Chapter 2
Metrics on Modular Spaces

Abstract In this chapter, we address the metrizability of modular spaces.

2.1 Modular Spaces

A pseudomodular w on X (cf. Fig. 1.2 on p. 5) induces an equivalence relation $\sim$ on X as follows: given $x, y \in X$,

$$x \sim y \quad \text{iff} \quad w^{x,y} \not\equiv \infty \quad \text{iff} \quad w_\lambda(x, y) < \infty \ \text{ for some } \ \lambda > 0,$$

where $\lambda = \lambda(x, y)$, possibly, depends on x and y. A modular space is any equivalence class with respect to $\sim$. More explicitly, let us fix an element $x^\circ \in X$. The set

$$X_w^* \equiv X_w^*(x^\circ) = \{x \in X : \exists\, \lambda = \lambda(x) > 0 \ \text{ such that } \ w_\lambda(x, x^\circ) < \infty\}$$

is called a *modular space* (around x°), and x° is called the *center* of X_w^* (x° is a representative of the equivalence class X_w^*). Note that $w^{x,y} \not\equiv \infty$ for all $x, y \in X_w^*$.

If w_{+0} and w_{-0} are the right and left regularizations of w, then (1.2.4) imply $X_{w_{+0}}^* = X_{w_{-0}}^* = X_w^*$.

Two more *modular spaces* (around x°) can be defined making use of other equivalence relations on X:

$$X_w^0 \equiv X_w^0(x^\circ) = \{x \in X : w_\lambda(x, x^\circ) \to 0 \ \text{ as } \ \lambda \to \infty\}$$

and

$$X_w^{\text{fin}} \equiv X_w^{\text{fin}}(x^\circ) = \{x \in X : w_\lambda(x, x^\circ) < \infty \ \text{ for all } \ \lambda > 0\}.$$

As above, $X_{w_{+0}}^0 = X_{w_{-0}}^0 = X_w^0$ and $X_{w_{+0}}^{\text{fin}} = X_{w_{-0}}^{\text{fin}} = X_w^{\text{fin}}$.

Clearly, $X_w^0 \subset X_w^*$ and $X_w^{\text{fin}} \subset X_w^*$ (with proper inclusions in general). However, if w is *convex*, then $X_w^0 = X_w^*$ (see Proposition 1.2.3(c)); moreover, note that this property is independent of the center x°, i.e., $X_w^0(x^\circ) = X_w^*(x^\circ)$ for all $x^\circ \in X$.

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Example 2.1.1. The inclusion relations between the three modular spaces are illustrated by the modular $w_\lambda(x,y) = g(\lambda)d(x,y)$ on a metric space (X,d) from (1.3.1):

$$X_w^* = \begin{cases} \{x^\circ\} & \text{if } g \equiv \infty, \\ X & \text{if } g \not\equiv \infty, \end{cases} \qquad X_w^0 = \begin{cases} \{x^\circ\} & \text{if } \lim_{\lambda\to\infty} g(\lambda) \neq 0, \\ X & \text{if } \lim_{\lambda\to\infty} g(\lambda) = 0, \end{cases}$$

and

$$X_w^{\mathrm{fin}} = \begin{cases} \{x^\circ\} & \text{if } g(\lambda) = \infty \text{ for some } \lambda > 0, \\ X & \text{if } g(\lambda) < \infty \text{ for all } \lambda > 0. \end{cases}$$

In particular, for modulars $\mathsf{w}_\lambda(x,y) = d(x,y)$ (nonconvex) and $w_\lambda(x,y) = d(x,y)/\lambda$ (convex) from Example 1.3.2(a), we have

$$X_{\mathsf{w}}^0 = \{x^\circ\} \subset X_{\mathsf{w}}^* = X_{\mathsf{w}}^{\mathrm{fin}} = X = X_w^0 = X_w^* = X_w^{\mathrm{fin}}.$$

In the sequel, by the *modular space* we mean the set X_w^* (the largest among the three) if not explicitly stated otherwise.

2.2 The Basic Metric

We begin by introducing the *basic (pseudo)metric* d_w^0 on the modular space X_w^*.

Theorem 2.2.1. *Let w be a (pseudo)modular on X. Set*

$$d_w^0(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \le \lambda\}, \qquad x,y \in X \quad (\inf \varnothing = \infty).$$

Then d_w^0 is an extended (pseudo)metric on X. Furthermore, if $x,y \in X$, $d_w^0(x,y) < \infty$ is equivalent to $x \sim y$, and so, d_w^0 is a (pseudo)metric on $X_w^* = X_w^*(x^\circ)$ (for any $x^\circ \in X$).

Proof. 1. Clearly, $d_w^0(x,y) \in [0,\infty]$, $d_w^0(x,x) = 0$, and $d_w^0(x,y) = d_w^0(y,x)$ for all $x,y \in X$. Now, suppose w is a modular on X, and $x,y \in X$ are such that $d_w^0(x,y) = 0$. The definition of d_w^0 implies $w_\mu(x,y) \le \mu$ for all $\mu > 0$. So, for all $\lambda > 0$ and $0 < \mu < \lambda$, we have from (1.2.1): $w_\lambda(x,y) \le w_\mu(x,y) \le \mu \to 0$ as $\mu \to +0$. Thus $w_\lambda(x,y) = 0$ for all $\lambda > 0$, and so, by axiom (i), $x = y$.

In order to prove the triangle inequality $d_w^0(x,y) \le d_w^0(x,z) + d_w^0(z,y)$ for all $x,y,z \in X$, we assume that $d_w^0(x,z)$ and $d_w^0(z,y)$ are finite (otherwise, the inequality is obvious). By the definition of d_w^0, given $\lambda > d_w^0(x,z)$ and $\mu > d_w^0(z,y)$, we find $w_\lambda(x,z) \le \lambda$ and $w_\mu(z,y) \le \mu$, and so, axiom (iii) implies

$$w_{\lambda+\mu}(x,y) \le w_\lambda(x,z) + w_\mu(z,y) \le \lambda + \mu.$$

It follows that $d_w^0(x,y) \le \lambda + \mu$, and it remains to take into account the arbitrariness of λ and μ as above.

2. If $d_w^0(x,y) < \infty$, then, for any $\lambda > d_w^0(x,y)$, we have $w_\lambda(x,y) \le \lambda < \infty$, which means that $x \sim y$. Conversely, suppose $x \sim y$, i.e., $w_\mu(x,y) < \infty$ for some $\mu > 0$. We set $\lambda = \max\{\mu, w_\mu(x,y)\}$. Since $\lambda \ge \mu$, the monotonicity (1.2.1) of w implies $w_\lambda(x,y) \le w_\mu(x,y) \le \lambda$, and so, $d_w^0(x,y) \le \lambda < \infty$.
3. Given $x, y \in X_w^*$, we have $x \sim y$, and so, $d_w^0(x,y) < \infty$. By step 1, this means that d_w^0 is a (pseudo)metric on X_w^*. □

The pair (X_w^*, d_w^0), being a (pseudo)metric space generated by the (pseudo)modular w, is called a *(pseudo)metric modular space*, and we will apply this terminology if we are interested in metric properties of X_w^* with respect to d_w^0 (or some other metric induced by w). We call X_w^* the *modular space* if the main concern is its modular properties (Sects. 4.2 and 4.3), which are outside the scope of metric properties.

Example 2.2.2. Suppose $w_\lambda(x,y) = g(\lambda)d(x,y)$ is the modular from (1.3.1), where $g : (0,\infty) \to [0,\infty]$ is a nonincreasing function, $g \not\equiv 0$, and $g \not\equiv \infty$. In the examples 1–6 below, we have $X_w^* = X$, and $x, y \in X$ and $\lambda_0 > 0$ are given.

1. If $g(\lambda) = 1/\lambda^p$ ($p \ge 0$), then $d_w^0(x,y) = (d(x,y))^{1/(p+1)}$.
2. Let $g(\lambda) = 1$ if $0 < \lambda < \lambda_0$, and $g(\lambda) = 0$ if $\lambda \ge \lambda_0$. Then w is nonstrict and nonconvex, and $d_w^0(x,y) = \min\{\lambda_0, d(x,y)\}$.
3. If $g(\lambda) = 1/\lambda$ for $0 < \lambda < \lambda_0$, and $g(\lambda) = 0$ for $\lambda \ge \lambda_0$, then w is nonstrict and convex, and $d_w^0(x,y) = \min\{\lambda_0, \sqrt{d(x,y)}\}$.
4. For $g(\lambda) = \max\{1, 1/\lambda\}$, we have: w is strict and nonconvex, and d_w^0 is given by $d_w^0(x,y) = \max\{d(x,y), \sqrt{d(x,y)}\}$.
5. If $g(\lambda) = \infty$ for $0 < \lambda < \lambda_0$, and $g(\lambda) = 0$ for $\lambda \ge \lambda_0$, then w is nonstrict and convex, and $d_w^0(x,y) = \lambda_0\delta(x,y)$, where δ is the discrete metric on X.
6. Putting $d = \delta$, for any function g as above, we have $d_w^0(x,y) = g^0\delta(x,y)$ with $g^0 = \inf\{\lambda > 0 : g(\lambda) \le \lambda\}$.

Remark 2.2.3. 1. If ρ is a classical modular on a real linear space X (cf. Sect. 1.3.3), the set $X_\rho = \{x \in X : \lim_{\alpha\to+0} \rho(\alpha x) = 0\}$ is called the *modular space* (with zero as its center). The modular space X_ρ is a linear subspace of X, and the functional $|\cdot|_\rho : X_\rho \to [0,\infty)$, given by $|x|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \le \varepsilon\}$, is an *F-norm* on X_ρ, i.e., given $x, y \in X_\rho$, it satisfies the conditions: (F.1) $|x|_\rho = 0$ iff $x = 0$; (F.2) $|-x|_\rho = |x|_\rho$; (F.3) $|x+y|_\rho \le |x|_\rho + |y|_\rho$; and (F.4) $|c_n x_n - cx|_\rho \to 0$ as $n \to \infty$ whenever $c_n \to c$ in $\mathbb{R}$ and $|x_n - x|_\rho \to 0$ as $n \to \infty$ (where $x_n \in X_\rho$ for $n \in \mathbb{N}$). The modular space X_w^0, which is a counterpart of X_ρ, does not play that significant role in our theory as X_ρ does in the classical theory of modulars (see also Remark 2.4.3(3)).

2. Under the assumptions of Proposition 1.3.5, where X is a real linear space and $\rho(x) = w_1(x,0)$, we also have: $X_\rho = X_w^0(0)$ is a linear subspace of X, and the functional $|x|_\rho = d_w^0(x,0)$, $x \in X_\rho$, is an F-norm on X_ρ.

In Theorem 2.2.1 (and Example 2.2.2(6)), we have encountered the quantity

$$g^0 = \inf\{\lambda > 0 : g(\lambda) \le \lambda\}, \tag{2.2.1}$$

evaluated at the nonincreasing function $g = w^{x,y} : (0,\infty) \to [0,\infty]$, which we denoted by $d_w^0(x,y) = (w^{x,y})^0$. This quantity is worth a more detailed study.

Lemma 2.2.4. *If $g : (0,\infty) \to [0,\infty]$ is a nonincreasing function, then $g^0 \in [0,\infty]$, and*

(a) $g^0 = \inf_{\lambda>0} \max\{\lambda, g(\lambda)\}$ (*where* $\max\{\lambda,\infty\} = \infty$ *for* $\lambda > 0$)*;*
(b) $g^0 < \infty$ *if and only if* $g \not\equiv \infty$ (*so,* $g^0 = \infty \Leftrightarrow g \equiv \infty$)*;*
(c) $g^0 \ne 0$ *if and only if* $g \not\equiv 0$ (*so,* $g^0 = 0 \Leftrightarrow g \equiv 0$).

Proof. 1. Let us prove inequality ($\le$) in (a) and implication ($\Leftarrow$) in (b). We may assume $g \not\equiv \infty$ (otherwise, (a) reads $\inf\varnothing = \infty$ and holds trivially). For each $\lambda > 0$ such that $g(\lambda) < \infty$, we set $\lambda_1 = \max\{\lambda, g(\lambda)\}$. Then $\lambda_1 \in (0,\infty)$, $g(\lambda) \le \lambda_1$, and since $\lambda \le \lambda_1$ and g is nonincreasing, $g(\lambda_1) \le g(\lambda)$. So, $g(\lambda_1) \le \lambda_1$. It follows that $g^0 \le \lambda_1 = \max\{\lambda, g(\lambda)\}$. This proves (b)($\Leftarrow$). Taking the infimum over all $\lambda > 0$ such that $g(\lambda) < \infty$ (or over all $\lambda > 0$), we establish the inequality $g^0 \le \dots$ in (a).

