A Propositional Tableaux Based Proof Calculus for Reasoning with Default Rules*

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Abstract. Since introduced by Reiter in his seminal 1980 paper: 'A Logic for Default Reasoning', the subject of reasoning with default rules has been extensively dealt with in the literature on nonmonotonic reasoning. Yet, with some notable exceptions, the same cannot be said about its proof theory. Aiming to contribute to the latter, we propose a tableaux based proof calculus for a propositional variant of Reiter's presentation of reasoning with default rules. Our tableaux based proof calculus is based on a reformulation of the semantics of Reiter's view of a *default theory*, i.e., a tuple comprised of a set of sentences and a set of default rules, as a *premiss structure*. In this premiss structure, sentences stand for definite assumptions, as normally found in the literature, and default rules stand for tentative assumptions, as opposed to rules of inference, as normally found in the literature. On this basis, a default consequence is defined as being such relative to a premiss structure, as is our notion of a default tableaux proof. In addition to its simplicity, as usual in tableaux based proof calculi, our proof calculus allows for the discovery of the non-existence of proofs by providing corresponding counterexamples.

1 Introduction

It is commonly recognized that the subject of reasoning with default rules, henceforth default reasoning, occupies a prominent role in the logical approach to non-monotonic reasoning. Introduced by Reiter in his seminal 1980 paper, 'A Logic for reasoning with Default Rules' (q.v. [1]), default reasoning has been extensively investigated from a syntactical and semantical point of view, with several variants to Reiter's original ideas being proposed (q.v. [2]).

On the other hand, the proof theoretical aspects of default reasoning seem to have received far less attention. More precisely, Reiter's own discussion on a proof theory for normal default rules, in the later sections of [1], does not

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necessarily formulate a proof calculus, for it gives no particular set of rules for constructing proof-like objects. Instead, for us, this discussion is best understood as another way of formally defining the concept of an extension, in this case for default rules that are normal. In the context of tableaux methods, works such as that of Risch in [3] and that of Amati et. al. in [4] are also focused on the concept of an extension, extending the work of Reiter by providing tableaux based definitions, and by proving some general properties, of its major variants. However, in and of themselves, neither [3] nor [4] present a tableaux based proof calculus, i.e., a mechanization of a consequence relation, for default reasoning.

In contrast, a noteworthy contribution in a rather traditional proof-theoretical line of research is the work of Bonatti and Olivetti in [5]. Therein, the authors present a sequent calculus for what they call *skeptical default logic*, a propositional variant of Reiter's presentation of default reasoning where default consequences are drawn *skeptically.* The work of Bonatti and Olivetti gains in interest for it introduces a complete mechanization of a consequence relation for default reasoning in proof-theoretical terms via the notion of an *anti-sequent calculus.*

In this work, also in a rather traditional proof-theoretical line of research, at least when seen from the perspective of a standard presentation of a tableaux method, we present a tableaux based proof calculus for a propositional variant of Reiter's presentation of default reasoning where default consequences are taken skeptically. More precisely, we reformulate the semantics of Reiter's view of a *default theory*, i.e., a tuple comprised of a set of sentences and a set of default rules, as a *premiss structure.* In this premiss structure, sentences stand for definite assumptions, as commonly found in the literature on default reasoning, and default rules stand for tentative assumptions, a departure from the common treatment of default rules as rules of inference normally found in the literature on default reasoning. It is on this basis that we propose our tableaux based proof calculus. In doing this, we have two main goals in mind. First, we aim at contributing to the mechanization of the notion of derivability for default reasoning. Second, we view the tableaux based proof calculus that is presented here as a first step towards an abstract definition of default tableaux proof calculi, i.e., one that is independent of the underlying logical system. To give an idea of the latter, a tableau method for a logic $\mathfrak{L}$ is a procedure for testing for the existence of models for sets of formulas of $\mathfrak{L}$ which can be used to construct canonical models by applying rules for decomposing formulas into their components in a structured and semantics preserving way. In the presence of negation,[1] a technique for building models can be understood as a refutation mechanism for the logic. This allows for tableaux methods to be used as proof calculi (for a set of sentences $\Gamma \cup \{\sigma\}$ of $\mathfrak{L}$, proving σ from Γ, $\Gamma \vdash_{\mathfrak{L}} \sigma$, requires us to check that there is no model of $\Gamma \cup \{\neg\sigma\}$). Model construction and provability as features of a tableaux method for a logic $\mathfrak{L}$ accommodate the use of default rules defined

[1] A logic $\mathfrak{L}$ defined on a language $\mathscr{L}$ is said to have negation if for any sentence σ in $\mathscr{L}$, there is a sentence σ' in $\mathscr{L}$, denoted as $\neg\sigma$, such that for any set Γ of sentences in $\mathscr{L}$, $\Gamma \models^{\mathfrak{L}} \sigma$ iff $\Gamma \not\models^{\mathfrak{L}} \neg\sigma$ (where $\models^{\mathfrak{L}}$ indicates semantic entailment in $\mathfrak{L}$).

on the language of $\mathfrak{L}$ in the form of premiss assumptions that are only used tentatively. These features set the context for a default tableaux method.

Structure of this work: §2 introduces the basics of a tableaux based proof calculus for classical propositional logic and a propositional variant of Reiter's presentation of default reasoning; §3 introduces our proposed tableaux based proof calculus for the propositional variant of Reiter's presentation of default reasoning in question; §4 discusses our ideas; lastly, §5 offers some conclusions and comment on some of the further work that we plan to undertake.

