. (I) (A) The proof is by induction. First, (A) is trivially true for $n = 1$. Assume that for some $n > 1$ we have shown that $(n-1)[\overline{AB}] < (n-1)[\overline{CD}]$; by hypothesis $[\overline{AB}] < [\overline{CD}]$; using Definition (1), Theorem FSEG.9(II), and Definition (1) again,

$$n[\overline{AB}] = (n-1)[\overline{AB}] \oplus [\overline{AB}] < (n-1)[\overline{CD}] \oplus [\overline{CD}] = n[\overline{CD}],$$

proving assertion (A).

(B) Again using induction, (B) is trivially true for $m = 0$. Assume that for some $m \geq 1$ it has been shown that $\frac{1}{2^{m-1}}[\overline{AB}] < \frac{1}{2^{m-1}}[\overline{CD}]$. Using Definition (2) above, let E be the point on $\overrightarrow{AB}$ such that $[\overline{AE}] = \frac{1}{2^{m-1}}[\overline{AB}]$, F be the point on $\overrightarrow{CD}$ such that $[\overline{CF}] = \frac{1}{2^{m-1}}[\overline{CD}]$, G be the midpoint of $\overline{AE}$, and H be the midpoint of $\overline{CF}$. Then $\frac{1}{2^m}[\overline{AB}] = [\overline{AG}]$ and $\frac{1}{2^m}[\overline{CD}] = [\overline{CH}]$. Since $\overline{AE} < \overline{CF}$, by Exercise NEUT.36 $\overline{AG} < \overline{CH}$ so $\frac{1}{2^m}[\overline{AB}] < \frac{1}{2^m}[\overline{CD}]$.

(C) By part (A) $n[\overline{AB}] < n[\overline{CD}]$ so by part (B) $\frac{n}{2^m}[\overline{AB}] < \frac{n}{2^m}[\overline{CD}]$.

(II) (a) $n([\overline{AB}] \oplus [\overline{CD}]) = n[\overline{AB}] \oplus n[\overline{CD}]$ is trivially true for $n = 1$. Assume that for some $n > 1$,

$$(n-1)([\overline{AB}] \oplus [\overline{CD}]) = (n-1)[\overline{AB}] \oplus (n-1)[\overline{CD}];$$

then

$$\begin{aligned} n([\overline{AB}] \oplus [\overline{CD}]) &= (n-1)([\overline{AB}] \oplus [\overline{CD}]) \oplus ([\overline{AB}] \oplus [\overline{CD}]) \\ &= ((n-1)[\overline{AB}] \oplus [\overline{AB}]) \oplus ((n-1)[\overline{CD}] \oplus [\overline{CD}]) \\ &= n[\overline{AB}] \oplus n[\overline{CD}]. \end{aligned}$$

(b) For $m = 1$, $\frac{1}{2^{m-1}}([\overline{AB}] \oplus [\overline{CD}]) = \frac{1}{2^{m-1}}[\overline{AB}] \oplus \frac{1}{2^{m-1}}[\overline{CD}]$ is trivially true. Assume that for some $m \geq 1$, it has been shown that $\frac{1}{2^{m-1}}([\overline{AB}] \oplus [\overline{CD}]) = \frac{1}{2^{m-1}}[\overline{AB}] \oplus \frac{1}{2^{m-1}}[\overline{CD}]$. Let J be the point such that $\frac{1}{2^{m-1}}[\overline{AB}] = [\overline{AJ}]$, K be the point such that $\frac{1}{2^{m-1}}[\overline{CD}] = [\overline{CK}]$, L be the midpoint of $\overline{AJ}$, and M be the midpoint of $\overline{CK}$. Then

$$\begin{aligned} \frac{1}{2^m}([\overline{AB}] \oplus [\overline{CD}]) &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2^{m-1}}\right)([\overline{AB}] \oplus [\overline{CD}]) \\ &= \tfrac{1}{2}\left(\left(\tfrac{1}{2^{m-1}}([\overline{AB}]\right) \oplus \left(\tfrac{1}{2^{m-1}}([\overline{CD}]\right)\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2^{m-1}}[\overline{AB}]\right) \oplus \tfrac{1}{2}\left(\tfrac{1}{2^{m-1}}[\overline{CD}]\right) \\ &= \tfrac{1}{2}[\overline{AJ}] \oplus \tfrac{1}{2}[\overline{CK}] = [\overline{AL}] \oplus [\overline{CM}] \\ &= \tfrac{1}{2^m}[\overline{AB}] \oplus \tfrac{1}{2^m}[\overline{CD}]. \end{aligned}$$

(c) Using parts (a) and (b) above,

$$\begin{aligned} \tfrac{n}{2^m}([\overline{AB}] \oplus [\overline{CD}]) &= \tfrac{1}{2^m}(n([\overline{AB}] \oplus [\overline{CD}])) = \tfrac{1}{2^m}(n[\overline{AB}] \oplus n[\overline{CD}]) \\ &= \tfrac{n}{2^m}[\overline{AB}] \oplus \tfrac{n}{2^m}[\overline{CD}]. \quad \square \end{aligned}$$

Exercise FSEG.4* If $\mathcal{S}$ and $\mathcal{T}$ are any free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{S} < \mathcal{T}$, then $(\mathcal{T} \oplus \mathcal{S}) \ominus \mathcal{S} = \mathcal{T}$ and $(\mathcal{T} \ominus \mathcal{S}) \oplus \mathcal{S} = \mathcal{T}$.

Exercise FSEG.4 Proof. By Definition FSEG.11 $(\mathcal{T} \oplus \mathcal{S}) \ominus \mathcal{S}$ is the free segment of $\mathcal{P}$ which when added to $\mathcal{S}$ yields $\mathcal{T} \oplus \mathcal{S}$ so this free segment is $\mathcal{T}$. By the same definition, $\mathcal{T} \ominus \mathcal{S}$ is the free segment of $\mathcal{P}$ which when added to $\mathcal{S}$ yields $\mathcal{T}$. So if it is added to $\mathcal{S}$ the result is $\mathcal{T}$. $\square$

Exercise FSEG.5* *Let* $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ be free segments of the neutral plane $\mathcal{P}$.

(A) If $\mathcal{U} < \mathcal{S}$ and $\mathcal{U} < \mathcal{T}$, then $(\mathcal{S} \oplus \mathcal{T}) \ominus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{T} = (\mathcal{T} \ominus \mathcal{U}) \oplus \mathcal{S}$.

(B) If $\mathcal{T} \oplus \mathcal{U} < \mathcal{S}$, then $\mathcal{S} \ominus (\mathcal{T} \oplus \mathcal{U}) = (\mathcal{S} \ominus \mathcal{T}) \ominus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \ominus \mathcal{T}$.

Exercise FSEG.5 Proof. (A) By Definition FSEG.11 $(\mathcal{S} \oplus \mathcal{T}) \ominus \mathcal{U}$ is the free segment of $\mathcal{P}$ which when added to $\mathcal{U}$ yields $\mathcal{S} \oplus \mathcal{T}$. By Theorem FSEG.8 and Exercise FSEG.4 $((\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{T}) \oplus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \oplus (\mathcal{T} \oplus \mathcal{U}) = (\mathcal{S} \ominus \mathcal{U}) \oplus (\mathcal{U} \oplus \mathcal{T}) = ((\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{U}) \oplus \mathcal{T} = \mathcal{S} \oplus \mathcal{T}$. Similarly $((\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus \mathcal{T}$ and $((\mathcal{T} \ominus \mathcal{U}) \oplus \mathcal{S}) \oplus \mathcal{U} = \mathcal{S} \oplus \mathcal{T}$. Hence $(\mathcal{S} \oplus \mathcal{T}) \ominus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{T} = (\mathcal{T} \ominus \mathcal{U}) \oplus \mathcal{S}$.

(B) Using Definition FSEG.11, $\mathcal{S} \ominus (\mathcal{T} \oplus \mathcal{U})$ is the free segment of $\mathcal{P}$ which when added to $\mathcal{T} \oplus \mathcal{U}$ yields $\mathcal{S}$.

Using the commutative and associative properties of the operation $\oplus$ we get $((\mathcal{S} \ominus \mathcal{T}) \ominus \mathcal{U}) \oplus (\mathcal{T} \oplus \mathcal{U}) = \mathcal{S}$ and $(\mathcal{T} \ominus \mathcal{U}) \ominus \mathcal{T} = \mathcal{T}$. Hence $\mathcal{S} \ominus (\mathcal{T} \oplus \mathcal{U}) = (\mathcal{S} \ominus \mathcal{T}) \ominus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \ominus \mathcal{T}$. $\square$

Exercise FSEG.6* Let $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ be free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{U} < \mathcal{S}$ and $\mathcal{U} < \mathcal{T}$. If $\mathcal{S} < \mathcal{T}$, then $\mathcal{S} \ominus \mathcal{U} < \mathcal{T} \ominus \mathcal{U}$.

Exercise FSEG.6 Proof. There exists a free segment $\mathcal{V}$ such that $\mathcal{T} = \mathcal{S} \oplus \mathcal{V}$, (*cf.* Theorem FSEG.9) so that $\mathcal{T} \ominus \mathcal{U} = (\mathcal{S} \oplus \mathcal{V}) \ominus \mathcal{U}$. By Exercise FSEG.4 $\mathcal{T} \ominus \mathcal{U} = \mathcal{S} \oplus \mathcal{V} \ominus \mathcal{U} = (\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{V}$. By Exercise FSEG.2 $\mathcal{S} \ominus \mathcal{U} < \mathcal{T} \ominus \mathcal{U}$. $\square$

Exercise FSEG.7* Let $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ be free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{S} \oplus \mathcal{U} < \mathcal{T} \oplus \mathcal{U}$, then $\mathcal{S} < \mathcal{T}$.

Exercise FSEG.7 Proof. By Exercise FSEG.2 $\mathcal{S} < \mathcal{S} \oplus \mathcal{U}$ and $\mathcal{T} < \mathcal{T} \oplus \mathcal{U}$. By Exercise FSEG.6 $\mathcal{S} \oplus \mathcal{U} < \mathcal{T} \oplus \mathcal{U}$ implies that $(\mathcal{S} \oplus \mathcal{U}) \ominus \mathcal{U} < (\mathcal{T} \oplus \mathcal{U}) \ominus \mathcal{U}$, so that $\mathcal{S} < \mathcal{T}$. $\square$

Exercise FSEG.8* Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{V}$ be free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{T} < \mathcal{S}$ and $\mathcal{V} < \mathcal{U}$, then $(\mathcal{S} \ominus \mathcal{T}) \oplus (\mathcal{U} \ominus \mathcal{V}) = (\mathcal{S} \oplus \mathcal{U}) \ominus (\mathcal{T} \oplus \mathcal{V})$.

Exercise FSEG.8 Proof. By Exercise FSEG.5(A) $(\mathcal{S} \ominus \mathcal{T}) \oplus (\mathcal{U} \ominus \mathcal{V}) = ((\mathcal{S} \ominus \mathcal{U}) \oplus \mathcal{T}) \ominus \mathcal{V} = ((\mathcal{S} \oplus \mathcal{U}) \ominus \mathcal{T}) \ominus \mathcal{V}$. By Exercise FSEG.5(B)$((\mathcal{S} \oplus \mathcal{U}) \ominus \mathcal{T}) \ominus \mathcal{V} = \mathcal{S} \oplus \mathcal{U} \ominus (\mathcal{T} \oplus \mathcal{V})$. $\square$

Exercise FSEG.9* Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{V}$ be free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{T} < \mathcal{S}$ and $\mathcal{V} < \mathcal{U}$, then $\mathcal{S} \ominus \mathcal{T} = \mathcal{U} \ominus \mathcal{V}$ iff $\mathcal{S} \oplus \mathcal{V} = \mathcal{T} \oplus \mathcal{U}$.

Exercise FSEG.9 Proof. (I: If $\mathcal{S} \ominus \mathcal{T} = \mathcal{U} \ominus \mathcal{V}$, then $\mathcal{S} \oplus \mathcal{V} = \mathcal{T} \oplus \mathcal{U}$.) If $\mathcal{S} \ominus \mathcal{T} = \mathcal{U} \ominus \mathcal{V}$, then using Exercise FSEG.4 and Exercise FSEG.5(A) $\mathcal{S} = (\mathcal{S} \ominus \mathcal{T}) \oplus \mathcal{T} = (\mathcal{U} \ominus \mathcal{V}) \oplus \mathcal{T} = (\mathcal{U} \oplus \mathcal{T}) \ominus \mathcal{V}$, so that $\mathcal{S} \oplus \mathcal{V} = ((\mathcal{U} \oplus \mathcal{T}) \ominus \mathcal{V}) \oplus \mathcal{V} = \mathcal{U} \oplus \mathcal{T}$.

(II: If $\mathcal{S} \oplus \mathcal{V} = \mathcal{T} \oplus \mathcal{U}$, then $\mathcal{S} \ominus \mathcal{T} = \mathcal{U} \ominus \mathcal{V}$.) If $\mathcal{S} \oplus \mathcal{V} = \mathcal{T} \oplus \mathcal{U}$, by Exercise FSEG.4 $\mathcal{S} = (\mathcal{S} \oplus \mathcal{V}) \ominus \mathcal{V} = (\mathcal{T} \oplus \mathcal{U}) \ominus \mathcal{V}$. Consequently, $\mathcal{S} \ominus \mathcal{T} = (\mathcal{T} \oplus \mathcal{U}) \ominus \mathcal{V}) \ominus \mathcal{T}$. By Exercise FSEG.5(A) and (B) this becomes $(\mathcal{T} \oplus \mathcal{U}) \ominus \mathcal{T}) \ominus \mathcal{V} = ((\mathcal{T} \ominus \mathcal{T}) \oplus \mathcal{U}) \ominus \mathcal{V} = \mathcal{U} \ominus \mathcal{V}$. $\square$

Exercise FSEG.10* If $\mathcal{S}$ and $\mathcal{T}$ are free segments of the neutral plane $\mathcal{P}$ such that $\mathcal{T} < \mathcal{S}$, then $\mathcal{S} \ominus \mathcal{T} < \mathcal{S}$ and $\mathcal{S} \ominus (\mathcal{S} \ominus \mathcal{T}) = \mathcal{T}$.

Exercise FSEG.10 Proof. (I) By Exercise FSEG.2 $\mathcal{S} < \mathcal{S} \oplus \mathcal{T}$. By Exercise FSEG.6 $\mathcal{S} \ominus \mathcal{T} < (\mathcal{S} \oplus \mathcal{T}) \ominus \mathcal{T}$. By Exercise FSEG.4 $(\mathcal{S} \oplus \mathcal{T}) \ominus \mathcal{T} = \mathcal{S}$, so $\mathcal{S} \ominus \mathcal{T} < \mathcal{S}$.

(II) By Definition FSEG.11 $\mathcal{T} \oplus (\mathcal{S} \ominus \mathcal{T}) = \mathcal{S}$. By Exercise FSEG.4 $(\mathcal{T} \oplus (\mathcal{S} \ominus \mathcal{T})) \ominus (\mathcal{S} \ominus \mathcal{T}) = \mathcal{T}$ so $\mathcal{T} = \mathcal{S} \ominus (\mathcal{S} \ominus \mathcal{T})$. $\square$

Exercise FSEG.11 If $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ are any free segments of the neutral plane $\mathcal{P}$, then $(\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{S})$ (the operation $\oplus$ is *associative* on the set $\mathbb{F}$ of free segments.

Exercise FSEG.12 Construct a theory FANG of free angles analogous to that developed in this chapter for free segments, based on the following definition: the **free angle** $FA(\angle BAC) = \{\angle XYZ | \angle XYZ \cong \angle BAC\}$.

Chapter 10: Exercises and Answers for Rotations about a Point of a Neutral Plane (ROT)

Exercise ROT.1* Let $\mathcal{P}$ be a neutral plane.

(A) If O is a point on $\mathcal{P}$, and $\mathcal{R}_O$ is the point reflection about O, then $\mathcal{R}_O(O) = O$ and $\mathcal{R}_O \circ \mathcal{R}_O = \imath$.

(B) If $\mathcal{L}$ and $\mathcal{M}$ are distinct lines on $\mathcal{P}$ and if $\alpha = \mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}$, then $\alpha^{-1} = \mathcal{R}_\mathcal{L} \circ \mathcal{R}_\mathcal{M}$.

(C) If G and H are points on $\mathcal{P}$ and if $\theta = \mathcal{R}_H \circ \mathcal{R}_G$, then $\theta^{-1} = \mathcal{R}_G \circ \mathcal{R}_H$.

Exercise ROT.1 Proof. (A) By Definition ROT.1 there exist lines $\mathcal{L}$ and $\mathcal{M}$ on $\mathcal{P}$ such that $\mathcal{L} \cap \mathcal{M} = \{O\}$, $\mathcal{L} \perp \mathcal{M}$, and $\mathcal{R}_O = \mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}$. By Definition NEUT.1(A) elementary theory of functions

$$\mathcal{R}_O(O) = (\mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L})(O) = \mathcal{R}_\mathcal{M}(\mathcal{R}_\mathcal{L}(O)) = \mathcal{R}_\mathcal{M}(O) = O.$$

By Corollary ROT.5 and Definition NEUT.1(C)

$$\begin{aligned}\mathcal{R}_O \circ \mathcal{R}_O &= (\mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}) \circ (\mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}) = (\mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}) \circ (\mathcal{R}_\mathcal{L} \circ \mathcal{R}_\mathcal{M}) \\ &= (\mathcal{R}_\mathcal{M} \circ (\mathcal{R}_\mathcal{L} \circ \mathcal{R}_\mathcal{L})) \circ \mathcal{R}_\mathcal{M} = (\mathcal{R}_\mathcal{M} \circ \imath) \circ \mathcal{R}_\mathcal{M} = \mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{M} = \imath.\end{aligned}$$

(B) If $\mathcal{L}$ and $\mathcal{M}$ are distinct lines on $\mathcal{P}$, and if $\alpha = \mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}$, then by Definition NEUT.1(C) and elementary theory of functions

$$\begin{aligned}(\mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{L}) \circ (\mathcal{R}_\mathcal{L} \circ \mathcal{R}_\mathcal{M}) &= (\mathcal{R}_\mathcal{M} \circ (\mathcal{R}_\mathcal{L} \circ \mathcal{R}_\mathcal{L})) \circ \mathcal{R}_\mathcal{M} \\ &= (\mathcal{R}_\mathcal{M} \circ \imath) \circ \mathcal{R}_\mathcal{M} = \mathcal{R}_\mathcal{M} \circ \mathcal{R}_\mathcal{M} = \imath.\end{aligned}$$

(C) By Part (A) and elementary theory of functions,

$$\begin{aligned}(\mathcal{R}_H \circ \mathcal{R}_G) \circ (\mathcal{R}_G \circ \mathcal{R}_H) &= (\mathcal{R}_H \circ (\mathcal{R}_G \circ \mathcal{R}_G)) \circ \mathcal{R}_H \\ &= (\mathcal{R}_H \circ \imath) \circ \mathcal{R}_H = \mathcal{R}_H \circ \mathcal{R}_H = \imath. \quad \square\end{aligned}$$

Exercise ROT.2* Let $\mathcal{P}$ be a neutral plane, O be a point on $\mathcal{P}$ and α be a rotation of $\mathcal{P}$ about O which is not a point reflection. If X is any member of $\mathcal{P} \setminus \{O\}$, then X, $\alpha(X)$, and O are noncollinear.

Exercise ROT.2 Proof. Let X be any member of $\mathcal{P} \setminus \{O\}$, and $\mathcal{K} = \overleftrightarrow{OX}$. By Theorem ROT.13 there exists a unique line $\mathcal{J}$ such that $O \in \mathcal{J}$ and $\alpha = \mathcal{R}_\mathcal{J} \circ \mathcal{R}_\mathcal{K}$. Thus $\alpha(X) = \mathcal{R}_\mathcal{J}(\mathcal{R}_\mathcal{K}(X)) = \mathcal{R}_\mathcal{J}(X)$ (*cf* Definition NEUT.1(A)). By Theorem ROT.2 O is the only fixed point of α, so $\alpha(X) \neq O$. By Definition NEUT.1(B) $\alpha(X)$(that is, $\mathcal{R}_\mathcal{J}(X)$) and X are on opposite sides of $\mathcal{J}$. If O, $\alpha(X)$ and X are collinear then $\mathcal{K} = \overleftrightarrow{X\mathcal{R}_\mathcal{J}(X)}$, and by Theorem NEUT.48(A) $\mathcal{K} = \overleftrightarrow{X\mathcal{R}_\mathcal{J}(X)} \perp \mathcal{J}$ so that α is a point reflection, in contradiction to our assumption. $\square$

Exercise ROT.3* Let $\mathcal{P}$ be a neutral plane, O be a point on $\mathcal{P}$, and α and β be rotations of $\mathcal{P}$ about O. If X is any member of $\mathcal{P} \setminus \{O\}$, then $\overrightarrow{O\alpha(X)} \cong \overrightarrow{O\beta(X)}$.

Exercise ROT.3 Proof. Since α and β are isometries of $\mathcal{P}$, by Theorem NEUT.15 $\alpha(\overrightarrow{OX}) = \overrightarrow{O\alpha(X)}$ and $\beta(\overrightarrow{OX}) = \overrightarrow{O\beta(X)}$ so by Definition NEUT.3(B) $\overrightarrow{OX} \cong \overrightarrow{O\alpha(X)}$ and $\overrightarrow{OX} \cong \overrightarrow{O\beta(X)}$. Since by Theorem NEUT.14 congruence is transitive, $\overrightarrow{O\alpha(X)} \cong \overrightarrow{O\beta(X)}$. $\square$

The following exercise shows that rotations (and half rotations, which we will meet in Chapter 13) behave as we expect them to—all points "rotate in the same direction."

Exercise ROT.4* Let O, X, and Y be noncollinear points on the neutral plane $\mathcal{P}$ and let α be a rotation of $\mathcal{P}$ about O which is not a point reflection; we note that α cannot be the identity, as was proved in Theorem ROT.2.

(A1) α rotates X and Y through congruent angles: $\angle XO\alpha(X) \cong \angle YO\alpha Y$.

