Chapter 3
Pullback and Forward Attractors of Nonautonomous Difference Equations

Peter Kloeden and Thomas Lorenz

Abstract In 1998 at the ICDEA Poznan the first author talked about pullback attractors of nonautonomous difference equations. That talk was published as [7] in the *Journal of Difference Equations & Applications* in 2000. Since then the theory of nonautonomous dynamical systems has been the topic of many papers and there are some new developments, in particular concerning the construction of forward nonautonomous attractors, that will be discussed here.

Keywords Nonautonomous diuffernece equation · Pullback attractor · Forward attractor

3.1 Introduction

There are many papers and now several books on the subject, e.g., the Springer Lecture Notes in Mathematics chapter on *Discrete-time nonautonomous dynamical systems* by Kloeden et al. [11] and the Springer Lecture Notes in Mathematics *Geometric theory of discrete nonautonomous dynamical systems* by Pötzsche [14] which deal explicitly with difference equations, while the monograph *Nonautonomous Dynamical Systems* by Kloeden and Rasmussen [12] considers both continuous and discrete nonautonomous dynamical systems. See also [1–4] and the references therein. The basic ideas will be recalled here and illustrated with examples, readers are referred to the literature for proofs and more detail. In particular a method for constructing forward nonautonomous attractors, which was recently presented by Kloeden and Lorenz [8], will be discussed here.

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3.2 Nonautonomous Difference Equations

A nonautonomous difference equations on a complete metric space (X, d_X) has the form

$$x_{n+1} = f_n\,(x_n) \tag{3.1}$$

with mappings $f_n : X \to X$ which may vary with time n. They are assumed to be continuous here. The following examples on $X = \mathbb{R}$ will be considered in the article:

$$x_{n+1} = f_n(x_n) := \frac{1}{2}x_n + g_n, \qquad x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|},$$

where $\{g_n\}_{n\in\mathbb{Z}}$ and $\{\lambda_n\}_{n\in\mathbb{Z}}$ are bounded sequences in $\mathbb{R}$.

Define $\mathbb{Z}^2_{\geq} := \{(n, n_0) \in \mathbb{Z}^2 : n \geq n_0\}$. The nonautonomous difference equation (3.1) generates a *solution mapping*

$$\phi : \mathbb{Z}^2_{\geq} \times X \to X$$

through iteration, i.e., $\phi(n, n_0, x_0) := f_{n-1} \circ \cdots \circ f_{n_0}(x_0)$ for all $n > n_0$ with $n_0 \in \mathbb{Z}$, and each $x_0 \in X$ with the initial value $\phi(n_0, n_0, x_0) := x_0$.

Solution mappings of nonautonomous difference equations (3.1) are one of the main motivations for the process formulation of an abstract nonautonomous dynamical system on a metric state space (X, d_X) and time set $\mathbb{Z}$.

Definition 3.1 *(Dafermos* [5], *Hale* [6]*)* A (discrete-time) process on a state space X is a mapping $\phi : \mathbb{Z}^2_{\geq} \times X \to X$, which satisfies the initial value, 2-parameter evolution and continuity properties:

(i) $\phi(n_0, n_0, x_0) = x_0$ for all $n_0 \in \mathbb{Z}$ and $x_0 \in X$,
(ii) $\phi(n_2, n_0, x_0) = \phi\,(n_2, n_1, \phi(n_1, n_0, x_0))$ for all $n_0 \leq n_1 \leq n_2$ in $\mathbb{Z}$ and $x_0 \in X$,
(iii) the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ of X into itself is continuous for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$.

The general nonautonomous case differs crucially from the autonomous in that the starting time n_0 is just as important as the time that has elapsed since starting, i.e., $n - n_0$. This has some *profound consequences* in terms of definitions and the interpretation of dynamical behaviour. Hence many of the concepts that have been developed and extensively investigated for autonomous dynamical systems in general and autonomous difference equations in particular are either *too restrictive* or *no longer valid* or meaningful (Fig. 3.1).