2 Preliminaries

2.1 Propositional Tableaux

Let $\mathscr{L}$ be the *standard propositional language* determined by a denumerable set of *propositional symbols* $p, q, \ldots$ and the *logical connectives* of: $\top$ and $\bot$ ('truth' and 'falsity'); $\neg$ ('negation'); $\wedge$, $\vee$, and $\supset$ ('conjunction', 'disjunction', and 'material implication'). Members of $\mathscr{L}$, indicated by lowercase Greek letters, are called *sentences*. A *substitution* is a mapping s from the propositional symbols of $\mathscr{L}$ into $\mathscr{L}$. It is a well-known result that any substitution s extends uniquely to all members of $\mathscr{L}$. A sentence σ is a *substitution instance* of another sentence σ' iff $\sigma = s(\sigma')$ for s a substitution. A sentence σ is: a *literal* if it is either a propositional variable or a negation thereof; of *linear type* if it is a substitution instance of $p \wedge q$, $\neg(p \vee q)$, $\neg(p \supset q)$, or $\neg\neg p$; of *branching type* if it is a substitution instance of $\neg(p \wedge q)$, $p \vee q$, or $p \supset q$. The lowercase Greek letters α and β indicate arbitrary sentences of linear and branching type, respectively. The *components* of a sentence α of linear type, and of a sentence β of branching type, indicated as α_1 and α_2, and as β_1 and β_2, respectively, are defined as usual – e.g., if α is a substitution instance of $p \wedge q$, then, its components are the corresponding substitution instances of p and q, respectively; if β is a substitution instance of $p \supset q$, then, its components are the corresponding substitution instances of $\neg p$ and q, respectively. The previous *unifying notation*, quoting Smullyan, "will save us considerable repetition of essentially the same arguments" (q.v. [6, pp. 20–21]).

Definition 1 (Tableau from Premisses). *Let σ be a sentence and Γ be a finite set of sentences; the set of all tableaux for σ with premisses in Γ is the smallest set of labeled trees T that satisfies the following conditions:*

R0 The unique one-node labeled tree with label $\{\sigma\} \cup \Gamma'$, where $\Gamma' \subseteq \Gamma$, is in T.
– *Let τ be in T, l be a leaf of τ with label Γ', and τ' a labeled tree:*
R1 If a sentence α of linear type belongs to Γ', and τ' is obtained from τ by adding a new node n' with label $\Gamma' \cup \{\alpha_1, \alpha_2\}$ as an immediate successor of l, then, τ' belongs to T.
R2 If a sentence β of branching type belongs to Γ', and τ' is obtained from τ by adding two new nodes n' and n'' with labels $\Gamma' \cup \{\beta_1\}$ and $\Gamma' \cup \{\beta_2\}$, respectively, as immediate successors of l, then, τ' belongs to T.

R3 For any sentence γ in Γ, if τ' is obtained from τ by adding a new node n' with label $\Gamma' \cup \{\gamma\}$ as immediate successors of l, then, τ' belongs to T.

A labeled tree τ is a tableau for σ with premisses in Γ iff it is a member of T.

Definition 1 emphasizes the view of tableau constructions as proof-theoretical objects, more precisely, *proof attempts*, i.e., we view a tableau for $\neg\sigma$ with premisses in Γ as an attempt at proving that σ is a *consequence* of the set of *premisses* Γ, with any closed tableau for $\neg\sigma$ with premisses in Γ being a successful proof attempt, i.e., a proof. This view of a proof is made precise in Definition 3 with the aid of Definition 2.

Definition 2 (Closed Tableau). *Let τ be a tableau for σ with premisses in Γ; a node n of τ with label Γ' is closed iff one of the following conditions holds:*

- $\{\bot, \neg\top\} \cap \Gamma \neq \emptyset$.
- $\{\sigma, \neg\sigma\} \subseteq \Gamma$ *for some sentence* σ.

The node n is open iff it is not closed. The tableau τ is closed iff all its leaf nodes are closed, otherwise τ is open.

Definition 3 (Proof). *Let σ be a sentence and Γ a finite set of sentences; a proof of σ from Γ is a closed tableau for $\neg\sigma$ with premisses in Γ. The sentence σ is provable from Γ iff there is a proof of σ from Γ. In addition, σ is a consequence of Γ, or follows from Γ, indicated by $\Gamma \vdash \sigma$, iff σ is provable from Γ.*

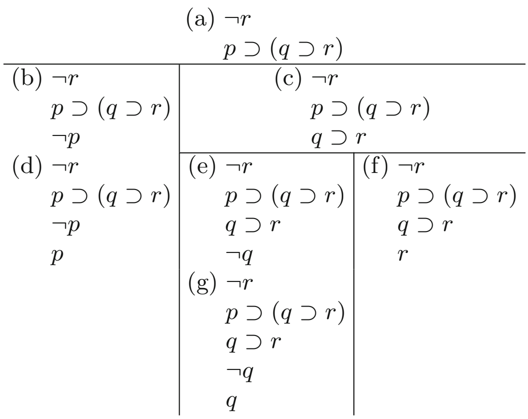

Fig. 1. Tableau for $\neg r$ with premisses in $\{p, q, p \supset (q \supset r)\}$

Fig. 1 depicts proof of r from $\{p, q, p \supset (q \supset r)\}$. In this figure, (a) is the initial node from which τ is constructed as per *R0* in Definition 1; nodes (b) and (c) are added as immediate successors of (a) as per *R2* in Definition 1; nodes (d) is added as an immediate successor of (b) as per *R3* in Definition 1; nodes (e) and (f) are added as immediate successors of (c) as per *R2* in Definition 1; and lastly, node (g) is added as an immediate successor of (e) as per *R3* in Definition 1.