2. Let us prove inequality ($\ge$) in (a) and implication ($\Rightarrow$) in (b). Suppose g^0 is finite. Given $\lambda_1 > g^0$, we have $g(\lambda_1) \le \lambda_1$, and so, $g \not\equiv \infty$. This establishes (b)($\Rightarrow$). Moreover (note that the monotonicity of g is not used),

$$\inf_{\lambda>0} \max\{\lambda, g(\lambda)\} \le \inf_{\lambda>0:\, g(\lambda)<\infty} \max\{\lambda, g(\lambda)\} \le \max\{\lambda_1, g(\lambda_1)\} = \lambda_1.$$

Passing to the limit as $\lambda_1 \to g^0$, we obtain the inequality $g^0 \ge \dots$ in (a).

3. (c)($\Rightarrow$) If $g \equiv 0$, then $g^0 = \inf(0,\infty) = 0$ (equivalently, if $g^0 \ne 0$, then $g \not\equiv 0$).

(c)($\Leftarrow$) Let $g^0 = 0$. Then $g(\mu) \le \mu$ for all $\mu > 0$. Given $\lambda > 0$, for any $0 < \mu < \lambda$, by virtue of the monotonicity of g, we get $0 \le g(\lambda) \le g(\mu) \le \mu$. Letting $\mu \to +0$, we find $g(\lambda) = 0$ for all $\lambda > 0$, i.e., $g \equiv 0$. In other words, we have shown that $g \not\equiv 0$ implies $g^0 \ne 0$. $\square$

Remark 2.2.5. It is seen from the proof of Lemma 2.2.4(a) that

$$g^0 = \inf\bigl\{\max\{\lambda, g(\lambda)\} : \lambda > 0 \text{ such that } g(\lambda) < \infty\bigr\} \in [0,\infty) \quad \text{if} \quad g \not\equiv \infty.$$

Following the same lines as in the proof of Lemma 2.2.4, it may be shown that $g^0 = \sup\{\lambda > 0 : g(\lambda) \ge \lambda\}$ ($\sup\varnothing = 0$) and $g^0 = \sup_{\lambda>0} \min\{\lambda, g(\lambda)\}$.

As a consequence of Theorem 2.2.1 and Lemma 2.2.4, we get the following

Corollary 2.2.6. $d_w^0(x,y) = \inf\limits_{\lambda>0} \max\{\lambda, w_\lambda(x,y)\}$*,* $x,y \in X$.

Given a nonincreasing function $g : (0,\infty) \to [0,\infty]$, we denote by g_{+0} and g_{-0} the right and left regularizations of g, defined (as in (1.2.2) and (1.2.3)) by:

$g_{+0}(\lambda) = g(\lambda + 0)$ and $g_{-0}(\lambda) = g(\lambda - 0)$ for all $\lambda > 0$. Functions g_{+0} and g_{-0} map $(0, \infty)$ into $[0, \infty]$ and are nonincreasing on $(0, \infty)$. Furthermore, g_{+0} is continuous from the right and g_{-0} is continuous from the left on $(0, \infty)$, and inequalities similar to (1.2.4) hold:

$$g(\lambda) \leq g(\lambda - 0) \leq g(\mu + 0) \leq g(\mu) \text{ in } [0, \infty] \text{ for all } 0 < \mu < \lambda. \tag{2.2.2}$$

Taking the above and (2.2.1) into account, we have

Lemma 2.2.7. *If $g : (0, \infty) \to [0, \infty]$ is nonincreasing, then $(g_{+0})^0 = g^0 = (g_{-0})^0$.*

Proof. Inequalities $(g_{+0})^0 \leq g^0 \leq (g_{-0})^0$ are consequences of the inclusions

$$\{\lambda > 0 : g(\lambda - 0) \leq \lambda\} \subset \{\lambda > 0 : g(\lambda) \leq \lambda\} \subset \{\lambda > 0 : g(\lambda + 0) \leq \lambda\},$$

which follow from (2.2.2). Now, we may assume that $g \not\equiv \infty$. Then $g_{+0} \not\equiv \infty$ and $g_{-0} \not\equiv \infty$, which ensures that g^0, $(g_{+0})^0$, and $(g_{-0})^0$ are finite.

Let us show that $g^0 \leq (g_{+0})^0$. Given $\lambda > (g_{+0})^0$, choose μ such that $(g_{+0})^0 < \mu < \lambda$. By (2.2.2) and definition of $(g_{+0})^0$, we get

$$g(\lambda) \leq g(\mu + 0) = g_{+0}(\mu) \leq \mu < \lambda.$$

Hence $g^0 \leq \lambda$. Since $\lambda > (g_{+0})^0$ is arbitrary, we find $g^0 \leq (g_{+0})^0$.

In order to show that $(g_{-0})^0 \leq g^0$, we let $\lambda > g^0$. Then, for any $\mu > 0$ such that $g^0 < \mu < \lambda$, inequalities (2.2.2) and definition of g^0 imply

$$(g_{-0})(\lambda) = g(\lambda - 0) \leq g(\mu) \leq \mu < \lambda.$$

Therefore $(g_{-0})^0 \leq \lambda$. Letting $\lambda \to g^0$, we get $(g_{-0})^0 \leq g^0$. □

Putting, for a (pseudo)modular w on X, $g = w^{x,y}$ in Lemma 2.2.7 and noting that $g_{\pm 0} = (w_{\pm 0})^{x,y}$ and $d^0_{w_{\pm 0}}(x, y) = (g_{\pm 0})^0$, we have

Corollary 2.2.8. $d^0_{w_{+0}}(x, y) = d^0_{w_{-0}}(x, y) = d^0_w(x, y)$ *for all $x, y \in X$.*

In particular, if w and w are (pseudo)modulars on X such that $w_{+0} = \mathsf{w}_{+0}$ or $w_{-0} = \mathsf{w}_{-0}$, then $d^0_w = d^0_{\mathsf{w}}$ on $X \times X$.

We conclude that the right and left regularizations of a (pseudo)modular w on X provide no new modular spaces as compared to X^*_w, X^0_w and X^{fin}_w (cf. Sect. 2.1) and no new (pseudo)metrics as compared to d^0_w.

Yet, in Sect. 2.5, we establish the existence of continuum many (equivalent) metrics on the modular space X^*_w.

This section is continued by studying the basic metric $d^0_w(x, y)$ at the level of the map $g \mapsto g^0$, applied later to nonincreasing functions $g = w^{x,y}$. Our next lemma clarifies the definition of g^0 and Lemma 2.2.7 and, along with (2.2.1), gives a method for evaluating g^0 in terms of solutions of certain inequalities.

Lemma 2.2.9 (inequalities for g^0). *Let $g : (0,\infty) \to [0,\infty]$ be a nonincreasing function with $0 < g^0 < \infty$ (i.e., $g \not\equiv 0$ and $g \not\equiv \infty$), and $\lambda > 0$. We have:*

(a) $g^0 < \lambda$ *if and only if* $g(\lambda - 0) < \lambda$;
(b) $g^0 > \lambda$ *if and only if* $g(\lambda + 0) > \lambda$;
(c) $g^0 = \lambda$ *if and only if* $g(\lambda + 0) \le \lambda \le g(\lambda - 0)$.

Proof. (a)($\Rightarrow$) Suppose $g^0 < \lambda$. Given λ_1 and λ_2 such that $g^0 < \lambda_1 < \lambda_2 < \lambda$, by the monotonicity of g, $g(\lambda_2) \le g(\lambda_1)$, and the definition of g^0 implies $g(\lambda_1) \le \lambda_1$. Hence $g(\lambda_2) \le \lambda_1$. Passing to the limits as $\lambda_1 \to g^0$ and $\lambda_2 \to \lambda$, we get $g(\lambda - 0) \le g^0$, where $g^0 < \lambda$, and so, $g(\lambda - 0) < \lambda$.

(a)($\Leftarrow$) By the assumption, $g(\lambda - 0) < \lambda$, where $g(\lambda - 0) = \lim_{\mu\to\lambda-0} g(\mu)$ and $\lambda = \lim_{\mu\to\lambda-0} \mu$. So, there exists μ_0 with $0 < \mu_0 < \lambda$ such that $g(\mu) < \mu$ for all μ with $\mu_0 \le \mu < \lambda$. By the definition of g^0, we find $g^0 \le \mu$, which implies $g^0 < \lambda$.

(b)($\Rightarrow$) Let $g^0 > \lambda$. For any λ_1 and λ_2 such that $g^0 > \lambda_2 > \lambda_1 > \lambda$, we have $g(\lambda_1) \ge g(\lambda_2) > \lambda_2$, where the last inequality follows from the definition of g^0: if, on the contrary, $g(\lambda_2) \le \lambda_2$, then $g^0 \le \lambda_2$, which contradicts the inequality $g^0 > \lambda_2$. Therefore $g(\lambda_1) > \lambda_2$. Letting $\lambda_2 \to g^0$ and $\lambda_1 \to \lambda$, we find $g(\lambda + 0) \ge g^0 > \lambda$.

(b)($\Leftarrow$) Since $\lim_{\mu\to\lambda+0} g(\mu) = g(\lambda + 0) > \lambda = \lim_{\mu\to\lambda+0} \mu$, there exists $\mu_0 > \lambda$ such that $g(\mu) > \mu$ for all μ with $\lambda < \mu \le \mu_0$. It follows that $g^0 \ge \mu$ (otherwise, if $g^0 < \mu$, then the definition of g^0 implies $g(\mu) \le \mu$, which is a contradiction). Since $\mu > \lambda$, we get $g^0 > \lambda$.

(c) The statement in (a) is equivalent to the following:

$$g^0 \ge \lambda \ \text{ if and only if } \ g(\lambda - 0) \ge \lambda, \tag{2.2.3}$$

and the one in (b) is equivalent to the assertion:

$$g^0 \le \lambda \ \text{ if and only if } \ g(\lambda + 0) \le \lambda. \tag{2.2.4}$$

From these two observations, (c) follows. □

Remark 2.2.10. (a) Actually, a little bit more is shown in the proof of Lemma 2.2.9: $g^0 < \lambda \Rightarrow g(\lambda - 0) \le g^0 < \lambda$ in (a), and $g^0 > \lambda \Rightarrow g(\lambda + 0) \ge g^0 > \lambda$ in (b).

(b) We have $g^0 = \inf\{\lambda > 0 : g(\lambda) < \lambda\} \equiv g^{0\prime}$ (cf. (2.2.1) and Lemma 2.2.4). In fact, this is clear if $g \equiv 0$ or $g \equiv \infty$, so let $0 < g^0 < \infty$. Since $\{\lambda > 0 : g(\lambda) < \lambda\} \subset \{\lambda > 0 : g(\lambda) \le \lambda\}$, we get $g^0 \le g^{0\prime}$. Now, given $\lambda > g^0$, inequalities (2.2.2) and Lemma 2.2.9(a) imply $g(\lambda) \le g(\lambda - 0) < \lambda$, and so, $g^{0\prime} \le \lambda$, which yields $g^{0\prime} \le g^0$.

(c) Assuming one-sided continuity of g on $(0,\infty)$, in view of (2.2.4) and (2.2.3), we get some useful particular cases of Lemma 2.2.9:

$g^0 \le \lambda \Leftrightarrow g(\lambda) \le \lambda$, provided g is continuous from the right;
$g^0 < \lambda \Leftrightarrow g(\lambda) < \lambda$, provided g is continuous from the left;
$g^0 = \lambda \Leftrightarrow g(\lambda) = \lambda$ (i.e., λ is a fixed point of g), provided g is continuous.

(d) To illustrate Lemma 2.2.9, consider $g : (0,\infty) \to (0,\infty)$ defined by: $g(\lambda) = 3$ if $0 < \lambda < 1$, $g(\lambda) = 2$ if $\lambda = 1$, and $g(\lambda) = 0$ if $\lambda > 1$. Clearly, g is nonincreasing and $g^0 = \inf(1,\infty) = 1$. Inequalities in Lemma 2.2.9(c) are of the form:

$$g(1+0) = 0 < g^0 = 1 < 3 = g(1-0).$$

Although strict inequality $g(1-0) = 3 > 1 = \lambda$ holds in (2.2.3), we have $g^0 = \lambda = 1$. Similarly, $g(1+0) = 0 < 1 = \lambda$ in (2.2.4) and $g^0 = 1 = \lambda$.