(A2) Let α and β be rotations of $\mathcal{P}$ about O which are not point reflections. Let X be a point of $\mathcal{P} \setminus \{O\}$ such that

$$\alpha(X) \in \operatorname{ins} \angle XO(\beta \circ \alpha(X)).$$

Then for any point $U \in \mathcal{P} \setminus \{O\}$,

$\angle UO\alpha(U) \cong \angle XO\alpha(X)$;

$\angle \alpha(U)O(\beta \circ \alpha)(U) \cong \angle \alpha(X)O(\beta \circ \alpha)(X)$;

$\angle UO(\beta \circ \alpha)(U) \cong \angle XO(\beta \circ \alpha)(U)$; and

$\alpha(U) \in \operatorname{ins} \angle UO(\beta \circ \alpha)(U)$.

(B) It cannot be true that both $\alpha(X) \in Y$-side $\overleftrightarrow{OX}$ and $\alpha(Y) \in X$-side $\overleftrightarrow{OY}$.

(C) It cannot be true that both $\alpha(X)$ is on the side of $\overleftrightarrow{OX}$ opposite Y and $\alpha(Y)$ is on the side of $\overleftrightarrow{OY}$ opposite X.

(D) $\alpha(X) \in Y$-side of $\overleftrightarrow{OX}$ iff $\alpha(Y)$ is in the side of $\overleftrightarrow{OY}$ opposite X; equivalently, $\alpha(Y) \in X$-side of $\overleftrightarrow{OY}$ iff $\alpha(X)$ is in the side of $\overleftrightarrow{OX}$ opposite Y.

(E) Let $W = \alpha(X)$, and $Z = \alpha(Y)$; let E be a point on the bisecting ray of $\angle XOW$ and F a point on the bisecting ray of $\angle YOZ$. Then $\angle EOX \cong \angle EOW \cong \angle FOY \cong \angle FOZ$.

(F) $E \in Y$-side of $\overleftrightarrow{OX}$ iff F is in the side of $\overleftrightarrow{OY}$ opposite X; equivalently, $F \in X$-side of $\overleftrightarrow{OY}$ iff E is in the side of $\overleftrightarrow{OX}$ opposite Y.

Exercise ROT.4 Proof. (A1) This is Theorem ROT.22. We repeat essentially the same proof with the current notation. Let γ be the rotation (see Theorem ROT.15) such that $\gamma(X) \in \overrightarrow{OY}$. $\gamma(\angle XO(\alpha(X))) = \angle\gamma(X)O\gamma(\alpha(X)) = \angle YO\gamma(\alpha(X))$. (*cf* Theorem NEUT.15). By Theorem ROT.21 $\gamma(\alpha(X)) = \alpha(\gamma(X)) \in \alpha(\overrightarrow{OY})$, and thus $\gamma(\angle XO\alpha(X)) = \angle YO\alpha(Y)$ and $\angle XOW = \angle XO\alpha(X) \cong \angle YO\alpha(Y) = \angle YOZ$ (*cf* Definition NEUT.6(B) (Congruence)).

(A2) From part (A1) we already know that $\angle XO\alpha(X) \cong \angle UO\alpha(U)$ and $\angle\alpha(X)O(\beta\circ\alpha)(X) \cong \alpha(U)O(\beta\circ\alpha)(U)$.

In the following we repeatedly use Theorem ROT.21 (commutativity of rotations about a point). Let γ be the rotation such that $\gamma(X) \in \overrightarrow{OU}$. Without loss of generality we may pick U so that $\gamma(X) = U$; then

$$(\gamma\circ\alpha)(X) = (\alpha\circ\gamma)(X) = \alpha(U)$$

and

$$(\gamma\circ\beta\circ\alpha)(X) = (\beta\circ\gamma\circ\alpha)(X) = (\beta\circ\alpha\circ\gamma)(X) = (\beta\circ\alpha)(U).$$

Now γ is an isometry, so by Theorem NEUT.15, $\alpha(U) = (\gamma\circ\alpha)(X)$ is a member of

$$\begin{aligned}\gamma(\operatorname{ins}\angle XO(\beta\circ\alpha)(X)) &= \operatorname{ins}\angle\gamma(X)O((\gamma\circ\beta\circ\alpha)(X))\\ &= \operatorname{ins}\angle UO(\beta\circ\alpha\circ\gamma)(X) = \operatorname{ins}\angle UO(\beta\circ\alpha(U))\end{aligned}$$

and hence by Exercise NEUT.40(A), $\angle UO(\beta\circ\alpha)(U) \cong \angle XO(\beta\circ\alpha)(X)$. This completes the proof of part (A2).

We consider parts (B) and (C) together.

Let $\overrightarrow{OD}$ be the bisecting ray of $\angle XO\alpha(X)$ and $\overrightarrow{OD'}$ the bisecting ray of $\angle YO\alpha(Y)$. Let D', X', and Y' be a points such that D–O–D', X–O–X', and Y–O–Y'. Without loss of generality, we may assume that $\overline{OX} \cong \overline{OX'} \cong \overline{OY} \cong \overline{OY'} \cong \overline{OD} \cong \overline{OD'}$. We will be referring a great deal to $\alpha(X)$ and $\alpha(Y)$, and since α is an isometry it will follow that $\overline{OX} \cong \overline{O\alpha(X)} \cong \overline{O\alpha(Y)}$. In all that follows we will use Theorem PSH.16 and Definition NEUT.70 repeatedly without further reference. We will assume the negations of (B) and (C) and eventually show contradictions in all cases.

(B) Assume the contrary, that $\alpha(X) \in Y$-side $\overleftrightarrow{OX}$ and $\alpha(Y) \in Y$-side $\overleftrightarrow{OY}$.

We know from part (A) that $\angle XO\alpha(X) \cong \angle YO\alpha(Y)$. Since D is on the line of symmetry for $\angle XOY$, by Theorem NEUT.39 $\angle DOX \cong \angle DOY$. By Theorem NEUT.75 (trichotomy for angles) (or Exercise PSH.32) there are three cases:

(Case B1: $\angle XO\alpha(X) < \angle XOD$) Then $\alpha(X) \in \operatorname{ins}\angle XOD$, so

$$\angle YO\alpha(Y) \cong \angle XO\alpha(X) < \angle XOD \cong \angle YOD$$

and hence $\alpha(Y) \in \operatorname{ins}\angle YOD$. By Exercise NEUT.40(B) $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$.

(Case B2: $\angle XO\alpha(X) \cong \angle XOD$) Then by Theorem NEUT.36 $\alpha(X) \in \overrightarrow{OD}$ and $\angle YOD \cong \angle XOD = \angle XO\alpha(X) \cong \angle YO\alpha(Y)$. Now by assumption D and $\alpha(Y)$ are on the same side of $\overleftrightarrow{OY}$ so by Theorem NEUT.36 $\alpha(Y) \in \overrightarrow{OD}$, and since $\overline{OD} \cong \overline{O\alpha(Y)}$ it follows from Property R.4 of Definition NEUT.2 that $\alpha(Y) = D = \alpha(X)$. But this is impossible because α is one-to-one.

(Case B3: $\angle XO\alpha(X) > \angle XOD$) Then $D \in \operatorname{ins}\angle XO\alpha(X)$ and $\angle YOD \cong \angle XOD < \angle XO\alpha(X) \cong \angle YO\alpha(Y)$, and $D \in \operatorname{ins}\angle YO\alpha(Y)$. Then by Exercise NEUT.40(B) $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$.

(C) Assume the contrary, that both $\alpha(X)$ is on the side of $\overleftrightarrow{OX}$ opposite Y and $\alpha(Y)$ is on the side of $\overleftrightarrow{OY}$ opposite X, $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$. In this part we will use Theorem NEUT.43 (supplements of congruent angles are congruent) repeatedly without further reference.

By Theorem NEUT.75 (trichotomy for angles) there are three cases:

(Case C1: $\angle XO\alpha(X) < \angle XOD'$) Then $\alpha(X) \in \operatorname{ins}\angle XOD'$; it follows that $\angle YO\alpha(Y) \cong \angle XO\alpha(X) < \angle XOD' \cong \angle YOD'$ (the last congruence is by supplements) and hence $\alpha(Y) \in \operatorname{ins}\angle YOD'$. By Exercise NEUT.40(B) $\angle D'O\alpha(X) \cong \angle D'O\alpha(Y)$, so that $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$, again by supplements.

(Case C2: $\angle XO\alpha(X) \cong \angle XOD'$) Then by Theorem NEUT.36 $\alpha(X) \in \overrightarrow{OD'}$. Since $\angle YOD \cong \angle XOD$, by supplements $\angle YOD' \cong \angle XOD' = \angle XO\alpha(X) \cong \angle YO\alpha(Y)$. Now by assumption D and $\alpha(Y)$ are on opposite sides of $\overleftrightarrow{OY}$, so that D' and $\alpha(Y)$ are on the same side of $\overleftrightarrow{OY}$. By Theorem NEUT.36 $\alpha(Y) \in \overrightarrow{OD'}$, and since $\overline{OD'} \cong \overline{O\alpha(Y)}$ it follows from Property R.4 of Definition NEUT.2 that $\alpha(Y) = D' = \alpha(X)$. But this is impossible because α is one-to-one.

(Case C3: $\angle XO\alpha(X) > \angle XOD'$) Then $D' \in \operatorname{ins}\angle XO\alpha(X)$. Since $\angle YOD \cong \angle XOD$, by supplements $\angle YOD' \cong \angle XOD'$. Then $\angle YOD' \cong \angle XOD' < \angle XO\alpha(X) \cong \angle YO\alpha(Y)$, and $D' \in \operatorname{ins}\angle YO\alpha(Y)$. By Exercise NEUT.40(B) $\angle D'O\alpha(X) \cong \angle D'O\alpha(Y)$, and again by supplements $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$.

We have now shown that in all the cases where we do not already have a contradiction (that is, B1, B3, C1, and C3) $\angle DO\alpha(X) \cong \angle DO\alpha(Y)$, and hence $\overleftrightarrow{OD}$ (the line of symmetry for $\angle XOY$) is the line of symmetry for $\angle\alpha(X)O\alpha(Y)$.

Let M be the midpoint of $\overline{XY}$. By Exercise NEUT.76(A) $\alpha(M)$ is the midpoint of $\overline{\alpha(X)\alpha(Y)}$. By Theorem NEUT.55, $M \in \overleftrightarrow{DO}$, and since $\overline{O\alpha(X)} \cong \overline{O\alpha(Y)}$, $\alpha(M) \in \overleftrightarrow{DO}$.

Either $\alpha(M) = O$ or M–O–$\alpha(M)$ or $\alpha(M) \in \overrightarrow{OM} = \overrightarrow{OD}$.

The first case gives us a contradiction immediately, because $\alpha(O) = O$ and α is one-to-one.

In the second case, since both M and $\alpha(M)$ are in $\overleftrightarrow{OD}$, α is a point reflection which is ruled out by hypothesis.

In the third case, $\alpha(M) \in \overrightarrow{OM}$ so that by Property R.4 of Definition NEUT.2 $\alpha(M) = M$ and α is the identity, which is ruled out by hypothesis. This completes the proof of parts (B) and (C).

(D) By parts (B) and (C), if $\alpha(X) \in Y$-side of $\overleftrightarrow{OX}$ then $\alpha(Y)$ cannot be in X-side of $\overleftrightarrow{OY}$ so must be in the side opposite X; conversely, if $\alpha(Y)$ is in the side of $\overleftrightarrow{OX}$ opposite X then $\alpha(X)$ cannot be in the side of $\overleftrightarrow{OX}$ opposite Y hence must be in the Y-side. Similar reasoning shows the other assertion.

(E) We know from Theorem NEUT.20(E) (also from Theorem NEUT.39) that $\angle EOX \cong \angle EOX$ and $\angle FOY \cong \angle FOZ$. By part (A) $\angle XOW \cong \angle YOZ$. By Exercise NEUT.39 $\angle EOX \cong \angle FOY$.

(F) If $\angle BAC$ is any angle and D is a point in its bisecting ray, D is on the B-side of $\overleftrightarrow{AC}$ and D is on the A-side of $\overleftrightarrow{AB}$. From part (D),

$E \in Y$-side of $\overleftrightarrow{OX}$ iff $\alpha(X) \in Y$-side of $\overleftrightarrow{OX}$) iff $\alpha(Y)$ is in the side of $\overleftrightarrow{OY}$ opposite X iff F is in the side of $\overleftrightarrow{OY}$ opposite X;

$F \in X$-side of $\overleftrightarrow{OY}$ iff $\alpha(Y) \in X$-side of $\overleftrightarrow{OY}$ iff $\alpha(X)$ is in the side of $\overleftrightarrow{OX}$ opposite Y iff E is in the side of $\overleftrightarrow{OX}$ opposite Y. □

Exercise ROT.5* Let $\mathcal{P}$ be a neutral plane, O be a point on $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{M}$ be lines on $\mathcal{P}$ through O which are not perpendicular to each other, Q and R be points on $\mathcal{L}$ such that Q–O–R, S and T be points on $\mathcal{M}$ such that S–O–T and ρ be the rotation $\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}$ about O. If we choose the notation (using Theorem NEUT.82) so that $\angle QOS$ is acute, then $\rho(Q)$ is the member of ins$\angle ROS$ such that Q and $\rho(Q)$ are on opposite sides of $\mathcal{M}$, $\angle SO\rho(Q) \cong \angle QOS$ and $\overline{O\rho(Q)} \cong \overline{OQ}$.

Exercise ROT.5 Proof. Since $\rho(Q) = \mathcal{R}_{\mathcal{M}}(Q)$ and $\mathcal{M} \perp \overline{Q\mathcal{R}_{\mathcal{M}}(Q)}$ (*cf* Theorem NEUT.48(A)), by Definition NEUT.1(B) Q and $\mathcal{R}_{\mathcal{M}}(Q)$ are on opposite sides of $\mathcal{M}$. Hence, by Axiom PSA, there exists a point G such that $\mathcal{M} \cap \overline{Q\mathcal{R}_{\mathcal{M}}(Q)} = \{G\}$. Since Q and R are on opposite sides of M, $\mathcal{R}_{\mathcal{M}}(Q)$ and R are on the same side of $\mathcal{M}$ (*cf* Theorem PSH.12 and Definition IB.11).

Since $\angle SOQ$ is acute, by Exercise NEUT.18 $G \in \overrightarrow{OS}$ so that by Theorem IB.14 $\overrightarrow{QG} = \overrightarrow{Q\mathcal{R}_{\mathcal{M}}(Q)} \subseteq S$-side of $\mathcal{L}$. Hence $\mathcal{R}_{\mathcal{M}}(Q)$ and S are on the same side of $\mathcal{L}$. Then $\mathcal{R}_{\mathcal{M}}(Q) \in \text{ins}\,\angle ROS$ (*cf* Definition PSH.36), and by Theorem NEUT.15 and Definition NEUT.1(A) $\mathcal{R}_{\mathcal{M}}(\angle QOS) = \angle\mathcal{R}_{\mathcal{M}}(Q)OS$ and $\mathcal{R}_{\mathcal{M}}(\overline{OQ}) = \overline{O\mathcal{R}_{\mathcal{M}}(Q)}$. By Definition NEUT.3(B) (congruence) $\angle QOS \cong \angle\mathcal{R}_{\mathcal{M}}(Q)OS = \angle\rho(Q)OS$ and $\overline{OQ} \cong \overline{O\mathcal{R}_{\mathcal{M}}(Q)} = \overline{O\rho(Q)}$. $\square$

Exercise ROT.6* Let A, B, and C be noncollinear points on the neutral plane $\mathcal{P}$ and let $\mathcal{L} = \overleftrightarrow{AB}$, $\mathcal{M} = \overleftrightarrow{AC}$, and $\mathcal{N} = \overleftrightarrow{BC}$. Then there exists a unique point G and a unique line $\mathcal{J}$ such that $C \in \mathcal{J}$ and $\mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{J}} \circ \mathcal{R}_G$.

Exercise ROT.6 Proof. Let $\mathcal{K} = \text{pr}(C, \overleftrightarrow{AB})$, and $G = \text{ftpr}(C, \overleftrightarrow{AB})$. By Theorem ROT.13 there exists a unique line $\mathcal{J}$ through C such that $\mathcal{R}_{\mathcal{J}} \circ \mathcal{R}_{\mathcal{K}} = \mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}}$. Then $\mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{J}} \circ \mathcal{R}_{\mathcal{K}} \circ \mathcal{R}_{\mathcal{L}}$. By Definition ROT.1 $\mathcal{R}_{\mathcal{K}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_G$, so that $\mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{J}} \circ \mathcal{R}_G$. $\square$

Exercise ROT.7* Let A and B be distinct points on the neutral plane $\mathcal{P}$. If M is the midpoint of $\overline{AB}$, then $\mathcal{R}_M(A) = B$.

Exercise ROT.7 Proof. Let $\mathcal{L} = \overleftrightarrow{AB}$ and $\mathcal{M} = \text{pr}(M, \mathcal{L})$. By Definition ROT.1 $\mathcal{R}_M = \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}$. Hence $\mathcal{R}_M(A) = \mathcal{R}_{\mathcal{M}}(\mathcal{R}_{\mathcal{L}}(A)) = \mathcal{R}_{\mathcal{M}}(A)$ so that $\mathcal{R}_{\mathcal{M}}(\overline{AM}) = \overline{\mathcal{R}_{\mathcal{M}}(A)\mathcal{R}_{\mathcal{M}}(M)} = \overline{\mathcal{R}_{\mathcal{M}}(A)M}$, and $\overline{AM} \cong \overline{\mathcal{R}_{\mathcal{M}}(A)M}$. Since M is the midpoint of $\overline{AB}$, $\overline{MB} \cong \overline{AM} \cong \overline{\mathcal{R}_{\mathcal{M}}(A)M}$; by Theorem NEUT.14, congruence is transitive, and $\overline{\mathcal{R}_{\mathcal{M}}(A)M} \cong \overline{MB}$. Now $\mathcal{R}_{\mathcal{M}}$ and B are on the same side of $\mathcal{L}$ so $\mathcal{R}_{\mathcal{M}} \in \overrightarrow{MB}$. By Property R.4 of Definition NEUT.2, $\mathcal{R}_M(A) = \mathcal{R}_{\mathcal{M}}(A) = B$. $\square$

Exercise ROT.8* Let $\mathcal{P}$ be a neutral plane, α be an isometry of $\mathcal{P}$ such that α has one and only one fixed point O, and for every member X of $\mathcal{P} \setminus \{O\}$, X–O–$\alpha(X)$ and $\overline{OX} \cong \overline{O\alpha(X)}$. Then α is the point reflection $\mathcal{R}_O$.

Exercise ROT.8 Proof. Since X–O–$\alpha(X)$, X–O–$\mathcal{R}_O(X)$, $\overline{OX} \cong \overline{O\mathcal{R}_O(X)}$, and congruence is transitive (*cf* Theorem NEUT.14) $\overline{OX} \cong \overline{O\alpha(X)}$ and $\overline{O\alpha(X)} \cong \overline{O\mathcal{R}_O(X)}$. By Theorem PSH.13 $\alpha(X) \in \overrightarrow{O\mathcal{R}_O(X)}$. Hence by Property R.4 of Definition NEUT.2 $\alpha(X) = \mathcal{R}_O(X)$, and $\alpha = \mathcal{R}_O$. $\square$

Exercise ROT.9* Let $\mathcal{P}$ be a neutral plane, O be a point on $\mathcal{P}$, ρ be the point reflection of $\mathcal{P}$ about O, $\mathcal{L}$ be a line on $\mathcal{P}$ through O which is ordered according to Definition ORD.1, and let X and Y be points on $\mathcal{L}$. Then $X < Y$ iff $\rho(Y) < \rho(X)$.

Exercise ROT.9 Proof. Let $\mathcal{M} = \mathrm{pr}(O, \mathcal{L})$. By Definition ROT.1 $\rho = \mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}}$. Hence $\rho|\mathcal{L}$ (the restriction of ρ to $\mathcal{L}$) is equal to $\mathcal{R}_{\mathcal{M}}|\mathcal{L}$. Since $\mathcal{R}_{\mathcal{M}}|\mathcal{L} = \mathcal{R}_O|\mathcal{L}$, by Exercise NEUT.69 $X < Y$ iff $\rho(Y) < \rho(X)$. $\square$

Exercise ROT.10* Let $\mathcal{P}$ be a neutral plane, A, B, and O noncollinear points on $\mathcal{P}$. Then there exists a unique rotation α of $\mathcal{P}$ about O such that $\angle AO\alpha(A) = \angle AOB$.

Exercise ROT.10 Proof. By Theorem ROT.15 there exists a unique rotation α such that $\alpha(\overrightarrow{OA}) = \overrightarrow{OB}$. Then by Theorem NEUT.15 $\angle AO\alpha(A) = \overrightarrow{OA} \cup \overrightarrow{O\alpha(A)} = \overrightarrow{OA} \cup \alpha(\overrightarrow{OA}) = \overrightarrow{OA} \cup \overrightarrow{OB} = \angle AOB$. $\square$

Chapter 11: Exercises and Answers for Euclidean Geometry Basics (EUC)

Exercise EUC.1* Prove Corollary EUC.4, using Theorem EUC.3: let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be distinct lines on the Euclidean plane $\mathcal{P}$ such that $\mathcal{M}$ and $\mathcal{N}$ intersect at a point O, $\mathcal{M}$ and $\mathcal{N}$ are not perpendicular to each other, and $\mathcal{L} \perp \mathcal{N}$; then $\mathcal{L}$ and $\mathcal{M}$ intersect at a point Q.

Exercise EUC.1 Proof. The contrapositive of Theorem EUC.3 says that if $\mathcal{M}$ and $\mathcal{N}$ are not perpendicular, then either $\mathcal{L} \not\perp \mathcal{N}$ or $\mathcal{L} \not\parallel \mathcal{M}$. By hypothesis $\mathcal{L} \perp \mathcal{N}$ so $\mathcal{L} \not\parallel \mathcal{M}$ and hence $\mathcal{L}$ intersects $\mathcal{M}$. $\square$

Exercise EUC.2* Using Definition PSH.31 and Remark PSH.12.1, prove Theorem EUC.6: A parallelogram is a rotund quadrilateral.