While finding a proof of σ from Γ is the same as finding that there are no models of $\Gamma \cup \{\neg\sigma\}$, the latter being a more common use for tableau constructions, we favor the view of tableau constructions as proof attempts for it more readily construes the method of tableaux as a *proof calculus*. It is a well-known result that such a proof calculus is both sound and complete with respect to the standard model theory of classical propositional logic (q.v. [6]).

Moreover, there are two properties of the previous presentation of the method of tableaux as a proof calculus that are worth noting: (i) it can be demonstrated that any attempt at proving that σ follows form Γ can be extended to a *successful* one if such a proof were to exist; and (ii) tableau constructions also make it possible to discover the *nonexistence of proofs* by looking at some particular tableau constructions. The second point is made precise below.

Definition 4 (Completed Tableau). *Let τ be a tableau for σ with premisses in Γ; a node n of τ with label Γ' is completed iff the following conditions are met:*

- *For any sentence α of linear type, if $\alpha \in \Gamma'$, then, $\{\alpha_1, \alpha_2\} \subseteq \Gamma'$.*
- *For any sentence β of linear type, if $\beta \in \Gamma'$, then, either $\beta_1 \in \Gamma'$ or $\beta_2 \in \Gamma'$.*
- *$\Gamma \subseteq \Gamma'$.*

The tableau τ is completed iff all leaf nodes of τ are completed.

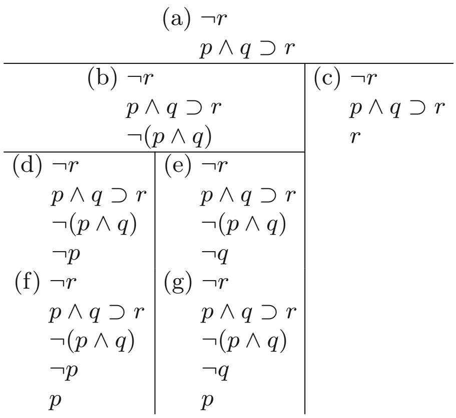

Fig. 2. Tableau for $\neg r$ with premisses in $\{p, p \wedge q \supset r\}$

From the perspective of a proof calculus, Definition 4 gains in interest for: (i) it indicates to us when to stop in the construction of a sought after proof; and

(ii) if a completed tableau is not closed, i.e., it has a leaf node that is open, then, the set of sentences labeling this node is satisfiable (q.v. node (g) in Fig. 2). This result, known as *Hintikka's lemma*, q.v., [6, pp. 26–28], indicates that the sought after proof does not exist (a result that will be used in the definition of a tableaux method for default reasoning presented in Section 3). Non-existence of proofs is made precise in Proposition 1.

Proposition 1. *Let τ be a tableau for $\neg\sigma$ with premisses in Γ; if τ has a leaf node that is open and complete, then, no expansion of τ results in a closed tableau for $\neg\sigma$ with premisses in Γ, i.e., a proof of σ from Γ.*

2.2 Reasoning with Default Rules

The set $\mathscr{D}$ of *all default rules* defined on the standard propositional language $\mathscr{L}$ is the set of all tuples

$$\frac{\pi : \rho}{\chi}$$

where $\{\pi, \rho, \chi\} \subseteq \mathscr{L}$. Members of $\mathscr{D}$, for inline formatting purposes displayed as $\pi : \rho \,/\, \chi$, are called *default rules*. In a default rule $\pi : \rho \,/\, \chi$, the sentences π, ρ, and χ are called: *prerequisite*, *justification*, and *consequent*, respectively. For a set of default rules Δ, $\Pi(\Delta)$ indicates the set of all prerequisites of the default rules in Δ, i.e., $\Pi(\Delta) = \{\pi \mid \pi : \rho \,/\, \chi \in \Delta\}$; $P(\Delta)$ indicates the set of all justifications of the default rules in Δ, i.e., $P(\Delta) = \{\rho \mid \pi : \rho \,/\, \chi \in \Delta\}$; and $X(\Delta)$ indicates the set of all consequents of the default rules in Δ, i.e., $X(\Delta) = \{\chi \mid \pi : \rho \,/\, \chi \in \Delta\}$.

Departing from the position sustaining that a default rule is a defeasible rule of inference, i.e., a rule of inference that is open to revision or annulment, commonly found in the literature on default reasoning, we view a default rule $\pi : \rho \,/\, \chi$ as indicating an assumption that is made tentatively: χ can be posited provided that π is fulfilled and that ρ is not established (ρ acts as a rebuttal condition). This view of default rules is based on the observation that they are not logic defining rules of inference, but, instead, they are premiss-like objects defined in the logic. On this basis, given a set of sentences Φ and a set of default rules Δ, we reformulate Reiter's view of $\langle\Phi, \Delta\rangle$ as a default theory, q.v. [1, p. 88], as a premiss structure. In this premiss structure, the sentences in Φ stand for *definite* assumptions and the default rules in Δ stand for *tentative* assumptions.

The notion of a default consequence δ of a premiss structure $\langle\Phi, \Delta\rangle$, indicated by $\langle\Phi, \Delta\rangle \mathrel{|\!\sim} \delta$, is then justified resorting to the notion of an extension. More precisely, a sentence δ is a default consequence of a premiss structure $\langle\Phi, \Delta\rangle$ iff for every extension E of $\langle\Phi, \Delta\rangle$, $E \vdash \delta$. In this respect, an extension is seen as an interpretation structure of a syntactical kind, i.e., the usual role of a model is taken up by an extension. The notion of an extension in question here is introduced in Definition 7 with the aid of Definitions 5 and 6. Several other variants of Reiter's notion of an extension are presented in [2].