Setting $g = w^{x,y}$ in Lemma 2.2.9 (for $x, y \in X_w^*$), we obtain the following important result for modulars w on X (cf. also Remark 2.2.10(a), (c)).

Theorem 2.2.11. *Let w be a (pseudo)modular on the set X, X_w^* be the modular space, $\lambda > 0$, and $x, y \in X_w^*$. Then we have:*

(a) *condition $d_w^0(x,y) < \lambda$ implies $w_{\lambda-0}(x,y) \le d_w^0(x,y) < \lambda$, and conversely, condition $w_{\lambda-0}(x,y) < \lambda$ implies $d_w^0(x,y) < \lambda$;*
(b) *inequality $d_w^0(x,y) > \lambda$ implies $w_{\lambda+0}(x,y) \ge d_w^0(x,y) > \lambda$, and conversely, inequality $w_{\lambda+0}(x,y) > \lambda$ implies $d_w^0(x,y) > \lambda$;*
(c) *equality $d_w^0(x,y) = \lambda$ is equivalent to $w_{\lambda+0}(x,y) \le \lambda \le w_{\lambda-0}(x,y)$.*

Under the continuity assumptions on w, additional equivalences hold:

(d) *if w is continuous from the right, then $d_w^0(x,y) \le \lambda \Leftrightarrow w_\lambda(x,y) \le \lambda$;*
(e) *if w is continuous from the left, then $d_w^0(x,y) < \lambda \Leftrightarrow w_\lambda(x,y) < \lambda$;*
(f) *if w is continuous on $(0,\infty)$, then $d_w^0(x,y) = \lambda \Leftrightarrow w_\lambda(x,y) = \lambda$.*

The conclusions of Theorem 2.2.11 are sharp (cf. Remark 2.2.10(d) and (1.3.1)).

Example 2.2.12. Let w be given by (1.3.2) with $h(\lambda) = \lambda^p$ ($p > 0$). Since w is continuous on $(0,\infty)$, by virtue of Theorem 2.2.11(f), the value $\lambda = d_w^0(x,y)$ with $x \ne y$ satisfies the equation $w_\lambda(x,y) = \lambda$, that is,

$$\lambda^{p+1} + d(x,y)\lambda - d(x,y) = 0. \tag{2.2.5}$$

If $p = 1$, then solving the corresponding quadratic equation, we get

$$d_w^0(x,y) = \frac{\sqrt{(d(x,y))^2 + 4d(x,y)} - d(x,y)}{2}. \tag{2.2.6}$$

For $p = 2$, the solution λ of the corresponding cubic equation (2.2.5) is given by Cardano's formula:

$$d_w^0(x,y) = \sqrt[3]{\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} - \sqrt[3]{-\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}, \tag{2.2.7}$$

where $a = d(x,y)$, and the square and cube roots of positive numbers have uniquely determined positive values. The solution by radicals of the fourth-order equation (for $p = 3$) can be obtained by Ferrari's method, and is left to the interested reader.

Note that, for any function h from (1.3.2), we have $d_w^0(x,y) < 1$.

In fact, if h is continuous on $(0,\infty)$, equality $w_\lambda(x,y) = \lambda$ is of the form $f(\lambda) = 0$, where $f(\lambda) = \lambda h(\lambda) - (1-\lambda)d(x,y)$, and $\lambda h(\lambda) \to 0$ as $\lambda \to +0$. Setting $\lambda h(\lambda) = 0$ if $\lambda = 0$, we find that f is continuous on $[0,\infty)$, $f(0) = -d(x,y) < 0$ (if $x \neq y$), and $f(1) = h(1) > 0$. By the Intermediate Value Theorem, $f(\lambda) = 0$ for some $0 < \lambda < 1$, and so, $d_w^0(x,y) = \lambda < 1$.

In the general case, we first show that if there exists $\mu > 0$ such that

$$w_{\lambda-0}(x,y) < \mu \text{ for all } \lambda > 0 \text{ and } x,y \in X, \text{ then } d_w^0(x,y) < \mu \text{ for all } x,y \in X.$$

Since $w_\lambda(x,y) \leq w_{\lambda-0}(x,y) < \mu$, and this holds for $\lambda = \mu$, we find $d_w^0(x,y) \leq \mu$. If we assume that $d_w^0(x,y) = \mu$ (for some $x \neq y$), then, by Theorem 2.2.11(b), we have $w_\lambda(x,y) \geq w_{\lambda+0}(x,y) > \lambda$ for all $0 < \lambda < d_w^0(x,y) = \mu$, and so, $w_{\mu-0}(x,y)$ is equal to $\lim_{\lambda\to\mu-0} w_\lambda(x,y) \geq \mu$, which contradicts the assumption. It remains to note that $w_{\lambda-0}(x,y) < 1 = \mu$ for our modular w from (1.3.2).

One more example of a (pseudo)metric from Theorem 2.2.1 is given by the quantity d_W^0 on the power set $\mathscr{P}(X)$ of X, where W is the Hausdorff pseudomodular on $\mathscr{P}(X)$ induced by a (pseudo)modular w on X. There are two ways of obtaining a distance function on $\mathscr{P}(X)$ starting from w on X, namely

$$w \text{ on } X \xrightarrow{\text{Theorem 2.2.1}} d_w^0 \text{ on } X \xrightarrow{\text{Appendix A.1}} D_{d_w^0} \text{ on } \mathscr{P}(X)$$

and

$$w \text{ on } X \xrightarrow{\text{Section 1.3.5}} W \text{ on } \mathscr{P}(X) \xrightarrow{\text{Theorem 2.2.1}} d_W^0 \text{ on } \mathscr{P}(X).$$

Fortunately, the resulting distance functions $D_{d_w^0}$ and d_W^0 coincide on $\mathscr{P}(X)$ as the following theorem asserts.

Theorem 2.2.13. *Let w be a (pseudo)modular on X, $D = D_{d_w^0}$ be the Hausdorff distance on $\mathscr{P}(X)$ generated by the extended (pseudo)metric d_w^0 on X, and W be the Hausdorff pseudomodular on $\mathscr{P}(X)$ induced by w. Then*

$$d_W^0(A,B) = D(A,B) \quad \textit{for all } A,B \in \mathscr{P}(X).$$

Proof. Since $d_W^0(\varnothing,\varnothing) = 0 = D(\varnothing,\varnothing)$, and $d_W^0(A,\varnothing) = \infty = D(A,\varnothing)$ for all $A \neq \varnothing$, we may assume that $A \neq \varnothing$ and $B \neq \varnothing$.

($\geq$) Suppose $d_W^0(A,B) = \inf\{\lambda > 0 : W_\lambda(A,B) \leq \lambda\}$ is finite, and $\lambda > d_W^0(A,B)$. Applying (1.2.4) and Theorem 2.2.11(a) (cf. also Remark 2.2.10(b)), we get

$$W_\lambda(A,B) = \max\{E_\lambda(A,B), E_\lambda(B,A)\} < \lambda,$$

and so, $E_\lambda(A,B) < \lambda$ and $E_\lambda(B,A) < \lambda$. By (1.3.12), we have $\inf_{y\in B} w_\lambda(x,y) < \lambda$ for all $x \in A$. So, for each $x \in A$ there exists $y_x \in B$ (depending also on λ) such that $w_\lambda(x,y_x) < \lambda$. The definition of d_w^0 gives $d_w^0(x,y_x) \le \lambda$. Since

$$\inf_{y\in B} d_w^0(x,y) \le d_w^0(x,y_x) \le \lambda \quad \text{for all } x \in A,$$

we get $e(A,B) = \sup_{x\in A} \inf_{y\in B} d_w^0(x,y) \le \lambda$. Similarly, $E_\lambda(B,A) < \lambda$ implies inequality $e(B,A) \le \lambda$. Therefore $D(A,B) = \max\{e(A,B), e(B,A)\} \le \lambda$ for all $\lambda > d_W^0(A,B)$, and so, $D(A,B) \le d_W^0(A,B) < \infty$.

($\le$) Let $D(A,B) < \infty$, and $\lambda > D(A,B)$ be arbitrary. Then $\lambda > e(A,B)$ as well as $\lambda > e(B,A)$. Inequality $\lambda > e(A,B) = \sup_{x\in A} \inf_{y\in B} d_w^0(x,y)$ implies that, given $x \in A$, $\lambda > \inf_{y\in B} d_w^0(x,y)$. So, for every $x \in A$ there exists $y_x \in B$ (also depending on λ) such that $\lambda > d_w^0(x,y_x)$. By the definition of d_w^0, we have $w_\lambda(x,y_x) \le \lambda$. Since

$$\inf_{y\in B} w_\lambda(x,y) \le w_\lambda(x,y_x) \le \lambda \quad \text{for all } x \in A,$$

we find $E_\lambda(A,B) = \sup_{x\in A} \inf_{y\in B} w_\lambda(x,y) \le \lambda$. Similarly, inequality $\lambda > e(B,A)$ implies $E_\lambda(B,A) \le \lambda$. Hence $W_\lambda(A,B) = \max\{E_\lambda(A,B), E_\lambda(B,A)\} \le \lambda$. The definition of d_W^0 yields $d_W^0(A,B) \le \lambda$ for all $\lambda > D(A,B)$, and so, $d_W^0(A,B) \le D(A,B) < \infty$. □

2.3 The Basic Metric in the Convex Case

Now we treat the case when a (pseudo)modular w on X is *convex*: w gives rise to an additional (pseudo)metric on the modular space X_w^* to be studied below.

We make use of the following observation. As we have seen in Remark 1.2.2(d), the convexity of a (pseudo)modular w on X is equivalent to the fact that the function $\hat{w}_\lambda(x,y) = \lambda w_\lambda(x,y)$ is a (pseudo)modular on X. On the other hand, if a function $\hat{w}$ on $(0,\infty) \times X \times X$ is initially given, then we have: $\hat{w}$ is a (pseudo)modular on X if and only if $w_\lambda(x,y) = \hat{w}_\lambda(x,y)/\lambda$ is a convex (pseudo)modular on X.

From Sect. 2.1, we find

$$X_{\hat{w}}^0 \subset X_w^0 = X_w^* = X_{\hat{w}}^* \quad \text{and} \quad X_{\hat{w}}^{\text{fin}} = X_w^{\text{fin}} \subset X_w^* = X_{\hat{w}}^*. \tag{2.3.1}$$

By Theorem 2.2.1, $\hat{w}$ generates a (pseudo)metric on X_w^* of the form

$$d_{\hat{w}}^0(x,y) = \inf\{\lambda > 0 : \hat{w}_\lambda(x,y) \le \lambda\} = \inf\{\lambda > 0 : w_\lambda(x,y) \le 1\}. \tag{2.3.2}$$

The last expression is given in terms of w and is denoted by $d_w^*(x,y)$.

Properties of d_w^* are gathered in the following theorem, where Theorem 2.2.1 and Corollary 2.2.6 are applied to $\hat{w}_\lambda(x,y) = \lambda w_\lambda(x,y)$ and expressed via w.

Theorem 2.3.1. *Let w be a convex (pseudo)modular on X. Then*

$$d_w^*(x,y) \equiv \inf\{\lambda > 0 : w_\lambda(x,y) \le 1\} = \inf_{\lambda>0} \max\{\lambda, \lambda w_\lambda(x,y)\}, \qquad x,y \in X, \tag{2.3.3}$$

is an extended (pseudo)metric on X (with $d_w^*(x,y) < \infty \Leftrightarrow x \sim y$), whose restriction to the modular space X_w^* is a (pseudo)metric on X_w^*.