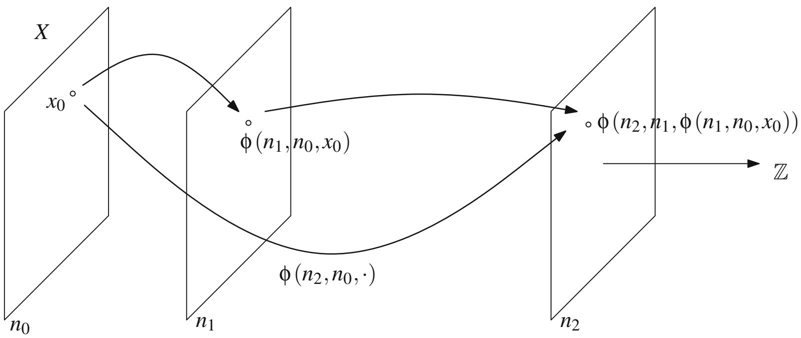


Fig. 3.1 Property (ii) of a discrete-time process ϕ

3.3 Invariant Sets and Attractors of Processes

Invariant sets and attractors are important regions of state space that characterise the long term behaviour of a dynamical system.

It is *too restrictive* to consider a single subset A of X to be invariant under ϕ in the sense that

$$\phi(n, n_0, A) = A, \quad \text{for all}(n, n_0) \in \mathbb{Z}^2_{\geq},$$

which is equivalent to $f_n(A) = A$ for every $n \in \mathbb{Z}$. For example, the trajectory $\{\chi^*_n : n \in \mathbb{Z}\}$ of a solution χ^*, i.e., an *entire solution*, that exists on all of $\mathbb{Z}$ is not invariant in such a sense. Note that an entire solution χ^* satisfies $\phi(n, n_0, \chi^*_{n_0}) = \chi^*_n$ for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$. The family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of *singleton subsets* $A_n := \{\chi^*_n\}$ of X for an entire solution χ^* satisfies

$$\phi\left(n, n_0, A_{n_0}\right) = A_n, \quad \text{for all } (n, n_0) \in \mathbb{Z}^2_{\geq}.$$

This suggests the following generalization of invariance for nonautonomous dynamical systems.

Definition 3.2 A family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty subsets of X is ϕ-invariant if

$$\phi\left(n, n_0, A_{n_0}\right) = A_n, \quad \text{for all } (n, n_0) \in \mathbb{Z}^2_{\geq},$$

or, equivalently, if $f_n(A_n) = A_{n+1}$ for all $n \in \mathbb{Z}$.

A ϕ-invariant family consists of entire solutions, [9].

Proposition 3.1 *A family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ is ϕ-invariant if and only if for every pair $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$ there exists an entire solution χ such that $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$.*

The (forward) convergence

$$d_X\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (n_0 \text{ fixed})$$

in the definition of *Lyapunov asymptotically stable* of an entire solution χ^* of a process ϕ does not provide convergence to a particular point $\chi_{n^*}^*$ for a given $n^* \in \mathbb{Z}$; it involves a *moving target*.

To obtain convergence to a particular point $\chi_{n^*}^*$ one has to *start progressively earlier*, i.e., use *pullback convergence*

$$d_X\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } n_0 \rightarrow -\infty \quad (n \text{ fixed}) \tag{3.2}$$

This contrasts with the usual *forward convergence*

$$d_X\left(\phi\left(n, n_0, x_0\right), \chi_n^*\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (n_0 \text{ fixed}) \tag{3.3}$$

Pullback convergence (3.2) makes use of *information about the past* of the nonautonomous dynamical system, while forward convergence (3.3) uses *information about the future*. Pullback and forward convergence are the same in autonomous systems since they depend only on the elapsed time $n - n_0$, but in nonautonomous dynamical systems they do not necessarily imply each other, as an example to be given later will show (Figs. 3.2 and 3.3).