Definition 5. *A set of default rules Δ is tentative w.r.t. a set of sentences Γ iff every $\pi : \rho \,/\, \chi \in \Delta$ is such that: (i) $\Gamma \vdash \pi$, and (ii) $\Gamma \cup X(\Delta) \nvdash \rho$.*

Example 1. The set of default rules $\{p : q \;/\; r, r : s \;/\; t\}$ is tentative w.r.t. the set of sentences $\{p\}$, but not w.r.t. the set of sentences $\{p, q\}$.

Definition 6. *A set of default rules Δ is sequentiable w.r.t. a set of sentences Φ iff there is a chain $\mathbf{C}$ of subsets of Δ ordered by inclusion such that: (i) $\emptyset \in \mathbf{C}$; (ii) let $\Delta' \in \mathbf{C}$ and $\delta \in \Delta \setminus \Delta'$, if $\Delta' \cup \{\delta\}$ is tentative w.r.t. $\Phi \cup X(\Delta')$, then $\Delta' \cup \{\delta\} \in \mathbf{C}$; and (iii) $\Delta = \bigcup_{\Delta' \in \mathbf{C}} \Delta'$.*

Example 2. The set of default rules $\{p : q \;/\; r, r : s \;/\; t\}$ is sequentiable w.r.t. the set of sentences $\{p\}$. The set of default rules $\{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\}$ is not sequentiable w.r.t. the set of sentences $\{p\}$.

Definition 7 (Extension). *Let Φ be a set of sentences and Δ be a set of default rules; the class $\mathscr{E}$ of extensions of $\langle \Phi, \Delta \rangle$ consists of all sets $\Phi \cup X(\Delta')$, where Δ' is a subset of Δ such that: (i) Δ' is sequentiable w.r.t. Φ; and (ii) for any other $\Delta'' \subseteq \Delta$ that is sequentiable w.r.t. Φ, if $\Delta' \subseteq \Delta''$, then, $\Delta'' = \Delta'$. A set E of sentences is an extension of $\langle \Phi, \Delta \rangle$ iff $E \in \mathscr{E}$.*

Example 3. The class of extensions associated to the premiss structure $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\} \rangle$ consists of the sets $E1$ and $E2$ defined as: $E_1 = \{p, p \supset (q \vee r \supset s), q \wedge t\}$, and $E_2 = \{p, p \supset (q \vee r \supset s), r \wedge u\}$.

Proposition 2 states two important properties that are satisfied by extensions if defined as in Definition 7.

Proposition 2. *For every premiss structure $\langle \Phi, \Delta \rangle$, the class $\mathscr{E}$ of extensions of $\langle \Phi, \Delta \rangle$ is not empty. Moreover, extensions, as in Definition 7, satisfy the property of semimonotonicity, i.e., for any two premiss structures $\langle \Phi, \Delta \rangle$ and $\langle \Phi, \Delta \cup \Delta' \rangle$, every extension of $\langle \Phi, \Delta \rangle$ is included in some extension of $\langle \Phi, \Delta' \rangle$.*

Examples 4 and 5 illustrate the way in which the notion of an extension justifies the notion of a default consequence.

Example 4. Let $\langle \Phi, \Delta \rangle$ be the premiss structure of Example 3, the sentence s is a default consequence of $\langle \Phi, \Delta \rangle$. To see why this is the case, observe that the class of extensions associated to this premiss structure is comprised of the extensions: $E_1 = \{p, p \supset (q \vee r \supset s), q \wedge t\}$, and $E_2 = \{p, p \supset (q \vee r \supset s), r \wedge u\}$. Immediately, $E_1 \vdash s$ and that $E_2 \vdash s$. Hence $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} s$.

Example 5. At the same time, observe that if $\langle \Phi, \Delta \rangle$ is as in Example 3, the sentence t is not a default consequence of $\langle \Phi, \Delta \rangle$. To see why this is the case, observe that, whereas $E_1 \vdash t$, $E_2 \nvdash t$. Hence $\langle \Phi, \Delta \rangle \mathrel{|\!\not\sim} t$.

It should be noted that, given the machinery presented above, determining whether a sentence is a default consequence of a premiss structure requires an enumeration-based approach, i.e., all extensions associated to the premiss structure in question must be constructed in order to check whether the alleged default consequence is indeed so (something that may be done by constructing suitable tableaux and checking whether they are closed, e.g., following the approaches proposed in [3] and in [4]). This enumeration-based approach is rather inefficient for two main reasons. First, constructing all extensions associated to a premiss structure is rather costly, the number of extensions associated with non-trivial premiss structures being exponential in the number of default rules. Second, enumerating all extensions associated to a premiss structure requires us to consider all default rules in this premiss structure. What is then needed is a systematization of the kind of reasoning involved in proving in all extensions, i.e., a *proof calculus* for default reasoning. In that respect, being able to check that a sentence is a default consequence of a premiss structure resorting only to a part of this premiss structure is a highly desirable feature of a proof calculus for default reasoning. Although this is not a trivially achieved, we incorporate it as a basic guiding feature in the tableaux based proof calculus that we present in Section 3.

