Preface

The theory of *metric spaces* was created by Fréchet [39] and Hausdorff [43] a century ago. In its basis is the notion of *distance* between any two points of a set. Usually (but not necessarily), the algebraic structure of the set does not play any role in the metric space analysis. If the set under consideration has a rich algebraic structure, e.g., it is a linear space, metrics, or distance functions, on the set can be defined by means of *norms*. In particular, the theory of *Banach spaces* [5] (i.e., complete normed linear spaces) is of fundamental importance in modern functional analysis. Despite the well-known Kuratowski's theorem [58] on the embedding of a metric space X into a Banach space (of bounded functions on X), the language of metric spaces is indispensable in expressing *nonlinear* properties of various phenomena and objects in metric spaces. Many results of the Banach space theory are extended to the *metric linear space* theory (Rolewicz [95]).

At the same time, a century ago, Lebesgue's theory [59] of measure and integral was developed, which is the theory of Banach space L_1 of summable functions equipped with the L_1-norm. Lebesgue's theory was extended by Riesz in his paper on L_p-spaces [94] ($1 < p < \infty$). In [87, 88], Orlicz defined and studied his famous normed *Orlicz spaces* L_φ of φ-summable functions, where φ is a convex (Orlicz) function on the reals $\mathbb{R}$. For nonconvex functions φ, F-normed L_φ-spaces were introduced by Masur and Orlicz [71] and Musielak and Orlicz [77] in the context of modular spaces. From different perspectives, the theory of Orlicz spaces is presented in many monographs, e.g., Adams [1], Krasnosel'skiĭ and Rutickiĭ [56], Lindenstrauss and Tzafriri [65], Maligranda [68], Musielak [75], and Rao and Ren [92, 93].

Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions. A general theory of *modular linear spaces* was founded by Nakano in his two monographs [80, 81], where he developed a spectral theory in semi-ordered linear spaces (vector lattices) and established the integral representation for projections acting in his modular space; Nakano's modulars on real linear spaces are convex functionals. Nonconvex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz [77] (we refer to Musielak [75] for a more comprehensive account). Orlicz spaces and modular linear spaces have already become classical tools in modern nonlinear functional analysis.

In spite of the significant generality of the modular spaces theory over linear spaces or spaces equipped with an additional algebraic structure, the notions of *modular* (which in particular extends the notion of norm) and the corresponding *modular linear space* are too restrictive. This is concerned, e.g., with the problems from the multivalued analysis such as the definition of metric functional spaces, characterization of set-valued superposition operators, (pointwise) selection principles, and existence of regular selections of multifunctions (Chistyakov [11–21]).

The purpose of this contribution is to develop a general theory of modulars on *arbitrary sets* and present a comprehensive background on metric and topological properties of the corresponding modular spaces. In our approach, a *modular* w on a set X is a parametrized family $w = \{w_\lambda\}_{\lambda>0}$ of functions of two variables of the form $w_\lambda : X \times X \to [0, \infty]$ satisfying certain natural axioms in both nonconvex and convex situations. On the one hand, if the family w is independent of parameter λ, we get the notion of (extended) metric w on X. On the other hand, if X is a linear space and $\rho : X \to [0, \infty]$ is a classical modular on X (in Nakano's or Musielak-Orlicz's sense), then the family w corresponding to $w_\lambda(x, y) = \rho((x-y)/\lambda)$ $(x, y \in X)$ defines a modular on X in our sense. Thus, our theory of modular spaces is consistent with the theories of metric spaces and modular linear spaces.

Depending on the context, modulars w allow different interpretations. For instance, the quantity $w_\lambda(x, y)$ may be thought of as a (absolute value of non-linear) *mean velocity* between points x and y in time $\lambda > 0$. It is known that in classical Newtonian mechanics, the deterministic principle says that the initial state of a mechanical system (the collections of its positions and velocities at a certain moment of time) determines uniquely the whole movement of the system. Accordingly, a modular w on X generates a distance function between any two points of X (actually, several distance functions can be defined on X). A subset of X, where the distance function assumes finite values (and so becomes a metric), is called a *modular space*. A natural (canonical) modular on a metric space (X, d) is given by $w_\lambda(x, y) = d(x, y)/\lambda$, which is the "real" mean velocity, and the induced distance function is $d(x, y)$ on the modular space X. In this way, we restore the original metric space by means of the canonical modular. More modulars can be considered on the metric space X, e.g., $w_\lambda(x, y) = d(x, y)/\lambda^p$ $(p \geq 0)$, or $w_\lambda(x, y) = \exp(d(x, y)/\lambda) - 1$, and they generate different distance functions on X.

Naturally, a modular space endowed with the generated metric is a *metric modular space* (hence the title of the book), and so, the standard metric space theory and its terminology apply to it. However, modulars are far from being metrics in general: they do not satisfy the usual triangle inequality. Having the ability to efficiently generate metrics, modulars might have been called *premetrics* or *prametrics*. However, the last two terms are already in use in topology (see Deza and Deza [36, p. 4]). So having in mind the connections with the modular linear theory, we adopt a more adequate term *metric modular* (for w as above) for its full name, or simply *modular* as its abbreviation (if no ambiguity with classical modulars arises).

Being dependent on the parameter λ, modulars give rise to a nonmetric, more weak convergence on a modular space, called the *modular convergence*. In the modular linear space theory, this notion was introduced by Musielak and Orlicz [77]. Correspondingly, the modular space is a topological space equipped with the *modular topology*, which, as a rule, is nonmetrizable. These notions are more subtle, and we postpone their definition and discussion until Chap. 4.

We apply the approach to the theory of modular spaces on arbitrary sets based on the author's papers [14, 16, 18–29], and [31]. We have added many new results and examples and made the exposition as self-contained as possible. The prerequisites for the reading of this book are some background knowledge of real and functional analysis, linear algebra, and rudiments of general topology. Thus the text, or much of it, is quite accessible to the university undergraduate students.

The plan of the exposition is as follows. The material is tacitly divided into two parts: Theory (Chaps. 1, 2, 3 and 4) and Applications (Chaps. 5–6). In Chap. 1, we define the notion of (metric) modulars, give their classification, obtain elementary properties, and present many examples useful in the sequel. In Chap. 2, we treat the metrizability of modular spaces: in contrast to the modular theory on linear spaces, where only few norms and F-norms are known, we define infinitely many metrics on our modular space and study their properties and relations between them. Further extensions of the notion of modular are given in Chap. 3, where we also study tools (transforms), by means of which new modulars can be produced. The most important are the right/left inverse modulars exhibiting the "duality" between modular spaces (Theorem 3.3.8). Chapter 4 is devoted to the classical topological aspects of the modular spaces, connected with the metric and modular convergences. In Chap. 5, a special $\mathbb{N}$-valued pseudomodular is introduced, whose induced modular spaces are the sets of all bounded and regulated mappings on an interval. This pseudomodular is crucial for obtaining a powerful pointwise selection principle, from which previously known pointwise selection principles follow, including Helly's theorem. The final Chap. 6 addresses some important classes of mappings of bounded generalized variation, which we interpret as the modular spaces for specifically constructed modulars. The results include the description of superposition operators acting in modular spaces, the existence of regular selections of set-valued mappings, the new interpretation of Lipschitzian and absolutely continuous mappings, and the existence of solutions to the Carathéodory-type ordinary differential equations in Banach spaces with the right-hand side from the Orlicz space.

Each chapter ends with Bibliographical Notes and Comments containing appropriate references, comments, and supplementary material.

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