Exercise EUC.2 Proof. Let $\square ABCD$ be a parallelogram; by Definition EUC.5 $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. By Remark PSH.12.1

A and B are on the same side of $\overleftrightarrow{DC}$,
D and C are on the same side of $\overleftrightarrow{AB}$,
A and D are on the same side of $\overleftrightarrow{BC}$, and
B and C are on the same side of $\overleftrightarrow{AD}$.

By Definition PSH.31 $\square ABCD$ is rotund. $\square$

Exercise EUC.3* Prove Corollary EUC.8: let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{J}$, and $\mathcal{K}$ be distinct lines on the Euclidean plane $\mathcal{P}$ such that $\mathcal{L}$ and $\mathcal{M}$ intersect at the point O, $\mathcal{L} \perp \mathcal{J}$, and $\mathcal{M} \perp \mathcal{K}$; then $\mathcal{J}$ and $\mathcal{K}$ intersect at a point Q.

Exercise EUC.3 Proof. This is essentially the contrapositive of Theorem EUC.7. If $\mathcal{J}$ and $\mathcal{K}$ were parallel, then by Theorem EUC.7 $\mathcal{L}$ and $\mathcal{M}$ would be parallel, contrary to the given fact that $\mathcal{L}$ and $\mathcal{M}$ intersect at the point O. Hence $\mathcal{J}$ and $\mathcal{K}$ are not parallel, and so intersect at a point Q. $\square$

Exercise EUC.4* Complete the proof of Theorem EUC.22: let A, B, and C be noncollinear points on the Euclidean plane $\mathcal{P}$; let M be the midpoint of $\overline{AB}$, and let $N \in \overline{AC}$ and $Q \in \overline{BC}$ be points such that $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$ and $\square BMNQ$ is a parallelogram. Then N is the midpoint of $\overline{AC}$.

Exercise EUC.4 Proof. Since $\square MNQB$ is a parallelogram, by Theorem EUC.12(A) $\overline{BQ} \cong \overline{MN}$, $\overline{BM} \cong \overline{QN}$, and $\angle NMB \cong \angle NQB$. By Definition NEUT.3(C) and Theorem NEUT.14 (congruence is transitive),

$\overline{AM} \cong \overline{MB} \cong \overline{QN}$. By Theorem NEUT.43 (supplements of congruent angles) $\angle AMN \cong \angle NQC$. By Theorem EUC.11(4) $\angle NAM \cong \angle CNQ$. Then by Theorem NEUT.65 (AEA) there exists an isometry α such that $\alpha(\triangle NQC) = \triangle AMN$ where $\alpha(Q) = M$, $\alpha(N) = A$, and $\alpha(C) = N$, so that $\triangle NQC \cong \triangle AMN$ and $\overline{AN} \cong \overline{CN}$; thus N is the midpoint of $\overline{AC}$. $\square$

Exercise EUC.5* Prove Corollary EUC.23: let $\mathcal{P}$ be a Euclidean plane and A, B, and C be noncollinear points on $\mathcal{P}$. If M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{AC}$, then $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$. Moreover, if L is the midpoint of $\overline{BC}$, then $\overline{BL} \cong \overline{MN}$.

Exercise EUC.5 Proof. Let $\mathcal{L} = \operatorname{par}(M, \overleftrightarrow{BC})$. By Theorem EUC.22 $\mathcal{L}$ and $\overleftrightarrow{AC}$ intersect at the midpoint N of $\overline{AC}$. By Axiom I.1 $\mathcal{L} = \overleftrightarrow{MN}$. Let $\mathcal{M} = \operatorname{par}(N, \overleftrightarrow{AB})$; by similar reasoning, $L \in \mathcal{M}$. By Definition EUC.5(B) $\square NMBL$ is a parallelogram and by Theorem EUC.12(A) $\overline{BL} \cong \overline{MN}$. $\square$

Exercise EUC.6* Prove Theorem EUC.32: complements of acute congruent angles are congruent.

Exercise EUC.6 Proof. Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{E}'$, and $\mathcal{F}'$ be acute angles where $\mathcal{E}$ is a complement of $\mathcal{F}$, $\mathcal{E}'$ is a complement of $\mathcal{F}'$, and $\mathcal{E} \cong \mathcal{E}'$. We need to prove that $\mathcal{F} \cong \mathcal{F}'$.

By Definition EUC.30, there exist points A, B, C, and D such that $D \in \operatorname{ins}\angle BAC$ and $\mathcal{E} \cong \angle BAD$, $\mathcal{F} \cong \angle CAD$ and $\angle BAC$ is right; also there exist points A', B', C', and D' such that $D' \in \operatorname{ins}\angle B'A'C'$ and $\mathcal{E}' \cong \angle B'A'D'$, $\mathcal{F}' \cong \angle C'A'D'$ and $\angle B'A'C'$ is right.

By Theorem NEUT.14 (congruence is transitive), since $\mathcal{E} \cong \mathcal{E}'$, $\angle BAD \cong \mathcal{E} \cong \mathcal{E}' \cong \angle B'A'D'$. To show $\mathcal{F} \cong \mathcal{F}'$ it is sufficient to show $\angle CAD \cong \angle C'A'D'$. By Theorem NEUT.38, there exists an isometry α such that $\alpha(\overrightarrow{AB}) = \overrightarrow{A'B'}$ and $\alpha(\overrightarrow{AD}) = \overrightarrow{A'D'}$. Since $D \in \operatorname{ins}\angle BAC$ by Definition PSH.36, $C \in D$-side of $\overleftrightarrow{AB}$ and hence $D \in C$-side of $\overleftrightarrow{AB}$; by a similar argument $D' \in C'$-side of $\overleftrightarrow{A'B'}$. By Theorem NEUT.15, $\alpha(C\text{-side of } \overleftrightarrow{AB}) =$ the $\alpha(C) = C'$-side of $\overleftrightarrow{A'B'}$ and since $D \in C$-side of $\overleftrightarrow{AB}$ then $\alpha(D) \in C'$-side $\overleftrightarrow{A'B'}$, as does D', as we have already seen.

By Corollary NEUT.44.2, $\alpha(\angle BAD) = \angle B'A'\alpha(D)$ is right. By Theorem NEUT.69 $\angle BAD \cong \angle B'A'D' \cong \angle B'A'\alpha(D)$. By Theorem NEUT.36, $\alpha(\overrightarrow{AD}) = \overrightarrow{A'\alpha(D)} = \overrightarrow{A'D'}$, so that $\alpha(\angle CAD) = \angle C'A'D'$, and $\mathcal{F} \cong \angle CAD \cong \angle C'A'D' \cong \mathcal{F}'$. $\square$

Exercise EUC.7* Prove Corollary EUC.3.1: let $\mathcal{R}_{\mathcal{M}}$ be the line reflection over $\mathcal{M}$, and let $\mathcal{L}$ be a fixed line for $\mathcal{R}_{\mathcal{M}}$. Then $\mathcal{N} \parallel \mathcal{L}$ iff $\mathcal{N}$ is a fixed line for $\mathcal{R}_{\mathcal{M}}$.

Exercise EUC.7 Proof. If $\mathcal{N}$ is a fixed line for $\mathcal{R}_{\mathcal{M}}$ then by Theorem NEUT.44 $\mathcal{N} \perp \mathcal{M}$ and by Theorem NEUT.48 $\mathcal{N} \parallel \mathcal{L}$. Conversely, if $\mathcal{N} \parallel \mathcal{L}$, since we know $\mathcal{L} \perp \mathcal{M}$, by Theorem EUC.3 $\mathcal{N} \perp \mathcal{M}$, and by Theorem NEUT.44, $\mathcal{N}$ is a fixed line for $\mathcal{R}_{\mathcal{M}}$. $\square$

Chapter 12: Exercises and Answers for Isometries of a Euclidean Plane (ISM)

Exercise ISM.1* Let $\mathcal{P}$ be a Euclidean plane.
(A) There is no translation τ of $\mathcal{P}$ such that $\tau \circ \tau = \imath$.
(B) For any translation τ of $\mathcal{P}$, $\tau \circ \tau$ is a translation.

Exercise ISM.1 Proof. Let τ be any translation of $\mathcal{P}$.

(A) By Theorem ISM.5 there exist parallel lines $\mathcal{L}$ and $\mathcal{M}$ on $\mathcal{P}$ such that $\tau = \mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}}$. If $\tau \circ \tau$ were equal to $\imath$, then we would have $\mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{M}}$. By Exercise NEUT.8 $\mathcal{R}_{\mathcal{L}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{R}_{\mathcal{L}}(\mathcal{M})}$. Hence we would have $\mathcal{R}_{\mathcal{R}_{\mathcal{L}}(\mathcal{M})} = \mathcal{R}_{\mathcal{M}}$. By Remark NEUT.1.2(B) $\mathcal{R}_{\mathcal{L}}(\mathcal{M})$ would be equal to $\mathcal{M}$. But $\mathcal{L} \parallel \mathcal{M}$, so by Exercise NEUT.1, $\mathcal{M}$ and $\mathcal{R}_{\mathcal{L}}(\mathcal{M})$ are subsets of opposite sides of $\mathcal{L}$, a contradiction. Thus $\tau \circ \tau \neq \imath$.

(B) By Theorem ISM.8(A) $\tau \circ \tau$ is either a translation or $\imath$. By part (A) $\tau \circ \tau$ is not $\imath$ so it must be a translation, having no fixed point by Definition CAP.6. $\square$

Exercise ISM.2* Let $\mathcal{P}$ be a Euclidean plane, σ and τ be translations of $\mathcal{P}$ such that $\mathcal{L}$ is a fixed line of σ, $\mathcal{M}$ is a fixed line of τ, $\mathcal{L}$ and $\mathcal{M}$ are not parallel, and let Q be any point on $\mathcal{P}$. Then $\square Q(\sigma(Q))(\tau(\sigma(Q)))(\tau(Q))$ is a parallelogram.

Exercise ISM.2 Proof. Since $\mathcal{L}$ and $\mathcal{M}$ are not parallel, by Theorem CAP.8(C) $\overleftrightarrow{Q\sigma(Q)} \parallel \overleftrightarrow{\tau(Q)\tau(\sigma(Q))}$ and $\overleftrightarrow{Q\tau(Q)} \parallel \overleftrightarrow{\sigma(Q)\sigma(\tau(Q))}$. By Theorem ISM.7(B) $\tau(\sigma(Q)) = \sigma(\tau(Q))$. By Definition EUC.5(B) $\square Q\sigma(Q)\tau(\sigma(Q))\tau(Q)$ is a parallelogram. $\square$

Exercise ISM.3* Let $\mathcal{P}$ be a Euclidean plane, A and B be distinct points on $\mathcal{P}$, and τ be a translation of $\mathcal{P}$ such that $\overleftrightarrow{AB}$ is not a fixed line of τ. Then $\overline{A\tau(A)}$ and $\overline{B\tau(B)}$ are opposite edges of a parallelogram.

Exercise ISM.3 Proof. By Theorem ISM.5 there exists a unique translation σ of $\mathcal{P}$ such that $\sigma(A) = B$, and its fixed line $\overleftrightarrow{AB}$ is not parallel to a fixed line of τ, so $\tau(B) = \tau(\sigma(A)) = (\tau \circ \sigma)(A)$. By Exercise ISM.2 $\overline{A\tau(A)}$ and $\overline{B\tau(B)}$ are opposite edges of the parallelogram $\square A\sigma(A)\tau(\sigma(A))\tau(A)$. $\square$

Exercise ISM.4* Let $\mathcal{P}$ be a Euclidean plane, $\mathcal{L}_1$ and $\mathcal{L}_2$ be parallel lines on $\mathcal{P}$, A_1 be a point on $\mathcal{L}_1$, A_2 be the point of intersection of $\mathrm{pr}\,(A_1, \mathcal{L}_1)$

and $\mathcal{L}_2$, and τ be the translation of $\mathcal{P}$ such that $\tau(A_1) = A_2$ (*cf* Theorem ISM.5). Then $\tau(\mathcal{L}_1) = \mathcal{L}_2$.

Exercise ISM.4 Proof. By Definition CAP.6 either $\tau(\mathcal{L}_1) \parallel \mathcal{L}_1$ or $\tau(\mathcal{L}_1) = \mathcal{L}_1$ that is $\tau(\mathcal{L}_1)\ \mathbb{PE}\ \mathcal{L}_1$. Since $\tau(A_1) = A_2$, $\tau(\mathcal{L}_1) \neq \mathcal{L}_1$. Therefore by Axiom PS, $\tau(\mathcal{L}_1)$ is the line through A_2 which is parallel to $\mathcal{L}$, *i.e.* $\tau(\mathcal{L}_1) = \mathcal{L}_2$. $\square$

Exercise ISM.5* Let $\mathcal{M}$ be a line on a Euclidean plane $\mathcal{P}$ and let σ be a translation along $\mathcal{M}$; that is, $\mathcal{M}$ is a fixed line for σ. Let $\mathcal{R}_\mathcal{M}$ be the reflection with axis $\mathcal{M}$. Then $\mathcal{R}_\mathcal{M} \circ \sigma = \sigma \circ \mathcal{R}_\mathcal{M}$.

Exercise ISM.5 Proof. Case 1. Let X be any member of $\mathcal{M}$. Since $\mathcal{M}$ is a fixed line of σ and is point-wise fixed for $\mathcal{R}_\mathcal{M}$, $\mathcal{R}_\mathcal{M}(\sigma(X)) = \sigma(X) = \sigma(\mathcal{R}_\mathcal{M}(X))$.

Case 2. (I) Let X be any member of $\mathcal{P} \setminus \mathcal{M}$. By Definition CAP.6 translations have no fixed points, so $\sigma(X) \neq X$; by Theorem CAP.8 the lines $\overleftrightarrow{X\sigma(X)}$ and $\overleftrightarrow{\mathcal{R}_\mathcal{M}(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$ are fixed lines for σ and are parallel to $\mathcal{M}$.

(II) By Theorem CAP.3, since $\overleftrightarrow{X\sigma(X)} \parallel \mathcal{M}$,
$$\mathcal{R}_\mathcal{M}(\overleftrightarrow{X\sigma(X)}) = \overleftrightarrow{\mathcal{R}_\mathcal{M}(X)\mathcal{R}_\mathcal{M}(\sigma(X))}$$
is also parallel to $\mathcal{M}$. (Here we have used Theorem NEUT.15.) Since $\mathcal{R}_\mathcal{M}(X)$ is a member of both $\overleftrightarrow{\mathcal{R}_\mathcal{M}(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$ and $\overleftrightarrow{\mathcal{R}_\mathcal{M}(X)\mathcal{R}_\mathcal{M}(\sigma(X))}$, and both these lines are parallel to $\mathcal{M}$, by Axiom PS they are the same.

(III) By Theorem NEUT.22 $\overleftrightarrow{X\mathcal{R}_\mathcal{M}(X)}$ and $\overleftrightarrow{\sigma(X)\mathcal{R}_\mathcal{M}(\sigma(X))}$ are fixed lines for $\mathcal{R}_\mathcal{M}$ and are parallel; by Theorem NEUT.44 they are perpendicular to (and intersect) $\mathcal{M}$; they are distinct lines because X does not belong to both of them.

(IV) By Theorem NEUT.15, $\sigma(\overleftrightarrow{X\mathcal{R}_\mathcal{M}(X)}) = \overleftrightarrow{\sigma(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$; by Definition CAP.6, $\overleftrightarrow{X\mathcal{R}_\mathcal{M}(X)} \parallel \overleftrightarrow{\sigma(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$. Since $\sigma(X)$ is a member of both $\overleftrightarrow{\sigma(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$ and $\overleftrightarrow{\sigma(X)\mathcal{R}_\mathcal{M}(\sigma(X))}$, and both these lines are parallel to $\overleftrightarrow{X\mathcal{R}_\mathcal{M}(X)}$, by Axiom PS they are the same. Thus both the points $\mathcal{R}_\mathcal{M}(\sigma(X))$ and $\sigma(\mathcal{R}_\mathcal{M}(X))$ belong to
$$\overleftrightarrow{\sigma(X)\mathcal{R}_\mathcal{M}(\sigma(X))} \cap \overleftrightarrow{\mathcal{R}_\mathcal{M}(X)\sigma(\mathcal{R}_\mathcal{M}(X))}$$
which intersection, by Exercise I.1, contains exactly one point; thus $\mathcal{R}_\mathcal{M}(\sigma(X)) = \sigma(\mathcal{R}_\mathcal{M}(X))$. $\square$

Exercise ISM.6 Prove, disprove, or improve: let $\mathcal{P}$ be a Euclidean plane, τ a translation, and $\mathcal{L}$ a line on $\mathcal{P}$. Then $\mathcal{R}_{\tau(\mathcal{L})} \circ \tau = \tau \circ \mathcal{R}_\mathcal{L}$.

Exercise ISM.7 In Theorem ISM.23 Case 2, create a simpler proof of the fact that $X < \tau_A(X)$.

Exercise ISM.8* Let $\mathcal{P}$ be a Euclidean plane, and let $\alpha = \mathcal{R}_{\mathcal{L}} \circ \tau$ be a glide reflection, where τ is a translation and $\mathcal{L}$ is the single fixed line for α according to Theorem ISM.13.

(A) If $\mathcal{N} \parallel \mathcal{L}$ then $\alpha(\mathcal{N}) \parallel \mathcal{L}$.

(B) If $\mathcal{N} \perp \mathcal{L}$ then $\alpha(\mathcal{N}) \perp \mathcal{L}$ and $\alpha(\mathcal{N}) \parallel \mathcal{N}$.

Exercise ISM.8 Proof. (A) By Definition CAP.6, $\tau(\mathcal{N}) \parallel \mathcal{N}$; by Exercise NEUT.1, $\alpha(\mathcal{N}) = \mathcal{R}_{\mathcal{L}}(\tau(\mathcal{N})) \parallel \mathcal{N} \parallel \mathcal{L}$.

(B) If $\mathcal{N} \perp \mathcal{L}$, then by Definition CAP.6 $\tau(\mathcal{N}) \parallel \mathcal{N}$ and by Theorem EUC.3 $\tau(\mathcal{N}) \perp \mathcal{L}$. By Theorem NEUT.44 $\tau(\mathcal{N})$ is a fixed line for $\mathcal{R}_{\mathcal{L}}$; hence $\alpha(\mathcal{N}) = \mathcal{R}_{\mathcal{L}}(\tau(\mathcal{N})) = \tau(\mathcal{N})$ is both parallel to $\mathcal{N}$, and perpendicular to $\mathcal{L}$. $\square$

Chapter 13: Exercises and Answers for Dilations of a Euclidean Plane (DLN)

Exercise DLN.1* Let O be a point on a Euclidean plane $\mathcal{P}$, and let α be a half-rotation of $\mathcal{P}$ about O. If X and Y are members of $\mathcal{P} \setminus \{O\}$ such that O, X, and Y are noncollinear, then $\angle XO\alpha(X) \cong \angle YO\alpha(Y)$.

Exercise DLN.1 Proof. Let ρ_α be the rotation of $\mathcal{P}$ about O associated with α. Let $\mathcal{L}$ be the line of symmetry of $\angle XO\rho_\alpha(X)$ and let $\mathcal{M}$ be the line of symmetry of $\angle YO\rho_\alpha(Y)$. Then by Theorem DLN.2(E) $\alpha(X) = \text{ftpr}\,(X, \mathcal{L})$ and $\alpha(Y) = \text{ftpr}\,(X, \mathcal{M})$. By Theorem NEUT.39 $\overrightarrow{O\alpha(X)}$ is the bisecting ray of $\angle XO\rho_\alpha(X)$ and $\overrightarrow{O\alpha(Y)}$ is the bisecting ray of $\angle YO\rho_\alpha(Y)$. The result then follows immediately from Exercise ROT.4(E). $\square$

Exercise DLN.2* Let O be a point on a Euclidean plane $\mathcal{P}$, and let α and β be half-rotations of $\mathcal{P}$ about O; let R, S, and T be members of $\mathcal{P} \setminus \{O\}$ such that $\alpha(R) = S$, $\beta(S) = T$, and $S \in \text{ins}\,\angle ROT$. Then for every member U of $\mathcal{P} \setminus \{O\}$ $\angle UO\alpha(U) \cong \angle ROS$, $\angle \alpha(U)O(\beta \circ \alpha)(U) \cong \angle SOT$ $\angle UO(\beta \circ \alpha)(U) \cong \angle ROT$, and $\alpha(U) \in \text{ins}\,\angle UO(\beta \circ \alpha)(U)$.

Exercise DLN.2 Proof. By Theorem ROT.15(A) let α' and β' be rotations such that $\alpha'(R) \in \overleftrightarrow{OS}$ and $\beta'(S) \in \overleftrightarrow{OT}$. Then the result follows immediately from Exercise ROT.4(A2). $\square$

Exercise DLN.3* Let O be a point on a Euclidean plane $\mathcal{P}$, and let δ_1 and δ_2 be dilations of $\mathcal{P}$ with fixed point O. Then $\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1$, *i.e.* the composition of dilations with a common fixed point is commutative.

Exercise DLN.3 Proof. By Theorem DLN.7 there exist half-rotations α_1, β_1, γ_1, α_2, β_2, and γ_2 of $\mathcal{P}$ about O such that $\delta_1 = \gamma_1^{-1} \circ \beta_1 \circ \alpha_1$ and $\delta_2 = \gamma_2^{-1} \circ \beta_2 \circ \alpha_2$ so that $\delta_2 \circ \delta_1 = (\gamma_2^{-1} \circ \beta_2 \circ \alpha_2) \circ (\gamma_1^{-1} \circ \beta_1 \circ \alpha_1)$.