3 Default Tableaux

Definition 8 introduces the basic elements of the tableaux based proof calculus for default reasoning, the notion of a *default tableau.*

Definition 8 (Default Tableau). *Let σ be a sentence, and Φ and Δ be finite sets of sentences and default rules, respectively; the set of all default tableaux for σ with premisses in $\langle \Phi, \Delta \rangle$ is the smallest set T_{dr} of labeled trees that satisfies the following conditions:*

R0 The unique one-node labeled tree with label $\langle \Phi \cup \{\sigma\}, \emptyset \rangle$ is in T_{dr}.
- *Let τ be in T_{dr}, l a leaf node of τ with label $\langle \Phi', \Delta' \rangle$, and τ' a labeled tree:*

R1 If a sentence α of linear type belongs to Φ', and τ' is obtained from τ by adding a new node n' with label $\langle \Phi' \cup \{\alpha_1, \alpha_2\}, \Delta' \rangle$ as an immediate successor of l, then, τ' is in T_{dr}.

R2 If a sentence β of branching type belongs to Φ', and τ' is obtained from τ by adding two new nodes n' and n'' with labels $\langle \Phi' \cup \{\beta_1\}, \Delta' \rangle$ and $\langle \Phi' \cup \{\beta_2\}, \Delta' \rangle$, respectively, as immediate successors of l, then, τ' is in T_{dr}.
- *Let n be a node of τ with label $\langle \Phi', \Delta' \rangle$:*

R3 For any default rule $\pi : \rho \,/\, \chi$ in Δ, if τ' is obtained from τ by adding a new node n' with label $\langle \Phi' \cup \{\chi\}, \Delta' \cup \{\pi : \rho \,/\, \chi\} \rangle$ as an immediate successor of n, then, τ' is in T_{dr} iff the following side conditions are satisfied:
   - *(a) there is a closed tableau for $\neg\pi$ with premisses in $\Phi \cup X(\Delta')$, and*
   - *(b) for every $\rho' \in P(\Delta') \cup \{\rho\}$, there is a tableau for $\neg\rho'$ with premisses in $\Phi \cup X(\Delta') \cup \{\chi\}$ that is both complete and open.*

A default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$ is a labeled tree τ in T_{dr}.

In order to understand the basic ideas underpinning the formulation of a default tableau, consider a situation in which we are required to prove that the sentence s is a default consequence of the premiss structure $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \,/\, q \wedge t, p : t \,/\, r \wedge u\} \rangle$. In attempting such a proof by refutation, we need to establish from the premiss structure in question that assuming $\neg s$ leads to a contradiction. As a first step, we may attempt this proof by appealing only to the sentences in $\{p, p \supset (q \vee r \supset s)\}$. Given this initial standpoint, we begin our proof with a labeled tree with a single node (a) labeled by $L_{(\mathrm{a})} = \{p, p \supset (q \vee r \supset s), \neg s\}$. Now, since $p \supset (q \vee r \supset s)$ belongs to $L_{(\mathrm{a})}$, we add as immediate successors of (a) nodes (b) and (c) labeled by $L_{(\mathrm{b})} = L_{(\mathrm{a})} \cup \{\neg p\}$ and $L_{(\mathrm{c})} = L_{(\mathrm{a})} \cup \{q \vee r \supset s\}$, respectively. Then, since $q \vee r \supset s$ belongs to $L_{(\mathrm{c})}$, we add as immediate successors of (c) nodes (d) and (e) labeled by $L_{(\mathrm{d})} = L_{(\mathrm{c})} \cup \{\neg(q \vee r)\}$ and $L_{(\mathrm{e})} = L_{(\mathrm{c})} \cup \{s\}$, respectively. Lastly, since $\neg(q \vee r)$ belongs to $L_{(\mathrm{d})}$, we add as an immediate successor of (d) a node (f) labeled by $L_{(\mathrm{f})} = L_{(\mathrm{d})} \cup \{\neg q, \neg r\}$. The previous default tableau construction steps yield the default tableau, a standard set of sentences labeled tableau, depicted in Fig. 3.

<table>
<tr><td colspan="3">(a) $p \supset (q \vee r \supset s)$
p
$\neg s$</td></tr>
<tr><td rowspan="2">(b) $p \supset (q \vee r \supset s)$
p
$\neg s$
$\neg p$</td><td colspan="2">(c) $p \supset (q \vee r \supset s)$
p
$\neg s$
$q \vee r \supset s$</td></tr>
<tr><td>(d) $p \supset (q \vee r \supset s)$
p
$\neg s$
$q \vee r \supset s$
$\neg(q \vee r)$
(f) $p \supset (q \vee r \supset s)$
p
$\neg s$
$q \vee r \supset s$
$\neg(q \vee r)$
$\neg q$
$\neg r$</td><td>(e) $p \supset (q \vee r \supset s)$
p
$\neg s$
$q \vee r \supset s$
s</td></tr>
</table>

Fig. 3. Default tableau for $\neg s$ with premisses in $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\}\rangle$

At this point, it may be observed that (b) and (e) are leaf nodes that are closed, and that (f) is a leaf node that is open and "completed". However, since we have not made use of the default rules in the premiss structure, (f) is only "completed" w.r.t. the tableau construction rules for classical propositional logic. Our proof is not done yet for we can proceed and use the default rules in the premiss structure. Given that, as per *R3* in Definition 8, the side conditions hold for applying $p : u \;/\; q \wedge t$ hold, we can add as an immediate successor of (f) a new node (g) labeled by $S_{(\mathrm{g})} = S_{(\mathrm{f})} \cup \{q \wedge t\}$, simultaneously recording that $p : u \;/\; q \wedge t$ has been used. Next, since $q \wedge t$ belongs to $S_{(\mathrm{g})}$, we can add as an immediate successor of (g) a new node (i) labeled by $S_{(\mathrm{i})} = S_{(\mathrm{g})} \cup \{q, t\}$. It is immediate to check that the leaf node (i) is closed and completed in a default tableau sense. (Given that, as per *R3* in Definition 8, the side conditions for $p : t \; / \; r \wedge u$ do not hold, this branch cannot be extended further.)