Furthermore, d_w^0 and d_w^* are nonlinearly equivalent *in the following sense: given $x,y \in X_w^*$, we have*

$$\min\bigl\{d_w^*(x,y), \sqrt{d_w^*(x,y)}\bigr\} \le d_w^0(x,y) \le \max\bigl\{d_w^*(x,y), \sqrt{d_w^*(x,y)}\bigr\}, \tag{2.3.4}$$

or, equivalently (written in a different way),

$$d_w^0(x,y) \cdot \min\bigl\{1, d_w^0(x,y)\bigr\} \le d_w^*(x,y) \le d_w^0(x,y) \cdot \max\bigl\{1, d_w^0(x,y)\bigr\}. \tag{2.3.5}$$

Only the second part of Theorem 2.3.1 is to be verified. For this, we need some precise inequalities for $d_w^* = d_{\hat{w}}^0$, which are reformulated from Theorem 2.2.11 (applied to $\hat{w}$) in terms of w and stated, for ease of reference, as

Theorem 2.3.2. *Let w be a convex (pseudo)modular on X, $\lambda > 0$, and $x,y \in X_w^*$. Then we have:*

(a) $d_w^*(x,y) < \lambda$ *implies* $w_{\lambda-0}(x,y) \le d_w^*(x,y)/\lambda < 1$, *and conversely,* $w_{\lambda-0}(x,y) < 1$ *implies* $d_w^*(x,y) < \lambda$;
(b) $d_w^*(x,y) > \lambda$ *implies* $w_{\lambda+0}(x,y) \ge d_w^*(x,y)/\lambda > 1$, *and conversely,* $w_{\lambda+0}(x,y) > 1$ *implies* $d_w^*(x,y) > \lambda$;
(c) $d_w^*(x,y) = \lambda$ *is equivalent to* $w_{\lambda+0}(x,y) \le 1 \le w_{\lambda-0}(x,y)$.

In addition, under the continuity assumptions on w, we get:

(d) $d_w^*(x,y) \le \lambda \Leftrightarrow w_\lambda(x,y) \le 1$, *provided w is continuous from the right;*
(e) $d_w^*(x,y) < \lambda \Leftrightarrow w_\lambda(x,y) < 1$, *provided w is continuous from the left;*
(f) $d_w^*(x,y) = \lambda \Leftrightarrow w_\lambda(x,y) = 1$, *provided w is continuous on* $(0,\infty)$.

Proof (of Theorem 2.3.1 (second part)). In steps 1 and 2, we show that inequalities $d_w^0(x,y) < 1$ and $d_w^*(x,y) < 1$ are equivalent, and if one of them holds, then

$$d_w^*(x,y) \le d_w^0(x,y) \le \sqrt{d_w^*(x,y)}. \tag{2.3.6}$$

Since $d_w^*(x,y) < 1$ implies $d_w^*(x,y) \le \sqrt{d_w^*(x,y)}$, inequality (2.3.6) proves (2.3.4).

1. Suppose $d_w^0(x,y) < 1$. Let us show that $d_w^*(x,y) \le d_w^0(x,y)$ (and so, $d_w^*(x,y) < 1$). In fact, for any number λ such that $d_w^0(x,y) < \lambda < 1$, the definition of d_w^0 gives

$w_\lambda(x,y) \le \lambda < 1$, whence, by the definition of d_w^*, $d_w^*(x,y) \le \lambda$. Passing to the limit as $\lambda \to d_w^0(x,y)$, we obtain the left-hand side inequality in (2.3.6).

2. Assume that $d_w^*(x,y) < 1$. Let us prove that $d_w^0(x,y) \le \sqrt{d_w^*(x,y)}$, which is the right-hand side inequality in (2.3.6) (and so, $d_w^0(x,y) < 1$). Since $d_w^*(x,y) \le \sqrt{d_w^*(x,y)} < 1$, for any λ such that $\sqrt{d_w^*(x,y)} < \lambda < 1$, inequalities (1.2.4) and, by virtue of convexity of w, Theorem 2.3.2(a) imply

$$w_\lambda(x,y) \le w_{\lambda-0}(x,y) \le \frac{d_w^*(x,y)}{\lambda} < \frac{\lambda^2}{\lambda} = \lambda.$$

By the definition of d_w^0, $d_w^0(x,y) \le \lambda$. Letting λ tend to $\sqrt{d_w^*(x,y)}$, we obtain the desired inequality.

As a consequence of steps 1 and 2, inequalities $d_w^0(x,y) \ge 1$ and $d_w^*(x,y) \ge 1$ are equivalent, as well. In steps 3 and 4, we show that if one of these inequalities holds, then

$$\sqrt{d_w^*(x,y)} \le d_w^0(x,y) \le d_w^*(x,y). \tag{2.3.7}$$

Since $d_w^*(x,y) \ge 1$ implies $d_w^*(x,y) \ge \sqrt{d_w^*(x,y)}$, (2.3.7) establishes (2.3.4).

3. Inequality $d_w^*(x,y) \ge 1$ implies $d_w^0(x,y) \le d_w^*(x,y)$: in fact, by the definition of d_w^*, $w_\lambda(x,y) \le 1$ for all $\lambda > d_w^*(x,y)$, and since $\lambda > 1$, $w_\lambda(x,y) < \lambda$. From the definition of d_w^0, we get $d_w^0(x,y) \le \lambda$. The assertion follows thanks to the arbitrariness of $\lambda > d_w^*(x,y)$.
4. Suppose $d_w^0(x,y) \ge 1$, and let us show that $\sqrt{d_w^*(x,y)} \le d_w^0(x,y)$, which is the left-hand side inequality in (2.3.7). Given $\lambda > d_w^0(x,y)$, we have $w_\lambda(x,y) \le \lambda$, and since $\lambda > 1$, $\lambda^2 > \lambda$. The convexity of w and (1.2.5) imply

$$w_{\lambda^2}(x,y) \le \frac{\lambda}{\lambda^2} w_\lambda(x,y) \le \frac{\lambda}{\lambda^2} \cdot \lambda = 1,$$

whence $d_w^*(x,y) \le \lambda^2$. Letting λ go to $d_w^0(x,y)$, we get $d_w^*(x,y) \le (d_w^0(x,y))^2$.

□

Remark 2.3.3. 1. If w is nonconvex, the quantity $d_w^*(x,y) \in [0,\infty]$ from (2.3.3) has only two properties: $d_w^*(x,x) = 0$, and $d_w^*(x,y) = d_w^*(y,x)$. It follows from (2) in this Remark that $d_w^*(x,y) = 0 \not\Rightarrow x = y$, and from (4)—that the triangle inequality may not hold for d_w^*.

2. The convexity of w is essential for inequalities (2.3.4) and (2.3.5): modular (1.3.2) is nonconvex, and d_w^0 is a well-defined metric on X (e.g., (2.2.6) and (2.2.7)), but, since $w_\lambda(x,y) < 1$ for all $\lambda > 0$, we have $d_w^*(x,y) = 0$ for all $x, y \in X$ (and, in particular, d_w^* is not a metric on X).
3. In the proof of Theorem 2.3.1, the implications in steps 1 and 3, which are of the form $d_w^0(x,y) < 1 \Rightarrow d_w^*(x,y) \le d_w^0(x,y)$, and $d_w^*(x,y) \ge 1 \Rightarrow d_w^0(x,y) \le d_w^*(x,y)$, do not rely on the convexity of w and are valid for those (pseudo)modulars w, for which the quantity $d_w^*(x,y)$ is well-defined. The example in (2) above is consistent with the former implication.

4. For the modular $w_\lambda(x,y) = d(x,y)/\lambda^p$ $(p > 0)$ from Example 2.2.2(1), we have $d_w^0(x,y) = (d(x,y))^{1/(p+1)}$ and $d_w^*(x,y) = (d(x,y))^{1/p}$, where we note that d_w^* is a metric on X if and only if w is convex, i.e., $p \geq 1$. So, for $p \geq 1$, setting $a = d(x,y)$, inequalities (2.3.6) and (2.3.7) assume the form:

$$a^{\frac{1}{p}} \leq a^{\frac{1}{p+1}} \leq a^{\frac{1}{2p}} \text{ if } 0 \leq a < 1, \text{ and } \quad a^{\frac{1}{2p}} \leq a^{\frac{1}{p+1}} \leq a^{\frac{1}{p}} \text{ if } a \geq 1.$$

5. Inequalities (2.3.4) are the best possible: see Example 2.3.5(1).

Remark 2.3.4. 1. If ρ is a classical convex modular on a real linear space X (cf. Sect. 1.3.3 and Remark 2.2.3), then the modular space X_ρ coincides with the set $X_\rho^* = \{x \in X : \rho(\alpha x) < \infty \text{ for some } \alpha > 0\}$, and the functional $\|x\|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\}$ $(x \in X_\rho^*)$ is a norm on $X_\rho = X_\rho^*$, which is *nonlinearly equivalent* to the F-norm $|x|_\rho$ in the same sense as in Theorem 2.3.1. Moreover, under the assumptions of Proposition 1.3.5, where X is a linear space and $\rho(x) = w_1(x,0)$, we have: $X_\rho^* = X_w^*(0) = X_\rho$ is a linear subspace of X, and the functional $\|x\|_\rho = d_w^*(x,0)$, $x \in X_\rho^*$, is a norm on X_ρ^*.
2. Similar to Corollary 2.2.8, if w is convex, then $d_{w+0}^* = d_{w-0}^* = d_w^*$ on $X \times X$. In fact, $(w^\wedge)_\lambda(x,y) \equiv \hat{w}_\lambda(x,y) = \lambda w_\lambda(x,y)$ is also a (pseudo)modular on X, and $(w_{\pm 0})^\wedge = (\hat{w})_{\pm 0} \equiv (w^\wedge)_{\pm 0}$, which can be seen as follows. Given $\lambda > 0$ and $x, y \in X$, (1.2.2) and (1.2.3) imply

$$\begin{aligned}\big((w_{\pm 0})^\wedge\big)_\lambda(x,y) &= \lambda(w_{\pm 0})_\lambda(x,y) = \lambda w_{\lambda\pm 0}(x,y) = \lim_{\mu\to\lambda\pm 0} \mu w_\mu(x,y)\\ &= \lim_{\mu\to\lambda\pm 0} (w^\wedge)_\mu(x,y) = (w^\wedge)_{\lambda\pm 0}(x,y) = \big((w^\wedge)_{\pm 0}\big)_\lambda(x,y).\end{aligned}$$

By virtue of (2.3.3) and (2.3.2), $d_w^* = d_{w^\wedge}^0$, and Corollary 2.2.8 yields

$$d_{w_{\pm 0}}^* = d_{(w_{\pm 0})^\wedge}^0 = d_{(w^\wedge)_{\pm 0}}^0 = d_{w^\wedge}^0 = d_w^*.$$

Example 2.3.5. Consider the modular $w_\lambda(x,y) = \varphi(d(x,y)/\lambda)$ from (1.3.5), where the function $\varphi : [0,\infty) \to [0,\infty]$ is nondecreasing and such that $\varphi(0) = 0$, $\varphi \not\equiv 0$, and $\varphi \not\equiv \infty$, (X,d) is a metric space, $x, y \in X_w^* = X$, and $\lambda > 0$.