By Theorem DLN.6, the composition of half-rotations is commutative; using this fact repeatedly, along with associativity, we can "pull" each factor of the second set of parentheses "through" the first set of parentheses so that

$$\begin{aligned}\delta_2 \circ \delta_1 &= (\gamma_1^{-1}) \circ (\gamma_2^{-1} \circ \beta_2 \circ \alpha_2) \circ (\beta_1 \circ \alpha_1)\\ &= (\gamma_1^{-1} \circ \beta_1) \circ (\gamma_2^{-1} \circ \beta_2 \circ \alpha_2) \circ (\alpha_1)\\ &= (\gamma_1^{-1} \circ \beta_1 \circ \alpha_1) \circ (\gamma_2^{-1} \circ \beta_2 \circ \alpha_2) = \delta_1 \circ \delta_2. \quad \square\end{aligned}$$

Exercise DLN.4* Let O be a point on a Euclidean plane $\mathcal{P}$, and let $\mathbb{D} = \{\alpha \mid \alpha$ be a dilation of $\mathcal{P}$ with fixed point O, or $\alpha = \imath\}$. Then under composition of mappings $\mathbb{D}$ is an abelian group.

Exercise DLN.4 Proof. By Theorem CAP.21 $\mathbb{D}$ is a group under composition of mappings. By Exercise DLN.3 that group is Abelian. □

Exercise DLN.5* Let O be a point on a Euclidean plane $\mathcal{P}$, and let δ be a dilation of $\mathcal{P}$ with fixed point O.

(I) If X and Y are members of $\mathcal{P} \setminus \{O\}$ such that O, X, and Y are noncollinear, then $\delta(X)$ and $\delta(Y)$ are on the same side of $\overleftrightarrow{XY}$.

(II) Let A be any member of $\mathcal{P} \setminus \{O\}$ and let X be any member of $\mathcal{P} \setminus \{O, A\}$.

(A) If O–A–$\delta(A)$, then O–X–$\delta(X)$.

(B) If O–$\delta(A)$–A, then O–$\delta(X)$–X.

(C) If $\delta(A)$–O–A, then $\delta(X)$–O–X.

(III) Let A be any member of $\mathcal{P} \setminus \{O\}$ and let X be any member of $\mathcal{P} \setminus \{O, A\}$.

(A) If $\delta(A) \in \overrightarrow{OA}$, then $\delta(X) \in \overrightarrow{OX}$.

(B) If A' is a point such that A'–O–A, X' is a point such that X'–O–X, and if $\delta(A) \in \overrightarrow{OA'}$, then $\delta(X) \in \overrightarrow{OX'}$.

(IV) Let A be any member of $\mathcal{P} \setminus \{O\}$ and let C be any member of $\mathcal{P} \setminus \overleftrightarrow{OA}$.

(A) If $\delta(A) \in \overrightarrow{OA}$, then $\delta(C)$ is on the C-side of $\overleftrightarrow{OA}$.

(B) If $\delta(A)$–O–A, then $\delta(C)$ is on the side of $\overleftrightarrow{OA}$ opposite the C-side.

Exercise DLN.5 Proof. (I) By Theorem CAP.1(A) $\delta(\overleftrightarrow{XY}) = \overleftrightarrow{\delta(X)\delta(Y)}$. By Theorem CAP.18(C) and Definition CAP.17 $\overleftrightarrow{XY} \parallel \delta(\overleftrightarrow{XY})$ so that $\overleftrightarrow{XY} \parallel \overleftrightarrow{\delta(X)\delta(Y)}$. By Theorem IB.10 and Exercise PSH.14, $\delta(X)$ and $\delta(Y)$ are on the same side of $\overleftrightarrow{XY}$.

(II) Let X be any member of $\mathcal{P} \setminus \{O, A\}$. Then by Definition CAP.10 $\overleftrightarrow{AX} \;\mathbb{PE}\; \overleftrightarrow{\delta(A)\delta(X)}$.

(Case 1: $X \in (\mathcal{P} \setminus \overleftrightarrow{OA})$.) By Theorem DLN.9(A), $\delta(X)$ is the point of intersection of $\overleftrightarrow{OX}$ with $\operatorname{par}(\delta(A), \overleftrightarrow{AX})$, so that the lines $\overleftrightarrow{A\delta(A)}$ and $\overleftrightarrow{X\delta(X)}$ are parallel. Conclusions (A), (B), and (C) are then immediate consequences of Exercise PSH.56.

(Case 2: $X \in (\overleftrightarrow{OA} \setminus \{O, A\})$.) Following the construction of Theorem DLN.9(B), we let Q be any point not on $\overleftrightarrow{OA}$. Part (A) locates $\delta(Q)$ as a point of $\overleftrightarrow{OQ}$. Since $A \notin \overleftrightarrow{OQ}$ we may apply Theorem DLN.9(A) again, locating $\delta(X)$

as the point of intersection of $\overleftrightarrow{OA}$ with par $(\delta(Q), \overleftrightarrow{QX})$, and the lines $\overleftrightarrow{QX}$ and $\overleftrightarrow{\delta(Q)\delta(X)}$ are parallel.

(A) If O–A–$\delta(A)$, then by Case 1, O–Q–$\delta(Q)$. By another application of Case 1 we get O–X–$\delta(X)$.

(B) If O–$\delta(A)$–A, then by Case 1 O–$\delta(Q)$–Q. Another application of Case 1 yields O–$\delta(X)$–X.

(C) If A–O–$\delta(A)$, then by Case 1, Q–O–$\delta(Q)$. Another application of Case 1 yields X–O–$\delta(X)$.

(III)(A) This is an immediate consequence of (A), (B), and Definition IB.4.

(B) This is an immediate consequence of (C) and Definition IB.4.

(IV) (A) By Part (II)(A) and (B) of this theorem, $\delta(A) \in \overrightarrow{OA}$ and $\delta(C) \in \overrightarrow{OC}$. Hence by Theorem IB.14 $\delta(C) \in C$-side of $\overleftrightarrow{OA}$.

(B) By part (III)(B) of this theorem, $\delta(C)$–O–C and by Definition IB.11 $\delta(C)$ and C are on opposite sides of $\overleftrightarrow{OA}$. $\square$

Exercise DLN.6* Let O be a point on a Euclidean plane $\mathcal{P}$; let δ be a dilation of $\mathcal{P}$ with fixed point O and let ρ be a rotation of $\mathcal{P}$ about O. Then $\rho^{-1} \circ \delta \circ \rho = \delta$ and $\delta^{-1} \circ \rho \circ \delta = \rho$.

Exercise DLN.6 Proof. By Theorem DLN.7(E) and associativity,

$$\rho^{-1} \circ (\delta \circ \rho) = \rho^{-1} \circ (\rho \circ \delta) = (\rho^{-1} \circ \rho) \circ \delta = \imath \circ \delta = \delta \text{ and}$$
$$\delta^{-1} \circ (\rho \circ \delta) = \delta^{-1} \circ (\delta \circ \rho) = (\delta^{-1} \circ \delta) \circ \rho = \imath \circ \rho = \rho. \quad \square$$

Exercise DLN.7* Let O be a point on a Euclidean plane $\mathcal{P}$; let δ be a dilation on $\mathcal{P}$ with fixed point O, and let $\mathcal{L}$ be any line on $\mathcal{P}$. Then

$$\mathcal{R}_{\mathcal{L}} \circ \delta = \delta \circ \mathcal{R}_{\delta^{-1}(\mathcal{L})}.$$

Exercise DLN.7 Proof. By Theorem CAP.21 δ^{-1} is a dilation of $\mathcal{P}$ with fixed point O. By Theorem DLN.15 $\delta \circ \mathcal{R}_{\delta^{-1}(\mathcal{L})} = \mathcal{R}_{\delta(\delta^{-1}(\mathcal{L}))} \circ \delta = \mathcal{R}_{\mathcal{L}} \circ \delta$. $\square$

Exercise DLN.8* Let O be a point on a Euclidean plane $\mathcal{P}$, and let δ be a dilation of $\mathcal{P}$ with fixed point O. Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be distinct lines on $\mathcal{P}$. Then $\delta \circ (\mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}) = (\mathcal{R}_{\delta(\mathcal{N})} \circ \mathcal{R}_{\delta(\mathcal{M})} \circ \mathcal{R}_{\delta(\mathcal{L})}) \circ \delta$.

Exercise DLN.8 Proof. By Theorem DLN.15

$$\begin{aligned}
\delta \circ (\mathcal{R}_{\mathcal{N}} \circ \mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}) &= (\delta \circ \mathcal{R}_{\mathcal{N}}) \circ (\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}) = (\mathcal{R}_{\delta(\mathcal{N})} \circ \delta) \circ (\mathcal{R}_{\mathcal{M}} \circ \mathcal{R}_{\mathcal{L}}) \\
&= (\mathcal{R}_{\delta(\mathcal{N})} \circ (\delta \circ \mathcal{R}_{\mathcal{M}})) \circ \mathcal{R}_{\mathcal{L}} \\
&= (\mathcal{R}_{\delta(\mathcal{N})} \circ (\mathcal{R}_{\delta(\mathcal{M})} \circ \delta)) \circ \mathcal{R}_{\mathcal{L}} \\
&= (\mathcal{R}_{\delta(\mathcal{N})} \circ \mathcal{R}_{\delta(\mathcal{M})}) \circ (\delta \circ \mathcal{R}_{\mathcal{L}})
\end{aligned}$$

$$= (\mathcal{R}_{\delta(\mathcal{N})} \circ \mathcal{R}_{\delta(\mathcal{M})}) \circ (\mathcal{R}_{\delta(\mathcal{L})} \circ \delta)$$
$$= (\mathcal{R}_{\delta(\mathcal{N})} \circ \mathcal{R}_{\delta(\mathcal{M})} \circ \mathcal{R}_{\delta(\mathcal{L})}) \circ \delta. \quad \square$$

Exercise DLN.9* Let O be a point on a Euclidean plane $\mathcal{P}$; let δ be a dilation of $\mathcal{P}$ with fixed point O, and let θ be an isometry of $\mathcal{P}$. Then there exists an isometry ψ of $\mathcal{P}$ such that $\theta \circ \delta = \delta \circ \psi$.

Exercise DLN.9 Proof. By elementary mapping theory, we know that $(\theta \circ \delta)^{-1} = \delta^{-1} \circ \theta^{-1}$. By Theorem NEUT.11 θ^{-1} is an isometry. By Theorem CAP.21 δ^{-1} is a dilation of $\mathcal{P}$. By Theorem DLN.16, there exists an isometry φ of $\mathcal{P}$ such that $\delta^{-1} \circ \theta^{-1} = \varphi \circ \delta^{-1}$. Taking inverses, we have

$$\theta \circ \delta = (\delta^{-1} \circ \theta^{-1})^{-1} = (\varphi \circ \delta^{-1})^{-1} = \delta \circ \varphi^{-1}.$$

Let $\psi = \varphi^{-1}$; by Theorem NEUT.11 this is an isometry; then $\theta \circ \delta = \delta \circ \psi$. $\square$

Exercise DLN.10 Using the construction of Theorem DLN.4, prove that for any half-rotation α, if A–B–C, then $\alpha(A)$–$\alpha(B)$–$\alpha(C)$.

Chapter 14: Exercises and Answers for A Line as an Ordered Field (OF)

Exercise OF.1* Let $\mathcal{P}$ be a Euclidean plane; let $\mathbb{L}$ be an ordered field on $\mathcal{P}$ with origin O, and τ_A be the translation of $\mathcal{P}$ such that $\tau_A(O) = A$, where A is any member of $\mathbb{L} \setminus \{O\}$. Then for every member X of $\mathbb{L}$, $\tau_A(X) = X \oplus A$.

Exercise OF.1 Proof. This is an immediate consequence of Definition OF.1(A) and (C). □

Exercise OF.2* Let $\mathcal{P}$ be a Euclidean plane; let $\mathbb{L}$ be an ordered field on $\mathcal{P}$ with origin O and unit U, (where $U \in (\mathbb{L} \setminus \{O\})$) and let δ_A be the dilation of $\mathcal{P}$ with fixed point O such that $\delta_A(U) = A$. Then for every member X of $\mathbb{L} \setminus \{O\}$, $\delta_A(X) = X \odot A$.

Exercise OF.2 Proof. This is an immediate consequence of Definition OF.1(B) and (D). □

Exercise OF.3* (A) If A, B, and C are members of the ordered field $\mathbb{L}$ (*cf.* Definition OF.1) such that $A \oplus C = B \oplus C$, then $A = B$.

(B) If A and B are members of $\mathbb{L}$ and if C is a member of $\mathbb{L} \setminus \{O\}$ such that $A \odot C = B \odot C$, then $A = B$.

Exercise OF.3 Proof. (A) By Theorem OF.2 there exists a member D of $\mathbb{L}$ such that $C \oplus D = O$. Since $(A \oplus C) \oplus D = (B \oplus C) \oplus D$, by the associative property for addition $A \oplus (C \oplus D) = B \oplus (C \oplus D)$, that is $A \oplus O = B \oplus O$, or $A = B$.

(B) There exists a member D of $\mathbb{L} \setminus \{O\}$, such that $C \odot D = U$. Since by the associative property for multiplication $(A \odot C) \odot D = A \odot (C \odot D)$ and $(B \odot C) \odot D = B \odot (C \odot D)$ and since $(A \odot C) \odot D = (B \odot C) \odot D$, $A \odot (C \odot D) = B \odot (C \odot D)$, that is $A \odot U = B \odot U$. Thus $A = B$. □

Exercise OF.4* (A) If A, B, and C are members of the field $\mathbb{L}$ such that $A \oplus B = A \oplus C$, then $B = C$.

(B) If A is a member of $\mathbb{L} \setminus \{O\}$, and if B and C are members of $\mathbb{L}$ such that $A \odot B = A \odot C$, then $B = C$.

Exercise OF.4 Proof. (A) By the commutative property for addition $B \oplus A = C \oplus A$, so by Exercise OF.3 $B = C$.

(B) By the commutative property for multiplication $B \odot A = C \odot A$, so by Exercise OF.3 $B = C$. $\square$

Exercise OF.5* Let A, B, and C be members of the field $\mathbb{L}$; then $(B \ominus A) \odot C = (B \odot C) \ominus (A \odot C)$.

Exercise OF.5 Proof. Using Definition OF.8, the commutative property of multiplication, and Theorem OF.6 (distributive property),

$$\begin{aligned}(B \ominus A) \odot C &= (B \oplus ({}^{\ominus}A)) \odot C = C \odot (B \oplus ({}^{\ominus}A)) \\ &= (C \odot B) \oplus (C \odot ({}^{\ominus}A)) = (B \odot C) \oplus (({}^{\ominus}A) \odot C).\end{aligned}$$

By Theorem OF.10(D) $({}^{\ominus}A) \odot C = {}^{\ominus}(A \odot C)$ so that this becomes

$$(B \odot C) \oplus^{\ominus} (A \odot C) = (B \odot C) \ominus (A \odot C),$$

as required. $\square$

Exercise OF.6* Let δ be a dilation of the Euclidean plane $\mathcal{P}$ with fixed point O, and let $\mathbb{L}$ be an ordered field with origin O and unit U. If K and T are any members of $\mathbb{L}$, then $\delta(K \odot T) = K \odot \delta(T)$.

Exercise OF.6 Proof. By Definition OF.1 and Exercise DLN.3

$$\begin{aligned}\delta(K \odot T) &= \delta(\delta_K(T)) = (\delta \circ \delta_K)(T) = (\delta_K \circ \delta)(T) \\ &= \delta_K(\delta(T)) = K \odot \delta(T). \quad \square\end{aligned}$$

Exercise OF.7* Let A and B be members of $\mathbb{L}$. Complete the proof of Theorem OF.11(A) by showing that $B \ominus A > O$ iff $({}^{\ominus}B) < ({}^{\ominus}A)$.

Exercise OF.7 Proof. By Theorem OF.10(A)(3) and (D) $O < B \ominus A = ({}^{\ominus}({}^{\ominus}B)) \ominus A = ({}^{\ominus}A) \ominus ({}^{\ominus}B)$. By the part of Theorem OF.11(A) already proved, this is true iff $({}^{\ominus}A) > ({}^{\ominus}B)$. $\square$

Exercise OF.8* Prove part D of Theorem OF.11: if $A < B$ and $C < O$, then $B \odot C < A \odot C$.

Exercise OF.8 Proof. If $A < B$, then by Theorem OF.11(A) $B \ominus A > O$. Since $C < O$ by Theorem OF.10(E) $C \odot (B \ominus A) < O$. By Exercise OF.5

$$O > C \odot (B \ominus A) = (C \odot B) \ominus (C \odot A);$$

by Theorem OF.11(A) $C \odot A > C \odot B$. $\square$

Exercise OF.9* Let A and B be negative members of $\mathbb{L}$. Then $A < B$ iff $|B| < |A|$.

Exercise OF.9 Proof. By Definition OF.13, $|A| = {}^{\ominus}A$ and $|B| = {}^{\ominus}B$. Then by Theorem OF.11(A) $A < B$ iff $|B| = {}^{\ominus}B < |A| = {}^{\ominus}A$. $\square$

Exercise OF.10 (A) Let $\mathbb{T} = \{\tau_A | A \in \mathbb{L}\}$; then the mapping $\alpha{:}A \to \tau_A$ is a bijection of $\mathbb{L}$ onto $\mathbb{T}$.

(B) Let $\mathbb{M} = \{\delta_A | A \in \mathbb{L}\}$; then the mapping $\mu{:}A \to \delta_A$ is a bijection of $\mathbb{L}$ onto $\mathbb{M}$; furthermore μ maps $\mathbb{L} \setminus \{O\}$ onto $\mathbb{M} \setminus \{O\}$.

Exercise OF.11* (This result is analogous to Theorem CAP.23) Let $\mathcal{P}$ be a Euclidean plane, and let $\mathbb{L}_1$ and $\mathbb{L}_2$ be parallel lines on $\mathcal{P}$, where $\mathbb{L}_1$ has been built into an ordered field with origin O_1 and unit U_1. Let O_2 be a point of $\mathbb{L}_2$, and let σ be the translation of $\mathcal{P}$ such that $\sigma(O_1) = O_2$. (The existence and uniqueness of this translation is guaranteed by Theorem ISM.5.) Let $A \in \mathbb{L}_1 \setminus \{O_1, U_1\}$. Then $\sigma \circ \delta_A \circ \sigma^{-1}$ is a dilation of $\mathcal{P}$ with fixed point O_2. In fact, $\sigma \circ \delta_A \circ \sigma^{-1} = \delta_{\sigma(A)}$ so that $\sigma \circ \delta_A = \delta_{\sigma(A)} \circ \sigma$.

Exercise OF.11 Proof. The statement of this exercise assumes that δ_A is the dilation with fixed point O_1 such that $\delta_A(U_1) = A$, and that $\delta_{\sigma(A)}$ is the dilation with fixed point O_2 such that $\delta_{\sigma(A)}(U_2) = \sigma(A)$, where $U_2 = \sigma(U_1)$.

By Definition CAP.6, $\sigma(\mathbb{L}_1) \parallel \mathbb{L}_1$, and since $O_2 \in \mathbb{L}_2$ and $\sigma(O_1) = O_2$, by the Parallel Axiom PS, $\sigma(\mathbb{L}_1) = \mathbb{L}_2$. Since σ^{-1} is a translation taking O_2 to O_1, it follows immediately from Theorem CAP.23(C) that $\sigma \circ \delta_A \circ \sigma^{-1}$ is a dilation of $\mathcal{P}$ with fixed point O_2. Moreover, $\sigma(\delta_A(\sigma^{-1}(U_2))) = \sigma(\delta_A(U_1)) = \sigma(A)$.

By Theorem CAP.24, $\delta_{\sigma(A)}$ is the only dilation with O_2 as a fixed point which maps U_2 to $\sigma(A)$, so that $\sigma \circ \delta_A \circ \sigma^{-1} = \delta_{\sigma(A)}$, and $\sigma \circ \delta_A = \delta_{\sigma(A)} \circ \sigma$. $\square$

Exercise OF.12* Let $\mathcal{P}$ be a Euclidean plane; let $\mathbb{L}$ be an ordered field on $\mathcal{P}$ with origin O and unit U, A be a member of $\mathbb{L} \setminus \{O, U\}$, and let τ_A and δ_A be as in Definition OF.1. Then $\delta_A \circ \tau_A = \tau_{\delta_A(A)} \circ \delta_A$.

Exercise OF.12 Proof. By Theorem CAP.13 $\delta_A \circ \tau_A \circ \delta_A^{-1}$ is a translation of $\mathcal{P}$. Then since O is a fixed point for δ_A^{-1},

$$(\delta_A \circ \tau_A \circ \delta_A^{-1})(O) = \delta_A(\tau_A(\delta_A^{-1}(O))) = \delta_A(\tau_A(O)) = \delta_A(A).$$

The translation $\tau_{\delta_A(A)}$ maps O to $\delta_A(A)$, and by Theorem CAP.9 is the only translation doing so. Therefore $\delta_A \circ \tau_A \circ \delta_A^{-1} = \tau_{\delta_A(A)}$. $\square$

Chapter 15: Exercises and Answers for Similarity on a Euclidean Plane (SIM)

Exercise SIM.1* Let $\mathcal{P}$ be a Euclidean plane and let $\mathcal{A}$ and $\mathcal{B}$ be free segments of $\mathcal{P}$.

(I) If $\mathcal{A} < \mathcal{B}$, then $\mathcal{A}^2 < \mathcal{B}^2$.

(II) If $\mathcal{A} > \mathcal{B}$, then $\mathcal{A}^2 > \mathcal{B}^2$.

Exercise SIM.1 Proof. (I) If $\mathcal{A} < \mathcal{B}$, then by Theorem SIM.9 and Definition SIM.10 $\mathcal{A}^2 < \mathcal{A} \odot \mathcal{B}$ and $\mathcal{A} \odot \mathcal{B} < \mathcal{B}^2$. By Theorem FSEG.7 (Transitivity for Free Segments), $\mathcal{A}^2 < \mathcal{B}^2$.

(II) If $\mathcal{A} > \mathcal{B}$, then by Definition FSEG.3 $\mathcal{B} < \mathcal{A}$. By part (I) $\mathcal{B}^2 < \mathcal{A}^2$, that is, $\mathcal{A}^2 > \mathcal{B}^2$. $\square$

Exercise SIM.2* Let $\mathcal{P}$ be a Euclidean plane and let $\mathcal{A}$ and $\mathcal{B}$ be free segments of $\mathcal{P}$. If $\mathcal{A}^2 = \mathcal{B}^2$, then $\mathcal{A} = \mathcal{B}$.