Fig. 3.2 Forward convergence $n \rightarrow \infty$

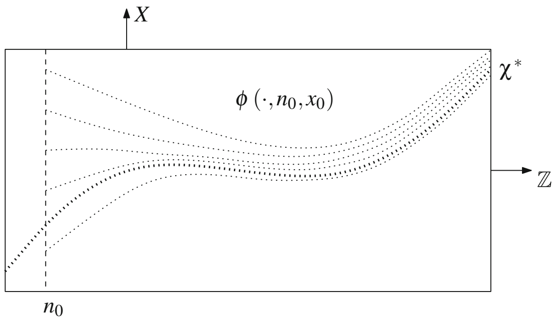
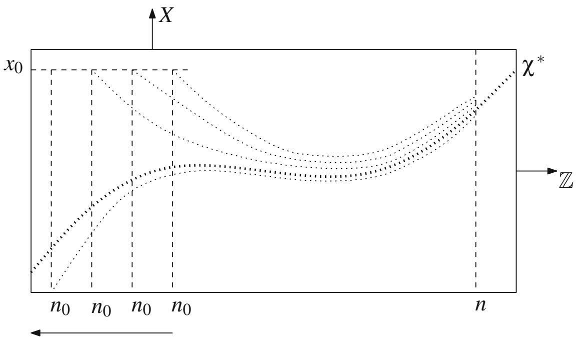


Fig. 3.3 Pullback convergence $n_0 \rightarrow -\infty$



3.4 An Example

The nonautonomous difference equation

$$x_{n+1} = \frac{1}{2}x_n + g_n$$

on $\mathbb{R}$ has the solution mapping (with $n := j + n_0$ here)

$$\phi(j + n_0, n_0, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-j+k} g_{n_0+k}.$$

Pullback convergence (with $n_0 := n - j$) gives

$$\phi(n, n - j, x_0) = 2^{-j}x_0 + \sum_{k=0}^{j} 2^{-k} g_{n-k} \to \sum_{k=0}^{\infty} 2^{-k} g_{n-k} \quad \text{as } j \to \infty,$$

since the infinite series here converges. The limiting entire solution χ^* is given by

$$\chi_n^* := \sum_{k=0}^{\infty} 2^{-k} g_{n-k}$$

for each $n \in \mathbb{Z}$. Since

$$\left|\phi(n, n_0, x_0) - \chi_n^*\right| = \frac{1}{2^n}\left|x_0 - \chi_{n_0}^*\right|$$

forward convergence also holds here.

3.5 Forward and Pullback Attractors

Forward and pullback convergences can be used to define *two distinct types* of nonautonomous attractors for a process ϕ.

Definition 3.3 A ϕ–invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a forward attractor if it forward attracts families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of X, i.e.,

$$\text{dist}_X\left(\phi\left(n, n_0, D_{n_0}\right), A_n\right) \to 0 \quad \text{as } n \to \infty \quad (n_0 \text{ fixed}) \tag{3.4}$$

and a pullback attractor if it pullback attracts families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of $\mathbb{R}^d$, i.e.,

$$\operatorname{dist}_X\left(\phi\left(n, n_0, D_{n_0}\right), A_n\right) \to 0 \quad \text{as } n_0 \to -\infty \quad (n \text{ fixed}) \tag{3.5}$$

Here

$$\operatorname{dist}_X(x, B) := \inf_{b \in B} \operatorname{dist}_X(x, b), \qquad \operatorname{dist}_X(A, B) := \sup_{a \in A} \operatorname{dist}_X(a, B)$$

for nonempty subsets A, B of X.

The existence of a pullback attractor follows from that of a pullback absorbing family in the following generalisation of the theorem for autonomous global attractors. The proof is simpler if the pullback absorbing family is assumed to be ϕ-positive invariant.

Definition 3.4 A family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called pullback absorbing if for every family $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of X and $n \in \mathbb{Z}$ there exists an $N(n, \mathscr{D}) \in \mathbb{N}$ such that

$$\phi\left(n, n_0, D_{n_0}\right) \subseteq B_n, \quad \text{for all } n_0 \leq n - N(n, \mathscr{D}).$$

It is said to be ϕ-positive invariant if $\phi\left(n, n_0, B_{n_0}\right) \subseteq B_n$ for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$.

The assumption about a ϕ-positively invariant pullback absorbing family is *not a serious restriction*, since one can always construct given a general pullback absorbing family.

Theorem 3.1 *Suppose that a process ϕ has a ϕ-positive invariant pullback absorbing family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$.*

Then there exists a global pullback attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ with component sets determined by

$$A_n = \bigcap_{j \geq 0} \phi\left(n, n - j, B_{n-j}\right) \quad \textit{for all } n \in \mathbb{Z}. \tag{3.6}$$

Moreover, if $\mathscr{A}$ is uniformly bounded, then it is unique.