1. Let $\varphi(u) = u^p$ $(p > 0)$. Then w is strict, convex if $p \geq 1$, and nonconvex if $0 < p < 1$. For any $p > 0$, we have

$$d_w^0(x,y) = (d(x,y))^{p/(p+1)} \quad \text{and} \quad d_w^*(x,y) = d(x,y).$$

To show that inequalities (2.3.4) are the best possible, we note that if $p = 1$, then $d_w^0(x,y) = \sqrt{d_w^*(x,y)}$, and if $p > 1$, then (w is convex and) we find

$$d_w^0(x,y) = (d_w^*(x,y))^{p/(p+1)} \to d_w^*(x,y) \quad \text{as} \quad p \to \infty.$$

2. Let w be the $(a, 0)$-modular from (1.3.9). If $a = \infty$, then w is nonstrict and convex, and we have: $d_w^0(x, y) = d_w^*(x, y) = d(x, y)$. Now, if $a > 0$, then w is nonstrict and nonconvex, and we have: $d_w^0(x, y) = \min\{a, d(x, y)\}$, $d_w^*(x, y) = 0$ if $a \le 1$, and $d_w^*(x, y) = d(x, y)$ if $a > 1$.
3. If $\varphi(u) = u$ for $0 \le u \le 1$, and $\varphi(u) = 1$ for $u > 1$, then the modular

$$w_\lambda(x, y) = 1 \text{ if } 0 < \lambda < d(x, y), \text{ and } w_\lambda(x, y) = \frac{d(x, y)}{\lambda} \text{ if } \lambda \ge d(x, y),$$

is strict and nonconvex, and $d_w^0(x, y) = \min\{1, \sqrt{d(x, y)}\}$.
4. Let $\varphi(u) = 0$ for $0 \le u \le 1$, and $\varphi(u) = u - 1$ for $u > 1$. We have:

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda} - 1 \text{ if } 0 < \lambda < d(x, y), \text{ and } w_\lambda(x, y) = 0 \text{ if } \lambda \ge d(x, y),$$

is nonstrict and convex, and (note that $d_w^0(x, y) < d(x, y)$ if $x \ne y$)

$$d_w^0(x, y) = \frac{\sqrt{1 + 4d(x, y)} - 1}{2} \quad \text{and} \quad d_w^*(x, y) = \frac{d(x, y)}{2}.$$

5. Suppose $\varphi(0) = 0$, $\varphi(u) = 1$ if $0 < u < 1$, and $\varphi(u) = u$ if $u \ge 1$. Given $\lambda > 0$ and $x, y \in X$, we have: $w_\lambda(x, y) = 0$ if $x = y$, and if $x \ne y$,

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda} \text{ if } 0 < \lambda \le d(x, y), \text{ and } w_\lambda(x, y) = 1 \text{ if } \lambda > d(x, y).$$

Then the modular w is strict and nonconvex, $d_w^0(x, y) = \max\{1, \sqrt{d(x, y)}\}$ if $x \ne y$, and $d_w^0(x, y) = 0$ if $x = y$.
6. Suppose φ is given by: $\varphi(u) = u$ if $0 \le u \le 1$, $\varphi(u) = 1$ if $1 < u < 2$, and $\varphi(u) = u - 1$ if $u \ge 2$. The corresponding modular w is strict and nonconvex, and we have: $d_w^0(x, y) = \sqrt{d(x, y)}$ if $d(x, y) \le 1$, $d_w^0(x, y) = 1$ if $1 < d(x, y) < 2$, and $d_w^0(x, y) = \frac{1}{2}(\sqrt{1 + 4d(x, y)} - 1)$ if $d(x, y) \ge 2$.

2.4 Modulars and Metrics on Sequence Spaces

Let (M, d) be a metric space, $X = M^{\mathbb{N}}$—the set of all sequences $x = \{x_n\}$ from M, and $x^\circ = \{x_n^\circ\} \subset M$—a given sequence (the center of a modular space). In this section, we study two special modulars defined on X.

1. The modular w from (1.3.10) with $\varphi(u) = u^p$ $(p > 0)$ and $h(\lambda) = \lambda^q$ $(q \ge 1)$ is strict and continuous, and it is convex if $p \ge 1$. The modular spaces (around x°) are given by

$$X_w^* = X_w^0 = X_w^{\text{fin}} = \left\{ x = \{x_n\} \in X : \sum_{n=1}^{\infty} (d(x_n, x_n^\circ))^p < \infty \right\}$$

(if $M = \mathbb{R}$ with metric $d(x,y) = |x-y|$ and $x^\circ = 0 = \{0\}_{n=1}^\infty$, then $X_w^*(0)$ is the usual space ℓ_p of all real p-summable sequences).

Let $H(\lambda) = \lambda(h(\lambda))^p = \lambda^{pq+1}$. The metric d_w^0 on X_w^* is of the form:

$$d_w^0(x,y) = H^{-1}\left(\sum_{n=1}^{\infty} (d(x_n, y_n))^p\right) = \left(\sum_{n=1}^{\infty} (d(x_n, y_n))^p\right)^{1/(pq+1)},$$

where $H^{-1}(\mu) = \mu^{1/(pq+1)}$ is the inverse function of H on $[0, \infty)$.

If $p \geq 1$, then w is convex, and we also have metric d_w^* on X_w^* of the form:

$$d_w^*(x,y) = h^{-1}\left(\left[\sum_{n=1}^{\infty} (d(x_n, y_n))^p\right]^{1/p}\right) = \left(\sum_{n=1}^{\infty} (d(x_n, y_n))^p\right)^{1/pq},$$

where $h^{-1} : [0, \infty) \to [0, \infty)$ is the inverse function of h (see Example 1.3.10, and Appendix A.1 concerning general superadditive functions h).

2. Given $\lambda > 0$ and $x = \{x_n\}$, $y = \{y_n\} \in X = M^{\mathbb{N}}$, we set

$$w_\lambda(x,y) = \sup_{n \in \mathbb{N}} \left(\frac{d(x_n, y_n)}{\lambda}\right)^{1/n}. \tag{2.4.1}$$

Proposition 2.4.1. *$w = \{w_\lambda\}_{\lambda > 0}$ is a strict nonconvex continuous modular on X.*

Proof. Axioms (i), (i$_s$), and (ii) are clear, and axiom (iii) follows from inequalities (1.3.11) with $\varphi(u) = u^{1/n}$ and $h(\lambda) = \lambda$.

In order to see that w is nonconvex, we show that $X_w^0(x^\circ) \neq X_w^*(x^\circ)$ for some $x^\circ \in X$ (cf. Sect. 2.1). Choose any $x^\circ \in M$ and $x \in M$, $x \neq x^\circ$, and let $x^\circ = \{x^\circ\}_{n=1}^\infty$ and $x = \{x\}_{n=1}^\infty$ also denote the corresponding constant sequences from X. Given $\lambda > d(x, x^\circ) > 0$, we find

$$w_\lambda(x, x^\circ) = \sup_{n \in \mathbb{N}} \left(\frac{d(x, x^\circ)}{\lambda}\right)^{1/n} = \lim_{n \to \infty} \left(\frac{d(x, x^\circ)}{\lambda}\right)^{1/n} = 1,$$

and so, $x \in X_w^*(x^\circ) \setminus X_w^0(x^\circ)$.

Let us show that $w_\lambda(x,y) \leq w_{\lambda+0}(x,y)$ and $w_{\lambda-0}(x,y) \leq w_\lambda(x,y)$ for all $\lambda > 0$ and $x, y \in X$, which, by virtue of inequalities (1.2.4), establish the continuity property of w. For any $n \in \mathbb{N}$ and $\mu > \lambda$, the definition of w implies

$$\left(\frac{d(x_n, y_n)}{\mu}\right)^{1/n} \leq w_\mu(x,y),$$

and so, as $\mu \to \lambda + 0$, we get

$$\left(\frac{d(x_n, y_n)}{\lambda}\right)^{1/n} \le w_{\lambda+0}(x, y).$$

Taking the supremum over all $n \in \mathbb{N}$, we obtain the first inequality above. Now, given $\lambda, \mu > 0$, we have

$$\begin{aligned} w_\mu(x, y) &= \sup_{n\in\mathbb{N}} \left(\frac{d(x_n, y_n)}{\lambda}\right)^{1/n} \cdot \left(\frac{\lambda}{\mu}\right)^{1/n} \le w_\lambda(x, y) \cdot \sup_{n\in\mathbb{N}} \big(\lambda/\mu\big)^{1/n} \\ &= w_\lambda(x, y) \cdot \max\big\{1, \lambda/\mu\big\}, \qquad x, y \in X. \end{aligned} \tag{2.4.2}$$

It follows that if $0 < \mu < \lambda$, then $w_\mu(x, y) \le w_\lambda(x, y) \cdot \lambda/\mu$, and so, passing to the limit as $\mu \to \lambda - 0$, we get $w_{\lambda-0}(x, y) \le w_\lambda(x, y)$. □

Note that (2.4.2) with $y = x^\circ$ proves that $X_w^{\text{fin}}(x^\circ) = X_w^*(x^\circ)$, and establishes the following characterization of this modular space in terms of sequences $x = \{x_n\}$ and $x^\circ = \{x_n^\circ\}$ themselves:

$$x \in X_w^*(x^\circ) \quad \text{if and only if} \quad w_1(x, x^\circ) = \sup_{n\in\mathbb{N}} \big(d(x_n, x_n^\circ)\big)^{1/n} < \infty. \tag{2.4.3}$$

The modular space $X_w^0(x^\circ)$ is characterized in the following way.

Proposition 2.4.2. *Given $x \in X$, $x \in X_w^0(x^\circ)$ if and only if* $\lim_{n\to\infty} \big(d(x_n, x_n^\circ)\big)^{1/n} = 0.$

Proof. Suppose $x \in X_w^0(x^\circ)$. Then $w_\lambda(x, x^\circ) \to 0$ as $\lambda \to \infty$, and so, for each $\varepsilon > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that

$$w_{\lambda_0}(x, x^\circ) = \sup_{n\in\mathbb{N}} \left(\frac{d(x_n, x_n^\circ)}{\lambda_0}\right)^{1/n} \le \varepsilon. \tag{2.4.4}$$

This inequality is equivalent to

$$\big(d(x_n, x_n^\circ)\big)^{1/n} \le (\lambda_0)^{1/n} \cdot \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.4.5}$$

Passing to the limit superior as $n \to \infty$, we get

$$\limsup_{n\to\infty} \big(d(x_n, x_n^\circ)\big)^{1/n} \le \varepsilon.$$

Due to the arbitrariness of $\varepsilon > 0$, $(d(x_n, x_n^\circ))^{1/n} \to 0$ as $n \to \infty$.

Now, assume that $(d(x_n, x_n^\circ))^{1/n} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $(d(x_n, x_n^\circ))^{1/n} < \varepsilon$ for all $n > n_0$. Setting

$$\lambda_1(\varepsilon) = \max\{1, 1/\varepsilon^{n_0}\} \cdot \max_{1 \le n \le n_0} d(x_n, x_n^\circ)$$

and noting that

$$d(x_n, x_n^\circ) = \frac{d(x_n, x_n^\circ)}{\varepsilon^n} \cdot \varepsilon^n \le \lambda_1(\varepsilon) \cdot \varepsilon^n \quad \text{for all} \quad 1 \le n \le n_0,$$

we obtain (2.4.5) with $\lambda_0 = \lambda_0(\varepsilon) = \max\{1, \lambda_1(\varepsilon)\}$. It follows that inequality (2.4.4) holds, whence, by virtue of (1.2.1), $w_\lambda(x, x^\circ) \le w_{\lambda_0}(x, x^\circ) \le \varepsilon$ for all $\lambda \ge \lambda_0$. This means that $w_\lambda(x, x^\circ) \to 0$ as $\lambda \to \infty$, i.e., $x \in X_w^0(x^\circ)$. □

The metric d_w^0 on the modular space $X_w^*(x^\circ)$ is given by

$$d_w^0(x, y) = \sup_{n \in \mathbb{N}} \big(d(x_n, y_n)\big)^{1/(n+1)}, \qquad x, y \in X_w^*(x^\circ). \tag{2.4.6}$$

Recalling that w is nonconvex, we note that $d_w^*(x, y) = \sup_{n \in \mathbb{N}} d(x_n, y_n)$ is only an extended metric on $X_w^*(x^\circ)$ and X (however, d_w^* is a metric on the set of all *bounded* sequences in M; see Remark 2.4.3 below).

Writing $x = \{x_n\} \in c(x^\circ)$ if $\lim_{n\to\infty} d(x_n, x_n^\circ) = 0$, and $x = \{x_n\} \in \ell_\infty(x^\circ)$ if $\sup_{n \in \mathbb{N}} d(x_n, x_n^\circ) < \infty$, we have the following (proper) inclusion relations:

$$X_w^0(x^\circ) \subset c(x^\circ) \subset \ell_\infty(x^\circ) \subset X_w^{\mathrm{fin}}(x^\circ) = X_w^*(x^\circ). \tag{2.4.7}$$

(Here $c(x^\circ)$ is the set of all sequences in M, which are *metrically equivalent* to $x^\circ = \{x_n^\circ\}$, and $\ell_\infty(x^\circ)$ is the set of all sequences in M, which are *bounded relative to* x°.) The first inclusion is a consequence of Proposition 2.4.2, and the third one is established as follows: if $b = \sup_{n \in \mathbb{N}} d(x_n, x_n^\circ) < \infty$, then, for all $\lambda > 0$, we have:

$$w_\lambda(x, x^\circ) = \sup_{n \in \mathbb{N}} \left(\frac{d(x_n, x_n^\circ)}{\lambda}\right)^{1/n} \le \sup_{n \in \mathbb{N}} \left(\frac{b}{\lambda}\right)^{1/n} = \max\{1, b/\lambda\} < \infty.$$

Remark 2.4.3. 1. If $x^\circ = \{x_n^\circ\}$ is a convergent sequence in M, then every sequence $x = \{x_n\} \in c(x^\circ)$ is also convergent in M (to the limit of x°), and if x° is *bounded* in M (i.e., $\sup_{n,m \in \mathbb{N}} d(x_n^\circ, x_m^\circ) < \infty$), then every $x \in \ell_\infty(x^\circ)$ is also bounded in M.