Exercise SIM.2 Proof. We prove the contrapositive: If $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A}^2 \neq \mathcal{B}^2$. If $\mathcal{A} \neq \mathcal{B}$, then by Theorem FSEG.5 (Trichotomy for Free Segments), either $\mathcal{A} < \mathcal{B}$, or $\mathcal{B} < \mathcal{A}$. If $\mathcal{A} < \mathcal{B}$, then by Exercise SIM.1 $\mathcal{A}^2 < \mathcal{B}^2$. If $\mathcal{B} < \mathcal{A}$, then by the same exercise, $\mathcal{B}^2 < \mathcal{A}^2$. $\square$

Exercise SIM.3* Let $\mathcal{P}$ be a Euclidean plane and let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be free segments of $\mathcal{P}$ such that $\mathcal{C} < \mathcal{B}$. Then $\mathcal{A} \odot (\mathcal{B} \ominus \mathcal{C}) = (\mathcal{A} \odot \mathcal{B}) \ominus (\mathcal{A} \odot \mathcal{C})$.

Exercise SIM.3 Proof. By Theorem SIM.9 $\mathcal{A} \odot \mathcal{C} < \mathcal{A} \odot \mathcal{B}$. By Theorem FSEG.10 $(\mathcal{A} \odot \mathcal{B}) \ominus (\mathcal{A} \odot \mathcal{C})$ exists. By Definition FSEG.11 $\mathcal{B} \ominus \mathcal{C}$ is the free segment $\mathcal{D}$ of $\mathcal{P}$ such that $\mathcal{C} \oplus \mathcal{D} = \mathcal{B}$. By Theorem SIM.8 $\mathcal{A} \odot \mathcal{B} = \mathcal{A} \odot (\mathcal{C} \oplus \mathcal{D}) = (\mathcal{A} \odot \mathcal{C}) \oplus (\mathcal{A} \odot \mathcal{D})$. By Definition FSEG.11 $\mathcal{A} \odot \mathcal{D} = (\mathcal{A} \odot \mathcal{B}) \ominus (\mathcal{A} \odot \mathcal{C})$, that is, $\mathcal{A} \odot (\mathcal{B} \ominus \mathcal{C}) = (\mathcal{A} \odot \mathcal{B}) \ominus (\mathcal{A} \odot \mathcal{C})$. $\square$

Exercise SIM.4* Let $\mathcal{P}$ be a Euclidean plane and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be free segments on $\mathcal{P}$. Then the following statements are equivalent to each other.

(1) $\mathcal{A} \odot \mathcal{D} = \mathcal{B} \odot \mathcal{C}$.

(2) $\mathcal{A} \odiv \mathcal{B} = \mathcal{C} \odiv \mathcal{D}$.

(3) $\mathcal{A} \odiv \mathcal{C} = \mathcal{B} \odiv \mathcal{D}$.

(4) $\mathcal{B} \odiv \mathcal{A} = \mathcal{D} \odiv \mathcal{C}$.

(5) $(\mathcal{A} \oplus \mathcal{B}) \odiv \mathcal{B} = (\mathcal{C} \oplus \mathcal{D}) \odiv \mathcal{D}$.

Exercise SIM.4 Proof. We use Theorem SIM.8 and Definition SIM.12 in the following reasoning:

$\mathcal{A} \odot \mathcal{D} = \mathcal{B} \odot \mathcal{C}$ iff $(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1})$; $(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1})$ iff $\mathcal{A} \odot \mathcal{C}^{-1} = \mathcal{B} \odot \mathcal{D}^{-1}$.

Also, $\mathcal{A} \odot \mathcal{C}^{-1} = \mathcal{B} \odot \mathcal{D}^{-1}$ iff $\mathcal{A} \odiv \mathcal{C} = \mathcal{B} \odiv \mathcal{D}$. $\mathcal{A} \odot \mathcal{D} = \mathcal{B} \odot \mathcal{C}$ iff $(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1})$.

$(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{C}^{-1} \odot \mathcal{D}^{-1})$ iff $\mathcal{A} \odot \mathcal{C}^{-1} = \mathcal{B} \odot \mathcal{D}^{-1}$ and $\mathcal{A} \odot \mathcal{C}^{-1} = \mathcal{B} \odot \mathcal{D}^{-1}$ iff $\mathcal{A} \odiv \mathcal{C} = \mathcal{B} \odiv \mathcal{D}$.

$\mathcal{A} \odot \mathcal{D} = \mathcal{B} \odot \mathcal{C}$ iff $(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{A}^{-1} \odot \mathcal{C}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{A}^{-1} \odot \mathcal{C}^{-1})$;

$(\mathcal{A} \odot \mathcal{D}) \odot (\mathcal{A}^{-1} \odot \mathcal{C}^{-1}) = (\mathcal{B} \odot \mathcal{C}) \odot (\mathcal{A}^{-1} \odot \mathcal{C}^{-1})$ iff $\mathcal{D} \odot \mathcal{C}^{-1} = \mathcal{B} \odot \mathcal{A}^{-1}$ iff $\mathcal{B} \odiv \mathcal{A} = \mathcal{D} \odiv \mathcal{C}$.

$\mathcal{A} \odiv \mathcal{B} = \mathcal{C} \odiv \mathcal{D}$ iff $(\mathcal{A} \odot \mathcal{B}^{-1}) \oplus \mathcal{U} = (\mathcal{C} \odot \mathcal{D}^{-1}) \oplus \mathcal{U}$;

$(\mathcal{A} \odot \mathcal{B}^{-1}) \oplus \mathcal{U} = (\mathcal{C} \odot \mathcal{D}^{-1}) \oplus \mathcal{U}$ iff $(\mathcal{A} \odot \mathcal{B}^{-1}) \oplus (\mathcal{B} \odot \mathcal{B}^{-1}) = (\mathcal{C} \odot \mathcal{D}^{-1}) \oplus (\mathcal{D} \odot \mathcal{D}^{-1})$;

$(\mathcal{A} \odot \mathcal{B}^{-1}) \oplus (\mathcal{B} \odot \mathcal{B}^{-1}) = (\mathcal{C} \odot \mathcal{D}^{-1}) \oplus (\mathcal{D} \odot \mathcal{D}^{-1})$ iff $(\mathcal{A} \oplus \mathcal{B}) \odot \mathcal{B}^{-1} = (\mathcal{C} \oplus \mathcal{D}) \odot \mathcal{D}^{-1}$; and

$(\mathcal{A} \oplus \mathcal{B}) \odot \mathcal{B}^{-1} = (\mathcal{C} \oplus \mathcal{D}) \odot \mathcal{D}^{-1}$ iff $(\mathcal{A} \oplus \mathcal{B}) \odiv \mathcal{B} = (\mathcal{C} \oplus \mathcal{D}) \odiv \mathcal{D}$. □

Exercise SIM.5* Let $\mathcal{P}$ be a Euclidean plane and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be free segments on $\mathcal{P}$ such that $\mathcal{A} < \mathcal{C}$, $\mathcal{B} < \mathcal{D}$, and $\mathcal{A} \odiv \mathcal{B} = \mathcal{C} \odiv \mathcal{D}$. Then $\mathcal{A} \odiv \mathcal{B} = (\mathcal{C} \ominus \mathcal{A}) \odiv (\mathcal{D} \ominus \mathcal{B})$.

Exercise SIM.5 Proof. Since $\mathcal{A} < \mathcal{C}$ and $\mathcal{B} < \mathcal{D}$, by Theorem SIM.9 $\mathcal{U} = \mathcal{A}^{-1} \odot \mathcal{A} < \mathcal{A}^{-1} \odot \mathcal{C} = \mathcal{C} \odiv \mathcal{A}$ and $\mathcal{U} = \mathcal{B}^{-1} \odot \mathcal{B} < \mathcal{B}^{-1} \odot \mathcal{D} = \mathcal{D} \odiv \mathcal{B}$. By Exercise SIM.4 $\mathcal{C} \odiv \mathcal{A} = \mathcal{D} \odiv \mathcal{B}$, so $(\mathcal{C} \odiv \mathcal{A}) \ominus \mathcal{U} = (\mathcal{D} \odiv \mathcal{B}) \ominus \mathcal{U}$. That is, $(\mathcal{C} \odiv \mathcal{A}) \ominus (\mathcal{A} \odiv \mathcal{A}) = (\mathcal{D} \odiv \mathcal{B}) \ominus (\mathcal{B} \odiv \mathcal{B})$, so that $(\mathcal{C} \ominus \mathcal{A}) \odiv \mathcal{A} = (\mathcal{D} \ominus \mathcal{B}) \odiv \mathcal{B}$. In Exercise SIM.4(4) substitute $\mathcal{C} - \mathcal{A}$ for $\mathcal{B}$, $\mathcal{D} - \mathcal{B}$ for $\mathcal{D}$, $\mathcal{A}$ for $\mathcal{A}$, and $\mathcal{B}$ for $\mathcal{C}$; reading the equivalent formulation from Exercise SIM.4(3) we have $\mathcal{A} \odiv \mathcal{B} = (\mathcal{C} \ominus \mathcal{A}) \odiv (\mathcal{D} \ominus \mathcal{B})$. □

Exercise SIM.6* Let $\mathcal{P}$ be a Euclidean plane and let A_1, B_1, C_1, A_2, B_2, and C_2, be points on $\mathcal{P}$ such that A_1, B_1, and C_1 are noncollinear and A_2, B_2, and C_2 are noncollinear. Furthermore, let $\mathcal{A}_1 = [\overline{B_1C_1}]$, $\mathcal{B}_1 = [\overline{A_1C_1}]$, $\mathcal{C}_1 = [\overline{A_1B_1}]$, $\mathcal{A}_2 = [\overline{B_2C_2}]$, $\mathcal{B}_2 = [\overline{A_2C_2}]$, and $\mathcal{C}_2 = [\overline{A_2B_2}]$. Then: $\angle B_1A_1C_1 \cong \angle B_2A_2C_2$ and $\angle C_1B_1A_1 \cong \angle C_2B_2A_2$ iff $\mathcal{A}_1 \odiv \mathcal{B}_1 = \mathcal{A}_2 \odiv \mathcal{B}_2$, $\mathcal{A}_1 \odiv \mathcal{C}_1 = \mathcal{A}_2 \odiv \mathcal{C}_2$, and $\mathcal{B}_1 \odiv \mathcal{C}_1 = \mathcal{B}_2 \odiv \mathcal{C}_2$.

Exercise SIM.6 Proof. By Theorem SIM.16 $\angle B_1A_1C_1 \cong \angle B_2A_2C_2$ and $\angle C_1B_1A_1 \cong \angle C_2B_2A_2$ iff $\mathcal{A}_1 \odiv \mathcal{A}_2 = \mathcal{B}_1 \odiv \mathcal{B}_2$, $\mathcal{A}_1 \odiv \mathcal{A}_2 = \mathcal{C}_1 \odiv \mathcal{C}_2$, and $\mathcal{B}_1 \odiv \mathcal{B}_2 = \mathcal{C}_1 \odiv \mathcal{C}_2$. However, by Exercise SIM.4 $\mathcal{A}_1 \odiv \mathcal{A}_2 = \mathcal{B}_1 \odiv \mathcal{B}_2$ iff $\mathcal{A}_1 \odiv \mathcal{B}_1 = \mathcal{A}_2 \odiv \mathcal{B}_2$; $\mathcal{A}_1 \odiv \mathcal{A}_2 = \mathcal{C}_1 \odiv \mathcal{C}_2$ iff $\mathcal{A}_1 \odiv \mathcal{C}_1 = \mathcal{A}_2 \odiv \mathcal{C}_2$; and $\mathcal{B}_1 \odiv \mathcal{B}_2 = \mathcal{C}_1 \odiv \mathcal{C}_2$ iff

$\mathcal{B}_1 \oslash \mathcal{C}_1 = \mathcal{B}_2 \oslash \mathcal{C}_2$. Hence $\angle B_1A_1C_1 \cong \angle B_2A_2C_2$ and $\angle C_1B_1A_1 \cong \angle C_2B_2A_2$ iff $\mathcal{A}_1 \oslash \mathcal{B}_1 = \mathcal{A}_2 \oslash \mathcal{B}_2$, $\mathcal{A}_1 \oslash \mathcal{C}_1 = \mathcal{A}_2 \oslash \mathcal{C}_2$, and $\mathcal{B}_1 \oslash \mathcal{C}_1 = \mathcal{B}_2 \oslash \mathcal{C}_2$. □

Exercise SIM.7* Let O be a point on a Euclidean plane $\mathcal{P}$, and let δ be a dilation of $\mathcal{P}$ with fixed point O, such that for every $X \neq O$, $\delta(X) \in \overrightarrow{OX}$. Let X and Y be distinct members of $\mathcal{P} \setminus \{O\}$ such that O, X, and Y are collinear. Then $[\overline{OX}] \oslash [\overline{O\delta(X)}] = [\overline{OY}] \oslash [\overline{O\delta(Y)}]$.

Exercise SIM.7 Proof. Let X' be any member of $\mathcal{P} \setminus \overleftrightarrow{OX}$, and let $Y' = \delta(X')$. By Theorem CAP.18 $Y' \in \overleftrightarrow{OX'}$. Then by Theorem DLN.9(B) $\delta(X)$ is the point such that $\operatorname{par}(Y', \overleftrightarrow{X'X}) \cap \overleftrightarrow{OX} = \{\delta(X)\}$ and $\delta(Y)$ is the point such that $\operatorname{par}(Y', \overleftrightarrow{X'Y}) \cap \overleftrightarrow{OX} = \{\delta(Y)\}$. That is to say, $\overleftrightarrow{X'X} \parallel \overleftrightarrow{Y'\delta(X)}$, and $\overleftrightarrow{X'Y} \parallel \overleftrightarrow{Y'\delta(Y)}$.

Applying Theorem SIM.13 to $\overleftrightarrow{X'X} \parallel \overleftrightarrow{Y'\delta(X)}$, we have $[\overline{OX'}] \oslash [\overline{OX}] = [\overline{OY'}] \oslash [\overline{O\delta(X)}]$; by Exercise SIM.4 $[\overline{OX'}] \oslash [\overline{OY'}] = [\overline{OX}] \oslash [\overline{O\delta(X)}]$. Applying the same reasoning to $\overleftrightarrow{X'Y} \parallel \overleftrightarrow{Y'\delta(Y)}$, we have $[\overline{OX'}] \oslash [\overline{OY'}] = [\overline{OY}] \oslash [\overline{O\delta(Y)}]$. Therefore

$$[\overline{OX}] \oslash [\overline{O\delta(X)}] = [\overline{OX'}] \oslash [\overline{OY'}] = [\overline{OY}] \oslash [\overline{O\delta(Y)}]. \quad \square$$

Chapter 16: Exercises and Answers for Axial Affinities of a Euclidean Plane (AX)

Exercise AX.1* Let $\mathcal{M}$ be a line on a Euclidean plane $\mathcal{P}$; let A and B be distinct points such that $\overleftrightarrow{AB} \parallel \mathcal{M}$. By Theorem AX.2 there exists a shear ψ with axis $\mathcal{M}$ such $\psi(A) = B$. Let $\mathcal{L}$ be a line parallel to $\mathcal{M}$; either $\mathcal{L} = \overleftrightarrow{AB}$ or $\mathcal{L} \parallel \overleftrightarrow{AB}$. Let $C = \text{ftpr}(A, \mathcal{M})$, let D be the point of intersection of $\overleftrightarrow{AC}$ and $\mathcal{L}$, and let E be the point of intersection of $\overleftrightarrow{BC}$ and $\mathcal{L}$. Then by Theorem AX.2 $\psi(D) = E$. Using Theorem ISM.5 let τ be the translation of $\mathcal{P}$ such that $\tau(D) = E$. Show that for every $X \in \mathcal{L}$, $\psi(X) = \tau(X)$. This shows that the action of a shear on a line parallel to its axis is the same as that of a translation.

Exercise AX.1 Proof. Note first that $\overleftrightarrow{AC} = \overleftrightarrow{DC} \perp \mathcal{M}$. Let X be any point on $\mathcal{L}$ and let T be the point of intersection of $\text{pr}(X, \mathcal{L})$ and $\mathcal{M}$. By Theorem NEUT.47(A) $\overleftrightarrow{XT} \parallel \overleftrightarrow{DC}$. Then again by Theorem AX.2 $\psi(X)$ is the point of intersection of $\text{par}(T, \overleftrightarrow{CE})$ and $\mathcal{L}$, that is, $\text{par}(T, \overleftrightarrow{CE}) = \text{par}(T, \overleftrightarrow{C\psi(D)})$. We have already defined τ so that $\tau(D) = E$.

Let $F = \text{ftpr}(E, \mathcal{M})$; by Theorem NEUT.47(A) $\overleftrightarrow{FE} \parallel \overleftrightarrow{DC}$. By Theorem ISM.23 $\tau(X)$ is the point of intersection of $\mathcal{L}$ and $\text{par}(F, \overleftrightarrow{CX})$. Using Theorem ISM.5 construct a translation σ such that $\sigma(D) = X$. Then by Theorem ISM.7 $\sigma \circ \tau = \tau \circ \sigma$. By Definition CAP.6 $\sigma(\overleftrightarrow{DC}) \parallel \overleftrightarrow{DC}$, which is parallel to $\overleftrightarrow{XT}$. Since $X \in \sigma(\overleftrightarrow{DC})$ by Axiom PS $\sigma(\overleftrightarrow{DC}) = \overleftrightarrow{XT}$. By Theorem NEUT.15 $\overleftrightarrow{\sigma(D)\sigma(C)} = \sigma(\overleftrightarrow{DC}) = \overleftrightarrow{XT}$. Since both T and $\sigma(C)$ belong to $\mathcal{M}$, $T = \sigma(C)$.

By similar reasoning, $\sigma(\overleftrightarrow{EC}) \parallel \overleftrightarrow{\psi(X)T}$, $\overleftrightarrow{\sigma(E)\sigma(C)} = \sigma(\overleftrightarrow{EC}) = \overleftrightarrow{\psi(X)T}$, and $\sigma(E) = \psi(X)$, because both $\sigma(E)$ and $\psi(X)$ belong to $\mathcal{L}$. Then $\tau(X) = \tau(\sigma(D)) = \sigma(\tau(D)) = \sigma(E) = \psi(X)$. □

Exercise AX.2* Let $\mathcal{P}$ be a Euclidean plane, and let φ be an axial affinity with axis $\mathcal{M}$ on $\mathcal{P}$, and let $\mathcal{L}$ be a line distinct from $\mathcal{M}$. Then $\mathcal{L}$ is a fixed line for φ iff for some $Q \notin \mathcal{M}$, $\mathcal{L} = \overleftrightarrow{Q\varphi(Q)}$.

Exercise AX.2 Proof. If $\mathcal{L}$ is a fixed line, by Theorem CAP.27(A) $\mathcal{L} = \overleftrightarrow{Q\varphi(Q)}$ for any point $Q \in \mathcal{L} \setminus \mathcal{M}$.

Conversely, suppose that $\mathcal{L} = \overleftrightarrow{Q\varphi(Q)}$ for some $Q \notin \mathcal{M}$. By Theorem AX.3(A) φ is either a stretch or a shear. If it is a stretch, by Definition AX.0 there exists a fixed line $\mathcal{N}$ which intersects $\mathcal{M}$. If $Q \in \mathcal{N}$ then $\varphi(Q) \in \mathcal{N}$ so that by Axiom I.1 $\mathcal{N} = \mathcal{L}$; thus $\mathcal{L}$ is a fixed line. If $Q \notin \mathcal{N}$ then by Axiom PS there exists a line $\mathcal{L}'$ such that $Q \in \mathcal{L}'$ and $\mathcal{L}' \parallel \mathcal{N}$; by Theorem CAP.27(B)

$\mathcal{L}'$ is a fixed line, and by part (A) of the same theorem, $\mathcal{L}' = \overleftrightarrow{Q\varphi(Q)} = \mathcal{L}$, so that $\mathcal{L}$ is a fixed line.

If φ is a shear, by Axiom PS there exists a line $\mathcal{L}'$ such that $\mathcal{L}' \parallel \mathcal{M}$ and $Q \in \mathcal{L}'$. By Definition AX.0 $\mathcal{L}'$ is a fixed line, and by Theorem CAP.27(A) and Axiom I.1 $\mathcal{L}' = \overleftrightarrow{Q\varphi(Q)} = \mathcal{L}$, and $\mathcal{L}$ is a fixed line. $\square$

Chapter 17: Exercises and Answers for Rational Points on a Line (QX)

Exercise QX.1* Let r be a nonzero rational number, and let $A \in \mathbb{L}\backslash\{O\}$.
(A) If A is positive, then rA is positive iff r is positive.
(B) If A is negative, then rA is negative iff r is positive.

Exercise QX.1 Proof. (A) If A is positive, then by Theorem QX.13(A) rA is positive if r is positive. If r is not positive, it is negative, because $r \neq 0$. Then by Theorem QX.13(B), rA is negative, that is, not positive. This is the contrapositive of the converse of the first statement.

(B) If A is negative, then by Theorem QX.13(C) rA is negative if r is positive. If r is not negative, it is positive. Then by Theorem QX.13(D), rA is positive, that is, not negative. This is the contrapositive of the converse of the first statement. $\square$

Exercise QX.2* Let A be a positive member of $\mathbb{L}$ and r and s be rational numbers. Then $rA < sA$ iff $r < s$.

Exercise QX.2 Proof. (I) If $r < s$, then $s - r > 0$ so by Theorem QX.13 (or Exercise QX.1) $(s - r)A > O$. Since by Theorem QX.11(A) $(s - r)A = (s + (-r))A = sA \oplus (^{\ominus}rA) = sA \ominus rA$, and this is positive, then by Theorem OF.11(A) $rA < sA$.

(II) If $rA < sA$, then by Theorem OF.11(A) $sA \ominus rA = (s - r)A$ is positive. If $s - r$ were negative, then $r - s$ would be positive and $rA > sA$ by part (I), contradicting the assumption. Hence $s-r$ is positive and so $r < s$. $\square$

Exercise QX.3* Let A be a negative member of $\mathbb{L}$ and r and s be rational numbers. Then $rA > sA$ iff $r < s$.