It is often asserted in the literature that there is *no counterpart* of this theorem for nonautonomous forward attractors. Such a result will, in fact, be given later.

3.6 Limitations of Pullback Attractors

Pullback attractors are based on the behaviour of a nonautonomous system in the past and may *not capture* the complete dynamics of a system formulated in terms of a process. This will be illustrated here through some simple examples from [10].

First consider the autonomous scalar difference equation

$$x_{n+1} = \frac{\lambda x_n}{1+|x_n|} \tag{3.7}$$

depending on a real parameter $\lambda > 0$. Its zero solution $x^* = 0$ exhibits a pitchfork bifurcation at $\lambda = 1$. The global dynamical behavior can be summarised as follows:

- If $\lambda \le 1$, then $x^* = 0$ is the only constant solution and is globally asymptotically stable. Thus $\{0\}$ is the global attractor of the autonomous dynamical system generated by the difference equation (3.7).
- If $\lambda > 1$, then there exist two additional nontrivial constant solutions given by $x_\pm := \pm(\lambda - 1)$. The zero solution $x^* = 0$ is now an unstable steady state solution and the symmetric interval $A = [x_-, x_+]$ is the global attractor (Fig. 3.4).

These constant solutions are the *fixed points* of the mapping $f(x) = \frac{\lambda x}{1+|x|}$.

3.6.1 Piecewise Autonomous Difference Equation

Consider now the piecewise autonomous equation

$$x_{n+1} = \frac{\lambda_n x_n}{1+|x_n|}, \qquad \lambda_n := \begin{cases} \lambda, & n \ge 0, \\ \lambda^{-1}, & n < 0 \end{cases} \tag{3.8}$$

for some $\lambda > 1$, which corresponds to a *switch* between the two autonomous problems of the form (3.7) at $n = 0$.

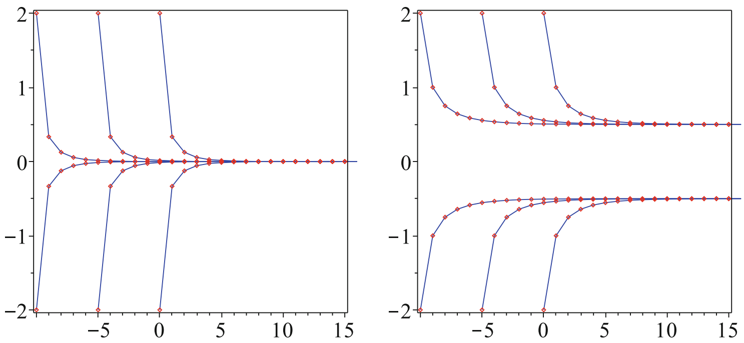


Fig. 3.4 Trajectories of the autonomous difference equation (3.7) with $\lambda = 0.5$ (*left*) and $\lambda = 1.5$ (*right*)

The zero solution of the resulting nonautonomous system is the only bounded entire solution, so the pullback attractor $\mathscr{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$. Note that the zero solution seems to be asymptotically stable for $n < 0$ and then unstable for $n \geq 0$.

The interval $[-(\lambda-1), (\lambda-1)]$ looks like a global attractor for the whole equation on $\mathbb{Z}$, but it is not really one since it is not invariant or minimal for $n < 0$.

The nonautonomous difference equation (3.8) is asymptotically autonomous in both directions, but the pullback attractor does not reflect the full limiting dynamics, in particular in the forward time direction (Fig. 3.5).

3.6.2 *Fully Nonautonomous Equation*

Instead of switching from one constant to another as above, let the parameters λ_n *increase monotonically* to $\bar{\lambda} > 1$. Then the nonautonomous problem

$$x_{n+1} = f_n(x_n) := \frac{\lambda_n x_n}{1 + |x_n|} \tag{3.9}$$

is *asymptotically autonomous* in both directions with the limiting autonomous systems given above, but is never equal to them.

Its pullback attractor $\mathscr{A}$ has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution, which is the only bounded entire solution. As above, the zero solution $x^* = 0$ seems to be asymptotically stable for $n < 0$ and then unstable for $n \geq 0$. However, the forward limit points for nonzero solutions are $\pm(\bar{\lambda} - 1)$, neither of which is a solution at all. In particular, they are not entire solutions of the process.