Notwithstanding, even though (i) is closed and completed, our proof that s is a default consequence of $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \; / \; q \wedge t, p : t \; / \; r \wedge u\}\rangle$ is still unfinished, the reason being that the addition of (g) as an immediate successor of (f) preempted the use of the other default rule in the premiss structure, i.e., of $p : t \; / \; r \wedge u$. Since the main idea underpinning a default tableau is that of systematizing a notion of provability in all extensions, a default proof should not depend on a particular selection of default rules to be applied. This means that we are required to check what would have been the case had we chosen to resort to $p : t \; / \; r \wedge u$ instead of $p : u \; / \; q \wedge t$. Thus, given that, as per *R3* in Definition 8, the side conditions for applying $p : t \; / \; r \wedge r$ hold, we need to add as an immediate successor of (f) a new node (h) labeled by $S_{(\mathrm{h})} = S_{(\mathrm{f})} \cup \{r \wedge u\}$; simultaneously recording that $p : t \; / \; r \wedge r$ has now been used. This branch can be completed, in a default tableau sense, by adding a new node (j) with label $S_{(\mathrm{j})} = S_{(\mathrm{h})} \cup \{r, u\}$ as an immediate successor of (h). These tableau construction steps yield the default tableau depicted in Fig. 4.

As with Definition 1, underpinning Definition 8 is the idea of emphasizing the view of default tableau constructions as proof-theoretical objects; more precisely, as *proof-attempts* (in this case, the focus is on proving that a sentence is a *default consequence* of a finite *premiss structure*). This view of a default tableau as a proof-theoretical object, and hence of the method of default tableau as a proof calculus, is made precise in Definition 11, with the aid of Definitions 9 and 10.

Definition 9 (Closedness). *Let τ be a default tableau for σ with premisses in $\langle\Phi, \Delta\rangle$; a node n of τ with label $\langle\Phi', \Delta'\rangle$ is closed (otherwise it is open) iff either of the following conditions holds:*

- $\{\bot, \neg\top\} \cap \Phi' \neq \emptyset$.
- $\{\sigma, \neg\sigma\} \subseteq \Phi'$ *for some sentence* σ.

The default tableau τ is closed iff its leaf nodes are closed (otherwise τ is open).

Definition 10 (d-Saturation). *Let τ be a default tableau for σ with premisses in $\langle\Phi, \Delta\rangle$; a node n of τ with label $\langle\Phi', \Delta'\rangle$ is d-branching iff it has an immediate successor a node n' with label $\langle\Phi'', \Delta''\rangle$ such that $\Delta' \subset \Delta''$. A d-branching node n of τ is d-saturated iff adding a new node n' with label $\langle\Phi' \cup \{\chi\}, \Delta' \cup \{\pi : \rho \; / \; \chi\}\rangle$ as an immediate successor of n, as per R3 in Definition 8, results in n having at least two immediate successors labeled with the same label. The default tableau τ is d-saturated iff all of its d-branching nodes are d-saturated.*

Example 6. Node (f) in Fig. 4 is both d-branching and d-saturated.

Definition 11 (Default Proof). *Let σ be a sentence, and Φ and Δ be finite sets of sentences and default rules, respectively; a default proof of σ from $\langle\Phi, \Delta\rangle$ is a closed and d-saturated default tableau for $\neg\sigma$ with premisses in $\langle\Phi, \Delta\rangle$. The sentence σ is provable from $\langle\Phi, \Delta\rangle$ iff there is a default proof of σ from $\langle\Phi, \Delta\rangle$.*

$$\vdots$$

(f) $p \supset (q \vee r \supset s)$
p
$\neg s$
$q \vee r \supset s$
$\neg(q \vee r)$
$\neg q$
$\neg r$

(g) $p \supset (q \vee r \supset s)$ $\langle p : u \;/\; q \wedge t\rangle$	(h) $p \supset (q \vee r \supset s)$ $\langle p : t \;/\; r \wedge u\rangle$
p	p
$\neg s$	$\neg s$
$q \vee r \supset s$	$q \vee r \supset s$
$\neg(q \vee r)$	$\neg(q \vee r)$
$\neg q$	$\neg q$
$\neg r$	$\neg r$
$q \wedge t$	$r \wedge u$
(i) $p \supset (q \vee r \supset s)$ $\langle p : u \;/\; q \wedge t\rangle$	(j) $p \supset (q \vee r \supset s)$ $\langle p : t \;/\; r \wedge u\rangle$
p	p
$\neg s$	$\neg s$
$q \vee r \supset s$	$q \vee r \supset s$
$\neg(q \vee r)$	$\neg(q \vee r)$
$\neg q$	$\neg q$
$\neg r$	$\neg r$
$q \wedge t$	$r \wedge u$
q	r
t	u

Fig. 4. Default tableau for $\neg s$ with premisses in $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\}\rangle$

The view of default tableau constructions as constituting a proof calculus for default reasoning conforms to the following rationale. For a sentence σ and a finite premiss structure $\langle \Phi, \Delta\rangle$, we can think of any default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta\rangle$ as an attempt at proving that σ is a default consequence of $\langle \Phi, \Delta\rangle$, with any default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta\rangle$ that is d-saturated and closed being a successful proof attempt, i.e., a proof that σ is a default consequence of $\langle \Phi, \Delta\rangle$. By way of example, the default tableau depicted in Fig. 4 constitutes a proof that s is a default consequence of $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\}\rangle$.