Exercise QX.3 Proof. (I) If $r < s$, then $s - r > 0$ so by Theorem QX.13 (or Exercise QX.1) $(s - r)A < O$. Since by Theorem QX.11(A) $(s - r)A = (s + (-r))A = sA \oplus (^{\ominus}rA) = sA \ominus rA$, and this is negative, then by Theorem OF.11(A) $rA > sA$.

(II) If $rA > sA$, then by Theorem OF.11(A) $sA \ominus rA = (s - r)A$ is negative. If $s - r$ were positive, then $r - s$ would be negative and $rA < sA$ by part (I), a contradiction. Hence $s - r$ is negative and so $r > s$. $\square$

Exercise QX.4* Let $\mathcal{P}$ be a Euclidean plane, $\mathbb{L}$ be an ordered field on $\mathcal{P}$ (*cf* Theorem Q.13), T be a member of $\mathbb{L}$ and r be a rational number. Then $(-r)T = {}^{\ominus}(rT)$.

Exercise QX.4 Proof. By Theorem QX.11(A), $(-r)T \oplus rT = (-r+r)T = 0T = O$. By Definition OF.4, $(-r)T = {}^{\ominus}(rT)$. $\square$

Exercise QX.5* Let $\mathbb{L}$ be an ordered field with origin O on a Euclidean plane $\mathcal{P}$, and let X and Y be positive members of $\mathbb{L}$. Then there exist non-collinear points A, B, and C on $\mathcal{P}$ such that $\frac{1}{2}X \odot Y$ is the area of $\triangle ABC$.

Exercise QX.5 Proof. Let C and E be distinct points on $\mathcal{P}$. By Theorem NEUT.67 (segment construction) there exists a unique point B on $\overrightarrow{CE}$ such that $[\overline{CB}] = [\overline{OX}]$. Let $\mathcal{L} = \mathrm{pr}(B, \overleftrightarrow{BC})$ (*cf* Definition NEUT.99(A)). Then by Theorem NEUT.67 there exists a point $A \in \mathcal{L}$ such that $[\overline{BA}] = [\overline{OY}]$. Then $\overline{CB}$ is an altitude of $\triangle ABC$, $\overline{AB}$ is the base for that altitude, and by Definition QX.22 the area of this triangle is $\frac{1}{2}X \odot Y$. $\square$

Chapter 18: Exercises and Answers for A Line as Real Numbers (REAL); Coordinatization of a Plane (RR)

Exercise REAL.1* Let A, B, C, and D be points on the Euclidean plane $\mathcal{P}$ such that $A \neq B$ and $C \neq D$. Then there exists a natural number n such that $\frac{[\overline{AB}]}{2^n} < [\overline{CD}]$.

Exercise REAL.1 Proof. By Theorem REAL.5 there exists a natural number n such that $n[\overline{CD}] > [\overline{AB}]$. Since for every natural number n, $2^n > n$, $2^n[\overline{CD}] > n[\overline{CD}]$, $2^n[\overline{CD}] > [\overline{AB}]$ so that $\frac{[\overline{AB}]}{2^n} < [\overline{CD}]$. $\square$

Exercise REAL.2* Let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ be a line in $\mathcal{P}$ having origin O and unit U. Then if T and V are positive members of $\mathbb{L}$, there exists a natural number n such that $\frac{1}{n}T < V$.

Exercise REAL.2 Proof. Since by Theorem REAL.9 the set $\{nV \mid n \in \mathbb{N}\}$ is unbounded above, there exists a natural number n such that $nV > T$. But then $\frac{1}{n}T < V$. $\square$

Exercise REAL.3* Let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ be a line in $\mathcal{P}$ having origin O and unit U. Then if T is a positive member of $\mathbb{L}$, $\{s \mid s \in \mathbb{Q} \text{ and } sU < T\}$ is bounded above.

Exercise REAL.3 Proof. Since by Corollary REAL.9.1 $\{tU \mid t \in \mathbb{Q}\}$ is unbounded above there exists a rational number h such that $hU > T$. Let s be any member of $\{t \mid t \in \mathbb{Q} \text{ and } tU < T\}$. Then $sU < T < hU$ so that by Exercise QX.2 $s < h$ and h is an upper bound of $\{s \mid s \in \mathbb{Q} \text{ and } sU < T\}$. $\square$

Exercise REAL.4* Prove Lemma REAL.4: let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ be an ordered field on $\mathcal{P}$ with origin O and unit U. If $\mathcal{E}$ is a subset of $\mathbb{L}$ which is bounded above, and $T > O$ is a member of $\mathbb{L}$, then $(\operatorname{lub}\mathcal{E}) \odot T = \operatorname{lub}(\mathcal{E} \odot T)$.

Exercise REAL.4 Proof. Suppose B is any upper bound for $\mathcal{E}$; then for every $A \in \mathcal{E}$, $B \geq A$. By Theorem OF.11(C), $B \odot T \geq A \odot T$ so that $B \odot T$ is an upper bound for the set $\mathcal{E} \odot T$. Since $\operatorname{lub}\mathcal{E}$ is an upper bound for $\mathcal{E}$, $(\operatorname{lub}\mathcal{E}) \odot T$ is an upper bound for $\mathcal{E} \odot T$, hence $(\operatorname{lub}\mathcal{E}) \odot T \geq \operatorname{lub}(\mathcal{E} \odot T)$.

By Theorem OF.10(E), $T^{-1} > O$, so that substituting $\mathcal{E} \odot T$ for $\mathcal{E}$ and T^{-1} for T in the calculation just above,

$$(\operatorname{lub}(\mathcal{E} \odot T)) \odot T^{-1} \geq \operatorname{lub}((\mathcal{E} \odot T) \odot T^{-1}) = \operatorname{lub} \mathcal{E},$$

and multiplying on the right by T we have

$$\operatorname{lub}(\mathcal{E} \odot T) = (\operatorname{lub}(\mathcal{E} \odot T)) \odot T^{-1} \odot T \geq (\operatorname{lub} \mathcal{E}) \odot T;$$

therefore $(\operatorname{lub} \mathcal{E}) \odot T = \operatorname{lub}(\mathcal{E} \odot T)$. $\square$

Exercise REAL.5* Prove Lemma REAL.24: let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ be an ordered field on $\mathcal{P}$ with origin O and unit U. Let $\mathcal{S}$ be a subset of $\mathbb{L}$ which is bounded above, and suppose A is an upper bound for $\mathcal{S}$. Then $A = \operatorname{lub} S$ iff the following property holds: for every $\epsilon > O$ in $\mathbb{L}$, there exists $x \in \mathbb{L}$ such that $x > A \ominus \epsilon$.

Exercise REAL.5 Proof. Assume that $A = \operatorname{lub} S$, and that the property does not hold; then there exists $\epsilon > O$ such that for all $X \in \mathcal{S}$, $X \leq A \ominus \epsilon$. Then $A \ominus \epsilon < A$ is an upper bound for $\mathcal{S}$, so that A is not the least upper bound.

Conversely, if the property holds and $A \neq \operatorname{lub} S$, since A is an upper bound, $A \geq \operatorname{lub} S$. Since A is not the least upper bound, there exists an upper bound B such that $A > B$. Let $\epsilon = A \ominus B$; then there exists $X \in \mathcal{S}$ such that $X > A \ominus \epsilon = A \ominus (A \ominus B) = B$. Then X is not an upper bound for $\mathcal{S}$, a contradiction. $\square$

Exercise REAL.6* Complete the proof of Case 4 of Theorem REAL.23: let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ an ordered field on $\mathcal{P}$ with origin O and unit U. Let $S > O$ be a member of $\mathbb{L}$. Then if $x < 0$ and $y > 0$ are irrational numbers, $x(yS) = (xy)S$.

Exercise REAL.6 Proof. Using, in succession, arithmetic, Theorem REAL.21(A), Case 2 of the proof of Theorem REAL.23, arithmetic, Theorem REAL.21(A), and Theorem OF.10(A), we have

$$\begin{aligned} x(yS) &= (-(-x))(yS) = {}^{\ominus}((-x)(yS)) = {}^{\ominus}(((-x)y)S) \\ &= {}^{\ominus}((-(xy))S) = {}^{\ominus}({}^{\ominus}((xy))S) = (xy)S. \quad \square \end{aligned}$$

Exercise REAL.7* Complete the proof of Theorem REAL.25, Case 3: let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ an ordered field on $\mathcal{P}$ with origin O and unit U. Let $S < O$ and $T > O$ be members of $\mathbb{L}$. If x is an irrational number, then $(xS) \odot T = x(S \odot T)$.

Exercise REAL.7 Proof. Using, in succession, Theorem REAL.21(C), Theorem OF.10(D), Case 1 of the proof of Theorem REAL.25, Theorem OF.10(D), and Theorem REAL.21(C), we have

$$\begin{aligned}(xS) \odot T &= ({}^{\ominus}(x({}^{\ominus}S))) \odot T = {}^{\ominus}((x({}^{\ominus}S)) \odot T)\\ &= {}^{\ominus}(x(({}^{\ominus}S) \odot T)) = {}^{\ominus}(x({}^{\ominus}(S \odot T)))\\ &= x(S \odot T). \quad \square\end{aligned}$$

Exercise REAL.8* Complete the proof of Case 5 of Theorem REAL.31: let $\mathcal{P}$ be a Euclidean/LUB plane, and let $\mathbb{L}$ an ordered field on $\mathcal{P}$ with origin O and unit U. If $x < 0$ and $y < 0$ are irrational numbers, and H is any member of $\mathbb{L}$, then $(x + y)H = xH \oplus yH$.

Exercise REAL.8 Proof. Applying, in order, Theorem REAL.21(A), arithmetic, Cases 3 and 4 of the proof of Theorem REAL.31, Theorem REAL.21(A), and Theorem OF.10(F), we have, since $-x > 0$ and $-y > 0$,

$$\begin{aligned}{}^{\ominus}(x + y)H &= -(x + y)H = ((-x) + (-y))H = (-x)H \oplus (-y)H\\ &= {}^{\ominus}(xH) \oplus {}^{\ominus}(yH) = {}^{\ominus}(xH \oplus yH).\end{aligned}$$

The result follows from Theorem OF.10(A). $\square$

Exercise REAL.9* (Alternative proof of Theorem REAL.32) Let x be any real number, and let S and T be members of $\mathbb{L}$. Prove, using Definition REAL.19 and other theorems from this chapter and previous ones, including Theorem REAL.21, that $x(S \oplus T) = xS \oplus xT$.

Exercise REAL.9 Proof. (Case 0: $x = 0$ or $S = O$ or $T = 0$.) If $x = 0$ then $x(S \oplus T) = O = xS \oplus xT$. If $S = O$ then $xS = O$ and $(x(S \oplus T) = xT = xS \oplus xT$. Similarly for $T = O$. Here we have used Definition REAL.19(A)(1).

(Case 1: x is a rational number.) This is Theorem QX.11(B).

(Case 2: x is irrational, $S > O$ and $T > O$.) Applying, in order, Definition REAL.19(A)(3), Theorem QX.11(A), Definition REAL.27(A), Theorem R.28(A), and Definition REAL.19(A)(3), we have

$$\begin{aligned}x(S \oplus T) &= \mathrm{lub}\{r(S \oplus T) \mid r < x\}\\ &= \mathrm{lub}\{rS \oplus rT \mid r < x\}\\ &= \mathrm{lub}(\{rS \mid r < x\} \oplus \{rT) \mid r < x\})\\ &= \mathrm{lub}\{rS \mid r < x\} \oplus \mathrm{lub}\{rT) \mid r < x\}\\ &= xS \oplus xT.\end{aligned}$$

(Case 3: x is irrational, $S < O$ and $T < O$.) Applying, in order, Theorem REAL.21(C), Theorem OF.10(F), Case 2 above, Theorem REAL.21(C), and Theorem OF.10(F), we have

$$\begin{aligned}{}^{\ominus}(x(S \oplus T)) &= x({}^{\ominus}(S \oplus T))\\ &= x(({}^{\ominus}S) \oplus ({}^{\ominus}T))\\ &= x({}^{\ominus}S) \oplus x({}^{\ominus}T))\end{aligned}$$

$$= {}^{\ominus}(xS) \oplus {}^{\ominus}(xT)$$
$$= {}^{\ominus}(xS \oplus xT).$$

By Theorem OF.10(A), $x(S \oplus T) = xS \oplus xT$.

(Case 4: x is irrational, one of S or T is a positive member of $\mathbb{L}$, and the other is negative.) Without loss of generality, we assume that $S > O$ and $T < O$.

(Subcase A: $S \oplus T > O$ and $T < O$.) Then ${}^{\ominus}T > O$. Applying Case 2 and Theorem REAL.21(C), we have

$$xS = x(S \oplus T \oplus ({}^{\ominus}T)) = x(S \oplus T) \oplus x({}^{\ominus}T) = x(S \oplus T) \oplus {}^{\ominus}(xT).$$

Adding xT to both sides, $xS \oplus xT = x(S \oplus T)$.

(Subcase B: $S \oplus T < O$ and $T > O$.) Then ${}^{\ominus}S < O$. Applying Case 3 and Theorem REAL.21(C), we have

$$xT = x({}^{\ominus}S \oplus S \oplus T) = x({}^{\ominus}S) \oplus x(S \oplus T) = {}^{\ominus}(xS) \oplus x(S \oplus T).$$

Adding xS to both sides, $xS \oplus xT = x(S \oplus T)$. □

Exercise RR.1* Complete the computations necessary to prove Remark RR.2(A) from Theorem ISM.8(A), that is, show that a Euclidean/LUB plane $\mathcal{P}$ is an abelian group under the operation $+$.

Exercise RR.1 Proof. Let A, B, and C be any points of $\mathcal{P}$, and let τ_A, τ_B, and τ_C be the translations in $\mathbb{T}$ such that $\tau_A(O) = A$, $\tau_B(O) = B$, and $\tau_C(O) = C$. We will freely use, without reference, the fact that the set of all such translations forms an abelian group under composition, as shown in Theorem ISM.8(A).

$A + B = (\tau_A \circ \tau_B)(O) \in \mathcal{P}$ since $\tau_A \circ \tau_B$ is a mapping of $\mathcal{P}$ to $\mathcal{P}$, so that $\mathcal{P}$ is closed under addition.

$A + (B + C) = (\tau_A \circ (\tau_B \circ \tau_C))(O) = ((\tau_A \circ \tau_B) \circ \tau_C)(O) = (A + B) + C$ so that addition is associative.

$A+B = (\tau_A \circ \tau_B)(O) = (\tau_B \circ \tau_A)(O) = B+A$, so addition is commutative.

$A + O = (\tau_A \circ \tau_O)(O) = (\tau_A \circ \imath)(O) = \tau_A(O) = A$ so O is the additive identity.

For any translation τ_A which maps O to A, there exists an inverse translation ${\tau_A}^{-1}$. If we define $-A = {\tau_A}^{-1}(O)$, then $A + (-A) = (\tau_A({\tau_A}^{-1}(O)) = O$ so that $-A$ is the additive inverse of A. □

Exercise RR.2* Prove Theorem RR.4: (A) For every $A \in \mathcal{P} \setminus \{O\}$, $\overleftrightarrow{OA} = \{xA \in \mathcal{P} \mid x \in \mathbb{R}\}$. That is, every line through the origin is the set of all scalar multiples of any point in that line which is distinct from O.

Moreover, if A and B are any points in $\mathcal{P}$ and x and y are any real numbers, (B) $x(yA) = (xy)A$, (C) $x(A+B) = xA + xB$, (D) $(x+y)A = xA + yA$, (E) $1A = A$, (F) $xA = O$ iff $x = 0$ or $A = O$ (or both).

Exercise RR.2 Proof. (A) $\overleftrightarrow{OA}$ is a line through the origin O and therefore is a fixed line for the dilation δ_x, and hence if $x \neq 0$, $xA \in \overleftrightarrow{OA}$. By Theorem REAL.35(A), for every $A' \in \overleftrightarrow{OA} \setminus \{O\}$, there exists a real number $t \neq 0$ such that $tU_1 = A'$ and a real number $s \neq 0$ such that $sU_1 = A$, so that $A' = \frac{t}{s}A$.

(B) This is Theorem REAL.23.

(C) If A, B, and O are collinear points, then (C) is Theorem REAL.32. If they are non-collinear, let δ_x be the dilation with fixed point O such that for every $A \in \mathcal{P} \setminus \{O\}$, $\delta_x(A) = xA$. Then both $\delta_x(B) = xB$ and $\delta_x(A + B) = x(A + B)$ (*cf* Theorem REAL.42). By Remark RR.2 and Exercise ISM.2, $A + B$ is the fourth corner of the parallelogram whose other corners are A, O, and B, that is, $\square AOB(A + B)$ is a parallelogram. Since all the lines $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, and $\overleftrightarrow{O(A+B)}$ are fixed lines for δ_x, $\delta_x(\angle AOB) = \angle AOB$, $\delta_x(\angle AO(A+B)) = \angle AO(A+B)$, and $\delta_x(\angle(A+B)OB) = \angle(A+B)OB$.

By Theorem DLN.14,

$$\delta_x(\angle OA(A+B)) \cong \angle OA(A+B) \text{ and}$$
$$\delta_x(\angle O(A+B)A) \cong \angle O(A+B)A.$$

By Theorem SIM.18

$$\triangle OA(A+B) \sim \triangle O(\delta_x(A))(\delta_x(A+B))$$

and hence by Definition CAP.17,

$$\overleftrightarrow{(\delta_x(A))(\delta_x(A+B))} \parallel \overleftrightarrow{A(A+B)} \text{ and } \overleftrightarrow{x(A)x(A+B)} \parallel \overleftrightarrow{A(A+B)}.$$

By similar reasoning

$$\overleftrightarrow{(\delta_x(B))(\delta_x(A+B))} \parallel \overleftrightarrow{B(A+B)} \text{ and } \overleftrightarrow{x(B)x(A+B)} \parallel \overleftrightarrow{B(A+B)}.$$

Since $\square AOB(A+B)$ is a parallelogram, so is $\square(xA)O(xB)(x(A+B))$. Again by Remark RR.2, $xA+xB$ is the fourth corner of this parallelogram, that is, $xA + xB = x(A + B)$.

(D)
$$\begin{aligned}
(x+y)A &= (x+y)(U \cdot A) \\
&= ((x+y)U) \cdot A && \text{by Theorem REAL.25} \\
&= (xU + yU) \cdot A && \text{by Theorem REAL.31} \\
&= xU \cdot A + yU \cdot A && \text{by Theorem OF.6} \\
&= xA + yA && \text{by Theorem REAL.25.}
\end{aligned}$$

(E) $1A = A$ is immediate from Definition QX.1(C).

(F) If $x = 0$ or $A = O$ (or both), $xA = O$ by Definition REAL.19(A)(1). If $xA = O$, then by Theorem REAL.25 $xA = x(U \cdot A) = xU \cdot A = O$ and by Theorem OF.10(H), $xU = O$ or $A = O$. If $xU = O$ then $x = 0$ by Corollary REAL.34(B). $\square$

Chapter 19—has no Exercises (AA)

Chapter 20: Exercises and Answers for Ratios of Sensed Segments (RS)

Exercise RS.1* If a, b, and x are real numbers, and $a \neq b$, $\dfrac{x-a}{b-x} \neq -1$.

Exercise RS.1 Proof. If $\dfrac{x-a}{b-x} = -1$ then $x-a = x-b$ so that $a = b$. $\square$

Exercise RS.2* Let a, b, x, and y be real numbers, and let $a \neq b$. Then if $\dfrac{x-a}{b-x} = \dfrac{y-a}{b-y}$, $x = y$.

Exercise RS.2 Proof. If $\dfrac{x-a}{b-x} = \dfrac{y-a}{b-y}$ then $(x-a)(b-y) = (y-a)(b-x)$ and $bx - xy - ab + ay = by - xy - ab + ax$ or $bx + ay = by + ax$, so that $b(x-y) = bx - by = ax - ay = a(x-y)$ and $(b-a)(x-y) = 0$. Since $b - a \neq 0$, $x - y = 0$ and $x = y$. $\square$

Exercise RS.3 Let A, B, and X be points on a line $\mathbb{L}$ in the Euclidean/LUB plane $\mathcal{P}$, where $A \neq B$. Make a graph of the function $f(X) = \dfrac{[AX\rangle}{[XB\rangle}$.

Exercise RS.4* If statement (2) of Ceva's theorem is true, that is if $\dfrac{[AF\rangle}{[FB\rangle} \cdot \dfrac{[BD\rangle}{[DC\rangle} \cdot \dfrac{[CE\rangle}{[EA\rangle} = 1$, then the number of exterior Cevians is either zero or two, the other Cevians being interior.

Exercise RS.4 Proof. Let A and B be two corners of a triangle, and suppose that $\overleftrightarrow{CF}$ is the Cevian through C, the third corner, where $F \in \overleftrightarrow{AB}$. Then by Remark RS.7(B)(3) $\overleftrightarrow{CF}$ is exterior iff $F \notin \overline{AB}$ iff F–A–B or A–B–F iff $\dfrac{[AF\rangle}{[FB\rangle} < 1$, and $\overleftrightarrow{CF}$ is interior iff $F \in \overline{AB}$ iff A–F–B iff $\dfrac{[AF\rangle}{[FB\rangle} > 1$.

Assume now that $\dfrac{[AF\rangle}{[FB\rangle} \cdot \dfrac{[BD\rangle}{[DC\rangle} \cdot \dfrac{[CE\rangle}{[EA\rangle} = 1$. Then either

(i) all the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, or $\overleftrightarrow{CF}$ are interior Cevians (none are exterior), in which case the product is positive, or

(ii) two are exterior and one is interior, in which case the product is positive.

If two of the Cevians are interior and one is exterior, then the product is negative which contradicts our assumption. $\square$

Chapter 21: Exercises and Answers for Consistency and Independence of Axioms (LM)(MLT)

Exercise LM.1 Using Definition LA.1(2) prove that $\overleftrightarrow{ABC} = \overleftrightarrow{CAB} = \overleftrightarrow{BCA} = \overleftrightarrow{CBA}$.