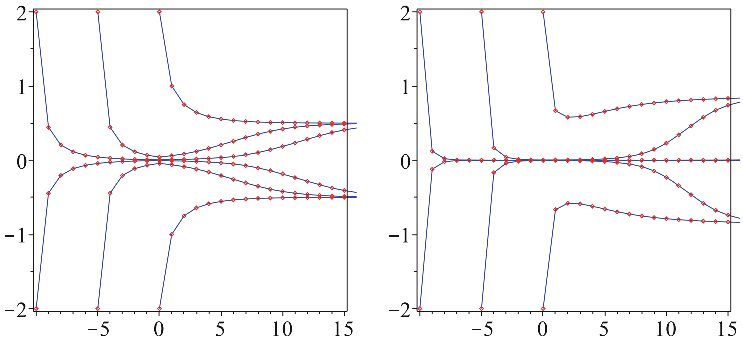


Fig. 3.5 Trajectories of the piecewise autonomous equation (3.8) with $\lambda = 1.5$ (*left*) and the asymptotically autonomous equation (3.9) with $\lambda_k = 1 + \frac{0.9k}{1+|k|}$ (*right*)

Remark 3.1 Pullback attraction alone *does not characterise* fully the bounded limiting behaviour of a nonautonomous system formulated as a process, in particular what happens in the future.

Note that the pullback attractor in the above examples is not a forward attractor.

3.7 Construction of Nonautonomous Forward Attractors

Recall that a ϕ-invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X is called a *forward attractor* of a process ϕ if it forward attracts all families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of nonempty bounded subsets of X, i.e.,

$$\lim_{n\to\infty} \mathrm{dist}_X \left(\phi(n, n_0, D_{n_0}), A_n\right) = 0, \qquad (\text{fixed } n_0) \tag{3.10}$$

The following important property of forward attractors holds. It is proved in [8] under the assumption that the state space X is locally compact. If not, the components sets B_n are just closed and bounded, in which case Theorem 3.2 still holds provided the process also satisfies a compactness or asymptotic compactness property.

Proposition 3.2 *A uniformly bounded forward attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ has a ϕ-positively invariant family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets, which is forward absorbing.*

Uniformly bounded means that $A_n \subset B$ for all $n \in \mathbb{Z}$ for some compact subset B of X.

The situation is somewhat more complicated for forward attractors than for pullback attractors due to some peculiarities of forward attractors [15], e.g., they need not be unique. For each $r \geq 0$ the process generated by

$$x_{n+1} = f_n(x_n) := \begin{cases} x_n, & n \leq 0, \\ \frac{1}{2}x_n, & n > 0 \end{cases} \tag{3.11}$$

has a forward attractor $\mathscr{A}^{(r)}$ with component subsets

$$A_n^{(r)} = \begin{cases} r[-1, 1], & n \leq 0, \\ \frac{1}{2^n} r[-1, 1], & n > 0 \end{cases} \tag{3.12}$$

These forward attractors are not pullback attractors.

The following theorem is a key observation for the construction of a forward attractor.

Theorem 3.2 *Suppose that a process ϕ on a X has a ϕ-positively invariant family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X.*

Then ϕ has a maximal ϕ-invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ in $\mathscr{B}$ of nonempty compact subsets determined by

$$A_n = \bigcap_{n_0 \le n} \phi\left(n, n_0, B_{n_0}\right) \quad \textit{for each} \quad n \in \mathbb{Z}. \tag{3.13}$$

In view of Proposition 3.2, the components sets of *any* uniformly bounded forward attractor can be constructed in this way. Note that nothing is assumed here about the dynamics *outside* of the family $\mathscr{B}$.

3.8 Conditions Ensuring Forward Convergence

The ϕ-invariant family $\mathscr{A} = \{A_n, n \in \mathbb{Z}\}$ constructed in Theorem 3.2 need *not* be a forward attractor, even when the ϕ-positively invariant family $\mathscr{B}$ is a forward absorbing family, e.g., consider any example of a pullback attractor that is not a forward attractor such as for the difference equations (3.8) and (3.9).