2. In the particular case when $M = \mathbb{R}$ with metric $d(x, y) = |x - y|$ and $x^\circ = 0$ is the zero sequence, we have: $c_0 = c(0)$ is the set of all real sequences convergent to zero, and $\ell_\infty = \ell_\infty(0)$ is the set of all bounded real sequences. The following examples are illustrative (see (2.4.7)): (a) $\{1/n\} \in c_0 \setminus X_w^0(0)$; (b) $\{2^n\} \in X_w^*(0) \setminus \ell_\infty$; (c) $\{2^{-n^2}\} \in X_w^0(0)$; (d) $\{2^{n^2}\} \notin X_w^*(0)$; (e) if $x = \{n\}$, then $x \in X_w^*(0)$, $d_w^0(x, 0) = \sup_{n \in \mathbb{N}} n^{1/(n+1)} < \infty$, while $d_w^*(x, 0) = \sup_{n \in \mathbb{N}} n = \infty$.

3. The classical F-norm $|x|_\rho = d_w^0(x,0) = \sup_{n\in\mathbb{N}} |x_n|^{1/(n+1)}$, corresponding to $\rho(x) = w_1(x,0)$ with w from (2.4.1) and $M = \mathbb{R}$, is well-defined for $x = \{x_n\}$ from $X_\rho = X_w^0(0) \subset c_0$ and satisfies conditions (F.1)–(F.4) from Remark 2.2.3. However, on the larger modular space $X_\rho^* = X_w^*(0)$ (see Remark 2.3.4(1)), the functional $|\cdot|_\rho$ does not satisfy the *continuity* condition (F.4): for instance, if $x = \{2^{n+1}\}_{n=1}^\infty$ and $\alpha_k = 1/k$, then $x \in X_\rho^* \setminus X_\rho$ and $\alpha_k \to 0$ as $k \to \infty$, but

$$|\alpha_k x|_\rho = \sup_{n\in\mathbb{N}} \big(\alpha_k \cdot 2^{n+1}\big)^{1/(n+1)} = 2 \sup_{n\in\mathbb{N}} \Big(\frac{1}{k}\Big)^{1/(n+1)} = 2 \quad \text{for all} \quad k \in \mathbb{N}.$$

2.5 Intermediate Metrics

In Theorem 2.2.1 and Corollary 2.2.6, we have seen two expressions for metric d_w^0 on X_w^* (see also Theorem 2.3.1 if w is convex). In this section, we define and study infinitely many metrics on the modular space X_w^*.

Theorem 2.5.1. *Let w be a (pseudo)modular on the set X. Given $0 \le \theta \le 1$ and $x, y \in X$, setting*

$$d_w^\theta(x,y) = \inf_{\lambda>0} \Big[(1-\theta)\max\big\{\lambda, w_\lambda(x,y)\big\} + \theta\big(\lambda + w_\lambda(x,y)\big)\Big], \tag{2.5.1}$$

we have: d_w^θ is an extended (pseudo)metric on X, and a (pseudo)metric on the modular space $X_w^* = X_w^*(x^\circ)$ for any $x^\circ \in X$, and the following (sharp) inequalities hold:

$$d_w^0(x,y) \le (1-\theta)d_w^0(x,y) + \theta d_w^1(x,y) \le d_w^\theta(x,y) \le d_w^1(x,y) \le 2d_w^0(x,y). \tag{2.5.2}$$

Proof. Clearly, $0 \le d_w^\theta(x,y) \le \infty$ for all $x, y \in X$ and $0 \le \theta \le 1$.

1. First, we prove our theorem for $\theta = 0$ and $\theta = 1$ simultaneously (for d_w^0, this is the second proof). Given $u, v \in [0,\infty]$, we denote by $u \oplus v$ either $\max\{u,v\}$ or $u + v$ (and $u \oplus v = \infty$ if $u = \infty$ or $v = \infty$). Then $d_w^0(x,y)$ and $d_w^1(x,y)$ are expressed by the formula:

$$d_w^\oplus(x,y) = \inf_{\lambda>0} \lambda \oplus w_\lambda(x,y), \qquad x, y \in X. \tag{2.5.3}$$

1a. If $x, y \in X_w^*$, then $d_w^\oplus(x,y) < \infty$. In fact, since $x \sim y$, there exists $\lambda_0 > 0$ such that $w_{\lambda_0}(x,y) < \infty$, and so, the set $\{\lambda \oplus w_\lambda(x,y) : \lambda > 0\} \setminus \{\infty\}$ is nonempty and bounded from below by 0 (i.e., is contained in $[0,\infty)$).

1b. Given $x \in X$, we have, by (i′), $\lambda \oplus w_\lambda(x,x) = \lambda \oplus 0 = \lambda$ for all $\lambda > 0$, and so, $d_w^\oplus(x,x) = \inf_{\lambda>0} \lambda = 0$. Now, suppose w is a modular. Let $x, y \in X$, and $d_w^\oplus(x,y) = 0$. If we show that $w_\lambda(x,y) = 0$ for all $\lambda > 0$, then axiom (i) will imply $x = y$. On the contrary, assume that $w_{\lambda_0}(x,y) \neq 0$ for some $\lambda_0 > 0$. Given $\lambda > 0$, we have two cases: if $\lambda \geq \lambda_0$, then

$$\lambda \oplus w_\lambda(x,y) \geq \lambda \oplus 0 = \lambda \geq \lambda_0,$$

and if $\lambda < \lambda_0$, then, by the monotonicity (1.2.1) of w, we find

$$\lambda \oplus w_\lambda(x,y) \geq 0 \oplus w_\lambda(x,y) = w_\lambda(x,y) \geq w_{\lambda_0}(x,y).$$

Hence $\lambda \oplus w_\lambda(x,y) \geq \min\{\lambda_0, w_{\lambda_0}(x,y)\} \equiv \lambda_1$ for all $\lambda > 0$. By the definition of $d_w^\oplus$, we get $d_w^\oplus(x,y) \geq \lambda_1 > 0$, which contradicts the assumption.

1c. Axiom (ii) for w implies the symmetry property of $d_w^\oplus$.

1d. Let us establish the triangle inequality $d_w^\oplus(x,y) \leq d_w^\oplus(x,z) + d_w^\oplus(z,y)$ for all $x,y,z \in X$. The inequality is clear if at least one summand on the right is infinite. So, we assume that both of them are finite. By (2.5.3), given $\varepsilon > 0$, there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda \oplus w_\lambda(x,z) \leq d_w^\oplus(x,z) + \varepsilon \quad \text{and} \quad \mu \oplus w_\mu(z,y) \leq d_w^\oplus(z,y) + \varepsilon.$$

Since $\oplus$ is max or $+$, (2.5.3) and axiom (iii) imply

$$\begin{aligned} d_w^\oplus(x,y) &\leq (\lambda+\mu) \oplus w_{\lambda+\mu}(x,y) \leq (\lambda+\mu) \oplus \big(w_\lambda(x,z) + w_\mu(z,y)\big) \qquad (2.5.4) \\ &\leq \big(\lambda \oplus w_\lambda(x,z)\big) + \big(\mu \oplus w_\mu(z,y)\big) \leq d_w^\oplus(x,z) + \varepsilon + d_w^\oplus(z,y) + \varepsilon. \end{aligned}$$

It remains to take into account the arbitrariness of $\varepsilon > 0$.

2. That d_w^θ is well-defined, nondegenerate (when w is a modular), and symmetric can be proved along the same lines as in steps 1a–1c. Let us show that d_w^θ satisfies the triangle inequality. Suppose $d_w^\theta(x,z)$ and $d_w^\theta(z,y)$ are finite. Given $\varepsilon > 0$, by virtue of (2.5.1), there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$(1-\theta)\max\{\lambda, w_\lambda(x,z)\} + \theta\big(\lambda + w_\lambda(x,z)\big) \leq d_w^\theta(x,z) + \varepsilon,$$

$$(1-\theta)\max\{\mu, w_\mu(z,y)\} + \theta\big(\mu + w_\mu(z,y)\big) \leq d_w^\theta(z,y) + \varepsilon.$$

Taking into account (2.5.1), axiom (iii) and the last inequality in (2.5.4), we get:

$$\begin{aligned} d_w^\theta(x,y) &\leq (1-\theta)\max\{\lambda+\mu, w_{\lambda+\mu}(x,y)\} + \theta\big(\lambda+\mu+w_{\lambda+\mu}(x,y)\big) \\ &\leq (1-\theta)\max\{\lambda+\mu, w_\lambda(x,z)+w_\mu(z,y)\} + \theta\big(\lambda+\mu+w_\lambda(x,z)+w_\mu(z,y)\big) \\ &\leq (1-\theta)\Big[\max\{\lambda, w_\lambda(x,z)\} + \max\{\mu, w_\mu(z,y)\}\Big] \\ &\quad + \theta\Big[\lambda + w_\lambda(x,z) + \mu + w_\mu(z,y)\Big] \end{aligned}$$

$$= \Big[(1-\theta)\max\big\{\lambda, w_\lambda(x,z)\big\} + \theta\big(\lambda + w_\lambda(x,z)\big)\Big]$$
$$+ \Big[(1-\theta)\max\big\{\mu, w_\mu(z,y)\big\} + \theta\big(\mu + w_\mu(z,y)\big)\Big]$$
$$\le d_w^\theta(x,z) + \varepsilon + d_w^\theta(z,y) + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, the triangle inequality for d_w^θ follows.

3. The inequalities $\max\{u,v\} \le u + v \le 2\max\{u,v\}$ for $u, v \ge 0$ imply

$$d_w^0(x,y) \le d_w^1(x,y) \le 2d_w^0(x,y) \quad \text{for all} \quad x,y \in X. \tag{2.5.5}$$

This proves also the first and fourth inequalities in (2.5.2). Since, for any $\lambda > 0$,

$$d_w^0(x,y) \le \max\{\lambda, w_\lambda(x,y)\} \quad \text{and} \quad d_w^1(x,y) \le \lambda + w_\lambda(x,y),$$

we find

$$(1-\theta)d_w^0(x,y) + \theta d_w^1(x,y) \le (1-\theta)\max\big\{\lambda, w_\lambda(x,y)\big\} + \theta\big(\lambda + w_\lambda(x,y)\big)$$
$$\le \lambda + w_\lambda(x,y),$$

which establishes the second and third inequalities in (2.5.2). □

The sharpness of inequalities (2.5.2) is elaborated in Examples 2.5.5 and 2.5.6.

Remark 2.5.2. Not only intermediate (pseudo)metrics d_w^θ between d_w^0 and d_w^1 can be introduced as in (2.5.1): given $\alpha, \beta \ge 0$ with $\alpha + \beta \ne 0$, we set

$$d_w^{\alpha,\beta}(x,y) = \inf_{\lambda>0}\Big[\alpha\max\big\{\lambda, w_\lambda(x,y)\big\} + \beta\big(\lambda + w_\lambda(x,y)\big)\Big], \qquad x,y \in X.$$

In this case, we have $d_w^{\alpha,\beta}(x,y) = (\alpha+\beta)d_w^\theta(x,y)$ with $\theta = \beta/(\alpha+\beta)$.