Exercise LM.1 Proof. Prove that $\overleftrightarrow{ABC} = \overleftrightarrow{CAB} = \overleftrightarrow{BCA} = \overleftrightarrow{CBA}$.

$$\overleftrightarrow{ABC} = \overleftrightarrow{CAB} = \overleftrightarrow{BCA} = \overleftrightarrow{CBA}$$

$$\begin{aligned}
\text{(A) } \overleftrightarrow{ABC} &= \{A + s(B-A) + t(C-A) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{C + (1-s-t)(A-C) + s(B-C) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{C + u(A-B) + v(C-B) \mid (u,v) \in \mathbb{F}^2\} = \overleftrightarrow{CAB}; \\
\text{(B) } \overleftrightarrow{ABC} &= \{A + s(B-A) + t(C-A) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{B + (1-s-t)(A-B) + t(C-B) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{B + v(C-B) + u(A-B) \mid (v,u) \in \mathbb{F}^2\} = \overleftrightarrow{BCA}; \\
\text{(C) } \overleftrightarrow{ABC} &= \{A + s(B-A) + t(C-A) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{C + s(B-C) + (1-s-t)(A-C) \mid (s,t) \in \mathbb{F}^2\} \\
&= \{C + u(B-C) + v(A-C) \mid (u,v) \in \mathbb{F}^2\} = \overleftrightarrow{CBA}. \quad \square
\end{aligned}$$

Exercise LM.2* Prove Theorem LA.3: distinct points A, B, and C are collinear iff $B - A$ and $C - A$ are linearly dependent.

Exercise LM.2 Proof. (I) If A, B, and C are collinear, then by Definition LA.1(1) there exists a number u such that $C = A + u(B - A)$, that is, $u(B - A) - (C - A) = O$. This means that $B - A$ and $C - A$ are linearly dependent.

(II) If $B - A$, and $C - A$ are linearly dependent, then there exists $(r, s) \in (\mathbb{F}^2 \setminus \{(0,0)\})$ such that $r(B - A) + s(C - A) = O$. If $r \neq 0$, then $B = A - \frac{s}{r}(C - A)$, *i.e.* $B \in \overleftrightarrow{AC}$. If $s \neq 0$, then $C = A - \frac{r}{s}(B - A)$, that is, $C \in \overleftrightarrow{AB}$. $\square$

Exercise LM.3* Prove Theorem LA.4: distinct points A, B, C, and D in $\mathbb{F}^3$ are coplanar iff $B - A$, $C - A$, and $D - A$ are linearly dependent.

Exercise LM.3 Proof. (I) If A, B, C, and D are coplanar, then by Definition LA.1(2) there exist numbers r and s such that $D = A + r(B - A) + s(C - A)$, *i.e.*, $r(B - A) + s(C - A) - (D - A) = O$. Since $1 \neq 0$, not all of 1, r, and s are 0, so that $B - A$, $C - A$, and $D - A$ are linearly dependent.

(II) If $B - A$, $C - A$, and $D - A$ are linearly dependent, then there exists $(u, v, w) \in (\mathbb{F}^3 \setminus \{(0,0,0)\}$ such that $u(B - A) + v(C - A) + w(D - A) = \bar{0}$.

If $u \neq 0$, then $B = A - \frac{v}{u}(C-A) - \frac{w}{u}(D-A)$, *i.e.*, $B \in \overleftrightarrow{ACD}$. If $v \neq 0$, then $C = A - \frac{u}{v}(B-A) - \frac{w}{v}(D-A)$, *i.e.*, $C \in \overleftrightarrow{ABD}$. If $w \neq 0$, then $D = A - \frac{u}{w}(B-A) - \frac{v}{w}(C-A)$, *i.e.*, $D \in \overleftrightarrow{ABC}$. In each case, all of A, B, C, and D are members of a single plane. $\square$

Exercise LM.4* Prove Theorem LA.5: if A and B are distinct points in $\mathbb{F}^3$, define, for each real number t, $\varphi(t) = A + t(B-A)$. Then φ is a one-to-one mapping of $\mathbb{F}$ onto $\overleftrightarrow{AB}$.

Exercise LM.4 Proof. (A) If $t = 0$, then $\varphi(0) = A$. If $t \neq 0$, then $\varphi(t) - A - t(B-A) = O$, so $\varphi(t) - A$ and $(B-A)$ are linearly dependent. By Theorem LA.3, $\varphi(t)$, A, and B are collinear, so $\varphi(t) \in \overleftrightarrow{AB}$.

(B) Note that

$$\varphi(t) - \varphi(s) = A + t(B-A) - (A + s(B-A)) = (t-s)(B-A).$$

Since A and B are distinct, $B - A \neq O$. Thus, if $t - s \neq 0$, $\varphi(t) - \varphi(s) \neq O$, so $\varphi(t) \neq \varphi(s)$ and φ is one-to-one.

(C) To show that φ is onto, let X be any member of $\overleftrightarrow{AB}$. If $X = A$, let $t = 0$. Then $\varphi(0) = A + 0(B-A) = A$.

If $X \neq A$, then $X - A \neq O$ and so by Theorem LA.3, $X - A$ and $B - A$ are linearly dependent. Hence there exists a member (u, v) of $\mathbb{F}^2 \setminus \{(0,0)\}$ such that $u(X-A) + v(B-A) = O$, or $v(B-A) = -u(X-A)$.

If u were equal to 0, then $v(B-A) = O$. Since $B - A \neq O$, v would equal 0. This would contradict the fact that at least one of the members u or v of the field $\mathbb{F}$ is different from 0. Hence $u \neq 0$, and

$$\varphi(\tfrac{-v}{u}) = A - \tfrac{v}{u}(B-A) = A - \tfrac{-u}{-u}(X-A) = X. \quad \square$$

Exercise LM.5 (A) Prove Theorem LA.15: (A) Two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ of $\mathbb{F}^2$ are linearly dependent iff $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. A solution is provided for this part.

(B) Three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ of $\mathbb{F}^3$ are linearly dependent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Exercise LM.5 Proof. (A) Suppose $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are linearly dependent; if $aA + bB = 0$ where not both $a = 0$ and $b = 0$, and $a = 0$, then $bB = O$ and $B = (0,0)$ so the determinant is zero; likewise if $b = 0$.

Suppose both a and b are non-zero; then let $c = \frac{a}{b} \neq 0$, so that $B = cA$, $b_1 = ca_1$ and $b_2 = ca_2$; then the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = a_1ca_2 - a_2ca_1 = 0.$$

On the other hand, if $0 = a_1b_2 - a_2b_1$, then $a_1b_2 = a_2b_1$; if $a_1 \neq 0$, $b_2 = \frac{b_1}{a_1}a_2$. Since $b_1 = \frac{b_1}{a_1}a_1$, A and B are linearly dependent. Similar proofs hold if $a_2 \neq 0$, $b_1 \neq 0$, or $b_2 \neq 0$.

(B) The proof is left to the reader. $\square$

Exercise LM.6* Prove Theorem LA.17: let a, b, c and d be members of $\mathbb{F}$, where at least one of a, b, c is non-zero; let $\mathcal{E}$ be the set of all points $(x_1, x_2, x_3) \in \mathbb{F}^3$ such that $ax_1 + bx_2 + cx_3 + d = 0$, as defined in Remark LA.16.

(A) $\mathcal{E}$ is a proper subset of $\mathbb{F}^3$.

(B) If $X = (x_1, x_2, x_3) \in \mathcal{E}$, there exist two other points $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$ in $\mathcal{E}$ such that X, Y, and Z are noncollinear, which is to say (by Theorem LA.4) that the vectors $Y - X$ and $Z - X$ are linearly independent.

Exercise LM.6 Proof. (A) If $d \neq 0$, $(0,0,0) \notin \mathcal{E}$; if $d = 0$, and (1) $a \neq 0$, then $(1,0,0) \notin \mathcal{E}$; if (2) $b \neq 0$, $(0,1,0) \notin \mathcal{E}$, and if (3) $c \neq 0$, then $(0,0,1) \notin \mathcal{E}$. Thus $\mathcal{E}$ is a proper subset of $\mathbb{F}^3$.

(B) If $a \neq 0$ and $b = c = 0$, let $Y = (y_1, y_2, y_3) = X + (0,1,0)$ and $Z = (z_1, z_2, z_3) = A + (0,0,1)$; then $a \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$ and $a \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0$ so that $ay_1 + by_2 + cy_3 + d = 0$ and $az_1 + bz_2 + cz_3 + d = 0$. Thus both Y and Z belong to $\mathcal{E}$, and $Y - X = (0,1,0)$ and $Z - X = (0,0,1)$, which are linearly independent, and X, Y, and Z are noncollinear. Similar arguments will show the result in case $b \neq 0$ and $a = c = 0$, and $c \neq 0$ and $a = b = 0$.

If $a \neq 0$ and $b \neq 0$ and $c = 0$, let $Y = (y_1, y_2, y_3) = X + (1, -\frac{a}{b}, 0)$ and $Z = (z_1, z_2, z_3) = X + (0,0,1)$; then $a \cdot 1 + b(-\frac{a}{b}) + 0 \cdot 0 = 0$ and $a \cdot 0 + b \cdot 0 + 0 \cdot 1 = 0$, so that $ay_1 + by_2 + cy_3 + d = 0$ and $az_1 + bz_2 + cz_3 + d = 0$, and both Y and Z belong to $\mathcal{E}$. Then $Y - X = (1, -\frac{a}{b}, 0)$ and $Z - X = (0,0,1)$, which are linearly independent, so that again by Theorem LA.3 (Exercise LM.2), X, Y, and Z are noncollinear. Similar arguments will show the result in case $a \neq 0$ and $c \neq 0$ and $b = 0$, and $b \neq 0$ and $c \neq 0$ and $a = 0$.

If a, b, and c are all non-zero, again let $X = (x_1, x_2, x_3) \in \mathcal{E}$ so that $ax_1 + bx_2 + cx_3 + d = 0$; let $Y = (y_1, y_2, y_3) = X + (1, -\frac{a}{b}, 0)$ and $Z = (z_1, z_2, z_3) = X + (1, 0, -\frac{a}{c})$; then $a \cdot 1 + b(-\frac{a}{b}) + 0 \cdot 0 = 0$ and $a \cdot 1 + b \cdot 0 + c(-\frac{a}{c}) = 0$, so that $ay_1 + by_2 + cy_3 + d = 0$ and

$az_1+bz_2+cz_3+d=0$ and both Y and Z belong to $\mathcal{E}$. Then $Y-X=(1,-\frac{a}{b},0)$ and $Z-X=(1,0,-\frac{a}{c})$, which are linearly independent, and X, Y, and Z are noncollinear. $\square$

Exercise LM.7* Prove Theorem LA.18: let $X=(x_1,x_2,x_3)$, $Y=(y_1,y_2,y_3)$, and $Z=(z_1,z_2,z_3)$ be noncollinear points in $\mathbb{F}^3$, so that $\overleftrightarrow{XYZ}$ is a plane as in Definition LA.1(2). Then there exist numbers a, b, c and d in $\mathbb{F}$, where not all of a, b, or c are zero, such that

$$\overleftrightarrow{XYZ}=\{(w_1,w_2,w_3)\mid aw_1+bw_2+cw_3+d=0\}.$$

Exercise LM.7 Proof. By Definition LA.1(2) $W=(w_1,w_2,w_3)$ is a point on $\overleftrightarrow{XYZ}$ iff there exist numbers s and t such that $W=X+s(Y-X)+t(Z-X)$ *i.e.* $(W-X)-s(Y-X)-t(Z-X)=O$. By Theorem LA.4 and Theorem LA.15 (Exercises LM.3 and LM.5) this equality holds iff

$$\begin{vmatrix} w_1-x_1 & y_1-x_1 & z_1-x_1 \\ w_2-x_2 & y_2-x_2 & z_2-x_2 \\ w_3-x_3 & y_3-x_3 & z_3-x_3 \end{vmatrix}=0.$$

Define $$det_1=\begin{vmatrix} y_2-x_2 & z_2-x_2 \\ y_3-x_3 & z_3-x_3 \end{vmatrix},\ det_2=\begin{vmatrix} y_1-x_1 & z_1-x_1 \\ y_3-x_3 & z_3-x_3 \end{vmatrix},\text{ and}$$

$$det_3=\begin{vmatrix} y_1-x_1 & z_1-x_1 \\ y_2-x_2 & z_2-x_2 \end{vmatrix}.$$

Expanding the first determinant by its first column, we have

$$\begin{aligned}&(w_1-x_1)det_1-(w_2-x_2)det_2+(w_3-x_3)det_3\\ &\quad=w_1det_1-w_2det_2+w_3det_3-x_1det_1+x_2det_2-x_3det_3\\ &\quad=aw_1+bw_2+cw_3+d=0,\end{aligned}$$

where $a=det_1$, $b=-det_2$, $c=det_3$, and $d=-x_1det_1+x_2det_2-x_3det_3$. Since these implications are all reversible, they show that $(w_1,w_2,w_3)\in\overleftrightarrow{XYZ}$ iff $aw_1+bw_2+cw_3+d=0$, that is to say

$$\overleftrightarrow{XYZ}=\{(w_1,w_2,w_3)\mid aw_1+bw_2+cw_3+d=0\}.\quad\square$$

Exercise LM.8* Prove Theorem LA.19: let a, b, c, and d be numbers in $\mathbb{F}$, where not all of a, b, or c are zero. Then the set

$$\mathcal{E}=\{(x_1,x_2,x_3)\mid ax_1+bx_2+cx_3+d=0\}$$

is a plane in $\mathbb{F}^3$ as defined by Definition LA.1(2).

Exercise LM.8 Proof. Let $W=(w_1,w_2,w_3)$ be any member of $\mathcal{E}$ so that $aw_1+bw_2+cw_3+d=0$. For any $X=(x_1,x_2,x_3)\in\mathcal{E}$, $X-W=(x_1-w_1,x_2-w_2,x_3-w_3)$ satisfies $a(x_1-w_1)+b(x_2-w_2)+c(x_3-w_3)+d-d=0$;

and if this is satisfied, then $X = X - W + W \in \mathcal{E}$. Thus, $\mathcal{E} - W = \{(x_1, x_2, x_3) \mid ax_1 + bx_2 + cx_3 = 0$.

We show that $\mathcal{E} - W$ is a vector space. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are members of this set, then $ax_1 + bx_2 + cx_3 = 0$ and $ay_1 + by_2 + cy_3 = 0$, so that $a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3) = 0$, so is a member of $\mathcal{E} - W$. Likewise, if z is any number, $zax_1 + zbx_2 + zcx_3 = 0$ so that $z(x_1, x_2, x_3) \in \mathcal{E} - W$. Therefore $\mathcal{E} - W$ is a vector space.

By Theorem LA.17(B) (Exercise LM.6) there exist vectors D and E in this space which are linearly independent, so that its dimension is ≥ 2. By Theorem LA.17(A), $\mathcal{E} - W$ is a proper subset of $\mathbb{F}^3$; by the Dimension Criterion of Chapter 1 Section 1.5, the dimension is 2; by Remark LA.9(C) it is a plane, and by part (D) of the same remark $\mathcal{E} = \mathcal{E} - W + W$ is a plane. $\square$

Exercise LM.9* Prove Theorem LB.4: for any numbers a, b, c, a', b', and c in $\mathbb{F}$, where at least one of a or b, and at least one of a' or b' is non-zero, then

(A) $\mathcal{L} = \{(x_1, x_2) \mid ax_1 + bx_2 + c = 0\} \neq \mathbb{F}^2$;

(B) there exist at least two distinct points in $\mathcal{L}$; and

(C) both $ax_1 + bx_2 + c = 0$ and $a'x_1 + b'x_2 + c' = 0$ are equations for $\mathcal{L}$ iff there exists a number $k \neq 0$ such that $a' = ka$, $b' = kb$, and $c' = kc$.

Exercise LM.9 Proof. (A) If $c \neq 0$, $(0, 0) \notin \mathcal{E}$; if $c = 0$ and $a \neq 0$, then $(1, 0) \notin \mathcal{E}$; if $c = 0$, and $b \neq 0$, then $(0, 1) \notin \mathcal{E}$. Thus $\mathcal{L}$ is a proper subset of $\mathcal{F}^2$.

(B) Suppose $X = (x_1, x_2) \in \mathcal{E}$; we show that there exists another point $Y = (y_1, y_2) \in \mathcal{L}$.

(1) If $a \neq 0$ and $b \neq 0$, let

$$Y = (y_1, y_2) = X + (-\tfrac{b}{a}, 1) = (x_1 - \tfrac{b}{a}, x_2 + 1);$$

then

$$\begin{aligned} ay_1 + by_2 + c &= a(x_1 - \tfrac{b}{a}) + b(x_2 + 1) + c \\ &= ax_1 + bx_2 + c + a(-\tfrac{b}{a}) + b \cdot 1 \\ &= ax_1 + bx_2 + c + (-b + b) = 0 + 0 = 0. \end{aligned}$$

Thus $Y \in \mathcal{L}$, and $Y \neq X$.

(2) If $a \neq 0$ and $b = 0$, let $Y = (y_1, y_2) = X + (0, 1)$; then $a \cdot 0 + 0 \cdot 1 = 0$ so that $ay_1 + by_2 + c = 0$. Thus $Y \in \mathcal{E}$, and $Y - X = (0, 1)$ so these are distinct points.

(3) If $a = 0$ and $b \neq 0$, let $Y = (y_1, y_2) = X + (1, 0)$; then $0 \cdot 1 + b \cdot 0 = 0$ so that $ay_1 + by_2 + c = 0$. Thus $Y \in \mathcal{E}$, and $Y - X = (1, 0)$ so these are distinct points.

(C) If for some $k \neq 0$ such that $a' = ka$, $b' = kb$, and $c' = kc$, and $ax_1 + bx_2 + c = 0$, then clearly $a'x_1 + b'x_2 + c' = 0$.

Conversely, suppose both $ax_1 + bx_2 + c = 0$ and $a'x_1 + b'x_2 + c' = 0$.

(1) If $c = 0$ then $(0, 0) \in \mathcal{L}$ and hence $c' = 0$; also not both a and b can be zero. If $a \neq 0$ and $b = 0$, $\mathcal{L} = \{(0, x_2) \mid x_2 \in \mathbb{F}\}$. Choose $x_2 = 1$; then $a' \cdot 0 + b' = 0$, so $b' = 0$ and a' can be any non-zero number.

Similarly, if $a = 0$ and $b \neq 0$ then $a' = 0$ and b' can be any non-zero number. If $a \neq 0$ and $b \neq 0$, the point $(-\frac{b}{a}, 1) \in \mathcal{L}$ so that $a'(-\frac{b}{a}) + b' = 0$, and $\frac{a'}{a} = \frac{b'}{b}$; we may let $k = \frac{a'}{a}$.

(2) If $c \neq 0$ then $c' \neq 0$. If $a \neq 0$ and $b = 0$, $(-\frac{c}{a}, 0) \in \mathcal{L}$ so that Then $a'(-\frac{c}{a}) + b' \cdot 0 + c' = 0$, and $\frac{a'}{a}(-c) + c' = 0$, or $\frac{a'}{a} = \frac{c'}{c}$; also, $(-\frac{c}{a}, 1) \in \mathcal{L}$ so that $a'(-\frac{c}{a}) + b' + c' = 0$ and hence $\frac{a'}{a} + (\frac{b'}{-c}) - \frac{c'}{c} = 0$, and $b' = 0$. We may let $k = \frac{a'}{a}$.

Similarly, if $c \neq 0$ and $a = 0$ and $b \neq 0$, then $a' = 0$ and $\frac{b'}{b} = \frac{c'}{c}$, and we may let $k = \frac{b'}{b}$.

Finally, if $c \neq 0$, $a \neq 0$, and $b \neq 0$, $(0, -\frac{c}{b}) \in \mathcal{L}$; then $a' \cdot 0 + b'(-\frac{c}{b}) + c' = 0$, and $\frac{b'}{b} = \frac{c'}{c}$. Likewise, $(-\frac{c}{a}, 0) \in \mathcal{L}$ and $a'(-\frac{c}{a}) + b' \cdot 0 + c' = 0$, so that $\frac{a'}{a} = \frac{c'}{c}$. In this case we can let $k = \frac{c'}{c}$. □

Exercise LM.10* Prove Theorem LB.5: let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be distinct points in $\mathbb{F}^2$, and let $\overleftrightarrow{XY}$ be the line containing both X and Y according to Definition LA.1(1). Then $\overleftrightarrow{XY} = \{(w_1, w_2) \mid aw_1 + bw_2 + c = 0\}$, where $a = y_2 - x_2$, $b = x_1 - y_1$, and $c = x_2(y_1 - x_1) - x_1(y_2 - x_2)$.

Exercise LM.10 Proof. $W = (w_1, w_2)$ is a point on $\overleftrightarrow{XY}$ iff there exists a number s such that $W = X + s(Y - X)$ *i.e.* $(W - X) - s(Y - X) = O$. By Theorem LA.3 and Theorem LA.15 (Exercise LM.2 and Exercise LM.5) this equality holds iff

$$\begin{aligned}\begin{vmatrix} w_1 - x_1 & y_1 - x_1 \\ w_2 - x_2 & y_2 - x_2 \end{vmatrix} &= (w_1 - x_1)(y_2 - x_2) - (w_2 - x_2)(y_1 - x_1) \\ &= (y_2 - x_2)w_1 - x_1(y_2 - x_2) - (y_1 - x_1)w_2 + x_2(y_1 - x_1) \\ &= (y_2 - x_2)w_1 + (x_1 - y_1)w_2 + (x_2(y_1 - x_1) - x_1(y_2 - x_2)) = 0.\end{aligned}$$

This is true iff $aw_1 + bw_2 + c = 0$, where $a = y_2 - x_2$, $b = x_1 - y_1$, and $c = x_2(y_1 - x_1) - x_1(y_2 - x_2)$. Since the implications are all reversible, they show that $(w_1, w_2) \in \overleftrightarrow{XY}$ iff $aw_1 + bw_2 + c = 0$. Therefore

$$\overleftrightarrow{XY} = \{(w_1, w_2) \mid aw_1 + bw_2 + c = 0\}. \quad \square$$

Exercise LM.11 Prove Theorem LB.6: let a, b, and c be numbers in $\mathbb{F}$, where at least one of a or b is non-zero. Then the set

$$\mathcal{E} = \{(x_1, x_2) \mid ax_1 + bx_2 + c = 0\}$$

is a line in $\mathbb{F}^2$ as defined by Definition LA.1(1).