Perhaps requiring a bit of explanation is the idea of a node of a default tableau being d-branching and d-saturated (q.v. Definition 10, exemplified by node (f) in Fig. 4). While there is no similar concept in the construction of a standard set labeled tableaux for classical propositional logic, its underpinning rationale may be understood by drawing the following correspondence. Suppose that in breaking down syntactically the sentences used in the construction of a tableau τ we find ourselves dealing with a sentence β of branching type, if instead of extending τ simultaneously with two nodes, whose labels correspond to the components of a sentence of branching type, for whatever reason, we were restricted to expand τ one node at a time, then, we would not be able to proceed solely at the level of leaves. In such a scenario, we would be required to take note of which one of the components of a sentence of branching type has been used in extending the tableau, and to consider what would be the case had we used the other component, i.e., construct the alternative branch (at the level of some intermediate node of τ). If a tableau is being constructed in this way, then, it would be completed, in a branching sense, once both components of a sentence of branching type have been used. Of course, this explanation is an elaborate way of describing what otherwise is an extremely simple construction which exhausts all possibilities for a sentence of branching type, i.e., "add two different nodes as immediate successors of another one". In this respect, there seems to be no rationale for its preference. However, the situation is rather different for default tableau constructions. In most cases it is necessary to have the flexibility of considering default rules one at a time – recall from the example shown in Section 3 how using one default rule prohibited the use of another, thus restricting the extensions being reasoned about. In such scenarios, d-saturation guarantees that all default rules have been considered (q.v. nodes (g) and (h) in Fig. 4).

The correctness of default tableau constructions as constituting a proof calculus for default reasoning is stated in Theorem 1.

Theorem 1 (Correctness). *For any sentence σ, and for any finite sets Φ and Δ of sentences and default rules, respectively, σ is provable from $\langle \Phi, \Delta \rangle$, i.e., there is a closed and d-saturated default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$, iff $\langle \Phi, \Delta \rangle \mathrel{|\!\!\sim} \sigma$, i.e., iff for every extension E of $\langle \Phi, \Delta \rangle$, $E \vdash \sigma$.*

Proof (sketch). Let τ be a default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$, and let l be any leaf node of τ with label $\langle \Gamma', \Delta' \rangle$; to be noted first is that: (i) $\Phi \cup X(\Delta')$ is included in some extension E of $\langle \Phi, \Delta \rangle$, and (ii) Γ is a leaf node of a tableau for $\neg\sigma$ with premisses in $\Phi \cup X(\Delta')$. In other words, if l is completed, constructing τ is equivalent to constructing an extension E of $\langle \Phi, \Delta \rangle$ together with a leaf node of a tableau for $\neg\sigma$ with premisses in E. If l is closed, then, every leaf node of a tableau for $\neg\sigma$ with premisses in E, where E is an extension of $\langle \Phi, \Delta \rangle$ which contains $\Phi \cup X(\Delta')$, is also closed, i.e., $E \vdash \sigma$. Semimontonicity and d-saturation guarantee that all extensions of $\langle \Phi, \Delta \rangle$ have been considered.

From a proof-theoretical perspective, the view of default tableau constructions as constituting a proof calculus further gains in interest for it makes it possible to discover the nonexistence of default proofs by inspecting some particular cases of proof attempts. For instance, the default tableau depicted in Fig. 5 indicates that t is not a default consequence of $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\} \rangle$.

Definition 12 (Completed). *Let τ be a default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$; a node n of τ with label $\langle \Phi', \Delta' \rangle$ is completed iff:*

- *For every sentence α of linear type in Φ', the components α_1 and α_2 of α are also in Φ'.*

– *For every sentence* β *of branching type in* Φ'*, at least one of the components* β_1 *or* β_2 *of* β *is in* Φ'*.*
– *For every default rule* $\pi : \rho \,/\, \chi$ *in* Δ*, if* $\pi : \rho \,/\, \chi$ *meets the side conditions of Definition 8(Rule c), then,* χ *is in* Φ' *and* $\pi : \rho \,/\, \chi$ *is in* Δ'*.*

The default tableau τ *is complete iff all of its leaf nodes are completed.*

The nonexistence of default proofs is made precise in Proposition 3 with the aid of Definition 12.

Proposition 3. *If a default tableau for* $\neg\sigma$ *with premisses in* $\langle \Phi, \Delta \rangle$ *has a complete leaf node that is also open, then,* σ *is not a default consequence of* $\langle \Phi, \Delta \rangle$*.*

Proof (sketch). Let τ *be a default tableau for* $\neg\sigma$ *with premisses in* $\langle \Phi, \Delta \rangle$*, and let* l *be any leaf node of* τ *with label* $\langle \Gamma', \Delta' \rangle$*; to be noted first is that: (i)* $\Phi \cup X(\Delta')$ *is included in some extension* E *of* $\langle \Phi, \Delta \rangle$*, and (ii)* Γ *is a leaf node of a tableau for* $\neg\sigma$ *with premisses in* $\Phi \cup X(\Delta')$*. If* l *is open and complete, then, there is a leaf node of a tableau for* $\neg\sigma$ *with premisses in* E*, where* E *is an extension of* $\langle \Phi, \Delta \rangle$ *which contains* $\Phi \cup X(\Delta')$*, that is open, i.e.,* $E \not\vdash \sigma$*. As a result,* $\langle \Phi, \Delta \rangle \not\vdash \sigma$*, i.e.,* σ *is not a default consequence of* $\langle \Phi, \Delta \rangle$*.*

In essence, a leaf node of a default tableau that is both complete and open constructs an extension from which the alleged default consequence does not follow. For the case of the default tableau depicted in Fig. 5, i.e., default tableau for $\neg t$ with premisses in $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \,/\, q \wedge t, p : t \,/\, r \wedge u\} \rangle$, said extension, the set $E_2 = \{p, p \supset (q \vee r \supset s), r \wedge u\}$, is obtained from the second component of the label of the leaf node (g) in Fig. 5 together with the set of sentences of the premiss structure in question. That t is not a consequence of this extension is also immediate from the information present in the leaf node (g) in Fig. 5: the first component of this node corresponds to a leaf node of a tableau for $\neg t$ with premisses in E_2.