Remark 2.5.3. Different binary operations $\oplus$ on $[0,\infty)$ can be used in formula (2.5.3) to define $d_w^\oplus(x,y)$, but then only the *generalized triangle inequality* holds:

$$d_w^\oplus(x,y) \le C\big(d_w^\oplus(x,z) + d_w^\oplus(z,y)\big) \quad \text{with} \quad C > 1. \tag{2.5.6}$$

This can be seen as follows. Suppose $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and, for some constant $C > 1$,

$$\varphi\Big(\frac{u+v}{C}\Big) \le \varphi(u) + \varphi(v) \le \varphi(u+v) \quad \text{for all} \quad u,v \ge 0. \tag{2.5.7}$$

(Here the right-hand side inequality is the superadditivity property of φ, which is satisfied, e.g., by any convex function φ; see Appendix A.1). Denoting by φ^{-1} the inverse function of φ and setting

$$u \oplus v = \varphi^{-1}\big(\varphi(u) + \varphi(v)\big) \quad \text{for all} \quad u, v \geq 0, \tag{2.5.8}$$

we find, from (2.5.7), that

$$u \oplus v \leq u + v \leq C(u \oplus v). \tag{2.5.9}$$

For instance, if $\varphi(u) = u^p$ with $p > 1$, then $u \oplus v = (u^p + v^p)^{1/p}$, and inequalities (2.5.9) hold with sharp constant $C = 2^{1-(1/p)}$, and if $\varphi(u) = \mathrm{e}^u - 1$, then $u \oplus v = \log(\mathrm{e}^u + \mathrm{e}^v - 1)$, and (2.5.9) hold with sharp constant $C = 2$. Now, in order to obtain (2.5.6), we take into account (2.5.3) and (2.5.9), and find that the right-hand side in (2.5.4) is less than or equal to

$$\begin{aligned}(\lambda + \mu) + \big(w_\lambda(x, z) + w_\mu(z, y)\big) &= \big(\lambda + w_\lambda(x, z)\big) + \big(\mu + w_\mu(z, y)\big)\\ &\leq C\big[\big(\lambda \oplus w_\lambda(x, z)\big) + \big(\mu \oplus w_\mu(z, y)\big)\big]\\ &\leq C\big[d_w^{\oplus}(x, z) + \varepsilon + d_w^{\oplus}(z, y) + \varepsilon\big], \quad \varepsilon > 0.\end{aligned}$$

The generalized triangle inequality (2.5.6) can also be obtained if, instead of $d_w^{\oplus}(x, y)$ from (2.5.3), we consider the quantity

$$d_w^{\oplus}(x, y) = \inf_{\lambda > 0} \big(\max\{\lambda, w_\lambda(x, y)\}\big) \oplus \big(\lambda + w_\lambda(x, y)\big)$$

with the operation $\oplus$ on $[0, \infty)$ of the form (2.5.8).

As in Corollary 2.2.8, the right w_{+0} and left w_{-0} regularizations of w do not produce new metrics of the form (2.5.1) in the following sense.

Proposition 2.5.4. *$d^\theta_{w_{+0}}(x, y) = d^\theta_{w_{-0}}(x, y) = d^\theta_w(x, y)$ for all $0 \leq \theta \leq 1$ and $x, y \in X$.*

Proof. For instance, let us verify this for $\theta = 1$. By virtue of (1.2.4), we have

$$\lambda + w_{\lambda+0}(x, y) \leq \lambda + w_\lambda(x, y) \leq \lambda + w_{\lambda-0}(x, y) \quad \text{for all} \quad \lambda > 0,$$

whence $d^1_{w_{+0}}(x, y) \leq d^1_w(x, y) \leq d^1_{w_{-0}}(x, y)$.

Let us show that $d^1_{w_{+0}}(x, y) \geq d^1_w(x, y)$. Suppose $d^1_{w_{+0}}(x, y) < \infty$, and $u > d^1_{w_{+0}}(x, y)$. Let $u > u_1 > d^1_{w_{+0}}(x, y)$. By (2.5.1) with $\theta = 1$, there exists $\lambda_1 > 0$ such that

$$\lim_{\lambda \to \lambda_1 + 0} \big(\lambda + w_\lambda(x, y)\big) = \lambda_1 + w_{\lambda_1+0}(x, y) \leq u_1 < u.$$

It follows that $\lambda_2 + w_{\lambda_2}(x, y) < u$ for some $\lambda_2 > \lambda_1$, which implies

$$d_w^1(x, y) = \inf_{\lambda>0} \big(\lambda + w_\lambda(x, y)\big) \le \lambda_2 + w_{\lambda_2}(x, y) < u,$$

and it remains to pass to the limit as $u \to d_{w+0}^1(x, y)$.

Now, we show that $d_w^1(x, y) \ge d_{w-0}^1(x, y)$. Let $d_w^1(x, y) < \infty$, and $u > d_w^1(x, y)$. Choose u_1 such that $u > u_1 > d_w^1(x, y)$. By (2.5.1) with $\theta = 1$, there exists $\mu_1 > 0$ such that $\mu_1 + w_{\mu_1}(x, y) \le u_1 < u$. It follows from (1.2.4) that

$$w_{\lambda_1-0}(x, y) \le w_{\mu_1}(x, y) < u - \mu_1 \quad \text{for all} \quad \lambda_1 > \mu_1,$$

and so,

$$d_{w-0}^1(x, y) \le \lambda_1 + w_{\lambda_1-0}(x, y) < \lambda_1 + u - \mu_1.$$

Passing to the limit as $\lambda_1 \to \mu_1 + 0$, we get $d_{w-0}^1(x, y) \le u$, and it remains to take into account the arbitrariness of u as above. □

Example 2.5.5 (metric d_w^1).

1. Let $w_\lambda(x, y) = \lambda^{-p} d(x, y)$ be of the form (1.3.1) with $p > 0$. By Example 2.2.2(1), $d_w^0(x, y) = (d(x, y))^{1/(p+1)}$.

   Let us calculate $d_w^1(x, y) = \inf_{\lambda>0} f(\lambda)$, where $f(\lambda) = \lambda + \lambda^{-p} d(x, y)$ (and $x \ne y$). The derivative $f'(\lambda) = 1 - p\lambda^{-p-1} d(x, y)$ vanishes at $\lambda_0 = (pd(x, y))^{1/(p+1)}$, $f'(\lambda) < 0$ if $0 < \lambda < \lambda_0$, and $f'(\lambda) > 0$ if $\lambda > \lambda_0$, and so, f attains the global minimum on $(0, \infty)$ at the point λ_0, which is equal to

   $$d_w^1(x, y) = f(\lambda_0) = \gamma(p) \cdot (d(x, y))^{1/(p+1)} \quad \text{for all} \quad x, y \in X,$$

   where

   $$\gamma(p) = (p + 1)p^{-p/(p+1)}, \qquad p > 0.$$

   Note that $1 < \gamma(p) \le 2$, $\gamma(p) = 2$ if and only if $p = 1$, and $\gamma(1/p) = \gamma(p)$. The inequalities for $\gamma(p)$ can be established directly by taking the logarithm and investigating the resulting function for extrema, or they follow from (2.5.5). In particular, if $p = 1$, the expressions for d_w^0 and d_w^1 are of the form:

   $$d_w^0(x, y) = \sqrt{d(x, y)} \quad \text{and} \quad d_w^1(x, y) = 2\sqrt{d(x, y)}, \quad x, y \in X.$$

2. Formulas for d_w^0 and d_w^1 above are valid in a somewhat more general case when a (pseudo)modular w on X is *p-homogeneous* with $p > 0$ in the sense that

   $$w_\lambda(x, y) = \lambda^{-p} w_1(x, y) \quad \text{for all } \lambda > 0 \text{ and } x, y \in X.$$

In this case, we have

$$d_w^0(x,y) = (w_1(x,y))^{1/(p+1)} \quad \text{and} \quad d_w^1(x,y) = \gamma(p)\cdot(w_1(x,y))^{1/(p+1)}. \tag{2.5.10}$$

One more example of a p-homogeneous modular w on a metric space (X,d) is given by $w_\lambda(x,y) = (d(x,y)/\lambda)^p = \lambda^{-p}w_1(x,y)$ (see Example 2.3.5(1)).

3. Given a metric space (X,d) and a convex function $\varphi : [0,\infty) \to [0,\infty)$ vanishing at zero only, we set (cf. (1.3.5))

$$w_\lambda(x,y) = \lambda\varphi\Big(\frac{d(x,y)}{\lambda}\Big), \qquad \lambda > 0, \quad x,y \in X.$$

Then w is a strict modular on X (cf. (1.3.8)), and since φ is increasing, continuous, and admits the continuous inverse φ^{-1}, we find

$$d_w^0(x,y) = \inf\{\lambda > 0 : \varphi(d(x,y)/\lambda) \le 1\} = d(x,y)/\varphi^{-1}(1).$$

In particular, if $\varphi(u) = u^p$ with $p > 1$, we have $d_w^0(x,y) = d(x,y)$, and taking into account that

$$w_\lambda(x,y) = \lambda\Big(\frac{d(x,y)}{\lambda}\Big)^p = \lambda^{-(p-1)}(d(x,y))^p = \lambda^{-(p-1)}w_1(x,y),$$

we conclude from (2.5.10) (replacing p there by $p-1$) that

$$d_w^1(x,y) = \gamma(p-1)\cdot(w_1(x,y))^{1/p} = p(p-1)^{(1-p)/p}\cdot d(x,y).$$

4. Setting $w_\lambda(x,y) = \mathrm{e}^{-\lambda}d(x,y)$ and following the same reasoning as in Example 2.5.5(1), we get

$$d_w^1(x,y) = \begin{cases} d(x,y) & \text{if } d(x,y) \le 1, \\ 1 + \log d(x,y) & \text{if } d(x,y) > 1, \end{cases} \qquad x,y \in X.$$

Example 2.5.6 (metric d_w^θ). In order to be able to calculate the value $d_w^\theta(x,y)$ from (2.5.1) explicitly for all $0 \le \theta \le 1$, here once again we consider the modular $w_\lambda(x,y) = \lambda^{-p}d(x,y)$ of the form (1.3.1) with $p > 0$. Since the cases $\theta = 0$ and $\theta = 1$ were considered in Example 2.5.5(1), we are left with the case when $0 < \theta < 1$ (in calculations below, we assume that $x \ne y$).

To begin with, we note that $d_w^\theta(x,y) = \inf_{\lambda>0} f(\theta,\lambda)$, where the function $f(\theta,\lambda)$ under the infimum sign in (2.5.1) is expressed as

$$f(\theta,\lambda) = \begin{cases} f_1(\lambda) \equiv w_\lambda(x,y) + \theta\lambda & \text{if } \lambda \le w_\lambda(x,y), \\ f_2(\lambda) \equiv \lambda + \theta w_\lambda(x,y) & \text{if } \lambda > w_\lambda(x,y), \end{cases}$$

with $f_1(\lambda) = \lambda^{-p}d(x,y) + \theta\lambda$ and $f_2(\lambda) = \lambda + \theta\lambda^{-p}d(x,y)$, and the inequality $\lambda \le w_\lambda(x,y) = \lambda^{-p}d(x,y)$ is equivalent to $\lambda \le \lambda_0 \equiv d_w^0(x,y) = (d(x,y))^{1/(p+1)}$. Hence

$$d_w^\theta(x,y) = \min\left\{\inf_{0<\lambda\le\lambda_0} f_1(\lambda),\ \inf_{\lambda>\lambda_0} f_2(\lambda)\right\}, \tag{2.5.11}$$

where we note that $f_1(\lambda_0) = f_2(\lambda_0) = \lambda_0(1+\theta)$.

The derivative $f_1'(\lambda) = -\lambda^{-p-1}pd(x,y) + \theta$ is equal to zero only at the point $\lambda_1 = \lambda_0(p/\theta)^{1/(p+1)}$, $f_1'(\lambda) < 0$ if $0 < \lambda < \lambda_1$, and $f_1'(\lambda) > 0$ if $\lambda > \lambda_1$, and so, the global minimum of f_1 on $(0,\infty)$ is attained at λ_1 and is equal to

$$f_1(\lambda_1) = \lambda_0\gamma(p)\theta^{p/(p+1)}.$$

Similarly, the derivative $f_2'(\lambda) = 1 - \lambda^{-p-1}\theta pd(x,y)$ is equal to zero at the point $\lambda_2 = \lambda_0(\theta p)^{1/(p+1)}$, $f_2'(\lambda) < 0$ for $0 < \lambda < \lambda_2$, and $f_2'(\lambda) > 0$ for $\lambda > \lambda_2$, and so, f_2 attains the global minimum on $(0,\infty)$ at λ_2, where it has the value

$$f_2(\lambda_2) = \lambda_0\gamma(p)\theta^{1/(p+1)}.$$

Given $p > 0$ and $0 \le \theta \le 1$, we have four cases: (I) $p \ge 1$ and $\theta \le 1/p$; (II) $p > 1$ and $1/p < \theta$; (III) $p < 1$ and $\theta \le p$; and (IV) $p < 1$ and $p < \theta$.