Exercise LM.11 Proof. Let $W = (w_1, w_2)$ be any member of $\mathcal{E}$ so that $aw_1 + bw_2 + c = 0$. For any $X = (x_1, x_2) \in \mathcal{E}$, $X - W = (x_1 - w_1, x_2 - w_2)$ satisfies $a(x_1 - w_1) + b(x_2 - w_2) + c - c = 0$ so that

$$\mathcal{E} - W \subseteq \{(x_1, x_2) \mid ax_1 + bx_2 = 0\}.$$

Conversely, if $Z = (z_1, z_2) \in \{(z_1, z_2) \mid az_1 + bz_2 = 0\}$, $Z = Z + W - W = (z_1 + w_1 - w_1, z_2 + w_2 - w_2)$ satisfies

$$\begin{aligned}
0 &= a(z_1 + w_1 - w_1) + b(z_2 + w_2 - w_2) \\
&= a(z_1 + w_1) - aw_1 + b(z_2 + w_2) - bw_2 \\
&= a(z_1 + w_1) + b(z_2 + w_2) - aw_1 - bw_2 \\
&= a(z_1 + w_1) + b(z_2 + w_2) - (aw_1 + bw_2) \\
&= a(z_1 + w_1) + b(z_2 + w_2) - (-c)
\end{aligned}$$

so that $a(z_1+w_1)+b(z_2+w_2)+c = 0$ and $Z+W \in \mathcal{E}$. Thus $Z = Z+W-W \in \mathcal{E} - W$, and $\mathcal{E} - W = \{(x_1, x_2) \mid ax_1 + bx_2 = 0\}$.

If $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are members of $\mathcal{E} - W$, then $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = 0$, so that $a(x_1 + y_1) + b(x_2 + y_2) = 0$ and hence $X + Y$ is a member of $\mathcal{E} - W$. Likewise, if $X \in \mathcal{E} - W$, and z is any number, $zax_1 + zbx_2 = 0$ so that $z(x_1, x_2) \in \mathcal{E} - W$.

According to the criterion in Chapter 1 Section 1.5 under the heading *Vector spaces*, $\mathcal{E} - W$ is a vector space. By Theorem LB.4(A), $\mathcal{E} - W$ is a proper subset of $\mathbb{F}^2$; by the *Dimension Criterion* of Chapter 1 Section 1.5, its dimension is 1; by Remark LA.8(C) it is a line according to Definition LA.1(1), and by part (D) of the same remark $\mathcal{E} = \mathcal{E} - W + W$ is a line. $\square$

Exercise LM.12 Prove Theorem LB.10: let

$$\mathcal{L} = \{(x_1, x_2) \mid a_1x_1 + b_1x_2 + c_1 = 0\} \text{ and}$$
$$\mathcal{M} = \{(x_1, x_2) \mid a_2x_1 + b_2x_2 + c_2 = 0\}$$

be two lines in $\mathbb{F}^2$. Then if they are c-perpendicular, they must intersect.

Exercise LM.12 Proof. By Theorem LB.8, $\mathcal{L} \perp \mathcal{M}$ iff $a_1a_2 + b_1b_2 = 0$. The two lines intersect iff there exists a point (x_1, x_2) such that both $a_1x_1 + b_1x_2 + c_1 = 0$ and $a_2x_1 + b_2x_2 + c_2 = 0$; by Cramer's Rule such a solution exists iff the determinant $a_1b_2 - a_2b_1 \neq 0$.

Therefore, we need to show that if $a_1a_2 + b_1b_2 = 0$, then $a_1b_2 - a_2b_1 \neq 0$. There are three cases:

Case 1: $a_1 = 0$; then $b_1 \neq 0$; since $a_1a_2 = -b_1b_2 = 0$, $b_2 = 0$ so that $a_2 \neq 0$; in this case $a_1b_2 - a_2b_1 = 0 + a_2b_1 \neq 0$. Interchanging a_1 with a_2 and b_2 with b_1, if $a_2 = 0$, $a_1b_2 - a_2b_1 = a_1b_2 + 0 \neq 0$.

Case 2: $b_1 = 0$; then $a_1 \neq 0$; since $a_1a_2 = -b_1b_2 = 0$, $a_2 = 0$ so that $b_2 \neq 0$; in this case $a_1b_2 - a_2b_1 = a_1b_2 + 0 \neq 0$. Again, interchanging a_1 with a_2 and b_2 with b_1, if $b_2 = 0$, $a_1b_2 - a_2b_1 = 0 - a_2b_1 \neq 0$.

Case 3: none of the coefficients a_1, a_2, b_1 and b_2 is zero. From $a_1a_2 + b_1b_2 = 0$ we get $a_1a_2 = -b_1b_2$, or $\frac{a_1}{b_1} = -\frac{b_2}{a_2}$. If $a_1b_2 - a_2b_1 = 0$, $a_1b_2 = a_2b_1$, and $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, so that $\frac{a_2}{b_2} = \frac{a_1}{b_1} = -\frac{b_2}{a_2}$.

Therefore $-{b_2}^2 = {a_2}^2$; since ${b_2}^2 \geq 0$ and ${a_2}^2 \geq 0$ this is true iff $b_2 = a_2 = 0$ which is a contradiction to our original assumption that none of the coefficients is zero. Therefore $a_1b_2 - a_2b_1 \neq 0$. □

Exercise LM.13* Show that the line $\mathcal{L}$ on $\mathbb{R}^2$ through the distinct points (u_1, u_2) and (v_1, v_2) is

$$\{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } (v_2 - u_2)(x_1 - u_1) - (v_1 - u_1)(x_2 - u_2) = 0\}.$$

Exercise LM.13 Proof. (I) By Remark LA.1(1), (x_1, x_2) belongs to the line in Model LM2 containing both (u_1, u_2) and (v_1, v_2) iff for some t,

$$(x_1, x_2) = (u_1, u_2) + t((v_1, v_2) - (u_1, u_2)).$$

Suppose that this holds, and $v_1 \neq u_1$; then $t = \frac{x_1 - u_1}{v_1 - u_1}$ and $x_2 = u_2 + \frac{x_1 - u_1}{v_1 - u_1}(v_2 - u_2)$ so that $(v_1 - u_1)(x_2 - u_2) - (v_2 - u_2)(x_1 - u_1) = 0$, that is,

$$(v_2 - u_2)(x_1 - u_1) - (v_1 - u_1)(x_2 - u_2) = 0.$$

A similar argument holds if $v_2 \neq u_2$. Then $ax_1 + bx_2 + c = 0$, where $a = v_2 - u_2$, $b = -(v_1 - u_1)$ and $c = (-u_1)(v_2 - u_2) + u_2(v_1 - u_1)$.

Now suppose that $ax_1 + bx_2 + c = 0$, where a, b, and c are defined as just above. By Theorem LB.6, we know that this is the formula of a line. We verify that it contains the points (u_1, u_2) and (v_1, v_2) by the following calculations:

$$(u_1, u_2) \in \mathcal{L}, \text{ since } (v_2 - u_2)(u_1 - u_1) - (v_1 - u_1)(u_2 - u_2) = 0, \text{ and}$$
$$(v_1, v_2) \in \mathcal{L}, \text{ since } (v_2 - u_2)(v_1 - u_1) - (v_1 - u_1)(v_2 - u_2) = 0.$$

Since by Axiom I.1 there is only one line containing both these points, this line must be the one with formula $ax_1 + bx_2 + c = 0$. □

Exercise LM.14* Show that for every member (x_1, x_2) on the line

$$\mathcal{L} = (x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } ax_1 + bx_2 + c = 0,$$

the formula for $\Phi(x_1, x_2)$ given in Definition LB.16 yields $\Phi(x_1, x_2) = (x_1, x_2)$. For a coordinate-free proof, see Theorem LC.23(A).

Exercise LM.14 Proof. If $ax_1 + bx_2 + c = 0$, then $\frac{(b^2-a^2)x_1 - 2abx_2 - 2ac}{a^2+b^2} = \frac{(b^2-a^2)x_1 - 2a(-ax_1-c) - 2ac}{a^2+b^2} = \frac{b^2x^2 - a^2x_1 + 2a^2x_1 + 2ac - 2ac}{a^2+b^2} = \frac{(a^2+b^2)x_1}{a^2+b^2} = x_1$ and $\frac{-2abx_1 + (a^2-b^2)x_2 - 2bc}{a^2+b^2} = \frac{-2b(-bx_2-c) + (a^2-b^2)x_2 - 2bc}{a^2+b^2} = \frac{(a^2+b^2)x_2}{a^2+b^2} = x_2$. □

Exercise LM.15* In the plane $\mathbb{F}$, if a line $\mathcal{L}$ is c-perpendicular to a line $\mathcal{M}$ and if line $\mathcal{M}$ and line $\mathcal{N}$ are parallel, then $\mathcal{L}$ is c-perpendicular to line $\mathcal{N}$.

Exercise LM.15 Proof.

Let $\mathcal{L} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } a_1x_1 + b_1x_2 + c_1 = 0\}$; since $\mathcal{M} \parallel \mathcal{N}$, there exist numbers a_2, b_2, c_2 and c_3 such that

$$\mathcal{M} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } a_2x_1 + b_2x_2 + c_2 = 0\}, \text{ and}$$
$$\mathcal{N} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } a_2x_1 + b_2x_2 + c_3 = 0\},$$

where $(a_1, b_1) \neq (0, 0)$ and $(a_2, b_2) \neq (0, 0)$.

By Theorem LB.8, $\mathcal{L}$ is c-perpendicular to $\mathcal{M}$ iff $a_1a_2 + b_1b_2 = 0$, which is true iff $\mathcal{L}$ is c-perpendicular to $\mathcal{N}$. □

Exercise LM.16* Let $\mathbb{F}$ be an ordered field, and let $\mathcal{R}_{\mathcal{L}} = \Phi$ be the mapping defined by Definition LB.16 and Definition LC.25 over the line

$$\mathcal{L} = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{F}^2 \text{ and } ax_1 + bx_2 + c = 0\}.$$

where $(a, b) \neq (0, 0)$. Define Γ_1 and Γ_2 to be the mappings such that $\mathcal{R}_{\mathcal{L}}(x_1, x_2) = (\Gamma_1(x_1, x_2), \Gamma_2(x_1, x_2))$. Then if $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are any points of $\mathbb{F}^2$,

$$\begin{aligned}&\big(\Gamma_1(x_1, x_2) - \Gamma_1(y_1, y_2)\big)^2 + \big(\Gamma_2(x_1, x_2) - \Gamma_2(y_1, y_2)\big)^2 \\ &= (x_1 - y_1)^2 + (x_2 - y_2)^2.\end{aligned}$$

In case $\mathbb{F}$ is algebraically closed, so that distance is defined, this says that $\mathrm{dis}^2(\mathcal{R}_{\mathcal{L}}(X), \mathcal{R}_{\mathcal{L}}(Y)) = \mathrm{dis}^2(X, Y)$.

Exercise LM.16 Proof. For any (x_1, x_2) and (y_1, y_2) in $\mathbb{F}^2$,

$$\begin{aligned}&\big(\Gamma_1(x_1, x_2) - \Gamma_1(y_1, y_2)\big)^2 \\ &= \Big(\tfrac{b^2-a^2}{a^2+b^2}x_1 - \tfrac{2ab}{a^2+b^2}x_2 - \tfrac{2ac}{a^2+b^2} - \tfrac{b^2-a^2}{a^2+b^2}y_1 + \tfrac{2ab}{a^2+b^2}y_2 + \tfrac{2ac}{a^2+b^2}\Big)^2 \\ &= \Big(\tfrac{b^2-a^2}{a^2+b^2}(x_1 - y_1) - \tfrac{2ab}{a^2+b^2}(x_2 - y_2)\Big)^2 \\ &= \Big(\tfrac{b^2-a^2}{a^2+b^2}\Big)^2(x_1 - y_1)^2 + \Big(\tfrac{2ab}{a^2+b^2}\Big)^2(x_2 - y_2)^2 \\ &\qquad + 2\Big(\tfrac{b^2-a^2}{a^2+b^2}\Big)\Big(\tfrac{-2ab}{a^2+b^2}\Big)(x_1 - y_1)(x_2 - y_2), \quad (*)\end{aligned}$$

and $\big(\Gamma_2(x_1,x_2)-\Gamma_2(y_1,y_2)\big)^2$

$$\begin{aligned}
&= \big(\tfrac{-2ab}{a^2+b^2}x_1+\tfrac{a^2-b^2}{a^2+b^2}x_2-\tfrac{2bc}{a^2+b^2}+\tfrac{2ab}{a^2+b^2}y_1-\tfrac{a^2-b^2}{a^2+b^2}y_2+\tfrac{2bc}{a^2+b^2}\big)^2\\
&= \left(\tfrac{-2ab}{a^2+b^2}(x_1-y_1)+\tfrac{a^2-b^2}{a^2+b^2}(x_2-y_2)\right)^2\\
&= \left(\tfrac{-2ab}{a^2+b^2}(x_1-y_1)\right)^2+\left(\left(\tfrac{a^2-b^2}{a^2+b^2}\right)(x_2-y_2)\right)^2\\
&\qquad +2\left(\tfrac{-2ab}{a^2+b^2}\right)(x_1-y_1)\left(\tfrac{a^2-b^2}{a^2+b^2}\right)(x_2-y_2). \quad (**)
\end{aligned}$$

Now add (*) and (**); their last terms are negatives of each other, so we have

$$\begin{aligned}
&\big(\Gamma_1(x_1,x_2)-\Gamma_1(y_1,y_2)\big)^2+\big(\Gamma_2(x_1,x_2)-\Gamma_2(y_1,y_2)\big)^2\\
&= \tfrac{b^4-2b^2a^2+a^4+4a^2b^2}{(a^2+b^2)^2}(x_1-y_1)^2+\tfrac{4a^2b^2+(a^2-b^2)^2}{(a^2+b^2)^2}(x_2-y_2)^2\\
&= \tfrac{b^4+2b^2a^2+a^4}{(a^2+b^2)^2}(x_1-y_1)^2+\tfrac{2a^2b^2+a^4+b^4}{(a^2+b^2)^2}(x_2-y_2)^2\\
&= \tfrac{(b^2+a^2)^2}{(a^2+b^2)^2}(x_1-y_1)^2+\tfrac{(a^2+b^2)^2}{(a^2+b^2)^2}(x_2-y_2)^2\\
&= (x_1-y_1)^2+(x_2-y_2)^2. \quad \square
\end{aligned}$$

Exercise MLT.1* Prove the uniqueness of the line found in Theorem MLT.3, which passes through both points A and B.

Exercise MLT.1 Proof. If a line $\mathcal{M}_m$ containing both A and B does not include the point P, then it intersects the y-axis at some point Q. If Q is above P, $sl(\overrightarrow{QA}) > sl(\overrightarrow{PA})$. Then $sl(\overrightarrow{QB}) > sl(\overrightarrow{PB}$ and these two rays are disjoint so that $B \notin \overrightarrow{QB}$, a contradiction. A similar proof will show that Q cannot be below P. $\square$

Exercise MLT.2* Prove Claim 1 of Theorem MLT.5.

Exercise MLT.2 Proof. (A) Suppose a line $\mathcal{L}_c$ in Model LM2R ($\mathbb{R}^2$) is not vertical. Let $C=(c_1,c_2)$ be a point that lies above $\mathcal{L}_c$ —so that there is a point $A=(a_1,a_2)\in\mathcal{L}_c$ with $a_1=c_1$ and $a_2<c_2$. Let $B=(b_1,b_2)$ be any point of $\mathcal{L}_c$ such that $B\neq A$. By Definition LA.1(1) $\mathcal{L}_c=\{A+s(B-A)\mid s\in\mathbb{R}\}$, and $\mathcal{L}_c=\overleftrightarrow{AB}_c$, the line in the coordinate plane containing A and B. By Theorem LC.18 the C-side of $\overleftrightarrow{AB}_c$ is the set

$$\mathcal{E}=\{A+s(B-A)+t(C-A)\mid (s,t)\in\mathbb{R}^2 \text{ and } t>0\}.$$

This is the set of all points that lie above the line $\mathcal{L}_c$. By Theorem LC.19, if C–A–C', the set of all points lying below $\mathcal{L}_c$ is its C'-side. It is quite obvious that every point of $\mathbb{R}^2$ is in $\mathcal{L}$ or one of these two sides.

Therefore, if X and Y are any two points, both lying above (or below) $\mathcal{L}_c$, $\overline{XY}_c\cap\mathcal{L}_c=\emptyset$. Moreover, if X and Y are points, one lying above and the other below $\mathcal{L}_c$, since Axiom PSA is true for Model LM2R, $\overline{XY}_c\cap\mathcal{L}_c\neq\emptyset$.

(B) Now suppose a line $\mathcal{L}_c$ in Model LM2R ($\mathbb{R}^2$) is vertical; that is, there exists a real number d such that $\mathcal{L}_c = \{(d, y) \mid y \in \mathbb{R}\}$. Again, let a point $C = (c_1, c_2)$ where $c_2 > d$. Then C lies to the right of $\mathcal{L}_c$. By reasoning similar to that of part (A), the C-side of $\mathcal{L}_c$ is the set of all points on the plane lying to the right of $\mathcal{L}_c$; if C–A–C', the set of all points lying to the left of $\mathcal{L}_c$ is the C'-side of $\mathcal{L}_c$; and every point of the plane is in $\mathcal{L}_c$ or one or the other of these two sides.

Therefore, if X and Y are any two points, both lying to the right of (or to the left of) $\mathcal{L}_c$, $\overline{XY}_c \cap \mathcal{L}_c = \emptyset$. Moreover, if X and Y are points, one lying to the right of $\mathcal{L}_c$, and the other to the left, since Axiom PSA is true for Model LM2R, $\overline{XY}_c \cap \mathcal{L}_c \neq \emptyset$. This shows that $\mathcal{E}$ and $\mathcal{F}$ are opposite sides of the line. $\square$

Exercise MLT.3* Prove that Case 4 of Claim 2 of the proof of Theorem MLT.5 leads to a contradiction.

Exercise MLT.3 Proof. Suppose both X and Y lie below $\mathcal{L}_m$, $x_1 < 0$, and $y_1 > 0$, and either $P = O$ or P lies above O. The slope $sl(\overrightarrow{PX}) > sl(lr(\mathcal{L}_m))$. (Thus if $\overleftrightarrow{XY}_m$ is of type N, so is $\mathcal{L}_m$.)

If $\overleftrightarrow{XY}_m$ is of type H or type P, $sl(\overrightarrow{PY}) = sl(\overrightarrow{PX}) > sl(lr(\mathcal{L}_m)) \geq sl(rr(\mathcal{L}_m))$. If $\overleftrightarrow{XY}_m$ is of type N, then $\mathcal{L}_m$ is of type N and $sl(\overrightarrow{PY}) = 2sl(\overrightarrow{PX}) > 2sl(lr(\mathcal{L}_m)) = sl(rr(\mathcal{L}_m)$. In either case, since P lies above O, all points of $\overrightarrow{PY}$ lie above $\mathcal{L}_m$, which is impossible, since Y is above that line. $\square$

Exercise MLT.4* Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be two points in Model MLT, where $x_1 < y_1$, and let d be any real number such that $x_1 < d < y_1$. Then there exists a real number e such that the point $Z = (d, e)$ is the point of intersection of $\mathcal{L}$ and $\overleftrightarrow{XY}_m$; also $Z \in \overline{XY}_m$. This proves that every non-vertical line intersects every vertical line.

Exercise MLT.4 Proof. (I) Suppose $\overleftrightarrow{XY}_m$ is a line of type H or type P, or it is a type N line and $0 < x_1 < d < y_1$ or $x_1 < d < y_1 < 0$ then $\overline{XY}_c = \overline{XY}_m$. Let $e = \dfrac{(d - x_1)(y_2 - x_2)}{y_1 - x_2}$; then $Z = (d, e)$ is the point of intersection of $\mathcal{L}$ and $\overleftrightarrow{XY}_m$. $Z \in \overline{XY}_m$ because X–Z–Y, since $0 < x_1 < d < y_1$.

(II) If $\overleftrightarrow{XY}_m$ is a type N line, and $x_1 < 0 < y_1$, then we have two cases:

Case 1: If $d = 0$, $\mathcal{L}$ is the y-axis, and the point of intersection of $\overleftrightarrow{XY}_m$ and $\mathcal{L}$ is the point $O = (0, e)$ where $e = \dfrac{ad + 2cb}{2c + a}$, as was calculated above in the proof of Theorem MLT.3 (showing that Axiom I.1 holds in Model MLT).

Case 2: If $d \neq 0$, first we locate the point $O = (0, e)$ of intersection with the y-axis, as in Case 1 above; then apply Part (I) above to calculate the intersection of whichever of the segments $\overline{OX}_c$ or $\overline{OY}_c$ intersects $\mathcal{L}$. Since both these are subsets of $\overline{XY}_m$, it follows that $\overline{XY}_m \cap \mathcal{L} \neq \emptyset$. $\square$

Exercise MLT.5* Prove that in Model MLT, every line parallel to a line of type N is a line of type N.

Exercise MLT.5 Proof. Let $\mathcal{L}$ be a line of type N; by Exercise MLT.4, every vertical line intersects $\mathcal{L}$; in particular, the y-axis intersects it at some point O. Let $\mathcal{M}$ be any line of type H or type P, which intersects the y-axis at a point P. If P is above O, then $\mathcal{M}$ intersects $\mathcal{L}$ at some point of the left ray $lr(\mathcal{L})$; if P is below O, these intersect at some point of the right ray $rr(\mathcal{L})$. Thus the only lines that do not intersect $\mathcal{L}$ are lines of type N. $\square$

Exercise MLT.6 Prove that the relation "$<$" defined (for lines of type N) in part (3) of Definition MLT.1(F) is an order relation according to Definition ORD.1. Note that this proof will involve Model MLT rays, which may lie partly in one side and partly on the other side of the y-axis (and hence don't look like the Model LM2R rays we considered in the text).

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