4 Discussion

One of the most concise descriptions of the rationale underlying tableau methods as proof methods is perhaps that provided by Fitting in [7]. In Fitting's terms, a tableau method is a formal proof procedure, existing in a variety of forms and for several logics, but always having certain characteristics. First, it is a refutation procedure. In order to prove that something is the case, the initial step is to begin with a syntactical expression intended to assert the contrary. Successive steps then syntactically break down this assertion into cases. Finally, there are impossibility conditions for closing cases. If all cases are closed, then, the initial assertion has been refuted. As a result, it is concluded that what had been taken not to be case is actually the case.

The kind of default tableau constructions presented here operate in the way just described. In order to prove that a sentence σ is a default consequence of a premiss structure $\langle \Phi, \Delta \rangle$, we begin with a syntactical expression intended to

(a) $p \supset (q \vee r \supset s)$
p
$\neg t$

(b) $p \supset (q \vee r \supset s)$
p
$\neg t$
$\neg p$

(c) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$

(d) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$
$\neg(q \vee r)$

(e) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$
s

(f) $p \supset (q \vee r \supset s)$ $\langle p : t \ / \ r \wedge u \rangle$
p
$\neg t$
$q \vee r \supset s$
s
$r \wedge u$

(g) $p \supset (q \vee r \supset s)$ $\langle p : t \ / \ r \wedge u \rangle$
p
$\neg t$
$q \vee r \supset s$
s
$r \wedge u$
r
u

Fig. 5. Default tableau for $\neg t$ with premisses in $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\} \rangle$

assert that this is not the case. In a default tableau, the set $\Phi \cup \{\neg\sigma\}$ is said syntactical expression. Next, we syntactically break down the sentences in this expression into their components according to rules *R1* or *R2* in Definition 8, i.e., depending on whether they are of linear or of branching type, respectively. *R3* in Definition 8 corresponds to our view of default rules as premiss-like objects and their corresponding usage in the construction of a default proof. Finally, the closedness and d-saturation of a default tableau indicate the impossibility conditions that are needed to establish whether what was asserted not to be the case, that σ is not a default consequence of $\langle \Phi, \Delta \rangle$, is actually the case; altogether establishing whether or not σ is a default consequence of $\langle \Phi, \Delta \rangle$.

The principles underpinning the definition and construction of a default tableau may also be understood in comparison with those intuitions underlying the definition and construction of a tableau for a set of sentences. For instance, classically, every leaf node of a tableau for σ with premisses in Γ may be taken as a partial syntactical description of a (canonical) model of Γ that is also a model of σ; leaf nodes that are closed indicate that this description is an impossibility, whereas leaf nodes that are open and complete indicate the contrary. In a default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$, the extensions of $\langle \Phi, \Delta \rangle$ play the role of models. In this respect, every leaf node of this default tableau may be taken as a partial description of an extension E of $\langle \Phi, \Delta \rangle$ that has been enlarged by incorporating σ into it; leaf nodes that are closed indicate that this enlargement is an impossibility, whereas leaf nodes that are open and complete indicate the contrary; d-saturation indicates that all extensions have been considered.

5 Conclusions and Further Work

In this work we have presented a tableaux based proof calculus for our reformulation of Reiter's original ideas on default reasoning. In summary and by way of conclusion, in formulating a suitable notion of a default proof, we established a proof-theoretical basis for mechanizing a consequence realtion for default reasoning. As a contribution to the proof theory of the latter, the main features of our presentation of a proof calculus for default reasoning are: (i) its simplicity, in that, as commented earlier on, it does not deviate from the standard presentation of a tableaux method; and (ii) the fact that, in certain cases, default proofs may only involve part of a premiss structure (something which is also true when it comes to showing their nonexistence). The advantages of (i) and (ii) are immediate.

Evidently, there is much yet to be done. It is more or less immediate that, in a worst case scenario, the complexity of a default proof inherits the complexity of a tableau proof for classical propositional logic, with the add-on of having to check for the application of all default rules. Definitely, tighter complexity bounds for default proofs are worthy of study. Moreover, insofar as its use is concerned, a machine implementation of the proof calculus that we have presented is a sought after feature. More interestingly, matters related to the development of strategies for systematizing default tableau proofs and properties of default tableau proofs must be investigated. An interesting direction for further research also concerns an exploration of some of the variants of Reiter's original presentation of default reasoning and how well our tableaux based proof calculus adapts to them. We view the latter as a first step towards an an abstract definition of default tableaux proof calculi, i.e., one that is independent of the underlying logical system. Additionally, the current presentation of the default tableau method sets the basis for a systematic construction of a model theory for a given default theory presentation as a fibred class of mathematical structures that happen to be models for theory presentations in the underlying logical language, where fibres are determined by the extensions constructed in each of the branches of the tableau. However, these are just some preliminary thoughts which have to be developed further.

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