Cases (I), (III). We have $p \ge 1 \ge \theta$ in case (I), and $p \ge \theta$ in case (III), and so, $\lambda_0 \le \lambda_1$. Since f_1 decreases on $(0,\lambda_1]$, the value $\inf_{\lambda\le\lambda_0} f_1(\lambda)$ is equal to $f_1(\lambda_0) = \lambda_0(1+\theta)$. Also, we have $\theta p \le 1$ in case (I), and $\theta p < 1$ in case (III), and so, $\lambda_2 \le \lambda_0$. Since f_2 increases on $[\lambda_2,\infty)$, the value $\inf_{\lambda>\lambda_0} f_2(\lambda)$ is equal to $f_2(\lambda_0) = \lambda_0(1+\theta)$. By virtue of (2.5.11), $d_w^\theta(x,y) = \lambda_0(1+\theta)$.

Case (II). As in case (I), since $p > 1 \ge \theta$, $\inf_{\lambda\le\lambda_0} f_1(\lambda) = \lambda_0(1+\theta)$. Furthermore, $\theta p > 1$ implies $\lambda_0 < \lambda_2$, where λ_2 is the point of minimum of f_2 on $[\lambda_0,\infty)$, and so,

$$\inf_{\lambda>\lambda_0} f_2(\lambda) = f_2(\lambda_2) < f_2(\lambda_0) = \lambda_0(1+\theta) = \inf_{\lambda\le\lambda_0} f_1(\lambda).$$

It follows from (2.5.11) that $d_w^\theta(x,y) = f_2(\lambda_2) = \lambda_0\gamma(p)\theta^{1/(p+1)}$.

Case (IV). Inequality $p < \theta$ implies $\lambda_1 < \lambda_0$, and since λ_1 is the point of minimum of f_1 on $(0,\lambda_0]$, we find

$$\inf_{\lambda\le\lambda_0} f_1(\lambda) = f_1(\lambda_1) < f_1(\lambda_0) = \lambda_0(1+\theta).$$

As in case (III), since $\theta p < 1$, $\inf_{\lambda>\lambda_0} f_2(\lambda) = \lambda_0(1+\theta)$. By (2.5.11), we conclude that $d_w^\theta(x,y) = f_1(\lambda_1) = \lambda_0\gamma(p)\theta^{p/(p+1)}$.

In this way, we have shown that

$$d_w^\theta(x,y) = (d(x,y))^{1/(p+1)} \cdot \begin{cases} 1+\theta & \text{if } 0 \le \theta \le 1/p \le 1 \text{ or} \\ & \quad 0 \le \theta \le p < 1, \\ \gamma(p)\theta^{1/(p+1)} & \text{if } 0 < 1/p < \theta \le 1, \\ \gamma(p)\theta^{p/(p+1)} & \text{if } 0 < p < \theta \le 1. \end{cases} \tag{2.5.12}$$

A few comments on this formula are in order. If $\theta = 0$ or $\theta = 1$, then it gives back the values $d_w^0(x,y)$ and $d_w^1(x,y)$ from Example 2.5.5(1). If $p > 1$ and $\theta = 1/p$ in the third line of (2.5.12), then $\gamma(p)\theta^{1/(p+1)} = 1+\theta$ (as in the first line). Similarly, if $p < 1$ and $\theta = p$ in the fourth line of (2.5.12), then $\gamma(p)\theta^{p/(p+1)} = 1+\theta$.

Note that, for any $p > 0$ and $0 \le \theta \le 1$, we have (cf. (2.5.2))

$$(1-\theta)d_w^0(x,y) + \theta d_w^1(x,y) = (1-\theta+\theta\gamma(p)) \cdot (d(x,y))^{1/(p+1)}.$$

For $p \ne 1$, we have $1 < \gamma(p) < 2$, so if (a) $p > 1$ and $0 < \theta < 1$, or (b) $p < 1$ and $0 < \theta \le p$, then $1-\theta+\theta\gamma(p) < 1+\theta$, and so,

$$(1-\theta)d_w^0(x,y) + \theta d_w^1(x,y) < d_w^\theta(x,y), \qquad x \ne y.$$

Now, if $p = 1$, then $\gamma(p) = 2$ and $1-\theta+\theta\gamma(p) = 1+\theta$, which imply

$$d_w^\theta(x,y) = (1+\theta)\sqrt{d(x,y)} = (1-\theta)d_w^0(x,y) + \theta d_w^1(x,y) \quad \text{for all } 0 \le \theta \le 1.$$

For a convex (pseudo)modular w on X, $\hat{w}_\lambda(x,y) = \lambda w_\lambda(x,y)$ is a (pseudo)modular on X, so setting $d_w^{\theta*} = d_{\hat{w}}^\theta$ and applying Theorem 2.5.1, we get

Theorem 2.5.7. *If w is a convex (pseudo)modular on X and $0 \le \theta \le 1$, then*

$$d_w^{\theta*}(x,y) = \inf_{\lambda>0}\Big[(1-\theta)\max\big\{\lambda, \lambda w_\lambda(x,y)\big\} + \theta\big(\lambda + \lambda w_\lambda(x,y)\big)\Big], \qquad x,y \in X,$$

is an extended (pseudo)metric on X and a (pseudo)metric on X_w^*, and

$$d_w^*(x,y) \le (1-\theta)d_w^*(x,y) + \theta d_w^{1*}(x,y) \le d_w^{\theta*}(x,y) \le d_w^{1*}(x,y) \le 2d_w^*(x,y),$$

where (see (2.3.3)) $d_w^*(x,y) = d_w^{0*}(x,y)$.

Remark 2.5.8. Given $0 \le \theta \le 1$, $d_w^\theta(x,y) < 1$ implies $d_w^{\theta*}(x,y) \le d_w^\theta(x,y)$. In fact, for any r such that $d_w^\theta(x,y) < r < 1$ there exists $\lambda = \lambda(r) > 0$ such that

$$(1-\theta)\max\{\lambda, w_\lambda(x,y)\} + \theta(\lambda + w_\lambda(x,y)) \le r < 1.$$

It follows that $\lambda = (1-\theta)\lambda + \theta\lambda < 1$,

$$\max\{\lambda, \lambda w_\lambda(x,y)\} \le \max\{\lambda, w_\lambda(x,y)\} \quad \text{and} \quad \lambda + \lambda w_\lambda(x,y) \le \lambda + w_\lambda(x,y),$$

and so,

$$d_w^{\theta *}(x,y) \le (1-\theta)\max\{\lambda, \lambda w_\lambda(x,y)\} + \theta(\lambda + \lambda w_\lambda(x,y)) \le r.$$

It remains to pass to the limit as $r \to d_w^\theta(x,y)$.

Example 2.5.9. Let $p \ge 1$ and $w_\lambda(x,y) = (d(x,y)/\lambda)^p$ be the p-homogeneous modular from Example 2.3.5(1). Then, by Example 2.5.5(1), (2),

$$d_w^1(x,y) = \gamma(p)\cdot(d(x,y))^{p/(p+1)} \quad \text{and} \quad d_w^{1*}(x,y) = \begin{cases} d(x,y) & \text{if } p=1, \\ \gamma(p-1)d(x,y) & \text{if } p>1. \end{cases}$$

2.6 Bibliographical Notes and Comments

Sections 2.1 and 2.2. Modular spaces X_w^* and X_w^0 were introduced in Chistyakov [22] and studied in [24, 25, 28]. The space X_w^0 is a counterpart of the classical modular space X_ρ defined in Musielak and Orlicz [77]; see Remark 2.2.3(1), in which the main results of [77] are briefly described. As condition (ρ.4) from Sect. 1.3.3 is crucial for defining the F-norm $|x|_\rho$ on X_ρ, axiom (iii) in Definition 1.2.1 is a proper tool to define the (pseudo)metric $d_w^0(x,y)$ on the space X_w^*, which is larger than X_w^0.

The properties of $d_w^0(x,y)$ are based on the properties of quantity g^0 from (2.2.1) (recall that $d_w^0(x,y) = (w^{x,y})^0$). This allows us to obtain an alternative expression for the (pseudo)metric $d_w^0(x,y)$ in Corollary 2.2.6.

Modular space X_w^{fin} is (natural and) new. Its role will be more clear below (see Theorem 3.3.8): some 'duality' holds between the modular spaces.

Corollary 2.2.8 was first established in Chistyakov [28].

Lemma 2.2.9 and Theorem 2.2.11 are sharp refinements of Theorem 2.10 from Chistyakov [24]. Counterparts of Theorem 2.2.11(d), (e) for classical modulars are presented in Maligranda [68, Theorem 1.4].

Theorem 2.2.13 is new.

Section 2.3. In the convex case, the results of the classical modular theory are presented in Remark 2.3.4(1). They were established by Nakano [81, Sect. 81], Musielak and Orlicz [78], and Orlicz [90] (for s-convex modulars with $0 < s \le 1$). For Orlicz modulars (i.e., integral modulars of the form $\rho(x) = \int_\Omega \varphi(|x(t)|)d\mu$), the norm $\|x\|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \le 1\}$ on X_ρ^* was considered by Morse and Transue [73] and Luxemburg [66]. Note that the norm $\|x\|_\rho$ is the Minkowski functional $p_A(x) = \inf\{\varepsilon > 0 : x/\varepsilon \in A\}$ of the convex set $A = \{x : \rho(x) \le 1\}$.

Furthermore, Musielak and Orlicz [78] proved inequalities of the form (2.3.6) and (2.3.7) for classical convex modulars ρ, and Orlicz [90] established the representation $\|x\|_\rho = \inf_{t>0} \sup\{t^{-1}, \rho(tx)t^{-1}\}$ (cf. the second equality in (2.3.3)).

The (pseudo)metric $d_w^*(x, y)$ on X_w^* was introduced in Chistyakov [22]. It is seen from the expressions for $d_w^*(x, y)$ and $\|x\|_\rho$ that $d_w^*(x, y)$ is a counterpart of the norm $\|x\|_\rho$. Interestingly, the idea of definition of $d_w^*(x, y) = (\hat{w}^{x,y})^0$ has no relation with the idea of Minkowski's functional of a convex set, and relies on g^0 from (2.2.1), however, by virtue of the 'embedding' (1.3.3), for convex modulars ρ on linear spaces, we get $\|x\|_\rho = d_w^*(x, 0)$ (see Remark 2.3.4(1)).

Section 2.4. The first modular stands for illustrative purposes—its idea is to generalize, in a straightforward way, the well-known space ℓ_p of p-summable sequences. The second modular (2.4.1), mentioned in [24, Example 3.2], is more interesting and studied in detail (see also Example 4.2.7(2)). Note that modular (2.4.1) can be obtained, via (1.3.3), from the classical modular $\rho(x) = \sup_{n\in\mathbb{N}} \sqrt[n]{|x_n|}$ for $x = \{x_n\} \in \mathbb{R}^{\mathbb{N}}$, see Rolewicz [95, Example 1.2.3].

Section 2.5. The whole material of Sect. 2.5 is new. Connections with the classical modular theory are as follows. Metric $d_w^\theta(x, y)$ from (2.5.1) for $\theta = 1$ is a counterpart of the F-norm $|x|_\rho^1 = \inf_{t>0}(1 + t\rho(tx))/t$, $x \in X_\rho$, from Koshi and Shimogaki [53], where inequality $|x|_\rho \le |x|_\rho^1 \le 2|x|_\rho$ of the form (2.5.5) was also established; here $|x|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \le \varepsilon\}$ is the Musielak-Orlicz F-norm.

The idea to define the operation $\oplus$ in (2.5.8) is taken from Musielak [74] and Musielak and Peetre [79] (see also Musielak [75, Sect. 3]).

The classical variant of Example 2.5.5 was elaborated in Maligranda [68, p. 4].

Metric $d_w^{\theta*}(x, y)$ from Theorem 2.5.7 for $\theta = 1$ is a counterpart of the Amemiya norm $\|x\|_\rho^1 = \inf_{t>0}(1 + \rho(tx))/t$, $x \in X_\rho^* = X_\rho$ (see Nakano [81, Sect. 81], Hudzik and Maligranda [48], Maligranda [68, p. 6], Musielak [75, Theorem 1.10]).

For more information about the modular theory on linear spaces and Orlicz spaces we refer to Adams [1], Kozlowski [55], Krasnosel'skiĭ and Rutickiĭ [56], Lindenstrauss and Tzafriri [65], Luxemburg [66], Maligranda [68], Musielak [75], Nakano [80, 81], Orlicz [89], Rao and Ren [92, 93], Rolewicz [95].

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