Another important observation, if somewhat obvious, is that here should be no ω-limit points from inside the family $\mathscr{B}$ that are not ω-limit points from inside the family $\mathscr{A}$. For each $n_0 \in \mathbb{Z}$, the forward ω-limit set with respect to $\mathscr{B}$ [13] is defined by

$$\omega_{\mathscr{B}}(n_0) := \bigcap_{m \ge n_0} \overline{\bigcup_{n \ge m} \phi(n, n_{0,n_0})},$$

The set $\omega_{\mathscr{B}}(n_0)$ is nonempty and compact as the intersection of nonempty nested compact subsets and

$$\lim_{n\to\infty} \mathrm{dist}_X \left(\phi(n, n_0, B_{n_0}), \omega_{\mathscr{B}}(n_0)\right) = 0, \quad (\text{fixed } n_0).$$

Since $A_{n_0} \subset B_{n_0}$ and $A_n = \phi(n, n_0, A_{n_0}) \subset \phi(n, n_0, B_{n_0})$

$$\lim_{n\to\infty} \mathrm{dist}_X \left(A_n, \omega_{\mathscr{B}}(n_0)\right) = 0, \quad (\text{fixed } n_0) \tag{3.14}$$

and since $\phi(n, n_0, B_{n_0}) \subset B_n$ for each $n \ge n_0$

$$\omega_{\mathscr{B}}(n_0) \subset \omega_{\mathscr{B}}(n_0') \subset B, \qquad n_0 \le n_0',$$

where the final inclusion is from the uniform boundedness of $\mathscr{B}$. Hence the set

$$\omega_{\mathscr{B}}^{\infty} := \overline{\bigcup_{n_0 \in \mathbb{Z}} \omega_{\mathscr{B}}(n_0)}$$

is nonempty and compact. From (3.14) it is clear that

$$\lim_{n \to \infty} \operatorname{dist}_X \left(A_n, \omega^\infty_{\mathscr{B}} \right) = 0. \tag{3.15}$$

The ω-limit points for dynamics starting inside the family of sets $\mathscr{A}$ are defined by

$$\omega^\infty_{\mathscr{A}} := \bigcap_{n_0 \in \mathbb{Z}} \overline{\bigcup_{n \geq n_0} A_n} = \bigcap_{n_0 \in \mathbb{Z}} \overline{\bigcup_{n \geq n_0} \phi(n, n_0, A_{n_0})} \subset B,$$

which is nonempty and compact as a family of nested compact sets. Obviously, $\omega^\infty_{\mathscr{A}} \subset \omega^\infty_{\mathscr{B}} \subset B$. The above examples show that inclusions here may be strict.

The following results are proved in [8].

Theorem 3.3 *$\mathscr{A}$ is forward attracting from within $\mathscr{B}$ if and only if $\omega^\infty_{\mathscr{A}} = \omega^\infty_{\mathscr{B}}$.*

Theorem 3.4 *The family $\mathscr{A}$ is forward attracting from within $\mathscr{B}$ if the rate of pullback convergence from within $\mathscr{B}$ to the components sets A_n of $\mathscr{A}$ is eventually uniform, i.e., for every $\varepsilon > 0$ there exist $\tau(\varepsilon) \in \mathbb{Z}$ and $N(\varepsilon) > 0$ such that for each $n \geq \tau(\varepsilon)$*

$$\operatorname{dist}_X \left(\phi(n, n_0, B_{n_0}), A_n \right) < \varepsilon \tag{3.16}$$

holds for all $n_0 \leq n - N(\varepsilon)$.

Then $\mathscr{A}$ will be a forward attractor if $\mathscr{B}$ is forward absorbing. The forward attractors of the difference equation (3.11) do not satisfy this property.

3.9 Final Remarks

Autonomous systems involve only the elapsed time $n - n_0$, so their attractors and limit sets exist in true time too. For nonautonomous systems, the pullback limit defines the component set A_n at each instant of actual time. On the other hand, the forward limit defining a forward attractor is different as it is the limit to the asymptotically distant future. Forward limiting objects do not have a similar dynamical meaning in actual time as in the autonomous or pullback cases. Nevertheless, when it exists, a forward attractor provides useful information about the dynamics as one approaches the distant future.

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